THE NONLINEAR NEUMANN PROBLEM AND SHARP WEIGHTED SOBOLEV INEQUALITIES

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Abstract. We prove sharp inequalities in weighted Sobolev spaces. Our approach is based on the blow-up technique applied to some nonlinear Neumann problems.

1. Introduction

The main purpose of this work is to prove some new weighted Sobolev inequalities. These inequalities are obtained by analyzing the asymptotic behaviour of solutions of nonlinear Neumann problems involving critical Sobolev exponent.

Let S denote the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^N)$ into $L^{2^\star}(\mathbb{R}^N)$, where $2^\star = \frac{2N}{N-2}$, $N \geq 3$, that is,

$$S = \inf \left\{ \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{\left(\int_{\mathbb{R}^N} |u|^{2^\star} \, dx \right)^{\frac{2}{2^\star}}} ; \, u \in D^{1,2}\left(\mathbb{R}^N\right) \setminus \{0\} \right\}. \tag{1}$$

It is known that the constant S is attained by

$$U(x) = \left[\frac{N(N-2)}{N(N-2) + |x|^2} \right]^{\frac{N-2}{2}}.$$

The function U, called an instanton, satisfies the equation

$$-\Delta U = U^{2^\star - 1} \ \text{ in } \ \mathbb{R}^N.$$

We also have $\int_{\mathbb{R}^N} |\nabla U|^2 \, dx = \int_{\mathbb{R}^N} U^{2^\star} \, dx = S^{\frac{N}{2}}$.

Let $\varepsilon > 0$ and $y \in \mathbb{R}^N$. We define

$$U_{\varepsilon,y}(x) = \varepsilon^{-\frac{N-2}{2}} U\left(\frac{x-y}{\varepsilon} \right).$$

Then any minimizer for S is of the form $U_{\varepsilon,y}$.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$. Let Q denote a Hölder continuous and positive function defined on $\bar{\Omega}$. Also let $Q_M = \max_{x \in \bar{\Omega}} Q(x)$ and $Q_m = \max_{x \in \partial\Omega} Q(x)$. In what follows we write $p + 1 = 2^\star$. In this paper the following inequalities are proved:

(I): Let $N \geq 5$. Suppose that $Q_M < 2^{\frac{2}{N-2}} Q_m$. Then there exists a constant $\Lambda_1(\Omega) > 0$ such that

$$\left(\int_\Omega Q(x) |u|^{p+1} \, dx \right)^{\frac{2}{p+1}} \leq \frac{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}{S} \int_\Omega |\nabla u|^2 \, dx + \Lambda_1(\Omega) \int_{\partial\Omega} u^2 \, dx$$

for every $u \in H^1(\Omega)$.

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(II): Let $N \geq 4$ and $\tau = \frac{2N}{N-1}$. Suppose that $Q_M \leq 2^{\frac{2}{N-2}} Q_m$. Then there exists a constant $\Lambda_2(\Omega) > 0$ such that

$$\left(\int_\Omega Q(x)|u|^{p+1}\,dx\right)^{\frac{2}{p+1}} \leq \frac{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}{S} \int_\Omega |\nabla u|^2\,dx + \Lambda_2(\Omega)\left(\int_\Omega |u|^\tau\,dx\right)^{\frac{2}{\tau}}$$

for every $u \in H^1(\Omega)$.

(III): Suppose $Q_M > 2^{\frac{2}{N-2}} Q_m$. Let $N \geq 5$ and $2 \leq \tau \leq \frac{2N}{N-1}$, or $N = 4$ and $2 < \tau \leq \frac{2N}{N-1}$. Then there exists a constant $\Lambda_3(\Omega) > 0$ such that

$$\left(\int_\Omega Q(x)|u|^{p+1}\,dx\right)^{\frac{2}{p+1}} \leq \frac{Q_M^{\frac{N-2}{N}}}{S} \int_\Omega |\nabla u|^2\,dx + \Lambda_3(\Omega)\left(\int_\Omega |u|^\tau\,dx\right)^{\frac{2}{\tau}}$$

for every $u \in H^1(\Omega)$.

These inequalities should be compared with the following inequalities established in the papers [6], [26]:

(A): There exists a constant $\lambda(\Omega) \geq \frac{N-2}{2} H(\Omega)$ such that

$$\left(\int_\Omega |u|^{p+1}\,dx\right)^{\frac{2}{p+1}} \leq \frac{2^{\frac{2}{N}}}{S} \int_\Omega |\nabla u|^2\,dx + \lambda(\Omega)\int_{\partial\Omega} u^2\,dx$$

for every $u \in H^1(\Omega)$, where $H(\Omega) = \max_{x\in\partial\Omega} H(x)$ and $H(x)$ denotes the mean curvature at $x \in \partial\Omega$.

(B): Let $\tau = \frac{2N}{N-1}$. Then there exists a constant $\tilde{\lambda}(\Omega) > 0$ such that

$$\left(\int_\Omega |u|^{p+1}\,dx\right)^{\frac{2}{p+1}} \leq \frac{2^{\frac{2}{N}}}{S} \int_\Omega |\nabla u|^2\,dx + \tilde{\lambda}(\Omega)\left(\int_\Omega |u|^\tau\,dx\right)^{\frac{2}{\tau}}$$

for every $u \in H^1(\Omega)$.

It is evident that none of these inequalities (I), (II), and (III) is a direct consequence of (A) and (B). We also point out that an inequality of type (III) in the case $Q_M \leq 2^{\frac{2}{N-2}} Q_m$ with $2 \leq \tau < \frac{2N}{N-1}$ is not possible. This will be clear from our analysis (see Proposition 2.7). The inequalities (I), (II), and (III) will be established by applying a blow-up technique to solutions of the following Neumann problems:

$$(1_\lambda) \quad \begin{cases} -\Delta u = Q(x)u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \lambda u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$(1_{\lambda,\tau}) \quad \begin{cases} -\Delta u + \lambda\left(\int_\Omega |u|^\tau\,dx\right)^{\frac{2}{\tau}-1} u^{\tau-1} & = Q(x)u^p \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where ν is the outward normal to $\partial\Omega$, $2 \leq \tau \leq \frac{2N}{N-1}$.

The proofs of the three inequalities are similar and proceed by contradiction. One assumes that least energy solutions of problems (1_λ) and $(1_{\lambda,\tau})$ exist for all positive λ and shows that they are close to some instantons. This enables one to give a lower bound for the energy of the solutions and to arrive at a contradiction.

In the case corresponding to inequality (I) the instantons concentrate at the boundary of Ω and therefore we can apply the arguments used in the proof of inequality (A). In the case corresponding to inequality (III) the instantons concentrate in the interior of Ω and the estimates are slightly different from the ones in the proof of inequality (B). In the case corresponding to inequality (II) the instantons either concentrate in the interior of Ω, or concentrate on the boundary of Ω, in which case the estimates are similar to those of (B).

This paper is organized as follows. Section 2 is concerned with the existence of least energy solutions of problems (1_λ) and $(1_{\lambda,\tau})$. In Section 3 we prove inequality (I) and in Section 4 we prove inequalities (II) and (III).

It is natural to ask if there exists a constant $\Lambda_4(\Omega) > 0$ such that

$$\left(\int_\Omega Q(x)|u|^{p+1}\,dx\right)^{\frac{2}{p+1}} \leq \frac{Q_M^{\frac{N-2}{N}}}{S}\int_\Omega |\nabla u|^2\,dx + \Lambda_4(\Omega)\int_{\partial\Omega} u^2\,dx,$$

for all $u \in H^1(\Omega)$, if $Q_M \geq 2^{\frac{2}{N-2}}Q_m$. We have not been able to answer this question.

2. Solvability of problems (1_λ) and $(1_{\lambda,\tau})$

Solutions of problems (1_λ) and $(1_{\lambda,\tau})$ will be obtained as minimizers on $H^1(\Omega)\setminus\{0\}$ of the functionals

$$J_\lambda(u) = \frac{\int_\Omega |\nabla u|^2\,dx + \lambda\int_{\partial\Omega} u^2\,dx}{\left(\int_\Omega Q(x)|u|^{p+1}\,dx\right)^{\frac{2}{p+1}}}$$

and

$$J_{\lambda,\tau}(u) = \frac{\int_\Omega |\nabla u|^2\,dx + \lambda\left(\int_\Omega |u|^\tau\,dx\right)^{\frac{2}{\tau}}}{\left(\int_\Omega Q(x)|u|^{p+1}\,dx\right)^{\frac{2}{p+1}}}.$$

We set

$$\begin{aligned} S_\lambda &= \inf\left\{J_\lambda(u);\ u \in H^1(\Omega)\setminus\{0\}\right\} \\ &= \inf\left\{\int_\Omega |\nabla u|^2\,dx + \lambda\int_{\partial\Omega} u^2\,dx;\ u \in V_Q\right\} \end{aligned}$$

and

$$\begin{aligned} S_{\lambda,\tau} &= \inf\left\{J_{\lambda,\tau}(u);\ u \in H^1(\Omega)\setminus\{0\}\right\} \\ &= \inf\left\{\int_\Omega |\nabla u|^2\,dx + \lambda\left(\int_\Omega |u|^\tau\,dx\right)^{\frac{2}{\tau}};\ u \in V_Q\right\}, \end{aligned}$$

where $V_Q = \left\{u \in H^1(\Omega);\ \int_\Omega Q(x)|u|^{p+1}\,dx = 1\right\}$.

To show that S_λ and $S_{\lambda,\tau}$ are achieved we need the following version of P.L. Lions' [14] concentration–compactness principle. Let $\{u_n\} \subset H^1(\Omega)$ be a weakly convergent sequence to u in $H^1(\Omega)$, $u_n \to u$ a.e., and such that $|u_n|^{p+1} \rightharpoonup \mu$ and $|\nabla u|^2 \rightharpoonup \tilde{\mu}$ weakly in the sense of measures. Then there exist numbers $\mu_j > 0$, $\tilde{\mu_j} > 0$ and points $x_j \in \bar{\Omega}$, $j \in J$, where J is at most a countable set, such that

$$\mu = |u|^{p+1} + \sum_{j\in J}\mu_j\delta_{x_j},$$

$$\tilde{\mu} \geq |\nabla u|^2 + \sum_{j\in J}\tilde{\mu}_j\delta_{x_j}.$$

Moreover, if $x_j \in \Omega$, then

$$S\big(\mu_j\big)^{\frac{N-2}{N}} \leq \tilde{\mu}_j \tag{2}$$

and if $x_j \in \partial\Omega$, then

$$\frac{S}{2^{\frac{2}{N}}}\big(\mu_j\big)^{\frac{N-2}{N}} \leq \tilde{\mu}_j. \tag{3}$$

The following lemmas give criteria for the existence of minima for J_λ and $J_{\lambda,\tau}$.

Lemma 2.1. *If*

$$S_\lambda < \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}} \tag{4}$$

and

$$Q_M < 2^{\frac{2}{N-2}} Q_m, \tag{5}$$

then S_λ is achieved.

Proof. Let $\{u_n\} \subset H^1(\Omega)$ be such that $\int_\Omega Q|u_n|^{p+1} = 1$, for each n, and $\int_\Omega |\nabla u_n|^2\, dx + \lambda \int_{\partial\Omega} u_n^2\, dx \to S_\lambda$ as $n \to \infty$. We may assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$, $u_n \to u$ in $L^2(\Omega)$ and $u_n \to u$ a.e. on Ω. Applying the concentration-compactness principle we can write

$$\int_\Omega Q|u|^{p+1} + \sum_{j\in J} Q(x_j)\mu_j = 1$$

and

$$S_\lambda = \lim_{n\to\infty} \left(\int_\Omega |\nabla u_n|^2\, dx + \lambda \int_{\partial\Omega} u_n^2\, dx \right) \geq \int_\Omega |\nabla u|^2\, dx + \lambda \int_{\partial\Omega} u^2\, dx + \sum_{j\in J} \tilde{\mu}_j.$$

Using (2), (3) and (5) we derive the following estimate from below for S_λ:

$$\begin{aligned}
S_\lambda &\geq \int_\Omega |\nabla u|^2\, dx + \lambda \int_{\partial\Omega} u^2\, dx + \sum_{x_j\in\Omega} \tilde{\mu}_j + \sum_{x_j\in\partial\Omega} \tilde{\mu}_j \\
&\geq S_\lambda \left(\int_\Omega Q|u|^{p+1}\, dx \right)^{\frac{2}{p+1}} + \sum_{x_j\in\Omega} \frac{S}{Q(x_j)^{\frac{N-2}{N}}} \left(\mu_j Q(x_j)\right)^{\frac{N-2}{N}} \\
&+ \sum_{x_j\in\partial\Omega} \frac{S}{2^{\frac{2}{N}} Q(x_j)^{\frac{N-2}{N}}} \left(\mu_j Q(x_j)\right)^{\frac{N-2}{N}} \\
&\geq S_\lambda \left(\int_\Omega Q|u|^{p+1}\, dx \right)^{\frac{2}{p+1}} + \sum_{x_j\in\Omega} \frac{S}{Q_M^{\frac{N-2}{N}}} \left(Q(x_j)\mu_j\right)^{\frac{N-2}{N}} \\
&+ \sum_{x_j\in\partial\Omega} \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}} \left(Q(x_j)\mu_j\right)^{\frac{N-2}{N}} \\
&\geq S_\lambda \left(\int_\Omega Q|u|^{p+1}\, dx \right)^{\frac{2}{p+1}} + \sum_{j\in J} \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}} \left(Q(x_j)\mu_j\right)^{\frac{N-2}{N}}.
\end{aligned}$$

Since S_λ satisfies (4) we must have $\mu_j = 0$ for all $j \in J$ and the result follows. □

Lemma 2.2. *If*

$$S_\lambda < \frac{S}{Q_M^{\frac{N-2}{N}}} \tag{6}$$

and

$$Q_M \geq 2^{\frac{2}{N-2}} Q_m, \tag{7}$$

then S_λ is achieved.

The same method can be used to obtain conditions guaranteeing the solvability of problem $(1_{\lambda,\tau})$.

Lemma 2.3. *If $2 \leq \tau \leq \frac{2N}{N-1}$,*

$$S_{\lambda,\tau} < \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}} \tag{8}$$

and (5) holds, then $S_{\lambda,\tau}$ is achieved.

Lemma 2.4. *If* $2 \le \tau \le \frac{2N}{N-1}$,

$$S_{\lambda,\tau} < \frac{S}{Q_M^{\frac{N-2}{N}}} \tag{9}$$

and (7) *holds, then* $S_{\lambda,\tau}$ *is achieved.*

If $Q \equiv 1$, the functionals J_λ and $J_{\lambda,\tau}$ will be denoted by I_λ and $I_{\lambda,\tau}$, respectively. Adimurthi and Mancini [1] proved that if $x_\circ \in \partial\Omega$, then

$$I_\lambda(U_{\varepsilon,x_\circ}) = \frac{S}{2^{\frac{2}{N}}} - A_N\left(\frac{N-2}{2}H(x_\circ) - \lambda\right)\varepsilon + O(\varepsilon^2),\ N \ge 5, \tag{10}$$

where $A_N > 0$ is a constant depending on N. Using this asymptotic formula we can give a condition on Q guaranteeing the validity of the inequality (4) in Lemma 2.1.

For this we need the following assumption on Q:

(**Q**) There exists a point $x_\circ \in \partial\Omega$ such that

$$Q(x_\circ) = Q_m \text{ and } |Q(x) - Q(x_\circ)| = o(|x - x_\circ|)$$

for x near $x_\circ$.

If (Q) holds, we have

$$J_\lambda(U_{\varepsilon,x_\circ}) = Q(x_\circ)^{-\frac{N-2}{N}} I_\lambda(U_{\varepsilon,x_\circ}) + o(\varepsilon)$$

and it follows from (10) that

$$J_\lambda(U_{\varepsilon,x_\circ}) = \frac{S}{2^{\frac{2}{N}}Q_m^{\frac{N-2}{N}}} - \frac{A_N}{Q_m^{\frac{N-2}{N}}}\left(\frac{N-2}{2}H(x_\circ) - \lambda\right)\varepsilon + o(\varepsilon). \tag{11}$$

We now observe that if $H(x_\circ) > 0$ and $\lambda < \frac{N-2}{2}H(x_\circ)$, then for sufficiently small $\varepsilon > 0$ we have

$$J_\lambda(U_{\varepsilon,x_\circ}) < \frac{S}{2^{\frac{2}{N}}Q_m^{\frac{N-2}{N}}},$$

which shows that (4) is satisfied for $\lambda < \frac{N-2}{2}H(x_\circ)$.

Proposition 2.5. *Let* $N \ge 5$. *Suppose that* $Q_M \le 2^{\frac{2}{N-2}}Q_m$ *and* (Q) *holds with* $H(x_\circ) > 0$. *Then problem* (1_λ) *has a solution for* $\lambda < \frac{N-2}{2}H(x_\circ)$. *Moreover,*

$$\lim_{\lambda\to\infty} S_\lambda = \frac{S}{2^{\frac{2}{N}}Q_m^{\frac{N-2}{N}}}.$$

Proposition 2.5 holds also for $N = 3, 4$ if one uses a suitable modification of (10) ([1], [5]). Note that, from the above discussion it is obvious that under the assumption of Proposition 2.5, $S_\lambda \le \frac{S}{2^{\frac{2}{N}}Q_m^{\frac{N-2}{N}}}$ for all $\lambda > 0$ and $S_\lambda < \frac{S}{2^{\frac{2}{N}}Q_m^{\frac{N-2}{N}}}$ for $\lambda < \frac{N-2}{2} = H(\Omega)$. The proof of the result on the asymptotic behavior of S_λ is standard.

In the case $Q_M > 2^{\frac{2}{N-2}}Q_m$, condition (6) from Lemma 2.2 is difficult to check. Obviously, it is satisfied for small $\lambda > 0$. By testing J_λ with $U_{\varepsilon,y}$, where $Q(y) = Q_M$ we easily show that $S_\lambda \le \frac{S}{Q_M^{\frac{N-2}{N}}}$ for all $\lambda > 0$.

Lemma 2.6. *If* $Q_M \ge 2^{\frac{2}{N-2}}Q_m$, *the condition* $S_\lambda < \frac{S}{Q_M^{\frac{N-2}{N}}}$ *is satisfied for small* $\lambda > 0$ *and* $\lim_{\lambda\to\infty} S_\lambda = \frac{S}{Q_M^{\frac{N-2}{N}}}$.

We now turn our attention to the functional $J_{\lambda,\tau}$ and problem $(1_{\lambda,\tau})$. By testing $I_{\lambda,\tau}$ with $U_{\varepsilon,x_\circ}$, we get for $N \geq 5$

$$I_{\lambda,\tau}(U_{\varepsilon,x_\circ}) = \frac{S}{2^{\frac{2}{N}}} - A_N H(x_\circ)\varepsilon + \lambda B \varepsilon^{\frac{2N}{\tau}-(N-2)} + o(\lambda \varepsilon^{\frac{2N}{\tau}-(N-2)}) + o(\varepsilon^2),$$

where $B > 0$ is a constant depending on N and τ. If (Q) holds and $H(x_\circ) > 0$, then

$$J_{\lambda,\tau}(U_{\varepsilon,x_\circ}) = Q_m^{-\frac{N-2}{N}} I_{\lambda,\tau}(U_{\varepsilon,x_\circ}) + o(\varepsilon).$$

Hence the condition (8) of Lemma 2.3 is satisfied for every $\lambda > 0$ provided $2 \leq \tau < \frac{2N}{N-1}$.

Proposition 2.7. *Let $N \geq 5$ and $Q_M \leq 2^{\frac{2}{N-2}} Q_m$. Suppose that* (Q) *holds, $H(x_\circ) > 0$ and $2 \leq \tau < \frac{2N}{N-1}$. Then problem $(1_{\lambda,\tau})$ has a solution for each $\lambda > 0$. Moreover, $\lim_{\lambda\to\infty} S_{\lambda,\tau} = \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N}{N-2}}}$ for $2 \leq \tau \leq \frac{2N}{N-1}$.*

Similarly, as in the case of Proposition 2.5, this continues to hold for $N = 3, 4$.
Finally,

Lemma 2.8. *If $Q_M \geq 2^{\frac{2}{N-2}} Q_m$, the condition $S_{\lambda,\tau} < \frac{S}{Q_M^{\frac{N-2}{N}}}$ is satisfied for small $\lambda > 0$ and $\lim_{\lambda\to\infty} S_{\lambda,\tau} = \frac{S}{Q_M^{\frac{N-2}{N}}}$ for $2 \leq \tau \leq \frac{2N}{N-1}$.*

To end this section we observe that from Propositions 2.5 and 2.7 and Lemma 2.8 we deduce a weak form of inequalities (I), (II) and (III). Namely, given a $\delta > 0$, there exist $\lambda_1 = \lambda_1(\Omega)$, $\lambda_2 = \lambda_2(\Omega)$ and $\lambda_3 = \lambda_3(\Omega)$ such that, for every $u \in H^1(\Omega)$,

$$\left(\int_\Omega Q(x)|u|^{p+1}\,dx\right)^{\frac{2}{p+1}} \leq \left(\frac{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}{S} + \delta\right) \int_\Omega |\nabla u|^2\,dx + \lambda_1 \int_{\partial\Omega} u^2\,dx \qquad (12)$$

and

$$\left(\int_\Omega Q(x)|u|^{p+1}\,dx\right)^{\frac{2}{p+1}} \leq \left(\frac{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}{S} + \delta\right) \int_\Omega |\nabla u|^2\,dx + \lambda_2 \left(\int_\Omega |u|^\tau\,dx\right)^{\frac{2}{\tau}} \qquad (13)$$

if $Q_M \leq 2^{\frac{2}{N-2}} Q_m$, and

$$\left(\int_\Omega Q(x)|u|^{p+1}\,dx\right)^{\frac{2}{p+1}} \leq \left(\frac{Q_M^{\frac{N-2}{N}}}{S} + \delta\right) \int_\Omega |\nabla u|^2\,dx + \lambda_3 \left(\int_\Omega |u|^\tau\,dx\right)^{\frac{2}{\tau}} \qquad (14)$$

if $Q_M \geq 2^{\frac{2}{N-2}} Q_m$.

3. Proof of inequality (I)

The proof of inequality (I) is by contradiction. Throughout this section we suppose that inequality (5) is satisfied. Assume that, for each $\lambda > 0$, $S_\lambda < \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}$. Let $\lambda_k \to \infty$. For each k there is a minimizer $u_k = u_{\lambda_k}$ of J_{λ_k} with $\int_\Omega Q u_k^{p+1}\,dx = 1$. It satisfies

$$(1_{u_k}) \qquad \begin{cases} -\Delta u_k = S_{\lambda_k} Q u_k^p & \text{in } \Omega, \\ \frac{\partial u_k}{\partial \nu} + \lambda_k u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Our aim is to show that u_k is close to some instanton U_{ε_k,P_k} with $P_k \to P_\circ$, $Q_m = Q(P_\circ)$. This in turn will contradict the inequality $S_{\lambda_k} < \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}$.

We start by setting

$$M_k := \max_{\Omega} u_k = u_k(P_k),$$

for some $P_k \in \bar{\Omega}$.

Lemma 3.1. *$M_k \to \infty$ and $u_k \rightharpoonup 0$ in $H^1(\Omega)$.*

Proof. Since $\{u_k\}$ is bounded in $H^1(\Omega)$ we may assume that $u_k \rightharpoonup u$ in $H^1(\Omega)$ and from (4) we get that $u_k \to 0$ in $L^2(\partial\Omega)$. Hence $u \in H^1_\circ(\Omega)$.
Assume that M_k is bounded. Then $\int_\Omega Qu^{p+1}\,dx = 1$ and by the lower semicontinuity of the norm with respect to weak convergence, we have

$$\int_\Omega |\nabla u|^2\,dx \leq \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}.$$

Since $u \in H^1_\circ(\Omega)$ we also have

$$\frac{\int_\Omega |\nabla u|^2\,dx}{\left(\int_\Omega Qu^{p+1}\,dx\right)^{\frac{2}{p+1}}} \geq \frac{S}{Q_M^{\frac{N-2}{N}}} > \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}$$

which is impossible. This shows that the sequence $\{M_k\}$ is unbounded.
Also $u_k \rightharpoonup 0$ in $H^1(\Omega)$, otherwise its weak limit u satisfies $0 < \int_\Omega Qu^{p+1}\,dx \leq 1$. The case $\int_\Omega Qu^{p+1}\,dx = 1$ is excluded by the above argument. If $0 < \int_\Omega Qu^{p+1}\,dx < 1$, we get a contradiction by applying the concentration-compactness principle. □

Let

$$\varepsilon_k = M_k^{-\frac{2}{N-2}}.$$

Lemma 3.2. *$\lambda_k \int_{\partial\Omega} u_k^2 \to 0$ and $\lambda_k \varepsilon_k \to 0$.*

Proof. Applying inequality (12) to u_k, we get

$$1 \leq \left(\frac{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}{S} + \delta\right) \int_\Omega |\nabla u_k|^2\,dx + \lambda_1 \int_{\partial\Omega} u_k^2\,dx.$$

Since $\int_{\partial\Omega} u_k^2\,dx \to 0$ we obtain that

$$\begin{aligned} 1 &\leq \left(\frac{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}{S} + \delta\right) \lim_{k\to\infty} \int_\Omega |\nabla u_k|^2\,dx \leq \left(\frac{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}{S} + \delta\right) \lim_{k\to\infty} J_{\lambda_k}(u_k) \\ &= \left(\frac{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}{S} + \delta\right) \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}. \end{aligned}$$

Since $\delta > 0$ is arbitrary we see that

$$\lim_{k\to\infty} \int_\Omega |\nabla u_k|^2\,dx = \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}$$

and $\lim_{k\to 0} \lambda_k \int_{\partial\Omega} u_k^2\,dx = 0$.
To prove the second assertion of the lemma, note that

$$\begin{aligned} \lambda_k \int_{\partial\Omega} u_k^2\,dx &= \lambda_k \varepsilon_k \int_{\partial\Omega} M_k^{\frac{2}{N-2}} u_k^2\,dx \\ &= \lambda_k \varepsilon_k \int_{\partial\Omega} M_k^q \frac{u_k^2}{M_k^2}\,dx \\ &\geq \lambda_k \varepsilon_k \int_{\partial\Omega} u_k^q\,dx, \end{aligned}$$

where $q = \frac{2(N-1)}{N-2}$. Therefore to complete the proof of the second assertion it is sufficient to show that

$$\liminf_{k\to\infty} \int_{\partial\Omega} u_k^q\,dx > 0.$$

We follow the argument used in [13] (see the proof of inequality (2.6) there). In the contrary case, assume that $\lim_{k\to\infty}\int_{\partial\Omega} u_k^q\,dx = 0$. Then we may assume that $u_k \rightharpoonup \bar u$ in $H^1_\circ(\Omega)$, up to a subsequence. We also have

$$1 \equiv \int_\Omega Q(x)u_k^{p+1}\,dx = \int_\Omega Q(x)|u_k-\bar u|^{p+1}\,dx + \int_\Omega Q(x)|\bar u|^{p+1}\,dx + o(1),$$

$$\int_\Omega Q(x)\bar u^{p+1}\,dx \le 1$$

and

$$\int_\Omega Q(x)|u_k-\bar u|^{p+1}\,dx \le 1+o(1).$$

Using the last three relations and the inequality

$$\left(\int_\Omega Q(x)u^{p+1}\right)^{\frac{2}{p+1}} \le \left(\frac{Q_M^{\frac{N-2}{N}}}{S}+\varepsilon\right)\int_\Omega |\nabla u|^2 + \frac{C}{\varepsilon}\left(\int_{\partial\Omega} u^q\right)^{\frac{2}{q}} \quad \text{for } u\in H^1(\Omega),$$

C a constant, due to Brézis and Lieb, we easily deduce that for every $\delta>0$ we have

$$\begin{aligned}
S_{\lambda_k} &= \int_\Omega |\nabla u_k|^2\,dx + \lambda_k\int_{\partial\Omega} u_k^2\,dx \\
&= \int_\Omega |\nabla(u_k-\bar u)|^2\,dx + \int_\Omega |\nabla\bar u|^2\,dx + \lambda_k\int_{\partial\Omega} u_k^2\,dx + o(1) \\
&\ge \left(\frac{S}{Q_M^{\frac{N-2}{N}}}-\delta\right)\left(\int_\Omega Q(x)|u_k-\bar u|^{p+1}\,dx\right)^{\frac{2}{p+1}} \\
&+ S_{\lambda_k}\left(\int_\Omega Q(x)\bar u^{p+1}\,dx\right)^{\frac{2}{p+1}} + \lambda_k\int_{\partial\Omega} u_k^2\,dx + o(1) \\
&\ge \left(\frac{S}{Q_M^{\frac{N-2}{N}}}-\delta\right)\int_\Omega Q(x)|u_k-\bar u|^{p+1}\,dx + S_{\lambda_k}\int_\Omega Q(x)\bar u^{p+1}\,dx \\
&+ \lambda_k\int_{\partial\Omega} u_k^2\,dx + o(1) \\
&= \left(\frac{S}{Q_M^{\frac{N-2}{N}}}-\delta-S_{\lambda_k}\right)\int_\Omega Q(x)|u_k-\bar u|^{p+1}\,dx + S_{\lambda_k} \\
&+ \lambda_k\int_{\partial\Omega} u_k^2\,dx + o(1).
\end{aligned}$$

Since $Q_M^{\frac{N-2}{N}} < 2^{\frac{2}{N}}Q_m^{\frac{N-2}{N}} = S/\lim S_{\lambda_k}$, choosing $\delta>0$ sufficiently small, we deduce that

$$\lim_{k\to\infty}\int_\Omega Q(x)|u_k-\bar u|^{p+1}\,dx = 0$$

and consequently $\int_\Omega Q(x)\bar u^{p+1}\,dx = 1$. This means that

$$\frac{\int_\Omega|\nabla\bar u|^2\,dx}{\left(\int_\Omega Q(x)\bar u^{p+1}\,dx\right)^{\frac{2}{p+1}}} \ge \frac{\int_\Omega|\nabla\bar u|^2\,dx}{Q_M^{\frac{N-2}{N}}\left(\int_\Omega \bar u^{p+1}\,dx\right)^{\frac{2}{p+1}}} > \frac{S}{Q_M^{\frac{N-2}{N}}} > \frac{S}{2^{\frac{2}{N}}Q_m^{\frac{N-2}{N}}},$$

since $\bar u\in H^1_\circ(\Omega)$. On the other hand, by the lower semicontinuity of the norm with respect to weak convergence, we have

$$\int_\Omega|\nabla\bar u|^2\,dx \le \frac{S}{2^{\frac{2}{N}}Q_m^{\frac{N-2}{N}}},$$

which is impossible. Therefore $\lambda_k\varepsilon_k\to 0$. □

Lemma 3.3. *Up to a subsequence, $P_k \to P_\circ$, where $P_\circ$ is such that $Q(P_\circ) = Q_m$. Moreover, $P_k \in \partial\Omega$ for large k,*

$$\lim_{k\to\infty} \int_\Omega \left| \nabla \left(u_k - \varepsilon_k^{-\frac{N-2}{2}} U \left(\frac{S^{\frac{1}{2}} Q_m^{\frac{1}{N}}}{2^{\frac{1}{N}}} \frac{\cdot - P_k}{\varepsilon_k} \right) \right) \right|^2 dx = 0 \tag{15}$$

and

$$\lim_{k\to\infty} \int_\Omega \left| u_k - \varepsilon_k^{-\frac{N-2}{2}} U \left(\frac{S^{\frac{1}{2}} Q_m^{\frac{1}{N}}}{2^{\frac{1}{N}}} \frac{\cdot - P_k}{\varepsilon_k} \right) \right|^{p+1} dx = 0. \tag{16}$$

Proof. Let $v_k(x) = \varepsilon_k^{\frac{N-2}{2}} u_k(\varepsilon_k x + P_k)$, for $x \in \Omega_k = \frac{\Omega - P_k}{\varepsilon_k}$. Then functions v_k are solutions of the Neumann problems

$$(1_{v_k}) \begin{cases} -\Delta v_k = S_{\lambda_k} Q(\varepsilon_k x + P_k) v_k^p & \text{in } \Omega_k, \\ \frac{\partial v_k}{\partial \nu} + \lambda_k \varepsilon_k v_k = 0 & \text{on } \partial\Omega_k, \\ 0 \le v_k(x) \le 1 & \text{in } \Omega_k, \\ v_k(0) = 1. \end{cases}$$

Passing to a subsequence, we can assume that $P_k \to P_\circ$, for some $P_\circ \in \bar{\Omega}$, and $\frac{\mathrm{dist}(P_k, \partial\Omega)}{\varepsilon_k}$ converges in the extended real line. Using elliptic regularity theory, we show that up to a subsequence, $v_k \to \omega$ in $C^2_{\mathrm{loc}}(\Omega_\infty)$, where $\Omega_\infty = \lim_{k\to\infty} \Omega_k$. Thus ω satisfies

$$\begin{cases} -\Delta \omega = \hat{S} Q(P_\circ) \omega^p & \text{in } \Omega_\infty, \\ \frac{\partial \omega}{\partial \nu} = 0 & \text{on } \partial\Omega_\infty, \\ 0 \le \omega \le 1 & \text{in } \Omega_\infty, \\ \omega(0) = 1. \end{cases}$$

where $\hat{S} = \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}$.

We now distinguish two cases: (i) $\frac{\mathrm{dist}(P_k, \partial\Omega)}{\varepsilon_k}$ converges to $+\infty$ and (ii) $\frac{\mathrm{dist}(P_k, \partial\Omega)}{\varepsilon_k}$ converges to a real number.

If case (i) occurs then $\Omega_\infty = \mathbb{R}^N$. By [12] (proof of Theorem 2.3 on p. 34) we see that $\omega(x) = U(\beta x)$, where $\beta^2 = Q(P_\circ)\hat{S}$. Since $\int_{\Omega_k} |\nabla v_k|^2\, dx \to \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}$, this yields

$$\beta^{2-N} S^{\frac{N}{2}} = \int_{\mathbb{R}^N} |\nabla \omega|^2\, dx \le \lim_{k\to\infty} \int_{\Omega_k} |\nabla v_k|^2\, dx = \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}.$$

Thus, $2Q_m^{\frac{N-2}{2^*} + \frac{N-2}{N}} \le Q(P_\circ)^{\frac{N-2}{2}}$. We must have that $2^{\frac{2}{N-2}} Q_m \le Q(P_\circ) \le Q_M$, which is impossible. Therefore (ii) prevails.

We can assume Ω_∞ is a half space, which we take to be $\mathbb{R}^N_+$. Notice that $P_\circ \in \partial\Omega$. Hence,

$$\beta^{2-N} \frac{S^{\frac{N}{2}}}{2} = \int_{\mathbb{R}^N_+} |\nabla \omega|^2\, dx \le \lim_{k\to\infty} \int_{\Omega_k} |\nabla v_k|^2\, dx \le \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}, \tag{17}$$

which implies $Q_m \le Q(P_\circ)$ and necessarily $Q(P_\circ) = Q_m$. Following the argument of the proof of Lemma 2.2 in [3], we check that $P_k \in \partial\Omega$ for large k.

Equalities (15) and (16) now easily follow. □

Let $E_\lambda(u) = \int_\Omega |\nabla u|^2\, dx + \lambda \int_{\partial\Omega} u^2\, dx$ for $u \in H^1(\Omega)$ and $w_k = Q_m^{\frac{1}{p+1}} u_k$. Notice that

$$E_{\lambda_k}(w_k) = Q_m^{\frac{N-2}{N}} E_{\lambda_k}(u_k) < \frac{S}{2^{\frac{2}{N}}} \tag{18}$$

and, by Lemma 3.3,

$$\lim_{k\to\infty}\left\|\nabla w_k-\nabla\sigma_k^{-\frac{N-2}{2}}S^{-\frac{N-2}{4}}2^{\frac{1}{p+1}}U\left(\frac{\cdot-P_k}{\sigma_k}\right)\right\|_2=0,$$

where $\sigma_k=\frac{\varepsilon_k 2^{\frac{1}{N}}}{S^{\frac{1}{2}}Q_m^{\frac{1}{N}}}$.

The sequence w_k satisfies the assumptions of Lemma 3.4 in [6]. With the aid of Lemmas 3.5-3.8 in [6] we deduce the estimate

$$E_{\lambda_k}(w_k)\geq\frac{S}{2^{\frac{2}{N}}}-A_N\left(\frac{N-2}{2}H(y_k)-\lambda_k\right)\varepsilon_k+O\big(\varepsilon_k^2\big)+O\big(\lambda_k\varepsilon_k^2\big),$$

where $A_N>0$ is a constant depending on N. Therefore there exists a $\bar\lambda>0$ such that $E_{\lambda_k}(w_k)>\frac{S}{2^{\frac{2}{N}}}$ for $\lambda_k\geq\bar\lambda$, which contradicts (18) and our assumption that $S_\lambda<\frac{S}{2^{\frac{2}{N}}Q_m^{\frac{N-2}{N}}}$, for each $\lambda>0$. The proof of inequality (I) is now complete.

4. Proof of inequalities (II) and (III)

As with inequality (I), the proofs of inequalities (II) and (III) are by contradiction. We start with the proof of (III). The proof is a generalization of [26]. The main difference is that now concentration occurs in the interior of Ω and the range of values of τ is $2\leq\tau\leq\frac{2N}{N-1}$.

Suppose $Q_M>2^{\frac{2}{N-2}}Q_m$. Assume that for each $\lambda>0$, $S_{\lambda,\tau}<S_\infty:=\frac{S}{Q_M^{\frac{N-2}{N}}}$. Let $\lambda_k\to\infty$. By Lemma 2.4, for each k there is a minimizer $u_k=u_{\lambda_k}$ of $J_{\lambda_k,\tau}$ with $\int_\Omega Qu_k^{p+1}\,dx=1$. The function u_k satisfies

$$(1_{\lambda_k,\tau})\begin{cases}-\Delta u_k+\lambda_k\left(\int_\Omega|u_k|^\tau\,dx\right)^{\frac{2}{\tau}-1}u_k^{\tau-1}=S_{\lambda_k,\tau}Qu_k^p & \text{in }\Omega,\\ \frac{\partial u_k}{\partial\nu}=0 & \text{on }\partial\Omega.\end{cases}$$

First of all we observe that

Lemma 4.1.

$$\lim_{k\to\infty}\lambda_k\left(\int_\Omega u_k^\tau\,dx\right)^{\frac{2}{\tau}}=0. \tag{19}$$

Proof. Since $J_{\lambda_k,\tau}(u_k)<S_\infty$, $\lambda_k\|u_k\|_\tau^2$ is bounded and $u_k\rightharpoonup 0$ in $H^1(\Omega)$. By inequality (14), given a $\delta>0$, there exists $\lambda_3>0$ such that

$$1=\left(\int_\Omega Q(x)|u_k|^{p+1}\,dx\right)^{\frac{2}{p+1}}\leq\left(\frac{Q_M^{\frac{N-2}{N}}}{S}+\delta\right)\int_\Omega|\nabla u_k|^2\,dx+\lambda_3\left(\int_\Omega|u_k|^\tau\,dx\right)^{\frac{2}{\tau}}.$$

Therefore

$$1\leq\left(\frac{Q_M^{\frac{N-2}{N}}}{S}+\delta\right)\lim_{k\to\infty}\int_\Omega|\nabla u_k|^2\,dx$$

and since δ is arbitrary,

$$\frac{S}{Q_M^{\frac{N-2}{N}}}\leq\lim_{k\to\infty}\|\nabla u_k\|_2^2\leq\lim_{k\to\infty}J_{\lambda_k,\tau}(u_k)=\frac{S}{Q_M^{\frac{N-2}{N}}}.$$

We conclude that $\lim_{k\to\infty}\|\nabla u_k\|_2^2=\frac{S}{Q_M^{\frac{N-2}{N}}}$ and $\lim_{k\to\infty}\lambda_k\left(\int_\Omega u_k^\tau\,dx\right)^{\frac{2}{\tau}}=0$. □

Set $M_k = \max_\Omega u_k = u_k(P_k)$ for some $P_k \in \bar\Omega$ and

$$v_k(x) = \varepsilon_k^{\frac{N-2}{2}} u_k(\varepsilon_k x + P_k),$$

for $x \in \Omega_k = \frac{\Omega - P_k}{\varepsilon_k}$, with

$$\varepsilon_k = M_k^{-\frac{2}{N-2}}.$$

Note that the functions v_k satisfy $0 \le v_k(x) \le 1$ and $v_k(0) = 1$. Define

$$\sigma = \frac{2N}{\tau} - (N-2). \tag{20}$$

Since $2 \le \tau \le \frac{2N}{N-1}$, the value σ satisfies $1 \le \sigma \le 2$. Lemma 4.1 implies

Lemma 4.2. $\lim_{k\to\infty} \lambda_k \varepsilon_k^\sigma = 0$.

Proof. Changing variables, we check that

$$\left(\int_\Omega u_k^\tau \, dx\right)^{\frac{2}{\tau}} = \varepsilon_k^\sigma \left(\int_{\Omega_k} v_k^\tau \, dx\right)^{\frac{2}{\tau}}. \tag{21}$$

But

$$\int_{\Omega_k} v_k^\tau \, dx \ge \int_{\Omega_k} v_k^{p+1} \, dx = \int_\Omega u_k^{p+1} \, dx \ge \frac{1}{Q_M} \int_\Omega Q u_k^{p+1} \, dx = \frac{1}{Q_M} > 0. \tag{22}$$

Combining (19), (21) and (22) we conclude that $\lim_{k\to\infty} \lambda_k \varepsilon_k^\sigma = 0$. □

In particular, $\varepsilon_k \to 0$ and $M_k \to \infty$. Next we verify that the sequence $\{u_k\}$ is close to a sequence of instantons concentrating in the interior of Ω.

Lemma 4.3. *If $Q_M > 2^{\frac{2}{N-2}} Q_m$, up to a subsequence, $P_k \to P_0$, where $Q(P_0) = Q_M$. Moreover,*

$$\lim_{k\to\infty} \int_\Omega \left| \nabla \left[u_k - \varepsilon_k^{-\frac{N-2}{2}} U\left(S^{\frac12} Q_M^{\frac1N} \frac{\cdot - P_k}{\varepsilon_k} \right) \right] \right|^2 dx = 0 \tag{23}$$

and

$$\lim_{k\to\infty} \int_\Omega \left| u_k - \varepsilon_k^{-\frac{N-2}{2}} U\left(S^{\frac12} Q_M^{\frac1N} \frac{\cdot - P_k}{\varepsilon_k} \right) \right|^{p+1} dx = 0. \tag{24}$$

Proof. The functions v_k are solutions of the Neumann problems

$$\begin{cases} -\Delta v_k + \lambda_k \varepsilon_k^\sigma \left(\int_{\Omega_k} |v_k|^\tau \, dx \right)^{\frac{2}{\tau}-1} v_k^{\tau-1} = S_{\lambda_k,\tau} Q(\varepsilon_k x + P_k) v_k^p & \text{in } \Omega_k, \\ \frac{\partial v_k}{\partial \nu} = 0 & \text{on } \partial\Omega_k, \\ 0 \le v_k(x) \le 1 & \text{in } \Omega_k, \\ v_k(0) = 1. \end{cases}$$

We can assume that $P_k \to P_\circ$, for some $P_\circ \in \bar\Omega$, and $\frac{\mathrm{dist}(P_k,\partial\Omega)}{\varepsilon_k}$ converges in the extended real line. Using elliptic regularity theory we show that, up to a subsequence, $v_k \to \omega$ in $C^2_{\mathrm{loc}}(\Omega_\infty)$, where $\Omega_\infty = \lim_{k\to\infty} \Omega_k$. The function ω satisfies

$$\begin{cases} -\Delta \omega = \hat S Q(P_\circ) \omega^p & \text{in } \Omega_\infty, \\ \frac{\partial \omega}{\partial \nu} = 0 & \text{on } \partial\Omega_\infty, \\ 0 \le \omega \le 1 & \text{in } \Omega_\infty, \\ \omega(0) = 1. \end{cases}$$

where $\hat S = \frac{S}{Q_M^{\frac{N-2}{N}}}$.

We distinguish two cases: (i) $\frac{\mathrm{dist}(P_k,\partial\Omega)}{\varepsilon_k}$ converges to a real number and (ii) $\frac{\mathrm{dist}(P_k,\partial\Omega)}{\varepsilon_k}$

converges to $+\infty$. In case (i) we assume that $\Omega_\infty = \mathbb{R}^N_+$. By [12] we see that $\omega(x) = U(\beta x)$, where $\beta^2 = Q(P_\circ)\hat{S}$. This yields

$$\beta^{2-N}\frac{S^{\frac{N}{2}}}{2} = \int_{\mathbb{R}^N_+} |\nabla\omega|^2\,dx \le \lim_{k\to\infty}\int_{\Omega_k} |\nabla v_k|^2\,dx = \frac{S}{Q_M^{\frac{N-2}{N}}}.$$

Thus $Q_M \le 2^{\frac{2}{N-2}}Q(P_\circ) \le 2^{\frac{2}{N-2}}Q_m$. So case (ii) prevails. Therefore $\Omega_\infty = \mathbb{R}^N$,

$$\beta^{2-N}S^{\frac{N}{2}} = \int_{\mathbb{R}^N} |\nabla\omega|^2\,dx \le \lim_{k\to\infty}\int_{\Omega_k} |\nabla v_k|^2\,dx = \frac{S}{Q_M^{\frac{N-2}{N}}}$$

and $Q_M \le Q(P_\circ)$. Hence $Q(P_\circ) = Q_M$. Equalities (23) and (24) follow. □

We now set $W(\,\cdot\,) = U\big(S^{\frac{1}{2}}Q_M^{\frac{1}{N}}\,\cdot\,\big)$ and

$$W_{\varepsilon,y}(\,\cdot\,) = \varepsilon^{-\frac{N-2}{2}}W\left(\frac{\cdot\, - y}{\varepsilon}\right)$$

for $y \in \mathbb{R}^N$ and $\varepsilon > 0$. Let

$$\mathcal{M} = \{CW_{\varepsilon,y};\, C \in \mathbb{R},\, \varepsilon > 0, y \in \bar{\Omega}\}$$

We use the notation $d(\varphi,\mathcal{M}) = \mathrm{dist}(\varphi,\mathcal{M}) = \inf\{||\nabla(\varphi-\psi)||_2;\, \psi \in \mathcal{M}\}$. The following lemma, together with the last one, guarantees the existence of an instanton closest to u_k, in the metric just defined.

Lemma 4.4. *Let $\delta > 0$ and $\{\varphi_l\} \subset H^1(\Omega)$ be such that $\varphi_l \rightharpoonup 0$ in $H^1(\Omega)$ and*

$$d(\varphi_l,\mathcal{M})^2 \le ||\nabla\varphi_l||_2^2 - 2\delta.$$

Then there exists $l_\circ$ such that for all $l \ge l_\circ$ $d(\varphi_l,\mathcal{M})$ is achieved by some $C_lW_{\varepsilon_l,y_l}$. If $y_l \to y$, with y an interior point of Ω and w_l is defined by

$$\varphi_l = C_lW_{\varepsilon_l,y_l} + w_l,$$

then, up to a subsequence,

(i) $\lim_{l\to\infty}\varepsilon_l = 0$;
(ii) *if $d(\varphi_l,\mathcal{M}) \to 0$ as $l \to \infty$, then $\lim_{l\to\infty} C_l = C_\circ \neq 0$;*
(iii) $\int_\Omega W^p_{\varepsilon_l,y_l}w_l\,dx = O\big(\varepsilon_l^{\frac{N-2}{2}}||w_l||_{H^1(\Omega)}\big)$;
(iv) $\int_\Omega W^{p-1}_{\varepsilon_l,y_l}w_l\frac{\partial}{\partial x_i}W_{\varepsilon_l,y_l}\,dx = O\big(\varepsilon_l^{\frac{N-2}{2}}||w_l||_{H^1(\Omega)}\big)$.

The proof is almost identical to that of Lemma 5.6 in [21] and is omitted (see also Lemma 3.1 in [3]).

It then follows from Lemma 4.4 that there exist sequences $\{C_kW_{\delta_k,y_k}\}$ which minimize $d(u_k,\mathcal{M})$. Equality (23) implies that

$$\lim_{k\to\infty}||\nabla(C_kW_{\delta_k,y_k}) - \nabla W_{\varepsilon_k,P_k}||_2 = 0.$$

We deduce

$$C_k \to 1, \quad y_k \to y = P_\circ \quad \text{and} \quad \frac{\varepsilon_k}{\delta_k} \to 1. \tag{25}$$

In particular, y is an interior point of Ω. We set

$$u_k = C_kW_{\delta_k,y_k} + w_k, \tag{26}$$

and define

$$W_k := W_{\delta_k,y_k}.$$

Obviously, from (23) and the definition of C_kW_k, we get $||\nabla w_k||_2 \to 0$. From (25), we get $||W_{\varepsilon_k,P_k} - C_kW_k||_{p+1} \to 0$, which together with (24) implies $||w_k||_{p+1} \to 0$.

Lemma 4.5. *There exists* $0 < \mu < 1$ *such that, for sufficiently large* k,

$$pSQ_M^{\frac{2}{N}} \int_\Omega W_k^{p-1} w_k^2 \, dx \le \mu\big(||\nabla w_k||_2^2 + \lambda_k ||w_k||_\tau^2\big).$$

This follows from Lemma 5.9 in [21].

The next lemma is due to Wang [20] and Zhu [26]. We give its proof since they do not present it for the whole range of values of τ we are interested in.

Lemma 4.6. *Let* $N \ge 5$ *and* $2 \le \tau \le \frac{2N}{N-1}$, *or* $N = 4$ *and* $2 < \tau \le \frac{2N}{N-1}$. *For any* $\gamma > 0$,

$$\int_\Omega W_k^{\tau-1} |w_k| \, dx \le o(1)\delta_k^{\frac{\tau\sigma}{2}} + \gamma \|w_k\|_\tau^\tau.$$

Proof. Choose $r \in \left]\max\left\{\frac{N}{N-2}\frac{1}{\tau-1}, \frac{2N}{N+2}\right\}, \frac{\tau}{\tau-1}\right[$. Note that this interval is nonempty since on the one hand $\min \frac{\tau}{\tau-1} = \frac{2N}{N+1} > \frac{2N}{N+2}$, and on the other hand $\frac{N}{N-2} < 2 \le \tau$ for $N \ge 5$ and $\tau > 2$ for $N = 4$. The conjugate exponent of r satisfies $r' \in]\tau, \frac{2N}{N-2}[$. By the Hölder inequality,

$$\begin{aligned}\int_\Omega W_k^{\tau-1}|w_k| \, dx &\le \left(\int_\Omega W_k^{(\tau-1)r} \, dx\right)^{\frac{1}{r}} \left(\int_\Omega |w_k|^{r'} \, dx\right)^{\frac{1}{r'}} \\ &\le C\delta_k^{-\frac{(\tau-1)(N-2)}{2}+\frac{N}{r}} \|w_k\|_{r'},\end{aligned}$$

and, by the interpolation inequality,

$$\|w_k\|_{r'} \le \|w_k\|_\tau^a \|w_k\|_{p+1}^{1-a},$$

where

$$\frac{1}{r'} = \frac{a}{\tau} + \frac{1-a}{p+1}, \qquad 0 < a < 1.$$

If $\gamma > 0$, there exists a $C(\gamma)$ such that

$$\begin{aligned}\int_\Omega W_k^{\tau-1}|w_k| \, dx &\le C\delta_k^{-\frac{(\tau-1)(N-2)}{2}+\frac{N}{r}} \|w_k\|_\tau^a \|w_k\|_{p+1}^{1-a} \\ &\le \gamma\|w_k\|_\tau^\tau + C(\gamma)\delta_k^{\left(-\frac{(\tau-1)(N-2)}{2}+\frac{N}{r}\right)\cdot\frac{\tau}{\tau-a}} \|w_k\|_{p+1}^{\frac{\tau(1-a)}{\tau-a}}. \quad (27)\end{aligned}$$

Now, from the definition of a,

$$a\left(\frac{1}{\tau} - \frac{1}{p+1}\right) = \frac{1}{r'} - \frac{1}{p+1}$$

and from the definition of σ (Eq. 20),

$$a\left(\frac{1}{\tau} - \frac{1}{p+1}\right) = \frac{a\sigma}{2N}.$$

Hence

$$\frac{a\sigma}{2N} = \frac{1}{r'} - \frac{1}{p+1}.$$

From this we get successively,

$$\frac{1}{r} - \frac{1}{N} = \frac{1}{2} - \frac{a\sigma}{2N},$$

$$\frac{1}{r} - \frac{(N-2)(\tau-1)}{2N} = \left(\frac{\tau}{2N} - \frac{a}{2N}\right)\sigma$$

and

$$\left(-\frac{(\tau-1)(N-2)}{2} + \frac{N}{r}\right) \cdot \frac{\tau}{\tau-a} = \frac{\tau\sigma}{2}.$$

Substituting into (27),

$$\int_\Omega W_k^{\tau-1}|w_k| \leq \gamma\|w_k\|_\tau^\tau + C(\gamma)\delta_k^{\frac{\tau\sigma}{2}}\|w_k\|_{p+1}^{\frac{\tau(1-a)}{\tau-a}}.$$

Since Eq. (24) implies $\|w_k\|_{p+1}^{\frac{\tau(1-a)}{\tau-a}} \to 0$ as $k\to\infty$, the proof is complete. □

We will now prove that for large k, $J_{\lambda_k,\tau}(u_k) > \frac{S}{Q_M^{\frac{N-2}{N}}}$, which contradicts Lemma 2.8 and proves inequality (III). So, we estimate the terms in $J_{\lambda_k,\tau}$.

By construction, $\int_\Omega Q|u_k|^{p+1}\,dx = 1$. However, it is convenient to give the following lower bound whose proof follows from (iii) of Lemma 4.4 and the arguments in [26]:

$$\left(\int_\Omega Q|u_k|^{p+1}\,dx\right)^{\frac{-2}{p+1}} \geq Q_M^{-\frac{N-2}{N}} C_k^{-2}\|W_k\|_{p+1}^{-2}\times$$
$$\times\left(1 - \frac{(p+\gamma_1)\int_\Omega W_k^{p-1}w_k^2\,dx}{C_k^2\|W_k\|_{p+1}^{p+1}} - C\delta_k^{\frac{N-2}{2}}\|w_k\|_{H^1(\Omega)} - C(\gamma_1)\|w_k\|_{H^1(\Omega)}^{p+1}\right)$$

where γ_1 is any positive number. Regarding $\|W_k\|_{p+1}^{-2}$, we have

$$\|W_k\|_{p+1}^{-2} \geq Q_M^{\frac{N-2}{N}} + O(\delta_k^N).$$

Inserting this estimate in the last inequality, we get the following lower bound for $\left(\int_\Omega Q|u_k|^{p+1}\,dx\right)^{\frac{-2}{p+1}}$:

$$C_k^{-2}\left(1 - \frac{(p+\gamma_1)\int_\Omega W_k^{p-1}w_k^2\,dx}{C_k^2\|W_k\|_{p+1}^{p+1}} - C\delta_k^{\frac{N-2}{2}}\|w_k\|_{H^1(\Omega)} - \right.$$
$$\left. - C(\gamma_1)\|w_k\|_{H^1(\Omega)}^{p+1} + O(\delta_k^N)\right).$$

To estimate $\left(\int_\Omega u_k^\tau\,dx\right)^{\frac{2}{\tau}}$, note first that

$$\begin{aligned}\left(\int_\Omega W_k^\tau\,dx\right)^{\frac{2}{\tau}} &= \delta_k^\sigma\left(\int_{\Omega_k} W^\tau\,dx\right)^{\frac{2}{\tau}}\\ &= C\delta_k^\sigma + o(\delta_k^\sigma)\end{aligned}$$

since $(N-2)\tau > N$. In fact, $\frac{N-2}{N} < 2 \leq \tau$ except when $N=4$, but in this case $\tau > 2$. So we can follow the argument in [26] to prove that for any $0<\gamma_2<1$ there is a constant $C(\gamma_2)$ such that

$$\left(\int_\Omega u_k^\tau\,dx\right)^{\frac{2}{\tau}} \geq \gamma_2\|w_k\|_\tau^2 + C(\gamma_2)\delta_k^\sigma.$$

To estimate $\int_\Omega|\nabla u_k|^2\,dx$, note that

$$\begin{aligned}\int_\Omega|\nabla W_k|^2\,dx &= \frac{1}{S^{\frac{N-2}{2}}Q_M^{\frac{N-2}{N}}}\int_\Omega|\nabla U_k|^2\,dx\\ &= \frac{S}{Q_M^{\frac{N-2}{N}}} + O\left(\delta_k^{N-2}\right).\end{aligned}$$

Therefore

$$\int_\Omega|\nabla u_k|^2\,dx = C_k^2\frac{S}{Q_M^{\frac{N-2}{N}}} + O\left(\delta_k^{N-2}\right) + \int_\Omega|\nabla w_k|^2\,dx.$$

We are now in a position to estimate $J_{\lambda_k,\tau}(u_k)$. Combining the previous estimates,

$$J_{\lambda_k,\tau}(u_k) \geq \left[\frac{S}{Q_M^{\frac{N-2}{N}}} + O\left(\delta_k^{N-2}\right) + \frac{\|\nabla w_k\|_2^2}{C_k^2} + \frac{\gamma_2\lambda_k\|w_k\|_\tau^2}{C_k^2} + \frac{C(\gamma_2)\lambda_k\delta_k^\sigma}{C_k^2}\right] \times$$
$$\left[1 - \frac{(p+\gamma_1)\int_\Omega W_k^{p-1}w_k^2\,dx}{C_k^2\|W_k\|_{p+1}^{p+1}} - C\delta_k^{\frac{N-2}{2}}\|w_k\|_{H^1(\Omega)} - C(\gamma_1)\|w_k\|_{H^1(\Omega)}^{p+1} + O(\delta_k^N)\right].$$

For k large, this is greater or equal than

$$\frac{S}{Q_M^{\frac{N-2}{N}}} + \frac{C(\gamma_2)\lambda_k\delta_k^\sigma}{C_k^2} + O\left(\delta_k^{N-2}\right) + \gamma_2\frac{\|\nabla w_k\|_2^2 + \lambda_k\|w_k\|_\tau^2}{C_k^2} -$$
$$- \frac{(p+\gamma_1)S\int_\Omega W_k^{p-1}w_k^2\,dx}{C_k^2 Q_M^{\frac{N-2}{N}}\|W_k\|_{p+1}^{p+1}}.$$

By Lemma 4.5, for k sufficiently large, the difference

$$\gamma_2\frac{\|\nabla w_k\|_2^2 + \lambda_k\|w_k\|_\tau^2}{C_k^2} - \frac{(p+\gamma_1)S\int_\Omega W_k^{p-1}w_k^2\,dx}{C_k^2 Q_M^{\frac{N-2}{N}}\|W_k\|_{p+1}^{p+1}}.$$

is greater than or equal to

$$\frac{pSQ_M^{\frac{2}{N}}}{C_k^2}\left(\frac{\gamma_2}{\mu} - \frac{p+\gamma_1}{p}\right)\int_\Omega W_k^{p-1}w_k^2\,dx,$$

for some $0 < \mu < 1$. Choosing γ_2 such that $\mu < \gamma_2 < 1$, and then γ_1 sufficiently small, we get

$$\frac{\gamma_2}{\mu} - \frac{p+\gamma_1}{p} > 0.$$

This implies that

$$J_{\lambda_k,\tau}(u_k) \geq \frac{S}{Q_M^{\frac{N-2}{N}}} + \frac{C(\gamma_2)}{C_k^2}\lambda_k\delta_k^\sigma + O\left(\delta_k^{N-2}\right).$$

Since $1 \leq \sigma \leq 2$, it follows that $\sigma \leq N-2$, since if $N = 4$ we do not allow τ (and hence σ) equal to 2. So, for sufficiently large k,

$$J_{\lambda_k,\tau}(u_k) > \frac{S}{Q_M^{\frac{N-2}{N}}},$$

which contradicts our initial assumption and implies inequality (III).

Now that we have proved inequality (III), let us **outline the proof of inequality (II).** Let $Q_M \leq 2^{\frac{2}{N-2}}Q_m$. One assumes that $S_{\lambda,\tau} < \frac{S}{2^{\frac{2}{N}}Q_m^{\frac{N-2}{N}}}$ for each $\lambda > 0$, picks $\lambda_k \to \infty$ and minimizers $u_k = u_{\lambda_k}$ of $J_{\lambda_k,\tau}$ with $\int_\Omega Qu_k^{p+1}\,dx = 1$, and starts to prove analogues of the previous lemmas. When one gets to the analogue Lemma 4.3, one proves that, up to a subsequence, $\{u_k\}$ either concentrates in the interior of Ω or concentrates on the boundary of Ω. In the former case concentration occurs at a point $P_\circ$ with $Q(P_\circ) = Q_M$ and one can repeat the argument given above to derive a contradiction. In the latter case concentration occurs at a point $P_\circ$ with $Q(P_\circ) = Q_m$.

Let $E_{\lambda,\tau}(u) = \int_\Omega|\nabla u|^2 + \lambda\left(\int_\Omega u^\tau\right)^{\frac{2}{\tau}}$ for $u \in H^1(\Omega)$ and $w_k = Q_m^{\frac{1}{p+1}}u_k$. Notice that

$$E_{\lambda_k,\tau}(w_k) = Q_m^{\frac{N-2}{N}}E_{\lambda_k}(u_k) < \frac{S}{2^{\frac{2}{N}}}. \tag{28}$$

Following the arguments in [26], one can prove that $E_{\lambda_k,\tau}(w_k) > \frac{S}{2^{\frac{2}{N}}}$ for λ_k sufficiently large, which contradicts (28) and proves inequality (II).

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