Mathematical Relativity, Spring 2022/23
Instituto Superior Técnico

Due March 16

1. Using the definition of covariant derivative, we showed in class that

$$\nabla\nabla Z(X, Y, \omega) = (\nabla_X \nabla_Y Z)(\omega) - (\nabla_{\nabla_X Y} Z)(\omega).$$

Check this equality by calculating both sides in local coordinates.

2. Recall that the nonzero Christoffel symbols for the Minkowski metric in spherical coordinates,

$$\eta = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),$$

are

$$\begin{aligned}
&\Gamma^r_{\theta\theta} = -r, \quad \Gamma^r_{\varphi\varphi} = -r\sin^2\theta, \\
&\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta, \\
&\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}, \quad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \cot\theta.
\end{aligned}$$

Consider the vector field

$$V = f(r)\partial_r.$$

Compute the tensor $\nabla^\mu V^\nu$.

3. We will show in class that

$$\frac{1}{2}(L_V g)_{\mu\nu} = \nabla_{(\mu} V_{\nu)}.$$

Use this equality to compute the deformation tensor $\nabla_{(\mu} V_{\nu)}$ of the vector field of exercise **2.**. Check your answer using the result of exercise **2.**.

Due March 23

4. Consider $\mathcal{M} = \mathbb{R} \times \mathbb{R}^+ \times S^1$ with metric

$$g = -V\, du^2 + 2\, du dr + r^2\, d\theta^2,$$

where $V = V(r) > 0$.

a) Using the definition of the Hodge dual, calculate $\star\, du$ and $\star\, dr$.

b) Using the definition of the Hodge dual, calculate $\star\star\, du$ and $\star\star\, dr$. Verify the formula $\star\star\eta = s(-1)^{k(n-k)}\eta$ when $\eta = du$ and when $\eta = dr$.

c) Verify the formula $\eta \wedge \xi = s(\star\eta, \xi)\varepsilon$ when $\eta = du$ and when $\eta = dr$.

d) Check that the frame

$$\left(\frac{1}{\sqrt{V}}\partial_u, \frac{1}{\sqrt{V}}(\partial_u + V\partial_r), \frac{1}{r}\partial_\theta\right)$$

is orthonormal and calculate its dual frame $(\omega^1, \omega^2, \omega^3)$. Use the property $\star\,\omega^1 = (\omega^1, \omega^1)\, \omega^2 \wedge \omega^3$ (and a similar formula for $\star\,\omega^2$) to confirm the result you obtained in **a)**.

e) Given a smooth function f defined on $\mathcal{M}$, calculate the Hodge laplacian of f, $\Delta f = (\delta d + d\delta) f = -s \star d \star df = \star d \star df$.

`Due April 3`

5. Draw the Penrose diagram for the Schwarzschild solution with negative mass. Do timelike geodesics hit the naked singularity at $r = 0$?

6. Consider the CDM model (FLRW with $\alpha > 0$, $\Lambda > 0$ and $k = 0$).

a) Given $\varepsilon > 0$, show that for a sufficiently large

$$\sqrt{\frac{\Lambda}{3}}\, a < \dot{a} < \sqrt{\frac{\Lambda}{3}(1+\varepsilon)}\, a$$

and that for a sufficiently small

$$\frac{\sqrt{2\alpha}}{\sqrt{a}} < \dot{a} < \frac{\sqrt{2\alpha(1+\varepsilon)}}{\sqrt{a}}$$

b) Show that $a(\tau)$ only goes to $+\infty$ when $\tau \nearrow +\infty$ and that there exists a finite value $\tau^\star$ such that $a(\tau) \searrow 0$ as $\tau \searrow \tau^\star$.

c) Show that the radial null outgoing geodesics $(\tau(\varsigma), \psi(\varsigma), \frac{\pi}{2}, 0)$ satisfy

$$\begin{aligned} \tau' &= \frac{c}{a(\tau)}, \\ \psi' &= \frac{c}{a^2(\tau)}. \end{aligned}$$

d) Let $\delta > 0$. Show that for sufficiently large ς,

$$\sqrt{\frac{3}{\Lambda}}(1-\delta)\ln\varsigma < \tau(\varsigma) < \sqrt{\frac{3}{\Lambda}}(1+\delta)\ln\varsigma.$$

e) Let $\delta > 0$. Show that (for an appropriate choice of affine parameter $\tilde{\varsigma}$) for $\tilde{\varsigma}$ sufficiently close to zero

$$(1-\delta)\tilde{\varsigma}^{\beta} \leq \tau(\tilde{\varsigma}) - \tau^{\star} \leq (1+\delta)\tilde{\varsigma}^{\beta}.$$

What is the value of β?

`Due April 11`

7. Consider an Oppenheimer-Snyder solution obtained by gluing a FLRW metric

$$-d\tau^2 + a^2(\tau)(d\psi^2 + \psi^2 dl^2_{\mathcal{S}^2})$$

satisfying Friedmann's equations with $k = \Lambda = 0$ and $\alpha > 0$ with a Schwarzschild metric along an hypersurface $\{\psi = \psi_0\}$ of FLRW. Determine the value of a at the center (in terms of α and ψ_0) that corresponds to a light-ray that goes to future timelike infinity i^+.

`Due April 18`

8. Consider the FLRW metric

$$g = -d\tau^2 + a^2(\tau)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right)$$

and the orthonormal frame

$$\begin{aligned}
\omega^0 &= d\tau, \\
\omega^r &= \frac{a}{\sqrt{1-kr^2}}\, dr, \\
\omega^\theta &= ar\, d\theta, \\
\omega^\varphi &= ar\sin\theta\, d\varphi.
\end{aligned}$$

a) Using Cartan's structure equations, check that

$$\begin{aligned}
\omega^r_0 &= \frac{\dot{a}}{\sqrt{1-kr^2}}\, dr, \\
\omega^\theta_0 &= \dot{a}r\, d\theta, \\
\omega^\varphi_0 &= \dot{a}r\sin\theta\, d\varphi, \\
\omega^\theta_r &= \sqrt{1-kr^2}\, d\theta, \\
\omega^\varphi_r &= \sqrt{1-kr^2}\sin\theta\, d\varphi, \\
\omega^\varphi_\theta &= \cos\theta\, d\varphi.
\end{aligned}$$

Moreover, check that

$$\begin{aligned}
\Omega^r_0 &= \frac{\ddot{a}}{a}\,\omega^0 \wedge \omega^r, \\
\Omega^\theta_0 &= \frac{\ddot{a}}{a}\,\omega^0 \wedge \omega^\theta, \\
\Omega^\varphi_0 &= \frac{\ddot{a}}{a}\,\omega^0 \wedge \omega^\varphi, \\
\Omega^\theta_r &= -\frac{k+\dot{a}^2}{a^2}\,\omega^r \wedge \omega^\theta, \\
\Omega^\varphi_r &= -\frac{k+\dot{a}^2}{a^2}\,\omega^r \wedge \omega^\varphi, \\
\Omega^\varphi_\theta &= -\frac{k+\dot{a}^2}{a^2}\,\omega^\theta \wedge \omega^\varphi.
\end{aligned}$$

Finally, check that

$$\begin{aligned}
R_{00} &= -\frac{3\ddot{a}}{a}, \\
R_{rr} &= R_{\theta\theta} = R_{\varphi\varphi} = 2\frac{k+\dot{a}^2}{a^2} + \frac{\ddot{a}}{a},
\end{aligned}$$

and

$$R = 6\left(\frac{k+\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right).$$

b) Using Einstein's equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu},$$

with $T = \rho\, d\tau \otimes d\tau$, check that

$$\frac{d}{d\tau}\left(\frac{a\dot{a}^2}{2} + \frac{ka}{2} - \frac{\Lambda}{6}a^3\right) = 0$$

and obtain Friedmann's equations.

`Due May 9`

9. Consider the Riemannian or Lorentzian metric

$$g = dt^2 + h_{ij}(t,x)dx^i dx^j.$$

Show that

a) The Christoffel symbols are

$$\Gamma^0_{ij} = -K_{ij}, \qquad \Gamma^i_{jk} = \bar{\Gamma}^i_{jk}, \qquad \Gamma^i_{0j} = K^i{}_j,$$

where $\bar{\Gamma}^i_{jk}$ are the Christoffel symbols of h and $K(t)$ is the second fundamental form of the hypersurface $t =$ constant.

b) The components of the Riemann tensor are

$$\begin{aligned} R_{0i0}{}^j &= -\frac{\partial}{\partial t} K^j{}_i - K_{il}K^{lj}, \\ R_{ij0}{}^l &= -\bar{\nabla}_i K^l{}_j + \bar{\nabla}_j K^l{}_i, \\ R_{ijl}{}^m &= \bar{R}_{ijl}{}^m - K_{il}K^m{}_j + K_{jl}K^m{}_i, \end{aligned}$$

where $\bar{\nabla}$ is the Levi-Civita connection of h and $\bar{R}_{ijl}{}^m$ are the components of the Riemann tensor of h.

c) The components of the Ricci tensor are

$$\begin{aligned} R_{00} &= -\frac{\partial}{\partial t} K^i{}_i - K_{ij}K^{ij}, \\ R_{0i} &= -\bar{\nabla}_i K^j{}_j + \bar{\nabla}_j K^j{}_i, \\ R_{ij} &= \bar{R}_{ij} - \frac{\partial}{\partial t} K_{ij} + 2K_{il}K^l{}_j - K^l{}_l K_{ij}, \end{aligned}$$

where $\bar{R}_{ij}$ are the components of the Ricci tensor of h.

d) The time derivative of the inverse of h is

$$\frac{\partial h^{ij}}{\partial t} = -2K^{ij}.$$

e) The scalar curvature is

$$R = \bar{R} - 2\frac{\partial}{\partial t} K^i{}_i - (K^i{}_i)^2 - K_{ij}K^{ij}, \tag{1}$$

where $\bar{R}$ is the scalar curvature of h.

f) The component G_{00} of the Einstein tensor is

$$G_{00} = \frac{1}{2}\left(-\bar{R} + (K^i{}_i)^2 - K_{ij}K^{ij}\right). \tag{2}$$

10. Let (M, g) be the quotient of the 2-dimensional Minkowski spacetime by the group of isometries generated by the map $(t, x) \mapsto (t + 1, x + 1)$. Show directly that (M, g) is not stably causal, i.e. it is not possible to define a global time function.

`Due May 16`

11. Consider $(\mathbb{R}^3, -dt^2 + dx^2 + dy^2)$ and the congruence of timelike geodesics through the x-axis with velocity

$$X = \frac{t\partial_t + y\partial_y}{\sqrt{t^2 - y^2}}.$$

Consider the orthonormal frame $\mathcal{F}$, given by

$$\left(X, \partial_x, \frac{y\partial_t + t\partial_y}{\sqrt{t^2 - y^2}} \right).$$

a) Write the second fundamental form $B_{\mu\nu}$ of X in the frame $\mathcal{F}$.
b) Without actually calculating $\nabla_X \frac{y\partial_t + t\partial_y}{\sqrt{t^2-y^2}}$, justify that $\frac{y\partial_t + t\partial_y}{\sqrt{t^2-y^2}}$ is parallel along each integral curve of X.
c) Write the spatial metric $h_{\mu\nu}$. Calculate the expansion, deformation and vorticity, and use these to decompose the second fundamental form.
d) Verify the Raychaudhuri equation.
e) Define an appropriate fundamental solution A of the Jacobi equation to encode the evolution of a general deviation vector orthogonal to X. Calculate the fundamental solution and check that $B = \dot{A}A^{-1}$.

`Due May 30`

12. Consider $\mathbb{R}^3$ with the Minkowski metric written in polar coordinates as

$$g = -dt^2 + dr^2 + r^2\, d\theta^2.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be periodic with period 2π and

$$X = f(\theta)(\partial_t + \partial_r).$$

Consider the frame

$$\mathcal{V} = \left(\partial_t, X, \frac{1}{r}\partial_\theta \right).$$

a) Verify that X is null geodesic.
b) Compute the second fundamental form $B^\mu{}_\nu$ in the coordinates corresponding to $\mathcal{V}$ by calculating $\nabla_{\partial_t} X$ and $\nabla_{\frac{1}{r}\partial_\theta} X$.
c) Determine the integral curves (t, r, θ) of X through $(0, 1, \theta_0)$ in terms of the affine parameter u. Express r in terms of t.

d) The vector field $Y = \partial_\theta$ is a Jacobi field $\left(\partial_\theta = \partial_{\theta_0} - u\frac{f'(\theta_0)}{f(\theta_0)}\partial_u\right)$. Note however that Y does not commute with X. Correct the equation $\nabla_X Y^\mu = B^\mu{}_\nu Y^\nu$ to take this into account and verify the corrected equation directly.

e) Write the expression for the metric g in the frame $\mathcal{V}$. Compute the covector $X_\flat$.

f) Compute the second fundamental form $B_{\mu\nu}$ in the coordinates corresponding to $\mathcal{V}$ by calculating $\nabla_{\partial_t} X_\flat$ and $\nabla_{\frac{1}{r}\partial_\theta} X_\flat$. To check your answer, verify that $B_{\mu\nu} = g_{\mu\gamma} B^\gamma{}_\nu$.

13. Use ideas similar to those leading to the proof of Hawking's singularity theorem to prove Myers's Theorem: if (M, g) is a complete Riemannian manifold such that there exists an $\varepsilon > 0$ so that $R_{\mu\nu} X^\mu X^\nu \geq \varepsilon g_{\mu\nu} X^\mu X^\nu$, then M is compact.

`Due June 15`

14. State why Hawking's Singularity Theorem and state why Penrose's Singularity Theorem do not apply to each of the following geodesically complete Lorentzian manifolds:

a) Minkowski's spacetime;
b) Einstein's spacetime;
c) de Sitter's spacetime;
d) Anti-de Sitter spacetime.

15. Calculate the following two expressions for the divergence of X in local coordinates

$$\operatorname{div} X = \frac{1}{\sqrt{|\det g|}} \partial_\mu \left(\sqrt{|\det g|} X^\mu \right) = \nabla_\mu X^\mu,$$

thereby checking that they agree.

16. Verify that the critical points of the action

$$I(\phi) = \frac{1}{2} \int (\operatorname{grad} \phi, \operatorname{grad} \phi) = \frac{1}{2} \int g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \sqrt{|\det g|}\, dx^0 dx^1 dx^2 dx^3$$

are the solutions of the wave equation

$$\nabla^\mu \nabla_\mu \phi = 0.$$

Do not worry about boundary terms.

17. Check that if ϕ satisfies the Euler-Lagrange equation for $L(x, \phi, \partial\phi)$, then

$$T^{\mu\nu}(\phi) := \frac{\partial L}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} L$$

satisfies $\nabla_\mu T^{\mu\nu} = 0$.