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NUMERICAL SOLUTION OF NONLINEAR SINGULAR BOUNDARY-VALUE PROBLEMS

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ABSTRACT

In the present paper we consider boundary-value problems for second order ODE's of the form

$$y''(x) = -g(x)\, x^p\, y(x)^q\,,\ \ 0 < x < 1,$$

where $p > -2$, $q < 0$ and g is positive and continuous on $[0, 1]$. Equations of this form are known as generalized Emden-Fowler equations and they arise in many applications, for example in fluid mechanics, chemical physics and physics of plasma. We search for positive solutions of this equation which satisfy the following boundary conditions

$$y(0) = 0, y(1) = 0.$$

We verify that the finite-difference method gives low convergence order when applied to the considered problem, due to the singularities of the solution. Therefore, we introduce a variable subtitution, which takes into account the behavior of the solution at both endpoints. Some numerical examples are analised and the results are discussed.

1. INTRODUCTION

In this paper we consider ordinary differential equations of the form

$$y''(x) = r(x)y(x)^q \tag{1.1}$$

where r is a known function, continuous and negative on $]0,1[$. Equations of this form are called generalized Emden-Fowler equations and they include many particular cases with physical interest (see [8] for a historical development) . Here we are interested in the case $q<0$ (negative exponent) and we assume that

$$r(x)=-g(x)x^p, \tag{1.2}$$

where p is a real number, $g\in C^1([0,1])$ and there exist positive constants m and M, such that

$$m=\min_{x\in[0,1]} g(x) \tag{1.3}$$

and

$$M=\max_{x\in[0,1]} g(x). \tag{1.4}$$

In particular, we shall consider the case $g(x)=C$. We shall look for positive solutions on $]0,1[$ which satisfy the boundary conditions:

$$y(0)=0, y(1)=0. \tag{1.5}$$

In [2] and [3], Lima and Carpentier have proposed numerical methods for the approximation of a similar problem, with boundary conditions $y'(0)=y(1)=0$ and $r(x)=\frac{1}{q}x$. The purposes of the present paper are to extend the methods, considered there,to the new boundary conditions and to the case where r has a different form. In Section 2, we shall introduce lower and upper solutions for these problems, based on the results of the cited papers. In [1],Habets and Zanolin have obtained existence and uniqueness results for this kind of boundary-value problems using the method of lower and upper solutions. We shall apply their results to our case.

In Section 3, we shall present iterative schemes for the approximation of problems of the classes (1.1)-(1.5), based on the knowledge of lower and upper solutions. By means of these iterative schemes, the nonlinear problems are reduced to sequences of linear ones. In Section 4, we use finite-difference schemes to discretize the resulting linear boundary-value problems and introduce a variable substitution in order to improve the accuracy of the method. Some numerical results are presented and discussed in Section 5. Finally, in the same section we present our conclusions about the effiicency of the proposed method.

2. UPPER AND LOWER SOLUTIONS

Let us consider the equation (1.1) with $r(x)=-Cx^p$ and boundary conditions

$$y(0)=0, \;\; y(1)=0. \tag{2.1}$$

The solution of this problem has singularities at both endpoints. Problems of this kind have been analysed in [4] and [5]; a more general problem was considered in [1]. Using the definition given in this last work, a function $\alpha \in C^2(]0,1[) \cap C([0,1])$ is said to be a lower solution of (1.1)-(2.1) if α satisfies

$$-\alpha"(x) + r(x)\alpha(x)^q \leq 0,\ 0 < x < 1; \tag{2.2}$$

and

$$\alpha(0) = 0;\ \alpha(1) = 0. \tag{2.3}$$

The definition of upper solution is obtained if we substitute $\leq$ by $\geq$ in (2.2).

We shall consider only the case when $q < -1$ and $-2 < p < -1 - q$, when the first derivative of the solution is unbounded at the singularities [7]. Let us look for upper and lower solutions in the form

$$\alpha(x) = Bx^{\gamma_1}(1-x)^{\gamma_2}, \tag{2.4}$$

where γ_1, γ_2 and B are adjustable parameters. The value of γ_1 may be obtained from the condition that α is asymptotically equivalent to the solution as $x \to 0+$, which gives us

$$\gamma_1 = \frac{2+p}{1-q}. \tag{2.5}$$

Analogously, the value of γ_2 may be obtained from the condition that α is asymptotically equivalent to the solution as $x \to 1-$, which gives us

$$\gamma_2 = \frac{2}{1-q}. \tag{2.6}$$

In the same way as in the previous case, we may obtain the bounds B_1 and B_2, such that if $B \leq B_1$, then α is a lower solution, and if $B \geq B_2$, α is an upper solution (see Table 2.1). If $\gamma_1 + \gamma_2 = 1$, we obtain

$$B_1 = B_2 = \left(\frac{C}{\gamma_1\gamma_2}\right)^{\frac{1}{1-q}}$$

and therefore, with $B = B_2 = B_1$, α is the exact solution of (1.1)–(2.1).

lower solutions	upper solutions	
$B_1 = a\,C^{\frac{1}{1-q}}$	$B_2 = b\,C^{\frac{1}{1-q}}$	if $\gamma_1 + \gamma_2 \leq 1$
$B_1 = b\,C^{\frac{1}{1-q}}$	$B_2 = a\,C^{\frac{1}{1-q}}$	if $\gamma_1 + \gamma_2 > 1$

Tab. 2.1 - bounds for the lower and upper solutions of problem (1.1)–(2.1). Here $a = \left(\frac{1}{\gamma_M(1-\gamma_M)}\right)^{\frac{1}{1-q}}$, $b = \left(\frac{\gamma_1+\gamma_2}{\gamma_1\gamma_2}\right)^{\frac{1}{1-q}}$, and $\gamma_M = \max(\gamma_1, \gamma_2)$.

Let us now consider the more general equation (1.1) with the boundary conditions (2.1). In this case, if we look for lower and upper and lower solutions in the form (2.4), we shall come to similar conclusions. Then the formulae of Table 2.1 remain valid if we replace C by m, in the expression for B_1, and by M, in the expression for B_2.

The existence and uniqueness of solution of the problem (1.1)–(2.1) using the results of Habets and Zanolin [1]. Actually, if we consider an upper solution α and an upper solution β of the form (2.4), then it may be verified that the conditions of theorem 1 of [1] are satisfied and therefore the problem has at last one solution y, such that $\alpha(x) \leq y(x) \leq \beta(x)$, $\forall x \in [0,1]$. Moreover, for $q < 0$ the right-hand side of (1.1) is a decreasing function with respect to y, for any $x \in]0,1[$ and therefore, we may assure that the considered solution is unique, by theorem 2 of [1].

Although the case where the derivative of the solution is continuous at one ore both endpoints is out of the scope of the present work, it should be noted that the method of lower and upper solutions may be also applied to that case. This was done in [2], where lower and upper solutions were obtained for generalized Emden-Fowler equations of different forms.

3. MONOTONE ITERATIVE METHODS

In [2], we have derived two monotone iterative schemes for the solution of (1.1), with boundary conditions $y'(0) = y(1) = 0$, based on the Picard and Newton methods. The generalization of these methods to the boundary conditions (1.5) and to the case where $r(x) = -g(x)x^p$ is straightforward. In the case of the Picard method, starting with an initial function y_0, we consider the sequence of linear boundary-value problems

$$y''_{\nu+1}(x) + qg(x)x^p\alpha(x)^{q-1}y_{\nu+1}(x) = \left(q\alpha(x)^{q-1} - y_\nu(x)^{q-1}\right) g(x)x^p y_\nu(x), \quad 0 < x < 1, \tag{3.1}$$

where α is a function such that $0 < \alpha(x) < y(x)$. The Picard iterates y_ν must satisfy the boundary conditions

$$y_\nu(0) = 0, y_\nu(1) = 0, \ \nu = 1, 2, \ldots. \tag{3.2}$$

When the Newton method is applied, we must solve linear boundary-value problems of the form

$$y''_{\nu+1}(x) + qg(x)x^p y_\nu(x)^{q-1}y_{\nu+1}(x) = (q-1)g(x)x^p y_\nu(x)^q, \ \ 0 < x < 1, \tag{3.3}$$

subject to the boundary conditions (3.2). The theory of monotone iterative methods, developed by Mooney [6], establishes sufficient conditions for the convergence of the Picard and Newton methods in the case of convex boundary value problems like these ones. Based on that theory, and taking account of the singularity of this particular case, it may be shown that the iterative schemes (3.1) and (3.3) converge to the true solution of (1.1)-(2.1) , if the initial approximation y_0 satisfies certain conditions. Let us rsume these results:

1. If y_0 is a lower solution of (1.1)-(2.1) then
   - $y_0(x) \leq y(x),\ \forall x \in [0,1]$
   - the Newton and the Picard iterates converge upwards.
2. If y_0 is an upper solution, then
   - $y_0(x) \geq y(x),\ \forall x \in [0,1]$;
   - the Picard iterates converge downwards.

4. DISCRETIZATION METHODS AND VARIABLE SUBSTITUTION

In the previous section, the nonlinear problem was reduced to a sequence of linear boundary-value problems. These problems may be easily discretized by means of finite-difference schemes, using central differences to approximate the derivatives. Let us introduce an uniform grid with stepsize h on $[0,1]$ and let us denote the gridpoints by $x_i = ih,\ i = 1,\ldots,N$, where $h = \frac{1}{N}$. We shall approximate each iterate $y_\nu(x)$ by a certain vector $y_\nu(x_i,h),\ i = 0,\ldots,N$. After discretization, the equations of the iterative methods reduce to linear systems of the following form:

$$\begin{aligned}
&y_{\nu+1}(x_0,h) = 0,\\
&\delta^2 y_{\nu+1}(x_i,h) + qC_1(x_i)^{q-1}x_i^p g(x_i) y_{\nu+1}(x_i,h) = x_i^p g(x_i) y_\nu(x_i,h)\left[-y_\nu(x_i,h)^{q-1} + qC_1(x_i)^{q-1}\right],\\
&i = 1,\ldots,N-1,\\
&y_{\nu+1}(x_N,h) = 0,
\end{aligned} \tag{4.1}$$

where

$$C_1(x) = \begin{cases} \alpha(x), \text{in the case of the Picard method;}\\ y_\nu(x), \text{in the case of the Newton method.}\end{cases} \tag{4.2}$$

Here we have used the common notation for the second central difference :

$$\delta^2 y_{\nu+1}(u_i,h) = \frac{y_{\nu+1}(x_{i+1},h) - 2y_{\nu+1}(x_i,h) + y_{\nu+1}(x_{i-1},h)}{h^2},\ i = 1,\ldots,N-1.$$

The scheme (4.1) was iterated until the following condition was satisfied:

$$\sqrt{\sum_{i=0}^{N} \left|y_{\nu+1}(x_i,h) - y_\nu(x_i,h)\right|^2} < \epsilon \tag{4.3}$$

where ϵ is the required precision. The number of iterations needed to satisfy this condition depends on the values of q,p and h. In the case of the examples which are presented in the next section, when $\epsilon = 10^{-6}$ this number is not greater than 5 for the Newton method.When the Picard method is used, the number of iterations needed to obtain the same precision may be significantly larger. Let ν_0 be the least value of ν for which the condition (4.3) is satisfied. For the sake of simplicity, we shall write $y(x_i,h)$ instead of

$y_{\nu_0}(u_i, h)$ and consider $y(x_i, h)$ as the finite-difference approximation to the solution of the considered nonlinear problem. As it is well-known, if $f \in C^4[a, b]$, then we have

$$|\delta^2 f(x, h) - f"(x)| \leq \mu\, h^2, \ \forall x \in [a, b], \tag{4.4}$$

where μ is a positive constant.

In the case of the considered boundary-value problems, the condition (4.4) is not satisfied by y on $[0, 1]$.

In the case of the boundary conditions $y(0) = y(1) = 0$, $y'(x)$ is not continuous at $x = 0$ and $x = 1$, for $q < -1$ and $p < -1 - q$. We shall consider all values of p and q which satisfy the mentioned contidtions and therefore the expected accuracy of the results depends on q and p. *A posteriori* estimates of the numerical results, which will be presented in the next section, have shown that, for some constant $M_{q,p}$,

$$\max_{i=0,\ldots,N} |y(x_i, h) - y(x_i)| \leq \bar{M}_{q,p}\, h^s, \tag{4.5}$$

where s is close to 1, if $p > 0$, and $s < 1$, if $p < 0$.

The accuracy of the numerical approximation may be significantly improved if we apply a suitable variable substitution and discretize the equation in the new variable. In order to find an adequate variable substitution for the present problem, we have analysed the behavior of the solution near the singularities.

As shown in Section 2,if

$$\gamma_1 = \frac{2 + p}{1 - q}$$

and

$$\gamma_2 = \frac{2}{1 - q},$$

the solution is asymptotically equivalent to

$$\alpha(x) = B x^{\gamma_1} (1 - x)^{\gamma_2}, \tag{4.6}$$

as $x \to 0+$ and $x \to 1-$. We want to transform α into a function which is asymptotically equivalent to t^2, as $t \to 0+$, and asymptotically equivalent to $(1 - t)^2$, as $t \to 1-$. With this aim, we introduce the variable substitution

$$t = \left(1 - (1 - x)^{\frac{\gamma_1}{2}}\right)^{\frac{\gamma 2}{2}}. \tag{4.7}$$

If we introduce this variable substitution into (3.1) and (3.3), we obtain linear equations of the form

$$\begin{aligned} &C_3(t)\, w''_{\nu+1}(t) \ + \ C_2(t)\, w'_{\nu+1}(t) \ + q\xi(t)^p\, g(\xi(t))\, C_1(\xi(t))^{q-1} w_{\nu+1}(t) \ = \\ &= \xi(t)^p g(\xi(t)) w_\nu(t) (q C_1(\xi(t))^{q-1} - w_\nu(t)^{q-1}), \ \ \nu = 0, 1, \ldots, \ \ \ t \in]0, 1[, \end{aligned} \tag{4.8}$$

where $w_\nu(t) = y_\nu(\xi(t))$,

$$C_3(t) = \frac{\gamma_1^2\gamma_2^2}{16} t^{\frac{2\gamma_2-4}{\gamma_2}} \left(1 - t^{\frac{2}{\gamma_2}}\right)^{\frac{2\gamma_1-4}{\gamma_1}} ,$$

$$C_2(t) = \frac{\gamma_1^2\gamma_2(\gamma_2-2)}{16} t^{\frac{\gamma_2-4}{\gamma_2}} \left(1 - t^{\frac{2}{\gamma_2}}\right)^{\frac{2\gamma_1-4}{\gamma_1}} + \frac{\gamma_1\gamma_2(2-\gamma_1)}{8} t^{\frac{\gamma_2-2}{2}} \left(1 - t^{\frac{2}{\gamma_2}}\right)^{\frac{\gamma_1-4}{\gamma_1}} ,$$

and

$$\xi(t) = 1 - \left(1 - t^{\frac{2}{\gamma_2}}\right)^{\frac{2}{\gamma_1}} ,$$

while C_1 is defined by (4.2). In this case, the initial approximmation w_0 for the iterative methods, according to Section 3, must be taken in the form $w_0(t) = \alpha(\xi(t))$, where α is a lower or an upper solution of the original problem.

Note that (4.8) can also be obtained by introducing the variable substitution into the equation (1.1) and applying the iterative method to the equation in the new variable.

If we discretize (4.8) using a finite-difference scheme of the same type, we obtain the linear system

$$w_{\nu+1}(0,h) = 0;$$

$$C_3(t_i)\,\delta^2 w_{\nu+1}(t_i,h) \;+\; C_2(t_i)\,\frac{w_{\nu+1}(t_{i+1},h) - w_{\nu+1}(t_{i-1},h)}{2h} \;+ q\xi(t_i)^p\; g(\xi(t_i))\, C_1(\xi(t_i))^{q-1} w_{\nu+1}(t_i) \;=$$

$$= \xi(t_i)^p g(\xi(t_i)) w_\nu(t_i,h)(qC_1(\xi(t_i))^{q-1} - w_\nu(t_i,h)^{q-1}),\;\; i = 1,2,\ldots,N-1$$

$$w_{\nu+1}(1,h) = 0;\;\; \nu = 0,1,\ldots. \tag{4.9}$$

Note that when the variable substitution is used, in general, a larger number of iteration is needed to obtain the same precision. In the examples below, for example, the number of Newton iterates changed between 12 and 17. However, as we shall see in the next section, this fact is compensated by the improvement of the convergence order.

5. NUMERICAL RESULTS

Computer codes were written to implement the methods described in the previous sections. Using these codes, we have performed some numerical experiments in order to test the efficiency of the proposed algorithms. We have considered 4 test cases, with $r(x) = -x^p$ and:

1. $p = -0.5$, $q = -3$;
2. $p = 0$, $q = -3$;
3. $p = 0.5$, $q = -3$;
4. $p = -0.5$, $q = -2$.

5. $p=-0.5,\ \ q=-2.5$.

For each test case, we have computed the numerical solutions using the Picard and the Newton methods. In order to evalute the convergence order of the finite-difference approximations, we have applied the scheme (4.8), as well as (4.9), with four stepsizes: $h_1=1/100, h_2=1/200, h_3=1/400$ and $h_4=1/800$.

The graphics of the numerical solutions obtained in the variable x for the first three test cases with $h=1/800$ are plotted in fig.1. The graphics of the corresponding solutions obtained with variable substitution are plotted in fig.2.

For each case, we have estimated the convergence order using the formula:

$$s(x_i)=-\log_2\frac{|w_\nu(x_i,h_4)-w_\nu(x_i,h_3)|}{|w_\nu(x_i,h_3)-w_\nu(x_i,h_2)|}.$$

In fig.3 ans fig.4 we compare the convergence order of the results obtained by the schemes (4.8) and (4.9) in the test cases 2 and 4. As it was expected, the convergence order of the results obtained in the new variable is about 2, for all the test cases, independently of t, due to the smoothness of the solutions on the whole interval.

Concerning the results in the variable x, the convergence order changes between 0.81 and 1.0, depending on the test cases and on the value of x.

Finally, in table 5.1 we present some results for the case $p=-0.5, q=-2.5$. Note that in this case we have $\gamma_1+\gamma_2=1$ and therefore the exact solution is known. This table shows the relative errors of the numerical results obtained at corresponding points in the old and in the new variable. Since the gridpoints x_i do not coincide with any gridpoints in the new variable, we had to interpolate the data in the new variable in order to obtain the approximations at the needed points. The displayed results confirm that the variable substitution improves significantly the accuracy of the finite-difference approximation.

x	(*)	(**)
0.1	0.0544	0.000107
0.2	0.0317	0.000111
0.3	0.0224	0.000119
0.4	0.0171	0.000128
0.5	0.0137	0.000141
0.6	0.0114	0.000157
0.7	0.0101	0.000179
0.8	0.0102	0.000209
0.9	0.0147	0.000249

Tab. 5.1. Relative errors of the numerical results for the test case 5 with $h=1/100$ (*) without variable substituion; (**)with variable substitution.

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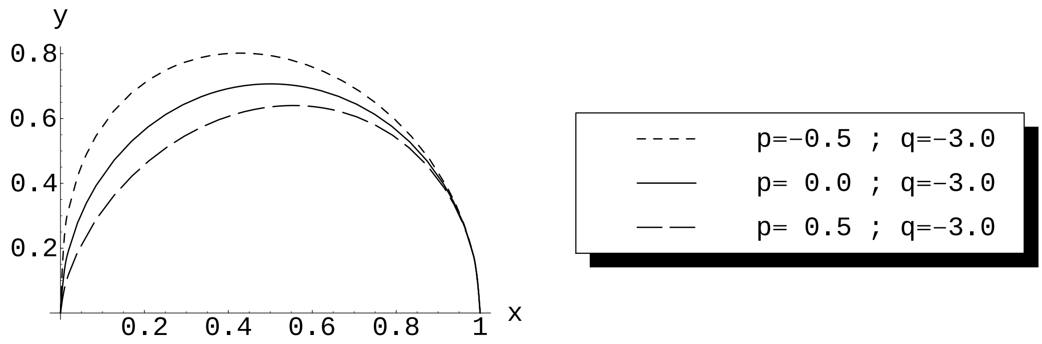
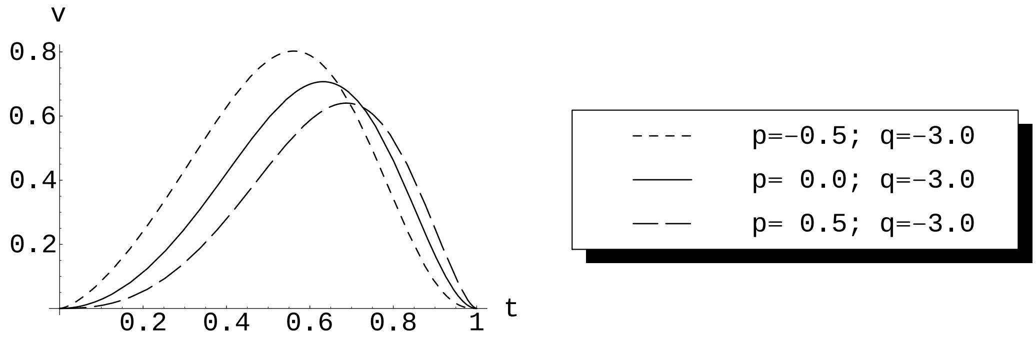


1 Figure. Graphics of the numerical solutions for the first three test cases in the old variable



2 Figure. Graphics of the numerical solutions for the first three test cases in the new variable

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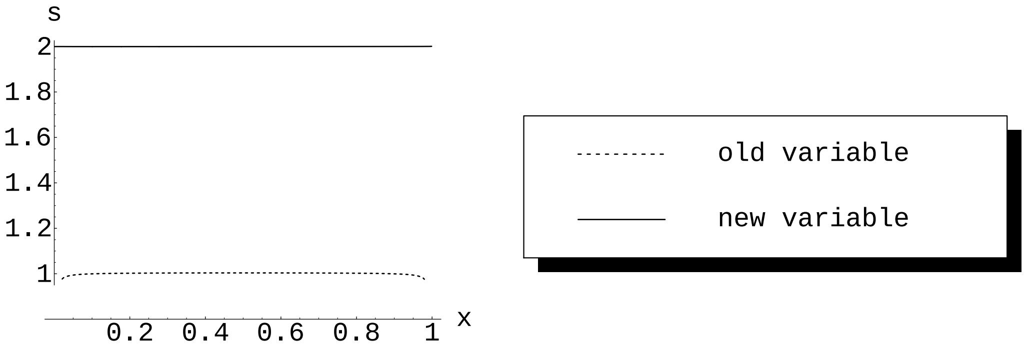
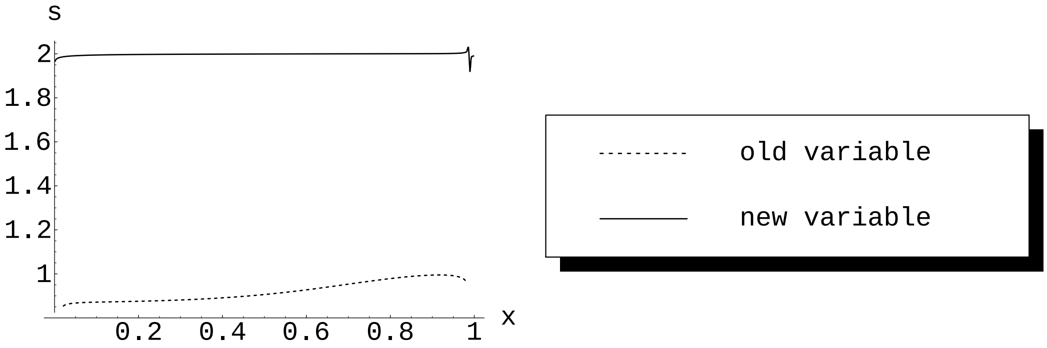


3 Figure. Graphics of the convergence order for the test case 2.



4 Figure. Graphics of the convergence order for the test case 4.