SYMMETRIC SOLUTIONS OF THE NEUMANN PROBLEM INVOLVING A CRITICAL SOBOLEV EXPONENT

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ABSTRACT. We study the effect of the coefficient of the critical nonlinearity for the Neumann problem on existence of symmetric least energy solutions. As a by-product we obtain two inequalities in symmetric Sobolev spaces involving a weighted critical Lebesgue norm and the H^1 norm.

1. Introduction

The purpose of this work is to investigate the existence and nonexistence of least energy solutions, having some symmetry properties, of the nonlinear Neumann problem

$$(I_\lambda)\qquad \begin{cases} -\Delta u+\lambda u=Qu^{2^\star-1}, & u>0, \quad \text{in } \Omega,\\ \frac{\partial u}{\partial \nu}=0 & \text{on } \partial\Omega, \end{cases}$$

under symmetric assumptions on the coefficient function Q and the domain Ω. The problem (I_λ) originates in the study of mathematical models in biological pattern formation theory governed by diffusion and cross-diffusion systems such as the Gierer and Meinhardt and the Keller and Segel models. In this respect the reader is referred to the survey article [16] by Ni.

The number $2^\star$ is the critical Sobolev exponent, $2^\star=\frac{2N}{N-2}$, and $\lambda>0$. The domain $\Omega\subset\mathbb{R}^N$, $N\geq 3$, is bounded, with smooth boundary, and G-invariant, where G is a finite subgroup of $O(\mathbb{N})$; that is, if $x\in\Omega$ and $g\in G$, then $gx\in\Omega$. The vector ν denotes the outward normal to $\partial\Omega$. The coefficient Q is Hölder continuous on $\bar\Omega$, nonconstant, nonnegative, and also G-invariant, that is

$$Q(x)=Q(gx),\qquad \text{for } x\in\bar\Omega \text{ and } g\in G.$$

Solutions will be obtained as minimizers of the constrained variational problem

$$m_{\lambda,G}:=\inf\{E_\lambda(u)|u\in H^1_G(\Omega), \textstyle\int_\Omega Q|u|^{2^\star}=1\},$$

where

$$E_\lambda(u):=\int_\Omega(|\nabla u|^2+\lambda u^2)$$

and

$$H^1_G(\Omega):=\{u\in H^1(\Omega)|\, u(\,\cdot\,)=u(g\,\cdot\,),\ \text{for every } g\in G\}.$$

If $Q\equiv 1$ on $\bar\Omega$, and with no symmetry assumptions, the problem (I_λ) has an extensive literature.

In the subcritical case, this problem has been studied by Lin, Ni, Takagi [14] and Ni, Takagi [17], [18]. They obtained existence of a least energy solution for λ sufficiently large. This solution has exactly one maximum point P_λ in $\bar\Omega$. $P_\lambda\in\partial\Omega$ and $H(P_\lambda)\to\max_{\partial\Omega}H$ as $\lambda\to\infty$, where H is the mean curvature of $\partial\Omega$ with respect to the outward normal.

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In the critical case, existence of a least energy solution was proved by Adimurthi and Mancini [1] and X.J. Wang [21]. Denoting by m_λ the corresponding least energy in this case,

$$m_\lambda := \inf \left\{ E_\lambda(u) \middle| \int_\Omega |u|^{2^\star} = 1 \right\},$$

and by S the best Sobolev constant,

$$S = \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^\star}\right)^{\frac{2}{2^\star}}},$$

they proved that

$$m_\lambda < \frac{S}{2^{\frac{2}{N}}} \quad \text{for } \lambda > 0.$$

and proved that this condition yields the existence of a minimizer. The least energy solutions are single-peaked in the sense that they attain their maximum at exactly one point $P_\lambda \in \partial\Omega$, for large λ. Moreover $H(P_\lambda) \to \max_{\partial\Omega} H$ as $\lambda \to \infty$.

Besides the study of least energy solutions, higher energy solutions have also been constructed. Adimurthi, Pacela and Yadava [3] proved existence of a solution concentrating at a strict local maximum of the mean curvature H, with $H > 0$, for $n \geq 7$. They also proved that solutions blow up at critical critical points of H as $\lambda \to \infty$. See also [4]. These results were extended to dimensions $n = 5$, 6 by Z.Q. Wang [23]. With the aid of the Lyapunov-Schmidt type reduction method, Adimurthi, Mancini and Yadava [2] proved existence of a solution concentrating at a nondegenerate critical point of the mean curvature, with positive mean curvature, as $\lambda \to \infty$, for $n \geq 6$.

More recently, Gui-Ghoussoub [13] gave a general construction of a multi-peaked solution having a finite number of peaks on the boundary, provided the mean curvature of the boundary has the same number of strict local maximums, with positive mean curvature, for $n \geq 5$. Rey [19] constructed solutions having a finite number of peaks on the boundary, provided the mean curvature of the boundary has the same number of nondegenerate critical points, with positive mean curvature, for $n \geq 6$.

In the case of symmetric domains, Z.Q. Wang [22], [24], [25], [26], proved existence of multi-peaked solutions belonging to symmetric Sobolev spaces.

Chabrowski and Willem [8] investigated the effect of the coefficient Q on the existence of least energy solutions.

In this work we study the effect of symmetries of the coefficient Q on the existence of multi-peaked solutions of (I_λ) in symmetric Sobolev spaces. This is done by combining the techniques used in [8] and [26]. As a by-product we obtain inequalities involving weighted Lebesgue norms with the critical Sobolev exponent and the H^1 norm.

The organization of this work is as follows. In Section 2 we give the minimum level $\hat{S}$ at which compactness of E_λ fails, and we give the asymptotic behavior of the infimum $m_{\lambda,G}$, as $\lambda \to \infty$. In Section 3 we examine the situation where concentration occurs at the boundary of Ω, as $\lambda \to \infty$, and prove there exist least energy solutions for all positive λ. In Section 4 we examine the situations where there exists a $\Lambda > 0$ such that least energy solutions exist for $0 < \lambda < \Lambda$ and least energy solutions do not exist for $\lambda > \Lambda$. The argument is by contradiction. If least energy solutions were to exist for all positive λ, then concentration would occur in the interior of Ω. Using an estimate due to Brezis and Nirenberg, this leads to a contradiction. In Section 5 we derive two Sobolev inequalities as corollaries of the results of Section 4. Finally, for the reader's convenience, in the Appendix we give the proof of Struwe's compactness lemma for this Neumann problem.

2. Preliminaries

First recall the case that $Q \equiv 1$ and $\Omega = \mathbb{R}^N$. The Talenti instanton

$$U(x) := \left(\frac{N(N-2)}{N(N-2) + |x|^2} \right)^{\frac{N-2}{2}}$$

minimizes the ratio $\frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} u^{2^\star}\right)^{\frac{2}{2^\star}}}$ among nonzero functions in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. The minimum is the Sobolev constant S. The instanton U satisfies

$$-\Delta U = U^{2^\star - 1}, \tag{1}$$

so that

$$\int_{\mathbb{R}^N} |\nabla U|^2 = \int_{\mathbb{R}^N} U^{2^\star} = S^{\frac{N}{2}}. \tag{2}$$

Let $\varepsilon > 0$. For later use, we define the rescaled function $U_{\varepsilon,P} := \varepsilon^{-\frac{N-2}{2}} U\left(\frac{\cdot - P}{\varepsilon}\right)$, which also satisfies equalities (1) and (2).

Now let Ω, Q and G be as in the Introduction. Let

$$Q_M := \max_{\bar\Omega} Q, \qquad Q_m := \max_{\partial\Omega} Q$$

and

$$k := \min\{\#G(x) | x \in \bar\Omega \setminus \{0\}\},$$

where $G(x)$ denotes the orbit of x and $\#G(x)$ is the cardinal number of $G(x)$. Define

$$E_\lambda(u) := \int_\Omega \left[|\nabla u|^2 + \lambda u^2 \right]$$

on $H^1(\Omega)$, and

$$V_G(\Omega) := \left\{ u \in H^1_G(\Omega) \,\middle|\, \int_\Omega Q|u|^{2^\star} = 1 \right\}$$

Let

$$m_{\lambda,G} := \inf_{u \in V_G(\Omega)} E_\lambda(u).$$

As stated in the Introduction, our main objective is to inquire about the existence of minimum for $m_{\lambda,G}$. Suppose that $0 \in \Omega$ and $Q(0) > 0$. Our first Lemma will concern the comparison of $m_{\lambda,G}$ with the quantity $\hat{S}$, which we now define. Let

$$\hat{S} := \min \left(\frac{S k^{\frac{2}{N}}}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}, \frac{S}{Q(0)^{\frac{N-2}{N}}}, \frac{S k^{\frac{2}{N}}}{Q_M^{\frac{N-2}{N}}} \right).$$

Note that if the minimum is equal to the first term, then

$$\textit{Case 1:} \qquad \begin{cases} Q_m \geq Q_M \Big/ 2^{\frac{2}{N-2}}, \\ Q_m \geq k^{\frac{2}{N-2}} Q(0) \Big/ 2^{\frac{2}{N-2}}. \end{cases}$$

Otherwise, either the minimum is equal to the second term,

$$\textit{Case 2:} \qquad \begin{cases} Q(0) > 2^{\frac{2}{N-2}} Q_m \Big/ k^{\frac{2}{N-2}}, \\ Q(0) \geq Q_M \Big/ k^{\frac{2}{N-2}}, \end{cases}$$

or the minimum is equal to the third term,

$$\textit{Case 3:} \qquad \begin{cases} Q_M > 2^{\frac{2}{N-2}} Q_m, \\ Q_M \geq k^{\frac{2}{N-2}} Q(0), \end{cases}$$

and, of course, these two last cases occur simultaneously if the second and third terms are equal and smaller than the first one.

Lemma 2.1. *For all $\lambda > 0$, $m_{\lambda,G} \leq \hat{S}$.*

Proof. The proof is standard. In the Case 2, concentrate an instanton at the origin, cut it off far away from the origin using a positive symmetric test function and multiply the product by a constant, so that the result belongs to the space $V_G(\Omega)$. The proof in Cases 1 and 3 is similar, but instead of one instanton use an invariant function obtained using k instantons, concentrated at points of maximum of Q_m and Q_M, respectively. □

The function $\lambda \mapsto m_{\lambda,G}$ is increasing and continuous. So, there exists a $\Lambda \in \mathbb{R}^+ \cup \{+\infty\}$ such that $m_{\lambda,G} < \hat{S}$ for $\lambda < \Lambda$ and $m_{\lambda,G} = \hat{S}$ for $\lambda \geq \Lambda$. The value Λ cannot be zero because $m_{\lambda,G} < \hat{S}$ for small λ, as can be seen by evaluating $E_\lambda(u)$ for u constant in $V_G(\Omega)$. So we give the following

Definition 2.2. $\Lambda := \inf\{\lambda : m_{\lambda,G} = \hat{S}\}$.

We now recall the *concentration-compactness principle* [15]. Let $u_n \rightharpoonup u$ in $H^1(\Omega)$, $u_n \to u$ a.e. on Ω and $|u_n|^{2^\star} \rightharpoonup \mu$, $|\nabla u_n|^2 \rightharpoonup \tilde{\mu}$ in the sense of measures on $\bar{\Omega}$. Then, there exists at most a countable set J, numbers $\mu_j > 0$ and $\tilde{\mu}_j > 0$ and points $x_j \in \bar{\Omega}$, $j \in J$, such that

$$\mu = |u|^{2^\star} + \sum_{j\in J} \mu_j \delta_{x_j},$$

$$\tilde{\mu} \geq |\nabla u|^2 + \sum_{j\in J} \tilde{\mu}_j \delta_{x_j}.$$

Moreover, if $x_j \in \Omega$, then $S(\mu_j)^{\frac{N-2}{N}} \leq \tilde{\mu}_j$ and if $x_j \in \partial\Omega$, then $\frac{S}{2^{\frac{2}{N}}}(\mu_j)^{\frac{N-2}{N}} \leq \tilde{\mu}_j$. The first inequality is a consequence of the Sobolev inequality and the second of the Cherrier inequality [9].

The next lemma gives a criterion for the existence of minimum for $m_{\lambda,G}$.

Lemma 2.3. *If $m_{\lambda,G} < \hat{S}$, then $m_{\lambda,G}$ is achieved.*

Remark. The proof we now give relies on the concentration-compactness principle. An alternative proof follows from Corollary A.7 of Struwe's Compactness Lemma in the Appendix.

Proof. Let $\{u_n\} \subset H^1_G(\Omega)$ be such that $\int_\Omega Q|u_n|^{2^\star} = 1$, for each n, and $E_\lambda(u_n) \to m_{\lambda,G}$. We may assume that $u_n \rightharpoonup u$ in $H^1_G(\Omega)$, $u_n \to u$ in $L^2(\Omega)$ and $u_n \to u$ a.e. on Ω. Applying the concentration-compactness principle we can write

$$\int_\Omega Q|u|^{2^\star} + \sum_{j\in J} Q(x_j)\mu_j = 1$$

and

$$m_{\lambda,G} = \lim_{n\to\infty} E_\lambda(u_n) \geq \int_\Omega (|\nabla u|^2 + \lambda u^2) + \sum_{j\in J} \tilde{\mu}_j.$$

Let j_0 be the index j such that $x_{j_0} = 0$, if such an index exists.

$$\int_\Omega Q|u|^{2^\star} + Q(0)\mu_{j_0} + \sum_{[x_j]\in(\bar{\Omega}\setminus\{0\})/G} \#G(x_j)Q(x_j)\mu_j = 1. \tag{3}$$

We have

$$\begin{aligned} m_{\lambda,G} &\geq \int_\Omega (|\nabla u|^2 + \lambda u^2) + \sum_{j\in J} \tilde{\mu}_j \\ &\geq m_{\lambda,G}\left(\int_\Omega Q|u|^{2^\star}\right)^{\frac{2}{2^\star}} + \sum_{x_j\in\Omega\setminus\{0\}} \tilde{\mu}_j + \tilde{\mu}_{j_0} + \sum_{x_j\in\partial\Omega} \tilde{\mu}_j \\ &\geq m_{\lambda,G}\left(\int_\Omega Q|u|^{2^\star}\right)^{\frac{2}{2^\star}} + \sum_{x_j\in\Omega\setminus\{0\}} \frac{Q(x_j)^{\frac{N-2}{N}}}{Q(x_j)^{\frac{N-2}{N}}} S\mu_j^{\frac{N-2}{N}} \\ &\quad + \frac{Q(0)^{\frac{N-2}{N}}}{Q(0)^{\frac{N-2}{N}}} S\mu_{j_0}^{\frac{N-2}{N}} + \sum_{x_j\in\partial\Omega} \frac{Q(x_j)^{\frac{N-2}{N}}}{Q(x_j)^{\frac{N-2}{N}}} \frac{S}{2^{\frac{2}{N}}} \mu_j^{\frac{N-2}{N}} \\ &\geq m_{\lambda,G}\left(\int_\Omega Q|u|^{2^\star}\right)^{\frac{2}{2^\star}} + \sum_{[x_j]\in(\Omega\setminus\{0\})/G} \frac{Sk^{\frac{2}{N}}}{Q_M^{\frac{N-2}{N}}} (\#G(x_j)Q(x_j)\mu_j)^{\frac{N-2}{N}} \\ &\quad + \frac{S}{Q(0)^{\frac{N-2}{N}}} (Q(0)\mu_{j_0})^{\frac{N-2}{N}} + \sum_{[x_j]\in\partial\Omega/G} \frac{Sk^{\frac{2}{N}}}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}} (\#G(x_j)Q(x_j)\mu_j)^{\frac{N-2}{N}} \end{aligned}$$

Recalling that $m_{\lambda,G} < \hat{S}$ and using Eq. (3), it follows that all the μ_j's are zero. So $\int_\Omega Q|u|^{2^\star} = 1$. Since $m_{\lambda,G} \geq \int_\Omega(|\nabla u|^2 + \lambda u^2) \geq m_{\lambda,G}$, we conclude that u is a minimizer for $m_{\lambda,G}$. □

So $\hat{S}$ is the minimum level at which compactness of E_λ fails.

Remark 2.4. *Obviously, $\lambda \mapsto m_{\lambda,G}$ is strictly increasing for $\lambda < \Lambda$ and $m_{\lambda,G}$ is not achieved for $\lambda > \Lambda$.*

The next lemma gives the asymptotic limit of $m_{\lambda,G}$ as $\lambda \to +\infty$.

Lemma 2.5. $\lim_{\lambda\to\infty} m_{\lambda,G} = \hat{S}$.

Proof. Once again, the proof is by contradiction and relies on the concentration-compactness principle. Suppose that the $\lim_{\lambda\to\infty} m_{\lambda,G} = S_\infty < \hat{S}$. Let (λ_j) be a sequence, $\lambda_j \to \infty$, and $u_j \in V_G(\Omega)$ be a minimizer for $m_{\lambda_j,G}$. The sequence (u_j) is bounded (say by S_∞) in $H^1(\Omega)$. From the inequality $\lambda_j \int_\Omega u_j^2 \leq S_\infty$, we deduce that $u_j \to 0$ in $L^2(\Omega)$. We may assume that $u_j \rightharpoonup 0$ in $H^1(\Omega)$, $u_j \to 0$ a.e. on Ω. By the concentration-compactness principle, there exists at most a countable set J, numbers $\mu_j > 0$ and $\tilde{\mu}_j > 0$ and points $x_j \in \bar{\Omega}$, $j \in J$, such that

$$\mu = \sum_{j\in J} \mu_j \delta_{x_j}$$

and

$$\tilde{\mu} \geq \sum_{j\in J} \tilde{\mu}_j \delta_{x_j}.$$

Let j_0 be the index j such that $x_{j_0} = 0$, if such an index exists. Then

$$Q(0)\mu_{j_0} + \sum_{[x_j]\in(\bar{\Omega}\setminus\{0\})/G} \#G(x_j)Q(x_j)\mu_j = 1 \tag{4}$$

and

$$\begin{aligned} S_\infty &\geq \sum_{j\in J}\tilde{\mu}_j \\ &\geq \sum_{[x_j]\in(\Omega\setminus\{0\})/G} \frac{Sk^{\frac{2}{N}}}{Q_M^{\frac{N-2}{N}}}(\#G(x_j)Q(x_j)\mu_j)^{\frac{N-2}{N}} \\ &\quad +\frac{S}{Q(0)^{\frac{N-2}{N}}}(Q(0)\mu_{j_0})^{\frac{N-2}{N}} + \sum_{[x_j]\in\partial\Omega/G} \frac{Sk^{\frac{2}{N}}}{2^{\frac{2}{N}}Q_m^{\frac{N-2}{N}}}(\#G(x_j)Q(x_j)\mu_j)^{\frac{N-2}{N}} \end{aligned}$$

Recalling that we are supposing $S_\infty < \hat{S}$ and using Eq. (4), it follows that all the μ_j's are zero, which is impossible. Therefore, $S_\infty = \hat{S}$. □

3. Existence of least energy solutions for all positive λ

It follows from the results of the previous section that there exists a least energy solution of (I_λ) for $\lambda < \Lambda$, and there does not exist a least energy solution of (I_λ) for $\lambda > \Lambda$. Therefore it is important to determine if Λ is finite or not. We start by proving that $\Lambda = \infty$ in Case 1.

Theorem 3.1. *Assume $N \geq 3$, and Ω has a smooth boundary. Suppose $Q_m \geq Q_M \Big/ 2^{\frac{2}{N-2}}$ and $Q_m \geq k^{\frac{2}{N-2}}Q(0) \Big/ 2^{\frac{2}{N-2}}$. Suppose also that $Q_m = Q(y)$ for some $y \in \partial\Omega$, with $\#G(y) = k$, $H(y) > 0$ and $|Q(x)-Q(y)| = o(|x-y|)$ as $x \to y$. Then $\Lambda = \infty$, i.e., $m_{\lambda,G} < \hat{S}$ for all $\lambda > 0$.*

Proof. Let $R > 0$ be small and φ be a radial C^∞-function such that

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \leq R/2, \\ 0 & \text{if } |x| \geq R. \end{cases}$$

Fixing y as in the statement of the theorem, Adimurthi and Mancini [1] proved that for sufficiently small $\varepsilon > 0$,

$$\frac{E_\lambda(U_{\varepsilon,y}\varphi(\cdot - y))}{||U_{\varepsilon,y}\varphi(\cdot-y)||^2_{L^{2^\star}(\Omega)}} <$$

$$\frac{S}{2^{\frac{2}{N}}} - \left\{ \begin{array}{l} A_N H(y)\varepsilon\log\frac{1}{\varepsilon} - a_N\lambda\varepsilon + O(\varepsilon) + o(\lambda\varepsilon),\ N=3, \\ A_N H(y)\varepsilon - a_N\lambda\varepsilon^2\log\frac{1}{\varepsilon} + O\left(\varepsilon^2\log\frac{1}{\varepsilon}\right) + o\left(\lambda\varepsilon^2\log\frac{1}{\varepsilon}\right),\ N=4, \\ A_N H(y)\varepsilon - a_N\lambda\varepsilon^2 + O(\varepsilon^2) + o\left(\lambda\varepsilon^2\right),\ N\geq 5, \end{array}\right.$$

where A_N and a_N are positive constants depending on N.

We now want to estimate $J_\lambda(U_{\varepsilon,y}\varphi(\cdot - y))$, where

$$J_\lambda(u) = \frac{\int_\Omega(|\nabla u|^2 + \lambda u^2)}{\left(\int_\Omega Q|u|^{2^\star}\right)^{\frac{2}{2^\star}}}.$$

To do that, we first show that

$$\int_\Omega Q(x)U^{2^\star}_{\varepsilon,y}(x)\,dx = \int_\Omega Q(y)U^{2^\star}_{\varepsilon,y}(x)\,dx + o(\varepsilon). \tag{5}$$

In fact, let $\delta > 0$. Fix ρ such that

$$|Q(x) - Q(y)| < \delta|x-y| \text{ for } |x-y| < \rho.$$

Then

$$\begin{aligned}\int_\Omega |Q(x)-Q(y)|U_{\varepsilon,y}^{2^\star}(x)\,dx &\le \int_{\Omega\cap B_\rho(y)} \delta|x-y|U_{\varepsilon,y}^{2^\star}(x)\,dx \\ &\quad + \int_{\Omega\setminus B_\rho(y)} |Q(x)-Q(y)|U_{\varepsilon,y}^{2^\star}(x)\,dx \\ &\le \delta\varepsilon \int_{B_{\frac{\rho}{\varepsilon}}(0)} |z|U^{2^\star}(z)\,dz + O(\varepsilon^N) \\ &\le C\delta\varepsilon + O(\varepsilon^N) \\ &\le C\delta\varepsilon,\end{aligned}$$

if ε is sufficiently small. This proves (5). Hence, it follows that

$$J_\lambda(U_{\varepsilon,y}\varphi(\cdot-y)) < \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}} - \begin{cases} \tilde{A}_N H(y)\varepsilon \log\frac{1}{\varepsilon} + O(\varepsilon), & N=3, \\ \tilde{A}_N H(y)\varepsilon + o(\varepsilon), & N\ge 4. \end{cases}$$

Choosing R small enough so as to obtain functions with disjoint supports, we get

$$J_\lambda\left(\sum_g U_{\varepsilon,gy}\varphi(\cdot-gy)\right) = k^{\frac{2}{N}} J_\lambda(U_{\varepsilon,y}\varphi(\cdot-y)),$$

which obviously implies the desired result. □

4. Nonexistence of least energy solutions for λ sufficiently large

In contrast to the result of the previous section, where $\Lambda=+\infty$, we now prove that $\Lambda<\infty$ in Cases 2 and 3. In the next lemma we suppose, by contradiction, that $\Lambda=+\infty$ and examine the behavior of a sequence of minima of $m_{\lambda_n,G}$ as $\lambda_n\to+\infty$ in the situation of Case 3. To simplify, we first consider the case that $0\notin\Omega$.

Lemma 4.1. *Let Ω be a smooth bounded G-invariant domain such that $0\notin\Omega$. Let Q be nonnegative and Hölder continuous in $\bar\Omega$ and suppose $Q_M > 2^{\frac{2}{N-2}}Q_m$. Let $\mathcal{M} := Q^{-1}(Q_M)$. Suppose that $m_{\lambda_n,G}$ is achieved by a nonnegative u_n where the λ_n form a sequence converging to ∞. Let P_n be a point where u_n achieves its maximum. Then, up to a subsequence, (P_n) converges to a point in $\mathcal{M}$ as $n\to\infty$ and there exists a sequence (ε_n), with $\varepsilon_n>0$ and $\varepsilon_n\to0$, such that*

$$\lim_{n\to\infty}\left\|\nabla u_n - \nabla\sum_g\left[\varepsilon_n^{-\frac{N-2}{2}} U\left(S^{\frac12}k^{\frac1N}Q_M^{\frac1N}\frac{\cdot\;-gP_n}{\varepsilon_n}\right)\right]\right\|_{L^2(\Omega)} = 0.$$

Note: $\varepsilon_n^{-\frac{N-2}{2}} = \max_{\bar\Omega} u_n$.

Proof. As we saw in Lemma 2.1,

$$m_{\lambda,G} \le \hat S := \frac{Sk^{\frac{2}{N}}}{Q_M^{\frac{N-2}{N}}}$$

for all $\lambda>0$. Since we are assuming that $m_{\lambda_n,G}(\Omega)$ is achieved for a sequence $\lambda_n\to\infty$, by Remark 2.4 $\Lambda=+\infty$ and $m_{\lambda,G}$ is strictly increasing in λ. Hence, $m_{\lambda,G}<\hat S$ for all $\lambda>0$. Recall that by Lemma 2.5 $m_{\lambda,G}\to\hat S$ as $\lambda\to\infty$. We can assume the functions u_n satisfy

$$\begin{cases} -\Delta u_n + \lambda_n u_n = m_{\lambda_n,G} Q u_n^{2^\star-1} & \text{in } \Omega, \\ u_n>0 & \text{in } \Omega, \\ \frac{\partial u_n}{\partial\nu}=0 & \text{on } \partial\Omega. \end{cases}$$

The maxima $M_n := \max_{\bar\Omega} u_n = u_n(P_n)$ satisfy

$$\lambda_n M_n \le m_{\lambda_n,G} Q(P_n) M_n^{2^\star-1} \quad\text{or}\quad M_n^{2^\star-2} \ge \frac{\lambda_n}{m_{\lambda_n,G}Q(P_n)},$$

since $Q(P_n) = 0$ implies $u_n \equiv 0$, which is impossible. So $M_n \to \infty$. Up to a subsequence we can assume $P_n \to P_0 \in \bar{\Omega}$. We note that there exist $g_0 =$id, g_1, ..., $g_{k-1} \in G$ and $\sigma > 0$ such that

$$\sigma \leq |g_i P_0 - g_j P_0|,$$

$i \neq j$, $i, j = 0, \ldots, k-1$. So, for n sufficiently large,

$$|g_i P_n - g_j P_n| \geq \frac{\sigma}{2}.$$

Let $A_n := \Omega \cap B_{\frac{\sigma}{4}}(P_n)$ and ε_n be such that $\varepsilon_n^{\frac{N-2}{2}} = \frac{1}{M_n}$, so that $\lambda_n \varepsilon_n^2 = \lambda_n M_n^{-(2^\star-2)} \leq m_{\lambda_n,G} Q(P_n) \leq \hat{S} Q_M$. We also define $v_n(x) := \varepsilon_n^{\frac{N-2}{2}} u_n(\varepsilon_n x + P_n)$ and $B_n := \frac{A_n - P_n}{\varepsilon_n}$. Then

$$\begin{cases} -\Delta v_n + \lambda_n \varepsilon_n^2 v_n = m_{\lambda_n,G} Q(\varepsilon_n x + P_n) v_n^{2^\star-1} & \text{in } B_n, \\ 0 < v_n \leq 1 & \text{in } B_n, \\ \frac{\partial v_n}{\partial \nu} = 0 & \text{on } \partial\left(\frac{\Omega - P_n}{\varepsilon_n}\right) \cap \partial B_n, \\ v_n(0) = 1. \end{cases}$$

Again up to a subsequence, we can assume that $\lambda_n \varepsilon_n^2 \to a \geq 0$ and that $\frac{\mathrm{dist}(P_n, \partial\Omega)}{\varepsilon_n}$ converges in the extended nonnegative real line. Let B be such that $\chi_B = \lim_{n\to\infty} \chi_{B_n} = \lim_{n\to\infty} \chi_{\frac{A_n - P_n}{\varepsilon_n}}$.

By elliptic regularity theory $v_n \in C^{2,\alpha}_{\mathrm{loc}}(\bar{B}_n)$, for some $\alpha > 0$ independent of n. Furthermore, by Theorems 15.3 (which deals with L^p estimates) and 7.3 (which deals with Schauder estimates) of [5], in each compact set we can find a bound for the $C^{2,\alpha}$ norm of v_n which is independent of n. We can also apply Lemma 6.37 of [11] to extend, if necessary, the functions v_n up to ∂B, to obtain functions w_n whose $C^{2,\alpha}$ norm is uniformly bounded. The functions w_n have a subsequence which is convergent in $C^2_{\mathrm{loc}}(\bar{B})$ to a function v satisfying

$$\begin{cases} -\Delta v + av = \hat{S} Q(P_0) v^{2^\star-1} & \text{in } B, \\ 0 \leq v \leq 1 & \text{in } B, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial B, \\ v(0) = 1. \end{cases}$$

Suppose that $\lim_{n\to\infty} \frac{\mathrm{dist}(P_n, \partial\Omega)}{\varepsilon_n} = d \in \mathbb{R}_0^+$. Then B is a half space and $P_0 \in \partial\Omega$. We extend v by reflection to $\mathbb{R}^N$ and call the extension v. We recall that u_n is bounded in $H^1(\Omega)$ and that the rescaling does not change the $L^{2^\star}$ norm or the L^2 norm of the gradient. By lower semicontinuity, the function v is in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. From the equation we conclude that $v \in L^2(\mathbb{R}^N)$. By Corollary B.4 of [27] (Pohozaev's identity in an unbounded domain), $a = 0$. Hence $v(x) = U(\beta x)$ where $\beta^2 = \hat{S} Q(P_0)$, or $\beta = \frac{S^{\frac{1}{2}} k^{\frac{1}{N}} Q(P_0)^{\frac{1}{2}}}{Q_M^{\frac{1}{2^\star}}}$. Now

$$\begin{aligned} \int_B |\nabla v|^2 &= \beta^{2-N} \int_B |\nabla U|^2 \\ &= \frac{1}{2} \beta^{2-N} S^{\frac{N}{2}} \\ &= \frac{1}{2} \frac{1}{k^{\frac{N-2}{N}}} \frac{Q_M^{\frac{N-2}{2^\star}}}{Q(P_0)^{\frac{N-2}{2}}} S. \end{aligned}$$

On the other hand,

$$\begin{aligned}\int_B |\nabla v|^2 &\le \liminf_{n\to\infty} \int_{B_n} |\nabla v_n|^2 \\ &= \liminf_{n\to\infty} \int_{A_n} |\nabla u_n|^2 \\ &\le \frac{1}{k} \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2 \\ &\le \frac{1}{k} \limsup_{n\to\infty} \int_\Omega |\nabla u_n|^2 \\ &\le \frac{1}{k} \lim_{n\to\infty} m_{\lambda_n,G} \\ &= \frac{1}{k} \frac{k^{\frac{2}{N}}}{Q_M^{\frac{N-2}{N}}} S.\end{aligned}$$

So $Q_M^{\frac{N-2}{2^\star}+\frac{N-2}{N}} \le 2Q(P_0)^{\frac{N-2}{2}} \le 2Q_m^{\frac{N-2}{2}}$, which implies $Q_M \le 2^{\frac{2}{N-2}} Q_m$ and contradicts our initial assumption.

Therefore $\lim_{n\to\infty} \frac{\operatorname{dist}(P_n,\partial\Omega)}{\varepsilon_n} = \infty$ and $B = \mathbb{R}^N$. As before, we get $v(x) = U(\beta x)$. Now

$$\begin{aligned}\int_B |\nabla v|^2 &= \beta^{2-N} \int_{\mathbb{R}^N} |\nabla U|^2 \\ &= \frac{1}{k^{\frac{N-2}{N}}} \frac{Q_M^{\frac{N-2}{2^\star}}}{Q(P_0)^{\frac{N-2}{2}}} S.\end{aligned}$$

Again,

$$\int_B |\nabla v|^2 \le \frac{1}{k} \frac{k^{\frac{2}{N}}}{Q_M^{\frac{N-2}{N}}} S.$$

So $Q_M^{\frac{N-2}{2^\star}+\frac{N-2}{N}} \le Q(P_0)^{\frac{N-2}{2}}$, which is equivalent to $Q_M \le Q(P_0)$. Therefore $P_0 \in \mathcal{M}$ and $\int_\Omega |\nabla u_n|^2 \to \hat{S}$.

Now we note that

$$\lim_{n\to\infty} ||\nabla v_n||^2_{L^2(B_n)} = ||\nabla[U(\beta\,\cdot\,)]||^2_{L^2(\mathbb{R}^N)} = \frac{\hat{S}}{k},$$

for $\beta = S^{\frac{1}{2}} k^{\frac{1}{N}} Q_M^{\frac{1}{N}}$. This implies

$$\lim_{n\to\infty} ||\nabla v_n - \nabla[U(\beta\,\cdot\,)]||_{L^2(B_n)} = 0, \tag{6}$$

as is shown below. Hence,

$$\lim_{n\to\infty} \left|\left|\nabla u_n - \nabla\left[\varepsilon_n^{-\frac{N-2}{2}} U\left(\beta \frac{\cdot\, - P_n}{\varepsilon_n}\right)\right]\right|\right|_{L^2(A_n)} = 0.$$

Similarly, since

$$\lim_{n\to\infty} ||\nabla u_n||^2_{L^2(\Omega)} = \hat{S},$$

we finally conclude that

$$\lim_{n\to\infty} \left|\left|\nabla u_n - \nabla \sum_g \left[\varepsilon_n^{-\frac{N-2}{2}} U\left(\beta \frac{\cdot\, - gP_n}{\varepsilon_n}\right)\right]\right|\right|_{L^2(\Omega)} = 0.$$

Proof of (6). Let $0 < \varepsilon < 1$.

• There exists a $R>0$ such that

$$\int_{\mathbb{R}^N\setminus B_R(0)} |\nabla[U(\beta\,\cdot\,)]|^2<\varepsilon.$$

Obviously,

$$\int_{B_R(0)} |\nabla[U(\beta\,\cdot\,)]|^2>\frac{\hat S}{k}-\varepsilon.$$

• Since v_n converges to $U(\beta\,\cdot\,)$ in $C^2_{\rm loc}(\mathbb{R}^N)$,

$$\int_{B_R(0)} |\nabla v_n-\nabla[U(\beta\,\cdot\,)]|^2<\varepsilon^2,$$

for n sufficiently big.

$$\bullet\ \frac{\hat S}{k}-\varepsilon \;<\; \int_{B_R(0)} |\nabla[U(\beta\,\cdot\,)]|^2$$

$$\le \int_{B_R(0)} |\nabla[U(\beta\,\cdot\,)]-\nabla v_n|^2+\int_{B_R(0)}|\nabla v_n|^2+$$

$$+2\int_{B_R(0)}(\nabla[U(\beta\,\cdot\,)]-\nabla v_n)\cdot\nabla v_n$$

$$\le \varepsilon^2+\int_{B_R(0)}|\nabla v_n|^2+2\varepsilon\sqrt{\frac{\hat S}{k}}.$$

So

$$\int_{B_R(0)}|\nabla v_n|^2\ge\frac{\hat S}{k}-c\varepsilon,$$

for some constant c. Since $\int_{B_n}|\nabla v_n|^2\le\frac{\hat S}{k}$,

$$\int_{B_n\setminus B_R(0)}|\nabla v_n|^2\le c\varepsilon.$$

• Combining the previous inequalities,

$$\begin{aligned}\int_{B_n}|\nabla v_n-\nabla[U(\beta\,\cdot\,)]|^2 &= \int_{B_n\cap B_R(0)}|\nabla v_n-\nabla[U(\beta\,\cdot\,)]|^2\\ &\quad+\int_{B_n\setminus B_R(0)}|\nabla v_n|^2+\int_{B_n\setminus B_R(0)}|\nabla[U(\beta\,\cdot\,)]|^2\\ &\quad+2\left(\int_{B_n\setminus B_R(0)}|\nabla v_n|^2\right)^{\frac12}\left(\int_{B_n\setminus B_R(0)}|\nabla[U(\beta\,\cdot\,)]|^2\right)^{\frac12}\\ &\le \varepsilon^2+(c+1)\varepsilon+2\sqrt{c}\varepsilon,\end{aligned}$$

for n big, which is what we wanted to prove. □

Now we suppose that $0\in\Omega$. Combining the arguments of Lemma 4.1 with the ones of Lemma 3.1 of [26] one easily gets the following three results, corresponding to Cases 2 and 3 above. The first result tells us that in Case 2, and when $\lambda_n\to+\infty$, the sequence (u_n), of minima of $m_{\lambda_n,G}$, is close to a sequence of instantons concentrating at the origin.

Lemma 4.2. *Let Ω be a smooth bounded G-invariant domain such that $0\in\Omega$. Let Q be nonnegative and Hölder continuous in $\bar\Omega$ and suppose $Q(0)>2^{\frac{2}{N-2}}Q_m\Big/k^{\frac{2}{N-2}}$ and $Q(0)>Q_M\Big/k^{\frac{2}{N-2}}$. Suppose that $m_{\lambda_n,G}$ is achieved by a nonnegative u_n where the λ_n form a sequence converging to ∞. Let P_n be a point where u_n achieves its* *maximum. Then, for n large, $P_n = 0$ and there exists a sequence (ε_n), with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$, such that*

$$\lim_{n\to\infty} \left|\left|\nabla u_n - \nabla \left[\varepsilon_n^{-\frac{N-2}{2}} U\left(S^{\frac{1}{2}} Q(0)^{\frac{1}{N}} \frac{\cdot}{\varepsilon_n}\right)\right]\right|\right|_{L^2(\Omega)} = 0. \tag{7}$$

The next lemma shows how concentration occurs in Case 3.

Lemma 4.3. *Let Ω be a smooth bounded G-invariant domain such that $0 \in \Omega$. Let Q be nonnegative and Hölder continuous in $\bar{\Omega}$ and suppose $Q_M > 2^{\frac{2}{N-2}} Q_m$ and $Q_M > k^{\frac{2}{N-2}} Q(0)$. Let $\mathcal{M} := Q^{-1}(Q_M)$. Suppose that $m_{\lambda_n,G}$ is achieved by a nonnegative u_n where the λ_n form a sequence converging to ∞. Let P_n be a point where u_n achieves its maximum. Then, up to a subsequence, (P_n) converges to a point in $\mathcal{M}$ as $n \to \infty$ and there exists a sequence (ε_n), with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$, such that*

$$\lim_{n\to\infty} \left|\left|\nabla u_n - \nabla \sum_g \left[\varepsilon_n^{-\frac{N-2}{2}} U\left(S^{\frac{1}{2}} k^{\frac{1}{N}} Q_M^{\frac{1}{N}} \frac{\cdot - gP_n}{\varepsilon_n}\right)\right]\right|\right|_{L^2(\Omega)} = 0. \tag{8}$$

Lemma 4.4. *Under the hypothesis of* Lemmas 4.2 *and* 4.3 *but assuming $Q_M = k^{\frac{2}{N-2}} Q(0) > 2^{\frac{2}{N-2}} Q_m$, either* (7) *or* (8) *hold.*

Next we state Lemma 4.7 of [24]. It allows us to compare the energy of the functions u_n with the energy of instantons.

Lemma 4.5. *Assume $N \geq 5$. Let $\lambda_n > 0$, $\lambda_n \to \infty$, $\sigma_n > 0$, $\sigma_n \to 0$, $P_n \in \Omega$, $P_n \to P_0$ with $P_0 \in \Omega$, and $v_n \in H^1(\Omega)$, $v_n \geq 0$, $v_n \rightharpoonup 0$ in $H^1(\Omega)$ be such that*

$$\lim_{n\to\infty} \left|\left|\nabla v_n - \nabla \left(\frac{U_{\sigma_n,P_n}}{||U_{\sigma_n,P_n}||_{L^{2^\star}(\mathbb{R}^N)}}\right)\right|\right|_{L^2(\Omega)} = 0.$$

If $E_{\lambda_n}(v_n) < S$, then there exist sequences (δ_n) and (y_n), $\delta_n > 0$, $y_n \in \Omega$, such that, modulo a subsequence, $\delta_n/\sigma_n \to 1$, $y_n \to P_0$,

$$E_{\lambda_n}(v_n) \geq E_{\lambda_n}\left(\frac{U_{\delta_n,y_n}}{||U_{\delta_n,y_n}||_{L^{2^\star}(\mathbb{R}^N)}}\right) + O(\delta_n^2) + o(\lambda_n \delta_n^2) \tag{9}$$

and $\lambda_n \delta_n^2 = O(\delta_n)$.

The next well-known Lemma, due to Brezis and Nirenberg [7], gives an estimate for the energy of an instanton.

Lemma 4.6. *Suppose $N \geq 5$. Let y be an interior point of Ω. There exists a constant $b_N > 0$, depending only on N, such that*

$$E_\lambda\left(\frac{U_{\delta,y}}{||U_{\delta,y}||_{L^{2^\star}(\mathbb{R}^N)}}\right) = S + b_N \lambda \delta^2 + O(\delta^2) + o(\lambda \delta^2);$$

$O(\,\cdot\,)$ and $o(\,\cdot\,)$ are uniform in λ and y as $\delta \to 0$ for $\lambda \geq 1$ and for y in a compact subset of Ω.

The main result in Case 2 is

Theorem 4.7. *Assume $N \geq 5$. Suppose $Q(0) > 2^{\frac{2}{N-2}} Q_m \Big/ k^{\frac{2}{N-2}}$ and $Q(0) > Q_M \Big/ k^{\frac{2}{N-2}}$. Then $\Lambda < \infty$.*

Proof. The proof is by contradiction. Suppose $\Lambda = \infty$. Choose a sequence $\lambda_n \to \infty$. For each n, since $m_{\lambda_n,G} < \hat{S}$, we can take $u_n \in V_G(\Omega)$ that minimizes E_{λ_n}. By

Lemma 4.2, there exists a sequence (ε_n), with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$, such that Eq. (7) holds. We define $v_n := Q(0)^{\frac{1}{2^\star}} u_n$. Note that

$$E_{\lambda_n}(v_n) = Q(0)^{\frac{2}{2^\star}} E_{\lambda_n}(u_n) < S$$

and

$$\lim_{n\to\infty} \left|\left|\nabla v_n - \nabla\left[\sigma_n^{-\frac{N-2}{2}} S^{-\frac{N-2}{4}} U\left(\frac{\cdot}{\sigma_n}\right)\right]\right|\right|_{L^2(\Omega)} = 0,$$

with $\sigma_n = \frac{\varepsilon_n}{S^{\frac{1}{2}} Q(0)^{\frac{1}{N}}}$. Using Lemma 4.5, with $P_n = 0$, we get sequences (δ_n) and (y_n), $\delta_n > 0$, $y_n \in \Omega$, such that, modulo a subsequence, $\delta_n/\sigma_n \to 1$, $y_n \to 0$ and Eq. (9) holds. Lemma 4.6 implies that $E_{\lambda_n}(v_n) > S$, for n large. We have reached a contradiction. Therefore $\Lambda < \infty$. □

On the other hand, the main result in Case 3 is

Theorem 4.8. *Assume $N \geq 5$. Suppose $Q_M > 2^{\frac{2}{N-2}} Q_m$ and $Q_M > k^{\frac{2}{N-2}} Q(0)$. Then $\Lambda < \infty$.*

Proof. Suppose $\Lambda = \infty$. Choose a sequence $\lambda_n \to \infty$. For each n, since $m_{\lambda_n,G} < \hat{S}$, we can take $u_n \in V_G(\Omega)$ that minimizes E_{λ_n}. By Lemma 4.3, there exists a sequence (ε_n), with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$, such that Eq. (8) holds. We define $v_n := k^{\frac{1}{2^\star}} Q_M^{\frac{1}{2^\star}} u_n$. Let σ and P_0 be as in Lemma 4.1. Note that

$$\begin{aligned}\int_{\Omega\cap B_{\frac{\sigma}{4}}(P_0)} \left[|\nabla v_n|^2 + \lambda_n v_n^2\right] &= k^{\frac{2}{2^\star}} Q_M^{\frac{2}{2^\star}} \int_{\Omega\cap B_{\frac{\sigma}{4}}(P_0)} \left[|\nabla u_n|^2 + \lambda_n u_n^2\right] \\ &< k^{\frac{2}{2^\star}} Q_M^{\frac{2}{2^\star}} \frac{\hat{S}}{k} = S\end{aligned}$$

and

$$\lim_{n\to\infty} \left|\left|\nabla v_n - \nabla\left[\sigma_n^{-\frac{N-2}{2}} S^{-\frac{N-2}{4}} U\left(\frac{\cdot - P_n}{\sigma_n}\right)\right]\right|\right|_{L^2\left(\Omega\cap B_{\frac{\sigma}{4}}(P_0)\right)} = 0,$$

with $\sigma_n = \frac{\varepsilon_n}{S^{\frac{1}{2}} k^{\frac{1}{N}} Q_M^{\frac{1}{N}}}$. Using Lemma 4.5, with Ω replaced by $\Omega \cap B_{\frac{\sigma}{4}}(P_0)$, we get sequences (δ_n) and (y_n), $\delta_n > 0$, $y_n \in \Omega$, such that, modulo a subsequence, $\delta_n/\sigma_n \to 1$, $y_n \to P_0$ and

$$\int_{\Omega\cap B_{\frac{\sigma}{4}}(P_0)} \left[|\nabla v_n|^2 + \lambda_n v_n^2\right] \geq O(\delta_n^2) + o(\lambda_n \delta_n^2)$$

$$+ \int_{\Omega\cap B_{\frac{\sigma}{4}}(P_0)} \left[\left|\nabla\left(\frac{U_{\delta_n,y_n}}{||U_{\delta_n,y_n}||_{L^{2^\star}(\mathbb{R}^N)}}\right)\right|^2 + \lambda_n\left(\frac{U_{\delta_n,y_n}}{||U_{\delta_n,y_n}||_{L^{2^\star}(\mathbb{R}^N)}}\right)^2\right].$$

Lemma 4.6 implies that $\int_{\Omega\cap B_{\frac{\sigma}{4}}(P_0)} \left[|\nabla v_n|^2 + \lambda_n v_n^2\right] > S$, for n large. Hence $E_{\lambda_n}(u_n) > k\frac{1}{k^{\frac{2}{2^\star}}}\frac{1}{Q_M^{\frac{2}{2^\star}}} S = \hat{S}$. We have reached a contradiction. Therefore $\Lambda < \infty$. □

In the case that the equality $Q_M = k^{\frac{2}{N-2}} Q(0)$ holds, by Lemma 4.4 we get

Theorem 4.9. *Assume $N \geq 5$. Suppose $Q_M = k^{\frac{2}{N-2}} Q(0) > 2^{\frac{2}{N-2}} Q_m$. Then $\Lambda < \infty$.*

5. Sobolev inequalities

From the results of Section 4, we derive two Sobolev inequalities. From Theorem 4.7 we get

Corollary 5.1. *Assume $N \geq 5$. Let Ω be a G-invariant domain with $0 \in \Omega$ and Q be Hölder continuous on $\bar{\Omega}$, G-invariant, nonnegative and satisfy $Q(0) > 2^{\frac{2}{N-2}} Q_m \Big/ k^{\frac{2}{N-2}}$, $Q(0) \geq Q_M \Big/ k^{\frac{2}{N-2}}$. Then there exists a $\Lambda(\Omega, Q)$ such that, for all $u \in H^1_G(\Omega)$,*

$$\left(\int_\Omega Q|u|^{2^\star}\right)^{\frac{2}{2^\star}} \leq \frac{Q(0)^{\frac{N-2}{N}}}{S}\left(\int_\Omega |\nabla u|^2 + \Lambda(\Omega, Q)\int_\Omega u^2\right).$$

From Theorem 4.8 we get

Corollary 5.2. *Assume $N \geq 5$. Let Ω be a G-invariant domain with $0 \in \Omega$ and Q be Hölder continuous on $\bar{\Omega}$, G-invariant, nonnegative and satisfy $Q_M > 2^{\frac{2}{N-2}} Q_m$, $Q_M \geq k^{\frac{2}{N-2}} Q(0)$. Then there exists a $\Lambda(\Omega, Q)$ such that, for all $u \in H^1_G(\Omega)$,*

$$\left(\int_\Omega Q|u|^{2^\star}\right)^{\frac{2}{2^\star}} \leq \frac{Q_M^{\frac{N-2}{N}}}{k^{\frac{2}{N}} S}\left(\int_\Omega |\nabla u|^2 + \Lambda(\Omega, Q)\int_\Omega u^2\right).$$

Remark. If $0 \notin \Omega$ the same result holds, provided one ignores the condition relating Q_M and $Q(0)$.

Appendix

For the reader's convenience, in this appendix we give a proof of Struwe's compactness Lemma [20] for our Neumann problem. We follow the argument of Theorem 8.13 in Willem's book [27]. See also Bahri and Lions [6] and Clapp [10].

Throughout this appendix we denote by N a fixed integer, $N \geq 3$. Let Ω be a smooth bounded domain of $\mathbb{R}^N$. Let a, $Q : \bar{\Omega} \to \mathbb{R}$ be Hölder continuous, with a positive and Q nonnegative and not identically equal to zero. We denote $Q_M = \max_{\bar{\Omega}} Q$. In H^1 we use the norm $||u||^2_{H^1} = |\nabla u|^2_{L^2} + |u|^2_{L^2}$. As usual, let $\mathcal{D}^{1,2}(\mathbb{R}^N) := \{u \in L^{2^\star}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ with norm $||u||^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla u|^2$.

Define

$$\varphi(u) := \int_\Omega \left[\frac{|\nabla u|^2}{2} + a\frac{u^2}{2} - Q\frac{|u|^{2^\star}}{2^\star}\right], \quad u \in H^1(\Omega),$$

and, for a smooth domain S of $\mathbb{R}^N$ and $\tilde{Q} \in C(\bar{S})$,

$$\psi_{\tilde{Q}}(u) := \int_S \left[\frac{|\nabla u|^2}{2} - \tilde{Q}\frac{|u|^{2^\star}}{2^\star}\right], \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

We will start by proving a few Lemmas. We will use the following assumption, which will later be satisfied.

(A): (y_n) is a sequence in $\bar{\Omega}$, $y_n \to y_0$, (λ_n) is a sequence in $\mathbb{R}^+$, $\lambda_n \to 0$, and $\frac{1}{\lambda_n}\operatorname{dist}(y_n, \partial\Omega)$ converges in the extended nonnegative real line.

Under (A) it follows that $\chi_{\frac{\Omega - y_n}{\lambda_n}} \to \chi_{S_0}$ pointwise, where S_0 is a half space or $S_0 = \mathbb{R}^N$; χ_S denotes the characteristic function of the set S.

Lemma A.1. *(Brezis-Lieb Lemma). Assume* (A). *If (u_n) is bounded in $L^p(\mathbb{R}^N)$, $1 \leq p < \infty$, and $u_n \to u$ a.e. on $\mathbb{R}^N$, then*

$$\lim_{n\to\infty}\left(\int_{\frac{\Omega-y_n}{\lambda_n}} Q(\lambda_n x + y_n)|u_n|^p - \int_{\frac{\Omega-y_n}{\lambda_n}} Q(\lambda_n x + y_n)|u_n - u|^p\right) = \int_{S_0} Q(y_0)|u|^p.$$

Proof. By Fatou's Lemma, $|u|_{L^p(\mathbb{R}^N)} < \infty$. For each $\varepsilon > 0$ there exists a $c(\varepsilon)$ such that, for all a, b in $\mathbb{R}$,

$$||a+b|^p - |a|^p| \le \varepsilon|a|^p + c(\varepsilon)|b|^p.$$

Taking $a = u_n - u$ and $b = u$,

$$\begin{aligned} f_n^\varepsilon &:= (|Q(\lambda_n x + y_n)|u_n|^p - Q(\lambda_n x + y_n)|u_n - u|^p - Q(y_0)|u|^p| - \\ &\qquad \varepsilon Q(\lambda_n x + y_n)|u_n - u|^p)^+ \\ &\le (|Q(\lambda_n x + y_n)|u_n|^p - Q(\lambda_n x + y_n)|u_n - u|^p| - \\ &\qquad \varepsilon Q(\lambda_n x + y_n)|u_n - u|^p)^+ + Q(y_0)|u|^p \\ &\le Q_M c(\varepsilon)|u|^p + Q_M|u|^p = Q_M(1 + c(\varepsilon))|u|^p. \end{aligned}$$

By Lebesgue's dominated convergence theorem, $\int_{\frac{\Omega - y_n}{\lambda_n}} f_n^\varepsilon \to 0$. Since

$$\begin{aligned} &|Q(\lambda_n x + y_n)|u_n|^p - Q(\lambda_n x + y_n)|u_n - u|^p - Q(y_0)|u|^p| \\ &\qquad \le f_n^\varepsilon + \varepsilon Q(\lambda_n x + y_n)|u_n - u|^p \end{aligned}$$

we obtain

$$\limsup_{n\to\infty} \int_{\frac{\Omega - y_n}{\lambda_n}} |Q(\lambda_n x + y_n)|u_n|^p - Q(\lambda_n x + y_n)|u_n - u|^p - Q(y_0)|u|^p| \le$$

$$0 + \varepsilon Q_M \sup |u_n - u|_{L^p(\mathbb{R}^N)}$$

Letting $\varepsilon \to 0$ the result follows. □

Note. The same proof shows that if (u_n) is bounded in $L^p(\Omega)$, $1 \le p < \infty$, and $u_n \to u$ a.e. on Ω, then

$$\lim_{n\to\infty} \left(\int_\Omega Q|u_n|^p - \int_\Omega Q|u_n - u|^p \right) = \int_\Omega Q|u|^p.$$

Lemma A.2. *Suppose* (A) *is satisfied and* $\chi_{\frac{\Omega - y_n}{\lambda_n}} \to \chi_{S_0}$. *Suppose also that* $u_n \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $u_n \rightharpoonup u$ *in* $\mathcal{D}^{1,2}(\mathbb{R}^N)$, *with* $u \in L^\infty_{loc}(\bar{S}_0)$. *Let* $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ *and* $w_n(x) := \lambda_n^{\frac{N-2}{2}} w(\lambda_n x + y_n)$. *Then, for any* $\varepsilon > 0$ *there exists a* $p \in \mathbb{N}$ *such that* $n > p$ *implies*

$$\begin{aligned} \Big| \int_{\frac{\Omega - y_n}{\lambda_n}} Q(\lambda_n x + y_n)|u_n|^{2^\star - 2} u_n w_n - \int_{\frac{\Omega - y_n}{\lambda_n}} Q(\lambda_n x + y_n)|u_n - u|^{2^\star - 2}(u_n - u) w_n \\ - \int_{\frac{\Omega - y_n}{\lambda_n}} Q(y_0)|u|^{2^\star - 2} u w_n \Big| < \varepsilon ||w||_{H^1(\Omega)} \end{aligned}$$

Proof. • By the mean value theorem

$$\Big| Q(\lambda_n x + y_n)|u_n|^{2^\star - 2} u_n - Q(\lambda_n x + y_n)|u_n - u|^{2^\star - 2}(u_n - u) \Big| =$$

$$(2^\star - 1) Q(\lambda_n x + y_n)|u_n - \theta_x u|^{2^\star - 2}|u| \le (2^\star - 1) Q_M [|u_n| + |u|]^{2^\star - 2}|u|,$$

with $\theta_x \in (0, 1)$. Let $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.

$$\begin{aligned} &\Big| \int_{\frac{\Omega - y_n}{\lambda_n} \setminus B_R(0)} Q(\lambda_n x + y_n)|u_n|^{2^\star - 2} u_n w_n \\ &\quad - \int_{\frac{\Omega - y_n}{\lambda_n} \setminus B_R(0)} Q(\lambda_n x + y_n)|u_n - u|^{2^\star - 2}(u_n - u) w_n \\ &\quad - \int_{\frac{\Omega - y_n}{\lambda_n} \setminus B_R(0)} Q(y_0)|u|^{2^\star - 2} u w_n \Big| \\ &\qquad \le c Q_M [|u_n|_{2^\star}^{2^\star - 2} + |u|_{2^\star}^{2^\star - 2}] \left(\int_{\frac{\Omega - y_n}{\lambda_n} \setminus B_R(0)} |u|^{2^\star} \right)^{\frac{1}{2^\star}} |w_n|_{L^{2^\star}(\frac{\Omega - y_n}{\lambda_n})} \\ &\qquad\quad + Q_M \left(\int_{\frac{\Omega - y_n}{\lambda_n} \setminus B_R(0)} |u|^{2^\star} \right)^{\frac{2^\star - 1}{2^\star}} |w_n|_{L^{2^\star}(\frac{\Omega - y_n}{\lambda_n})} \end{aligned}$$

For every $\varepsilon > 0$, there exists $R > 0$ such that for every $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ the last expression is less than $\frac{\varepsilon}{2}||w||_{H^1(\Omega)}$.

• Let $M := \sup_{S_0 \cap B_R(0)} |u|$.

$$\Big| Q(\lambda_n x + y_n)|u_n|^{2^\star - 2} u_n - Q(\lambda_n x + y_n)|u_n - u|^{2^\star - 2}(u_n - u) \Big| \le$$

$$Q_M(2^\star-1)(|u_n|+M)^{2^\star-2}M.$$

Suppose $1\le p<\frac{N}{2}$. Since $u_n\to u$ in $L^{(2^\star-2)p}_{\rm loc}$ because of Rellich's theorem, Krasnoselski's theorem implies that

$$\begin{aligned} z_n &:= Q(\lambda_n x+y_n)|u_n|^{2^\star-2}u_n-Q(\lambda_n x+y_n)|u_n-u|^{2^\star-2}(u_n-u)\\ &\quad -Q(y_0)|u|^{2^\star-2}u\\ &\to 0\end{aligned}$$

in $L^p(S_0\cap B_R(0))$, and in particular for $p=\frac{9N}{20}$, so $\int_{B_R(0)}|z_n|^{\frac{2N}{5}}\chi_{\frac{\Omega-y_n}{\lambda_n}}\to 0$. Now

$$\begin{aligned}&\Big|\int_{\frac{\Omega-y_n}{\lambda_n}\cap B_R(0)}Q(\lambda_n x+y_n)|u_n|^{2^\star-2}u_nw_n\\ &\quad-\int_{\frac{\Omega-y_n}{\lambda_n}\cap B_R(0)}Q(\lambda_n x+y_n)|u_n-u|^{2^\star-2}(u_n-u)w_n\\ &\quad-\int_{\frac{\Omega-y_n}{\lambda_n}\cap B_R(0)}Q(y_0)|u|^{2^\star-2}uw_n\Big|\\ &\qquad\le \big|Q(\lambda_n x+y_n)|u_n|^{2^\star-2}u_n-Q(\lambda_n x+y_n)|u_n-u|^{2^\star-2}(u_n-u)\\ &\qquad\qquad -Q(y_0)|u|^{2^\star-2}u\big|_{L^{\frac{2N}{5}}\left(\frac{\Omega-y_n}{\lambda_n}\cap B_R(0)\right)}|w_n|_{L^{\frac{2N}{2N-5}}\left(\frac{\Omega-y_n}{\lambda_n}\cap B_R(0)\right)}\end{aligned}$$

and $|w_n|_{L^{\frac{2N}{2N-5}}\left(\frac{\Omega-y_n}{\lambda_n}\cap B_R(0)\right)}\le c||w||_{H^1(\Omega)}$.

• Combining the two previous results, we get the assertion in the statement of the lemma.

Note. The same proof shows that if $u_n\rightharpoonup u$ in $H^1(\Omega)$, with $u\in L^\infty(\Omega)$, then

$$Q(x)|u_n|^{2^\star-2}u_n-Q(x)|u_n-u|^{2^\star-2}(u_n-u)\to Q(x)|u|^{2^\star-2}u$$

in $H^{-1}(\Omega)$. □

Lemma A.3. *Suppose that*

$$\begin{aligned}&u_n\rightharpoonup u &&\textit{in } H^1(\Omega),\\ &u_n\to u &&\textit{a.e. on } \Omega,\\ &\varphi(u_n)\to c, &&\\ &\varphi'(u_n)\to 0 &&\textit{in } H^{-1}(\Omega).\end{aligned}$$

Then $\varphi'(u)=0$ *and* $v_n:=u_n-u$ *is such that*

$$\begin{aligned}&||v_n||^2_{H^1(\Omega)}=||u_n||^2_{H^1(\Omega)}-||u||^2_{H^1(\Omega)}+o(1),\\ &\psi_Q(v_n)\to c-\varphi(u),\\ &\psi'_Q(v_n)\to 0 \qquad \textit{in } H^{-1}(\Omega).\end{aligned}$$

Proof. • Since $v_n\rightharpoonup 0$ in $H^1(\Omega)$,

$$\begin{aligned}||v_n||^2_{H^1(\Omega)} &=\int_\Omega|\nabla u_n-\nabla u|^2+\int_\Omega|u_n-u|^2\\ &=||u_n||^2_{H^1(\Omega)}+||u||^2_{H^1(\Omega)}-2\int_\Omega\nabla u_n\nabla u-2\int_\Omega u_nu\\ &=||u_n||^2_{H^1(\Omega)}-||u||^2_{H^1(\Omega)}+o(1)\end{aligned}$$

• Since $v_n\rightharpoonup 0$ in $H^1(\Omega)$, (v_n) is bounded in $L^{2^\star}(\Omega)$, $v_n^2\rightharpoonup 0$ in $L^{\frac{N}{N-2}}$ and so $\int_\Omega av_n^2\to 0$. Using the Brezis-Lieb Lemma,

$$\begin{aligned}\psi_Q(v_n) &=\int_\Omega\left[\frac{|\nabla u_n-\nabla u|^2}{2}-Q\frac{|u_n-u|^{2^\star}}{2^\star}\right]\\ &=\int_\Omega\left[\frac{|\nabla u_n|^2}{2}-\frac{|\nabla u|^2}{2}-Q\frac{|u_n|^{2^\star}}{2^\star}+Q\frac{|u|^{2^\star}}{2^\star}\right]+o(1)\\ &=\varphi(u_n)-\varphi(u)-\int_\Omega au_n^2+\int_\Omega au^2+o(1)\\ &=\varphi(u_n)-\varphi(u)-\int_\Omega av_n^2+o(1)\\ &=\varphi(u_n)-\varphi(u)+o(1).\end{aligned}$$

• $\varphi'(u_n)\to 0$ in $H^{-1}(\Omega)$ implies that, for all $w\in H^1(\Omega)$,

$$0=\lim_{n\to\infty}\varphi'(u_n)w=\lim_{n\to\infty}\int_\Omega\left[\nabla u_n\nabla w+au_nw-Q|u_n|^{2^\star-2}u_nw\right]=\varphi'(u)w,$$

because $u_n \rightharpoonup u$ in $H^1(\Omega)$. So $\varphi'(u) = 0$. This implies that $u \in C^2(\bar\Omega)$. On the other hand, the preceding lemma implies that, for all $w \in H^1(\Omega)$,

$$\begin{aligned}
\psi_Q'(v_n)w &= \textstyle\int_\Omega \left[\nabla v_n \nabla w - Q|v_n|^{2^\star-2} v_n w\right] \\
&= \textstyle\int_\Omega \left[\nabla u_n \nabla w - \nabla u \nabla w - Q|u_n|^{2^\star-2} u_n w + Q|u|^{2^\star-2} u w\right] \\
&\quad + o(||w||_{H^1(\Omega)}) \\
&= \varphi'(u_n)w - \varphi'(u)w - \textstyle\int_\Omega a u_n w + \int_\Omega a u w + o(||w||_{H^1(\Omega)}) \\
&= o(||w||_{H^1(\Omega)}).
\end{aligned}$$

□

Lemma A.4. *Assume* (A) *and* $\chi_{\frac{\Omega - y_n}{\lambda_n}} \to \chi_{S_0}$. *Suppose* $u_n \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $v_n(x) := \lambda_n^{\frac{N-2}{2}} u_n(\lambda_n x + y_n)$ *and*

$$\begin{aligned}
&v_n \rightharpoonup v && \text{in } \mathcal{D}^{1,2}(\mathbb{R}^N), \\
&v_n \to v && \text{a.e on } \mathbb{R}^N, \\
&\psi_Q(u_n) \to c, && \\
&\psi_Q'(u_n) \to 0 && \text{in } H^{-1}(\Omega).
\end{aligned}$$

Then

$$-\Delta v = Q(y_0)|v|^{2^\star-2} v, \ \text{in } S_0, \qquad \frac{\partial v}{\partial \nu} = 0 \ \text{on } \partial S_0,$$

and the sequence

$$w_n(x) := u_n(x) - \lambda_n^{\frac{2-N}{2}} v\left(\frac{x - y_n}{\lambda_n}\right)$$

satisfies

$$\begin{aligned}
&||w_n||^2_{H^1(\Omega)} = ||u_n||^2_{H^1(\Omega)} - |\nabla v|^2_{L^2(S_0)} + o(1), \\
&\psi_Q(w_n) \to c - \psi_{Q(y_0)}(v), \\
&\psi_Q'(w_n) \to 0 \ \text{in } H^{-1}(\Omega),
\end{aligned}$$

where the integral in $\psi_{Q(y_0)}(v)$ *is computed in* S_0.

Proof. • Note that $v\left(\frac{x-y_n}{\lambda_n}\right) \in L^{2^\star}(\Omega) \subset L^2(\Omega)$. Since $v_n \rightharpoonup v$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$,

$$\begin{aligned}
||w_n||^2_{H^1(\Omega)} &= \textstyle\int_\Omega |\nabla w_n|^2 + \int_\Omega |w_n|^2 \\
&= \textstyle\int_{\frac{\Omega-y_n}{\lambda_n}} |\nabla (v_n - v)|^2 + |u_n - \lambda_n^{\frac{2-N}{2}} v\left(\frac{x-y_n}{\lambda_n}\right)|^2_{L^2(\Omega)} \\
&= \textstyle\int_{\frac{\Omega-y_n}{\lambda_n}} |\nabla v_n|^2 - \int_{S_0} |\nabla v|^2 + |u_n|^2_{L^2(\Omega)} + o(1) \\
&= \textstyle\int_\Omega |\nabla u_n|^2 + |u_n|^2_{L^2(\Omega)} - \int_{S_0} |\nabla v|^2 + o(1) \\
&= ||u_n||^2_{H^1(\Omega)} - |\nabla v|^2_{L^2(S_0)} + o(1).
\end{aligned}$$

• The Brezis-Lieb Lemma yields,

$$\begin{aligned}
\psi_Q(w_n) &= \textstyle\int_\Omega \frac{|\nabla w_n|^2}{2} - \int_\Omega Q \frac{|w_n|^{2^\star}}{2^\star} \\
&= \textstyle\int_{\frac{\Omega-y_n}{\lambda_n}} \frac{|\nabla (v_n - v)|^2}{2} - \int_{\frac{\Omega-y_n}{\lambda_n}} Q(\lambda_n x + y_n) \frac{|v_n - v|^{2^\star}}{2^\star} \\
&= \textstyle\int_{\frac{\Omega-y_n}{\lambda_n}} \frac{|\nabla v_n|^2}{2} - \int_{\frac{\Omega-y_n}{\lambda_n}} Q(\lambda_n x + y_n) \frac{|v_n|^{2^\star}}{2^\star} - \\
&\quad - \textstyle\int_{S_0} \frac{|\nabla v|^2}{2} + \int_{S_0} Q(y_0) \frac{|v|^{2^\star}}{2^\star} + o(1) \\
&= \psi_Q(u_n) - \psi_{Q(y_0)}(v) + o(1) \\
&= c - \psi_{Q(y_0)}(v) + o(1).
\end{aligned}$$

• For $z \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ let $z_n = \lambda_n^{\frac{2-N}{2}} z\left(\frac{x-y_n}{\lambda_n}\right)$. Since $\psi_Q'(u_n) \to 0$ in $H^{-1}(\Omega)$ and $v_n \rightharpoonup v$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$,

$$\begin{aligned}
0 &= \lim \psi_Q'(u_n) z_n = \lim \textstyle\int_\Omega [\nabla u_n \nabla z_n - Q|u_n|^{2^\star-2} u_n z_n] \\
&= \lim \textstyle\int_{\frac{\Omega-y_n}{\lambda_n}} [\nabla v_n \nabla z - Q(\lambda_n x + y_n)|v_n|^{2^\star-2} v_n z] \\
&= \textstyle\int_{S_0} [\nabla v \nabla z - Q(y_0)|v|^{2^\star-2} v z],
\end{aligned}$$

i.e., $\psi'_{Q(y_0)}(v) = 0$. Therefore $v \in C^2(\bar{S}_0)$. Let $z \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ and $z_n(x) := \lambda_n^{\frac{N-2}{2}} z(\lambda_n x + y_n)$. By Lemma A.2,

$$\begin{aligned} |\psi'_Q(w_n)z| &= |\psi'_{Q(\lambda_n x+y_n)}(v_n - v)z_n| \\ &= |\psi'_{Q(\lambda_n x+y_n)}(v_n)z_n - \psi'_{Q(y_0)}(v)z_n| + o(||z||_{H^1(\Omega)}) \\ &= |\psi'_{Q(\lambda_n x+y_n)}(v_n)z_n| + o(||z||_{H^1(\Omega)}) \\ &= |\psi'_Q(u_n)z| + o(||z||_{H^1(\Omega)}) \\ &= o(||z||_{H^1(\Omega)}). \end{aligned}$$

□

Lemma A.5. *Suppose S is a half space and B_1 is a ball with radius one whose center belongs to $\bar{S}$. Let $\tilde{Q} \in C(S)$, $\tilde{Q}_M := \sup \tilde{Q} < \infty$, $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ and $v \in \mathcal{D}(B_1)$. There is a constant σ such that*

$$\int_S \tilde{Q}v^2|u|^{2^\star} \leq \sigma \tilde{Q}_M^{\frac{N-2}{N}} \left(\int_{S\cap \text{supp } v} \tilde{Q}|u|^{2^\star} \right)^{\frac{2}{N}} \int_S |\nabla(vu)|^2.$$

Proof. By Poincaré's inequality,

$$\begin{aligned} \int_S \tilde{Q}v^2|u|^{2^\star} &= \int_S [(\tilde{Q}^{\frac{2}{N}}|u|^{\frac{4}{N-2}})(\tilde{Q}^{\frac{N-2}{N}} u^2 v^2)] \\ &\leq \left(\int_{S\cap \text{supp } v} \tilde{Q}|u|^{2^\star} \right)^{\frac{2}{N}} \left(\int_S \tilde{Q}|vu|^{2^\star} \right)^{\frac{N-2}{N}} \\ &\leq \sigma \tilde{Q}_M^{\frac{N-2}{N}} \left(\int_{S\cap \text{supp } v} \tilde{Q}|u|^{2^\star} \right)^{\frac{2}{N}} \int_S |\nabla(vu)|^2. \end{aligned}$$

□

Finally, we can prove the main result of this appendix.

Theorem A.6. *Suppose the conditions in the second paragraph of this appendix are satisfied. Let (u_n) be a sequence in $H^1(\Omega)$ such that*

$$\varphi(u_n) \to c \qquad \text{and} \qquad \varphi'(u_n) \to 0 \text{ in } H^{-1}(\Omega).$$

Then, replacing (u_n) by a subsequence, if necessary, there exist

- *a function $v_0 \in H^1(\Omega)$ such that*

$$-\Delta v_0 + a v_0 = Q|v_0|^{2^\star-2}v_0,$$

- *m points $y_0^i \in \Omega$ and m sets S_i, $i = 1, \dots, m$, where each S_i is either a half space or $\mathbb{R}^N$*
- *for each $i = 1, \dots, m$, a function $v_i \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that*

$$-\Delta v_i = Q(y_0^i)|v_i|^{2^\star-2}v_i \text{ in } S_i, \qquad \frac{\partial v_i}{\partial \nu} = 0 \text{ on } \partial S_i,$$

- *and sequences (y_n^i), (λ_n^i), $i = 1, \dots, m$, satisfying $y_n^i \in \bar{\Omega}$, $y_n^i \to y_0^i$, $\lambda_n^i > 0$, $\lambda_n^i \to 0$,*

$$\frac{1}{\lambda_n^i} dist(y_n^i, \partial\Omega) \to d \in \mathbb{R}_0^+ \qquad \text{or} \qquad \frac{1}{\lambda_n^i} dist(y_n^i, \partial\Omega) \to \infty,$$

satisfying

$$\begin{aligned} &\left|\left|u_n - v_0 - \sum_{i=1}^m (\lambda_n^i)^{\frac{2-N}{2}} v_i \left(\frac{x-y_n^i}{\lambda_n^i}\right)\right|\right|_{H^1(\Omega)} \to 0, \\ &||u_n||^2_{H^1(\Omega)} \to ||v_0||^2_{H^1(\Omega)} + \sum_{i=1}^m |\nabla v_i|^2_{L^2(S_i)}, \\ &\varphi(v_0) + \sum_{i=1}^m \psi_{Q(y_0^i)}(v_i) = c. \end{aligned}$$

If $u_n \geq 0$ a.e. on Ω for all n, then $v_0 \geq 0$ and $v_i > 0$ for $i = 1, \dots, m$.

Proof. • For n big,

$$\begin{aligned}\sup\varphi(u_n)+||u_n||_{H^1(\Omega)} &\geq \varphi(u_n)-\frac{1}{2^\star}\varphi'(u_n)u_n\\ &= \frac{1}{N}\int_\Omega |\nabla u_n|^2+a|u_n|^2\\ &\geq \frac{c}{N}||u_n||^2_{H^1(\Omega)}.\end{aligned}$$

So $||u_n||_{H^1(\Omega)}$ is bounded. We extend each u_n to $H^1(\mathbb{R}^N)$, with $||u_n||_{H^1(\mathbb{R}^N)} \leq C||u_n||_{H^1(\Omega)} \leq C$.

• We assume that $u_n \rightharpoonup v_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $u_n \to v_0$ a.e. on $\mathbb{R}^N$. By Lemma A.3, $\varphi'(v_0)=0$, and $u_n^1 := u_n - v_0$ is such that

$$\begin{aligned}&||u_n^1||^2_{H^1(\Omega)} = ||u_n||^2_{H^1(\Omega)} - ||v_0||^2_{H^1(\Omega)} + o(1),\\ &\psi_Q(u_n^1)\to c-\varphi(v_0),\\ &\psi'_Q(u_n^1)\to 0 \qquad \text{in } H^{-1}(\Omega).\end{aligned}$$

• If $\int_\Omega Q|u_n^1|^{2^\star}\to 0$, then $u_n^1\to 0$ in $H^1(\Omega)$ and the proof is complete. If $\int_\Omega Q|u_n^1|^{2^\star}\not\to 0$, then choose $0<\delta<\left(2^{\frac{N}{2}}\sigma^{\frac{N}{2}}Q_M^{\frac{N-2}{2}}\right)^{-1}$ sufficiently small, so that

$$\int_\Omega Q|u_n^1|^{2^\star}>\delta.$$

Consider the Levy concentration function

$$\mathcal{Q}_n(r) := \sup_{y\in\bar\Omega}\int_{\Omega\cap B_r(y)} Q|u_n^1|^{2^\star}.$$

Since $\mathcal{Q}_n(0)=0$, $\mathcal{Q}_n(\infty)>\delta$ and $\mathcal{Q}_n$ is continuous, there exist sequences (y_n^1) and (λ_n^1) such that $y_n^1\in\bar\Omega$ and $\lambda_n^1>0$ for all n, and

$$\delta=\sup_{y\in\bar\Omega}\int_{\Omega\cap B_{\lambda_n^1}(y)} Q|u_n^1|^{2^\star} = \int_{\Omega\cap B_{\lambda_n^1}(y_n^1)} Q|u_n^1|^{2^\star}.$$

We may assume $y_n^1\to y_0^1\in\bar\Omega$, $\lambda_n^1\to\lambda_0^1\geq 0$ and $\frac{1}{\lambda_n}\mathrm{dist}(y_n,\partial\Omega)$ converges in the extended nonnegative real line. Then there exists a set S_1 such that $\chi_{\frac{\Omega-y_n^1}{\lambda_n^1}}\to\chi_{S_1}$.

Let $v_n^1(x) := (\lambda_n^1)^{\frac{N-2}{2}}u_n^1(\lambda_n^1 x+y_n^1)$. Since $||u_n^1||_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = ||v_n^1||_{\mathcal{D}^{1,2}(\mathbb{R}^N)}$, we may assume that $v_n^1\rightharpoonup v_1$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. Note that

$$\delta=\sup_{y\in\frac{\Omega-y_n^1}{\lambda_n^1}}\int_{\frac{\Omega-y_n^1}{\lambda_n^1}\cap B_1(y)} Q(\lambda_n^1x+y_n^1)|v_n^1|^{2^\star} = \int_{\frac{\Omega-y_n^1}{\lambda_n^1}\cap B_1(0)} Q(\lambda_n^1x+y_n^1)|v_n^1|^{2^\star}.$$

• Suppose $v_1=0$. Then $v_n^1\to 0$ in $L^2_{\mathrm{loc}}(S_1)$. We wish to prove $\nabla v_n^1\to 0$ in $L^2_{\mathrm{loc}}(S_1)$. Let $h\in\mathcal{D}(\mathbb{R}^N)$, with support contained in a ball of radius one. We have

$$\begin{aligned}\int_{\frac{\Omega-y_n^1}{\lambda_n^1}}|\nabla(hv_n^1)|^2 &= \int_{\frac{\Omega-y_n^1}{\lambda_n^1}}\nabla v_n^1\nabla(h^2v_n^1)+\int_{\frac{\Omega-y_n^1}{\lambda_n^1}}(v_n^1)^2|\nabla h|^2\\ &= \int_{\frac{\Omega-y_n^1}{\lambda_n^1}}\nabla v_n^1\nabla(h^2v_n^1)+o(1)\\ &= \int_{\frac{\Omega-y_n^1}{\lambda_n^1}} Q(\lambda_n^1x+y_n^1)h^2|v_n^1|^{2^\star}+\psi'_{Q(\lambda_n^1x+y_n^1)}(v_n^1)(h^2v_n^1)+o(1)\\ &= \int_{\frac{\Omega-y_n^1}{\lambda_n^1}} Q(\lambda_n^1x+y_n^1)h^2|v_n^1|^{2^\star}+o(1);\end{aligned}$$

using Lemma A.5 and appropriate diffeomorphisms,

$$\begin{aligned}
&\leq \sigma Q_M^{\frac{N-2}{N}} \left(\int_{\frac{\Omega - y_n^1}{\lambda_n^1} \cap \mathrm{supp}\, h} Q(\lambda_n^1 x + y_n^1)|v_n^1|^{2^\star} \right)^{\frac{2}{N}} \times \\
&\quad \times \int_{\frac{\Omega - y_n^1}{\lambda_n^1}} |\nabla(h v_n^1)|^2 + o(1) \\
&\leq \sigma Q_M^{\frac{N-2}{N}} \delta^{\frac{2}{N}} \int_{\frac{\Omega - y_n^1}{\lambda_n^1}} |\nabla(h v_n^1)|^2 + o(1) \\
&= \frac{1}{2} \int_{\frac{\Omega - y_n^1}{\lambda_n^1}} |\nabla(h v_n^1)|^2 + o(1).
\end{aligned}$$

Therefore $\nabla v_n^1 \to 0$ in $L^2_{\mathrm{loc}}(S_1)$, and, since $v_n^1 \to 0$ in $L^2_{\mathrm{loc}}(S_1)$, $v_n^1 \to 0$ in $L^{2^\star}_{\mathrm{loc}}(S_1)$. This contradicts $\int_{\frac{\Omega - y_n^1}{\lambda_n^1} \cap B_1(0)} Q(\lambda_n^1 x + y_n^1)|v_n^1|^{2^\star} = \delta > 0$. We conclude that $v_1 \neq 0$.

• If $\lambda_0^1 > 0$, then $u_n^1 \rightharpoonup 0$ in $H^1(\Omega)$ and $||u_n^1||_{H^1(\mathbb{R}^N)} \leq C$ implies $v_n^1 \rightharpoonup 0$ in $H^1(S_1)$. This is a contradiction. So $\lambda_0^1 = 0$ and S_1 is a half space or $S_1 = \mathbb{R}^N$.

• By Lemma A.4, the function v_1 satisfies

$$-\Delta v_1 = Q(y_0^1)|v_1|^{2^\star - 2} v_1 \text{ in } S_1, \qquad \frac{\partial v_1}{\partial \nu} = 0 \text{ on } \partial S_1.$$

If $S_1 = \mathbb{R}^N$,

$$S|v_1|^2_{L^{2^\star}(\mathbb{R}^N)} \leq |\nabla v_1|^2_{L^{2^\star}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} Q(y_0^1)|v_1|^{2^\star}.$$

Hence

$$\int_{\mathbb{R}^N} |v_1|^{2^\star} \geq \left(\frac{S}{Q(y_0^1)} \right)^{\frac{N}{2}}$$

and

$$\psi(v_1) = \frac{Q(y_0^1)}{N} \int_{\mathbb{R}^N} |v_1|^{2^\star} \geq \frac{1}{N} \frac{S^{\frac{N}{2}}}{Q(y_0^1)^{\frac{N-2}{2}}} \geq \frac{1}{N} \frac{S^{\frac{N}{2}}}{Q_M^{\frac{N-2}{2}}}.$$

On the other hand, if S_1 is a half space,

$$\int_{S_1} |v_1|^{2^\star} \geq \frac{1}{2} \left(\frac{S}{Q(y_0^1)} \right)^{\frac{N}{2}}$$

and

$$\psi(v_1) = \frac{Q(y_0^1)}{N} \int_{S_1} |v_1|^{2^\star} \geq \frac{1}{2N} \frac{S^{\frac{N}{2}}}{Q_m^{\frac{N-2}{2}}}.$$

• Also by Lemma A.4, the sequence

$$u_n^2(x) := u_n^1(x) - (\lambda_n^1)^{\frac{2-N}{2}} v_1 \left(\frac{x - y_n^1}{\lambda_n^1} \right)$$

satisfies

$$\begin{aligned}
&||u_n^2||^2_{H^1(\Omega)} = ||u_n||^2_{H^1(\Omega)} - ||v_0||^2_{H^1(\Omega)} - |\nabla v_1|^2_{L^2(S_1)} + o(1), \\
&\psi_Q(u_n^2) \to c - \varphi(v_0) - \psi_{Q(y_0^1)}(v_1), \\
&\psi'_Q(u_n^2) \to 0 \text{ in } H^{-1}(\Omega),
\end{aligned}$$

where the integrals in $\psi_{Q(y_0^1)}(v_1)$ are computed over S_1.

• We iterate the above procedure to find sequences (y_n^i), (λ_n^i), (y_0^i), (λ_0^i), (S_i) and (v_i) as in the statement of the Theorem. Since $|\nabla v_i|^2_{L^2(S_i)} \geq \frac{S^{\frac{N}{2}}}{2Q_M^{\frac{N-2}{2}}}$, the iteration must end at some finite index m.

• Suppose now that $u_n \geq 0$ for all n. Then clearly $v_0 \geq 0$. Using the functionals

$$\begin{aligned}\varphi(u) &:= \int_\Omega \left[\frac{|\nabla u|^2}{2} + a\frac{u^2}{2} - Q\frac{u_+^{2^\star}}{2^\star}\right], \qquad u \in H^1(\Omega),\\ \psi_Q(u) &:= \int_S \left[\frac{|\nabla u|^2}{2} - Q\frac{u_+^{2^\star}}{2^\star}\right], \qquad u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\end{aligned}$$

and the maximum principle, we conclude that the v_i's with $i \geq 1$ are positive and that they are necessarily instantons or half-instantons. □

Corollary A.7. *Let*

$$V_G(\Omega) := \left\{ u \in H^1_G(\Omega) \,\middle|\, \int_\Omega Q|u|^{2^\star} = 1 \right\}$$

and

$$E_\lambda(u) := \int_\Omega \left[|\nabla u|^2 + \lambda u^2\right].$$

Let

$$m_{\lambda,G} := \inf_{u\in V_G(\Omega)} E_\lambda(u).$$

If $m_{\lambda,G} < \hat{S} = \min\left(\frac{Sk^{\frac{2}{N}}}{2^{\frac{2}{N}}Q_m^{\frac{N-2}{N}}}, \frac{S}{Q(0)^{\frac{N-2}{N}}}, \frac{Sk^{\frac{2}{N}}}{Q_M^{\frac{N-2}{N}}}\right)$, *then* $m_{\lambda,G}$ *is achieved.*

Proof. Let (u_n) be a minimizing sequence. Without loss of generality, we can suppose that $u_n \geq 0$. By the Ekeland variational principle (see Theorem 8.5 in [27]), (u_n) is a Palais-Smale sequence for

$$\hat{\varphi}(u) := \int_\Omega \left[\frac{|\nabla u|^2}{2} + \lambda\frac{u^2}{2} - m_{\lambda,G}Q\frac{|u|^{2^\star}}{2^\star}\right], \quad u \in H^1(\Omega).$$

By Theorem A.6, there exists a function $v_0 \in H^1(\Omega)$ satisfying

$$-\Delta v_0 + \lambda v_0 = m_{\lambda,G}Q|v_0|^{2^\star-2}v_0,$$

m points $y_0^i \in \Omega$, m sets S_i, where each S_i is either a half space or $\mathbb{R}^N$, and m functions $v_i \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ satisfying

$$-\Delta v_i = m_{\lambda,G}Q(y_0^i)|v_i|^{2^\star-2}v_i, \qquad \frac{\partial v_i}{\partial \nu} = 0 \text{ on } \partial S_i,$$

such that

$$\hat{\varphi}(v_0) + \sum_{i=1}^m \psi_{m_{\lambda,G}Q(y_0^i)}(v_i) = \frac{m_{\lambda,G}}{N}. \tag{10}$$

We easily conclude (cf. [12]) that

$$v_i = \left(\frac{1}{m_{\lambda,G}Q(y_0^i)}\right)^{\frac{N-2}{4}} U.$$

Multiplying Eq. (10) by N and taking into account the invariance of $V_G(\Omega)$,

$$E_\lambda(v_0) + l_1\left(m_{\lambda,G}Q(0)\right)^{\frac{2-N}{2}} S^{\frac{N}{2}} + \sum_{[y_0^i]\in(\Omega\setminus\{0\})/G} \#G(y_0^i)\left(m_{\lambda,G}Q(y_0^i)\right)^{\frac{2-N}{2}} S^{\frac{N}{2}} +$$

$$+ \sum_{[y_0^i]\in\partial\Omega/G} \#G(y_0^i)\left(m_{\lambda,G}Q(y_0^i)\right)^{\frac{2-N}{2}} \frac{S^{\frac{N}{2}}}{2} = m_{\lambda,G},$$

where l_1 is either 1 or 0, according to if there exists a $y_0^i = 0$ or not. If any of the v_i's, with $i \geq 1$, is nonzero, then

$$l_1 \left(m_{\lambda,G} Q(0)\right)^{\frac{2-N}{2}} S^{\frac{N}{2}} + \sum_{[y_0^i]\in(\Omega\setminus\{0\})/G} k \left(m_{\lambda,G} Q_M\right)^{\frac{2-N}{2}} S^{\frac{N}{2}} +$$

$$+ \sum_{[y_0^i]\in\partial\Omega/G} k \left(m_{\lambda,G} Q_m\right)^{\frac{2-N}{2}} \frac{S^{\frac{N}{2}}}{2} \leq m_{\lambda,G}.$$

Solving for $m_{\lambda,G}$, we get $\hat{S} \leq m_{\lambda,G}$, which contradicts our initial assumption. Therefore all the v_i's with $i \geq 1$ are zero, and it follows, again from Theorem A.6, that u_n converges to v_0 in $H^1(\Omega)$. But then, of course, $v_0 \in V_G(\Omega)$ and $E_\lambda(v_0) = m_{\lambda,G}$. □

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