

**Mathematical Relativity, Spring 2018/19**  
**Instituto Superior Técnico**

1. Starting from

$$R_{\alpha\beta\mu\nu}Z^\nu = 2\nabla_{[\alpha}\nabla_{\beta]}Z_\mu,$$

deduce the components of the Riemann curvature tensor in terms of the Christoffel symbols.

2. Recall that the nonzero Christoffel symbols for the Minkowski metric in spherical coordinates,

$$\eta = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

are

$$\begin{aligned}\Gamma_{\theta\theta}^r &= -r, & \Gamma_{\varphi\varphi}^r &= -r\sin^2\theta, \\ \Gamma_{r\theta}^\theta &= \Gamma_{\theta r}^\theta = \frac{1}{r}, & \Gamma_{\varphi\varphi}^\theta &= -\sin\theta\cos\theta, \\ \Gamma_{r\varphi}^\varphi &= \Gamma_{\varphi r}^\varphi = \frac{1}{r}, & \Gamma_{\theta\varphi}^\varphi &= \Gamma_{\varphi\theta}^\varphi = \cot\theta.\end{aligned}$$

Consider the vector field

$$V = f(r)\partial_r.$$

Compute the tensor  $\nabla^\mu V^\nu$ .

3. Show that

$$\frac{1}{2}(LVg)_{\mu\nu} = \nabla_{(\mu}V_{\nu)}.$$

Use this equality to compute the deformation tensor  $\nabla_{(\mu}V_{\nu)}$  of the vector field of exercise 2.. Check your answer using the result of exercise 2..

4. Let  $M$  be a Riemannian or Lorentzian manifold, and let  $s$  be equal to 1 in the first case and  $-1$  in the second case. Let also  $\eta$  and  $\xi$  be  $k$ -forms on  $M$ , and  $\theta$  be a  $(n-k)$  for on  $M$ . Denote by  $\varepsilon$  the volume form on  $M$ . Show that

$$\begin{aligned}\theta \wedge \eta &= s(*\theta, \eta)\varepsilon, \\ \xi \wedge *\eta &= (\xi, \eta)\varepsilon, \\ **\eta &= s(-1)^{k(n-k)}\eta,\end{aligned}$$

using bilinearity and the expression for the Hodge dual of a form in a positively oriented coframe.

5. Let  $r^*$  be the tortoise coordinate

$$r^* = r + 2m \ln |r - 2m|,$$

and  $u$  and  $v$  be the null coordinates

$$\begin{aligned} u &= t - r^*, \\ v &= t + r^* \end{aligned}$$

for the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 dS^2$$

Verify that in the coordinates

$$\begin{aligned} \tilde{u} &= \tanh\left(\frac{u}{8m}\right), \\ \tilde{v} &= \tanh\left(\frac{v}{8m}\right), \end{aligned}$$

the metric can be extended continuously to the event horizon.

6. Draw the Penrose diagram for the Schwarzschild solution with negative mass. Do timelike geodesics hit the naked singularity at  $r = 0$ ?

7. Consider the Riemannian or Lorentzian metric

$$g = dt^2 + h_{ij}(t, x) dx^i dx^j.$$

Show that

a) The Christoffel symbols are

$$\Gamma_{ij}^0 = -K_{ij}, \quad \Gamma_{jk}^i = \bar{\Gamma}_{jk}^i, \quad \Gamma_{0j}^i = K_j^i,$$

where  $\bar{\Gamma}_{jk}^i$  are the Christoffel symbols of  $h$  and  $K(t)$  is the second fundamental form of the hypersurface  $t = \text{constant}$ .

b) The components of the Riemann tensor are

$$\begin{aligned} R_{0i0}{}^j &= -\frac{\partial}{\partial t} K_j^i - K_{il} K^{lj}, \\ R_{ij0}{}^l &= -\bar{\nabla}_i K_j^l + \bar{\nabla}_j K_i^l, \\ R_{ijl}{}^m &= \bar{R}_{ijl}{}^m - K_{il} K_j^m + K_{jl} K_i^m, \end{aligned}$$

where  $\bar{\nabla}$  is the Levi-Civita connection of  $h$  and  $\bar{R}_{ijl}{}^m$  are the components of the Riemann tensor of  $h$ .

c) The components of the Ricci tensor are

$$\begin{aligned} R_{00} &= -\frac{\partial}{\partial t}K^i{}_i - K_{ij}K^{ij}, \\ R_{0i} &= -\bar{\nabla}_i K^j{}_j + \bar{\nabla}_j K^j{}_i, \\ R_{ij} &= \bar{R}_{ij} - \frac{\partial}{\partial t}K_{ij} + 2K_{il}K^l{}_j - K^l{}_l K_{ij}, \end{aligned}$$

where  $\bar{R}_{ij}$  are the components of the Ricci tensor of  $h$ .

d) The time derivative of the inverse of  $h$  is

$$\frac{\partial h^{ij}}{\partial t} = -2K^{ij}.$$

e) The scalar curvature is

$$R = \bar{R} - 2\frac{\partial}{\partial t}K^i{}_i - (K^i{}_i)^2 - K_{ij}K^{ij}, \quad (1)$$

where  $\bar{R}$  is the scalar curvature of  $h$ .

f) The component  $G_{00}$  of the Einstein tensor is

$$G_{00} = \frac{1}{2}(-\bar{R} + (K^i{}_i)^2 - K_{ij}K^{ij}). \quad (2)$$

**8.** Construct an Oppenheimer-Snyder solution of the Einstein equations with a positive cosmological constant using a ball of dust taken from a hyperbolic universe glued to a portion of the Schwarzschild de Sitter spacetime.

**9.** Let  $(M, g)$  be a stably causal spacetime and  $h$  be an arbitrary symmetric  $(2, 0)$ -tensor field with compact support. Show that for sufficiently small  $|\varepsilon|$  the tensor field  $g_\varepsilon = g + \varepsilon h$  is still a Lorentzian metric on  $M$ , and  $(M, g_\varepsilon)$  satisfies the chronology condition.

**10.** Let  $(M, g)$  be the quotient of the 2-dimensional Minkowski spacetime by the discrete group of isometries generated by the map  $(t, x) \mapsto (t + 1, x + 1)$ . Show that  $(M, g)$  satisfies the chronology condition, but there exist arbitrarily small perturbations of  $(M, g)$  (in the sense of exercise **9.**) which do not.

**11.** Let  $(\Sigma, h)$  be a 3-dimensional Riemannian manifold. Show that the spacetime  $(M, g) = (\mathbb{R} \times \Sigma, -dt \otimes dt + h)$  is globally hyperbolic if and only if  $(\Sigma, h)$  is complete.

**12.** Show that the following spacetimes are globally hyperbolic:

- a) the Minkowski spacetime;
- b) the FLRW spacetimes;
- c) the region  $r > 2m$  of the Schwarzschild spacetime;
- d) the region  $r < 2m$  of the Schwarzschild spacetime;
- e) the maximal analytical extension of the Schwarzschild spacetime.

13. Consider  $\mathbb{R}^3$  with the Minkowski metric written in polar coordinates as

$$g = -d\tau^2 + dr^2 + r^2 d\theta^2.$$

- a) Using the geodesic Lagrangian obtained from the metric, compute the Christoffel symbols in these coordinates.
- b) Verify that

$$X := \frac{1}{\sqrt{3}} \left( 2\partial_\tau + \left(\frac{\tau}{2r}\right)\partial_r + \sqrt{1 - \left(\frac{\tau}{2r}\right)^2} \frac{1}{r}\partial_\theta \right)$$

is unit timelike and geodesic.

- c) Compute the second fundamental form  $B_{\mu\nu}$  using the coordinates corresponding to the basis  $(\partial_\tau, \partial_r, \partial_\theta)$ .
- d) Compute the spatial metric  $h_{\mu\nu} = g_{\mu\nu} + X_\mu X_\nu$  using the coordinates corresponding to the basis  $(\partial_\tau, \partial_r, \partial_\theta)$ .
- e) Verify that  $(X, E_1, E_2)$  is an orthonormal basis, where

$$E_1 = -\frac{1}{\sqrt{3}} \left( \partial_\tau + \frac{\tau}{r}\partial_r + \frac{2}{r}\sqrt{1 - \left(\frac{\tau}{2r}\right)^2}\partial_\theta \right),$$

and

$$E_2 = \sqrt{1 - \left(\frac{\tau}{2r}\right)^2}\partial_r - \frac{\tau}{2r^2}\partial_\theta.$$

It will follow from the computations below that  $E_1$  and  $E_2$  are parallel along  $X$ .

- f) Express  $B_{\mu\nu}$  in the basis  $(X, E_1, E_2)$ .
- g) Compute the expansion  $\theta$ , the deformation  $\sigma_{\mu\nu}$  and the vorticity  $\omega_{\mu\nu}$  using the coordinates corresponding to the basis  $(X, E_1, E_2)$ .
- h) Verify that

$$X \cdot \left( \frac{1}{r\sqrt{1 - \left(\frac{\tau}{2r}\right)^2}} \right) = 0.$$

- i) Determine the integral curves  $(\tau, r, \theta)$  of  $X$  through  $(0, r_0, \theta_0)$ , parametrized in terms of the affine parameter  $u$ . Express  $\tau$  and  $r$  in terms of

$\theta$ . Check that the lines which form the congruence are given by the intersection of the hyperboloids

$$x^2 + y^2 - \frac{\tau^2}{4} = r_0^2$$

with the planes

$$\cos \theta_0 x + \sin \theta_0 y = r_0.$$

j) Compute the change of basis matrix  $A$  such that

$$\begin{bmatrix} \partial_r \\ \partial_\theta \end{bmatrix} = A \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix}.$$

k) Verify that along the line of the congruence through  $(0, r_0, \theta_0)$  we have

$$E_1 = -\frac{1}{\sqrt{3}} \left( \partial_\tau + 2 \sin(\theta - \theta_0) \partial_r + 2 \cos(\theta - \theta_0) \frac{1}{r} \partial_\theta \right),$$

and

$$E_2 = \cos(\theta - \theta_0) \partial_r - \sin(\theta - \theta_0) \frac{1}{r} \partial_\theta.$$

Express  $E_1$  and  $E_2$  in the basis  $(\partial_\tau, \partial_x, \partial_y)$ . What is  $B$  along the line?

l) Center your attention on the line  $L$  corresponding to  $r_0 = 1$  and  $\theta_0 = 0$ . The vector fields  $Y_1 = \partial_x$  and  $Y_2 = \partial_\theta$  are Jacobi fields along  $L$ . Write  $Y_1$  and  $Y_2$  in the basis  $(X, E_1, E_2)$ . Check directly that  $Y_1$  and  $Y_2$  satisfy the equation

$$\frac{d}{du} Y^\mu = B^\mu{}_\nu Y^\nu.$$

14. Apply Hawking's Singularity Theorem to the CDM model. Be careful to guarantee that the SEC is satisfied.

15. Consider  $\mathbb{R}^3$  with the Minkowski metric written in polar coordinates as

$$g = -dt^2 + dr^2 + r^2 d\theta^2.$$

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be periodic with period  $2\pi$  and

$$X = f(\theta)(\partial_t + \partial_r).$$

Consider the frame

$$\mathcal{V} = \left( \partial_t, X, \frac{1}{r} \partial_\theta \right).$$

a) Verify that  $X$  is null geodesic.

- b) Compute the second fundamental form  $B^\mu{}_\nu$  in the coordinates corresponding to  $\mathcal{V}$  by calculating  $\nabla_{\partial_t} X$  and  $\nabla_{\frac{1}{r}\partial_\theta} X$ .
- c) Determine the integral curves  $(t, r, \theta)$  of  $X$  through  $(0, 1, \theta_0)$  in terms of the affine parameter  $u$ . Express  $r$  in terms of  $t$ .
- d) The vector field  $Y = \partial_\theta$  is a Jacobi field  $\left(\partial_\theta = \partial_{\theta_0} - u \frac{f'(\theta_0)}{f(\theta_0)} \partial_u\right)$ . Note however that  $Y$  does not commute with  $X$ . Correct the equation  $\nabla_X Y^\mu = B^\mu{}_\nu Y^\nu$  to take this into account and verify the corrected equation directly.
- e) Write the expression for the metric  $g$  in the frame  $\mathcal{V}$ . Compute the covector  $X_b$ .
- f) Compute the second fundamental form  $B_{\mu\nu}$  in the coordinates corresponding to  $\mathcal{V}$  by calculating  $\nabla_{\partial_t} X_b$  and  $\nabla_{\frac{1}{r}\partial_\theta} X_b$ . To check your answer, verify that  $B_{\mu\nu} = g_{\mu\gamma} B^\gamma{}_\nu$ .

16. Consider the metric

$$g = -dt^2 + a^2(t)(d\psi^2 + f^2(\psi)(d\theta^2 + \sin^2 \theta d\varphi^2))$$

defined on  $M$ . When  $M = \mathbb{R} \times S^3$ ,  $f(\psi) = \sin \psi$ ; when  $M = \mathbb{R} \times \mathbb{R}^3$ ,  $f(\psi) = \psi$ ; when  $M = \mathbb{R} \times H^3$ ,  $f(\psi) = \sinh \psi$ .

- a) Using the geodesic Lagrangian obtained from the metric, compute the Christoffel symbols in the coordinates  $(t, \psi, \theta, \phi)$ .
- b) Verify that the null vector field

$$X = \frac{1}{a} \partial_t + \frac{1}{a^2} \partial_\psi$$

is geodesic.

- c) Compute

$$\nabla_{\partial_\theta} X \quad \text{and} \quad \nabla_{\partial_\varphi} X.$$

Use these to obtain the expansion of the congruence of null geodesics tangent to  $X$ .

- d) Let  $\Sigma$  be the sphere  $t = t_0$  and  $\psi = \psi_0$ . Denote by  $\xi_r$  the flow of  $X$ . The 2-dimensional surface  $\xi_r(\Sigma)$  is parametrized by

$$(\theta, \varphi) \mapsto (t(r), \psi(r), \theta, \varphi).$$

Here  $t(r)$  is determined by

$$r = \int_{t_0}^t a(s) ds$$

and  $\psi(r)$  satisfies

$$\psi = \psi_0 + \int_0^r \frac{1}{a^2(t(s))} ds.$$

Define

$$\tilde{A}(t, \psi) = a^2(t) f^2(\psi)$$

and

$$A(r) = \tilde{A}(t(r), \psi(r)).$$

Note that

$$\frac{dA}{dr} = X \cdot \tilde{A}.$$

Use the area element of  $\xi_r(\Sigma)$ , which is

$$A(r) \sin \theta d\theta \wedge d\varphi,$$

to confirm the result you obtained above for the expansion of the congruence of null geodesics.

**17.** Use ideas similar to those leading to the proof of Hawking's singularity theorem to prove Myers's Theorem: if  $(M, g)$  is a complete Riemannian manifold such that there exists an  $\varepsilon > 0$  so that  $R_{\mu\nu} X^\mu X^\nu \geq \varepsilon g_{\mu\nu} X^\mu X^\nu$ , then  $M$  is compact. Can these ideas be used to prove a singularity theorem in Riemannian geometry?

**18.** Explain why Hawking's Singularity Theorem and explain why Penrose's Singularity Theorem do not apply to each of the following geodesically complete Lorentzian manifolds:

- a) Minkowski's spacetime;
- b) Einstein's spacetime;
- c) de Sitter's spacetime;
- d) Anti-de Sitter spacetime.

**19.** Check that Penrose's Singularity Theorem can be applied to a FLRW flat universe with  $\alpha > 0$  and  $\Lambda > 0$ .

**20.** Write the wave equation for the FLRW models,  $\square\phi = \nabla^\mu \nabla_\mu \phi = 0$ .

**21.** Compute the Komar mass of the Schwarzschild solution.

**22.** Let  $h$  be the spherically symmetric Riemannian metric defined in  $R^3$  by

$$h = \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2),$$

where  $m$  is a smooth function in  $[0, +\infty[$  such that  $m(0) = m'(0) = m''(0) = 0$ ,  $m(r) < \frac{r}{2}$  for all  $r > 0$ , there exist  $\varepsilon$  and  $M$  in  $\mathbb{R}^+$  such that

$$\lim_{r \rightarrow +\infty} m(r) = M$$

and

$$m'(r) = O\left(\frac{1}{r^{1+\varepsilon}}\right) \text{ as } r \rightarrow \infty,$$

and

$$m''(r) = O\left(\frac{1}{r^2}\right) \text{ as } r \rightarrow \infty.$$

a) Check that in Cartesian coordinates we have

$$h_{ij} = \delta_{ij} + \frac{\frac{2m(r)}{r^3}}{1 - \frac{2m(r)}{r}} x^i x^j.$$

b) Check that  $h$  has ADM mass  $M$ .

c) Knowing that the scalar curvature of  $h$  is  $\bar{R} = \frac{4m'(r)}{r^2}$ , verify that  $h$  is asymptotically flat.

d) Relaxing the assumption on the second derivative of  $m$  to  $m''(r) = O(r^\alpha)$  as  $r \rightarrow \infty$ , for what values of  $\alpha$  is  $h$  asymptotically flat?

e) Verify the rigidity statement of the Positive Mass Theorem for the metric  $h$ .

f) Assume now that  $m$  is constant, so that  $h$  defines a Riemannian metric only in the region where  $r > 2m$ . Compute the second fundamental form of the sphere  $r = r_0$  and show that when  $r_0 = 2m$  the sphere is a minimal surface. Check the Penrose inequality.

**23.** Consider the 3-dimensional Riemannian manifold  $M$  equal to the graph of the function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ , with the metric induced by the Euclidean metric of  $\mathbb{R}^4$ . Let  $c$  be a constant and  $S = S_c = \{(p, c) = (x, y, z, c) \in M : f(p) = c\}$  be a 2-dimensional surface such that  $\nabla f(p) \neq 0$  for all  $(p, c) \in M$ . Suppose  $(u, v)$  are coordinates on  $S$  such that

$$\partial_u \cdot \partial_u = 1, \quad \partial_v \cdot \partial_v = 1, \quad \partial_u \cdot \partial_v = 0.$$

a) Check that the (unique up to sign) unit normal to  $S$  in  $TM$  is

$$X = \frac{\left(\frac{\nabla f}{|\nabla f|}, |\nabla f|\right)}{\sqrt{1 + |\nabla f|^2}}.$$



b) Let  $\tilde{\nabla}$  denote the Levi-Civita connection of  $\mathbb{R}^4$ . Denote by

$$v_1 = \tilde{\nabla}_{\partial_u}(\nabla f, 0), \quad v_2 = \tilde{\nabla}_{\partial_v}(\nabla f, 0),$$

and by

$$\alpha_1 = v_1 \cdot (\nabla f, 0), \quad \alpha_2 = v_2 \cdot (\nabla f, 0).$$

Calculate  $\tilde{\nabla}_{\partial_u}X$  and  $\tilde{\nabla}_{\partial_v}X$  in terms of the  $v_i$ ,  $\alpha_i$  and  $\nabla f$ . Simplify your answer and check directly that the vectors you obtained are orthogonal to  $X$  (which is obvious from  $X \cdot X = 1$ ).

c) Let  $\bar{\nabla}$  denote the Levi-Civita connection on  $M$ . Compute  $\partial_u \cdot \bar{\nabla}_{\partial_u}X$ ,  $\partial_v \cdot \bar{\nabla}_{\partial_u}X$ ,  $\partial_u \cdot \bar{\nabla}_{\partial_v}X$  and  $\partial_v \cdot \bar{\nabla}_{\partial_v}X$  in terms of  $\partial_u \cdot v_i$ ,  $\partial_v \cdot v_i$  and  $|\nabla f|$ . Write the second fundamental form  $\text{II}$  of  $S$  in  $M$ .

d) Denote by  $\nabla f$  the column vector

$$\nabla f = \begin{bmatrix} \partial_x f \\ \partial_y f \\ \partial_z f \end{bmatrix}.$$

Check that the metric on  $M$  is given by

$$g = I + \nabla f (\nabla f)^T$$

in the coordinates  $(x, y, z)$ , i.e.

$$ds^2 = \begin{bmatrix} dx & dy & dz \end{bmatrix} g \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}.$$

e) Let  $(u, v, w)$  be coordinates on  $M$ , and  $l$  be a parametrization of  $M$ , such that  $u$  and  $v$  are as above, and let  $w$  be such that it vanishes on  $S$  and

$$\partial_w = \frac{\partial_x f \partial_x + \partial_y f \partial_y + \partial_z f \partial_z}{|\nabla f| \sqrt{1 + |\nabla f|^2}}.$$

Denote by

$$Dl = [\partial_u \ \partial_v \ \partial_w] = \begin{bmatrix} \partial_u l^1 & \partial_v l^1 & \partial_w l^1 \\ \partial_u l^2 & \partial_v l^2 & \partial_w l^2 \\ \partial_u l^3 & \partial_v l^3 & \partial_w l^3 \end{bmatrix}.$$

What is the matrix that represents the metric on  $M$  in the coordinates  $(u, v, w)$ ? What does it reduce to over points that belong to  $S$ ?

f) Is it true that

$$\frac{1}{2} L_{\partial_w} g \Big|_{TS} = \text{II}?$$

Justify your answer.

g) Use your answer to e) to show that on points that belong to  $S$

$$\frac{1}{2}L_{\partial_w}g \Big|_{TS} = \begin{bmatrix} du & dv & dw \end{bmatrix} \frac{1}{2}\partial_w[(Dl)^T Dl] \begin{bmatrix} du \\ dv \\ dw \end{bmatrix} \Big|_{TS} .$$

24. Consider the surfaces  $\Sigma_t$  obtained from the boundary  $\partial K \equiv \Sigma_0$  of a compact convex set  $K$  contained in  $\mathbb{R}^3$  by flowing a distance  $t$  along the unit normal. Let  $A(t)$  be the area of  $\Sigma_t$  and  $V(t)$  be the volume bounded by  $\Sigma_t$ . As we saw in class,

$$\dot{A}(0) = \int_{\partial K} \text{tr } \kappa,$$

where  $\kappa$  is the second fundamental form of  $\Sigma$ .

- a) Prove that  $\ddot{A}(t) = 8\pi$ . (Suggestion: use equalities (1) and (2) above, and the Gauss-Bonnet Theorem).
- b) Deduce that

$$V(t) = \frac{4\pi}{3}t^3 + \frac{\dot{A}(0)}{2}t^2 + A(0)t + V(0).$$

- c) Use the isoperimetric inequality,  $A^3(t) \geq 36\pi V^2(t)$ , to prove Minkowski's inequality

$$\dot{A}(0) \geq \sqrt{16\pi A(0)}.$$