Mathematical Relativity, Spring 2017/18
Instituto Superior Técnico

1. Starting from

$$R_{\alpha\beta\mu\nu}Z^{\nu} = 2\nabla_{[\alpha}\nabla_{\beta]}Z_{\mu},$$

deduce the components of the Riemann curvature tensor in terms of the Christoffel symbols.

2. Recall that the nonzero Christoffel symbols for the Minkowski metric in spherical coordinates,

$$\eta = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),$$

are

$$\Gamma^{r}_{\theta\theta} = -r, \quad \Gamma^{r}_{\varphi\varphi} = -r\sin^2\theta,$$
$$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r}, \quad \Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta,$$
$$\Gamma^{\varphi}_{r\varphi} = \Gamma^{\varphi}_{\varphi r} = \frac{1}{r}, \quad \Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \cot\theta.$$

Consider the vector field

$$V = f(r)\partial_r.$$

Compute the tensor $\nabla^{\mu}V^{\nu}$.

3. Show that

$$\frac{1}{2}(L_V g)_{\mu\nu} = \nabla_{(\mu}V_{\nu)}.$$

Use this equality to compute the deformation tensor $\nabla_{(\mu}V_{\nu)}$ of the vector field of exercise **2.**. Check your answer using the result of exercise **2.**.

4. Let M be a Riemannian or Lorentzian manifold, and let s be equal to 1 in the first case and -1 in the second case. Let also η and ξ be k-forms on M, and θ be a $(n-k)$ for on M. Denote by ε the volume form on M. Show that

$$\begin{aligned} \theta \wedge \eta &= s(*\theta, \eta)\varepsilon, \\ \xi \wedge *\eta &= (\xi, \eta)\varepsilon, \\ **\eta &= s(-1)^{k(n-k)}\eta, \end{aligned}$$

using bilinearity and the expression for the Hodge dual of a form in a positively oriented coframe.

5. Let r^* be the tortoise coordinate

$$r^* = r + 2m \ln |r - 2m|,$$

and u and v be the null coordinates

$$\begin{aligned} u &= t - r^*, \\ v &= t + r^* \end{aligned}$$

for the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 dS^2$$

Verify that in the coordinates

$$\begin{aligned} \tilde{u} &= \tanh\left(\frac{u}{8m}\right), \\ \tilde{v} &= \tanh\left(\frac{v}{8m}\right), \end{aligned}$$

the metric can be extended continuously to the event horizon.

6. Draw the Penrose diagram for the Schwarzschild solution with negative mass. Do timelike geodesics hit the naked singularity at $r = 0$?

7. Consider the Riemannian or Lorentzian metric

$$g = dt^2 + h_{ij}(t, x) dx^i dx^j.$$

Show that

a) The Christoffel symbols are

$$\Gamma^0_{ij} = -K_{ij}, \qquad \Gamma^i_{jk} = \bar{\Gamma}^i_{jk}, \qquad \Gamma^i_{0j} = K^i{}_j,$$

where $\bar{\Gamma}^i_{jk}$ are the Christoffel symbols of h and $K(t)$ is the second fundamental form of the hypersurface $t =$ constant.

b) The components of the Riemann tensor are

$$\begin{aligned} R_{0i0}{}^j &= -\frac{\partial}{\partial t} K^j{}_i - K_{il} K^{lj}, \\ R_{ij0}{}^l &= -\bar{\nabla}_i K^l{}_j + \bar{\nabla}_j K^l{}_i, \\ R_{ijl}{}^m &= \bar{R}_{ijl}{}^m - K_{il} K^m{}_j + K_{jl} K^m{}_i, \end{aligned}$$

where $\bar{\nabla}$ is the Levi-Civita connection of h and $\bar{R}_{ijl}{}^m$ are the components of the Riemann tensor of h.

c) The components of the Ricci tensor are

$$\begin{aligned} R_{00} &= -\frac{\partial}{\partial t}K^i{}_i - K_{ij}K^{ij}, \\ R_{0i} &= -\bar{\nabla}_i K^j{}_j + \bar{\nabla}_j K^j{}_i, \\ R_{ij} &= \bar{R}_{ij} - \frac{\partial}{\partial t}K_{ij} + 2K_{il}K^l{}_j - K^l{}_l K_{ij}, \end{aligned}$$

where $\bar{R}_{ij}$ are the components of the Ricci tensor of h.

d) The time derivative of the inverse of h is

$$\frac{\partial h^{ij}}{\partial t} = -2K^{ij}.$$

e) The scalar curvature is

$$R = \bar{R} - 2\frac{\partial}{\partial t}K^i{}_i - (K^i{}_i)^2 - K_{ij}K^{ij}, \qquad (1)$$

where $\bar{R}$ is the scalar curvature of h.

f) The component G_{00} of the Einstein tensor is

$$G_{00} = \frac{1}{2}\left(-\bar{R} + (K^i{}_i)^2 - K_{ij}K^{ij}\right). \qquad (2)$$

8. Construct an Oppenheimer-Snyder solution of the Einstein equations with a positive cosmological constant using a ball of dust taken from a hyperbolic universe glued to a portion of the Schwarzschild de Sitter spacetime.

9. Let (M, g) be a stably causal spacetime and h be an arbitrary symmetric $(2, 0)$-tensor field with compact support. Show that for sufficiently small $|\varepsilon|$ the tensor field $g_\varepsilon = g + \varepsilon h$ is still a Lorentzian metric on M, and (M, g_ε) satisfies the chronology condition.

10. Let (M, g) be the quotient of the 2-dimensional Minkowski spacetime by the discrete group of isometries generated by the map $(t, x) \mapsto (t + 1, x + 1)$. Show that (M, g) satisfies the chronology condition, but there exist arbitrarily small perturbations of (M, g) (in the sense of exercise **9.**) which do not.

11. Let (Σ, h) be a 3-dimensional Riemannian manifold. Show that the spacetime $(M, g) = (\mathbb{R} \times \Sigma, -dt \otimes dt + h)$ is globally hyperbolic if and only if (Σ, h) is complete.

12. Show that the following spacetimes are globally hyperbolic:

a) the Minkowski spacetime;
b) the FLRW spacetimes;
c) the region $r > 2m$ of the Schwarzschild spacetime;
d) the region $r < 2m$ of the Schwarzschild spacetime;
e) the maximal analytical extension of the Schwarzschild spacetime.

13. Consider $\mathbb{R}^3$ with the Minkowski metric written in polar coordinates as

$$g = -d\tau^2 + dr^2 + r^2\, d\theta^2.$$

a) Using the geodesic Lagrangian obtained from the metric, compute the Christoffel symbols in these coordinates.

b) Verify that

$$X := \frac{1}{\sqrt{3}}\left(2\partial_\tau + \left(\frac{\tau}{2r}\right)\partial_r + \sqrt{1-\left(\frac{\tau}{2r}\right)^2}\frac{1}{r}\partial_\theta\right)$$

is unit timelike and geodesic.

c) Compute the second fundamental form $B_{\mu\nu}$ using the coordinates corresponding to the basis $(\partial_\tau, \partial_r, \partial_\theta)$.

d) Compute the spatial metric $h_{\mu\nu} = g_{\mu\nu} + X_\mu X_\nu$ using the coordinates corresponding to the basis $(\partial_\tau, \partial_r, \partial_\theta)$.

e) Verify that (X, E_1, E_2) is an orthonormal basis, where

$$E_1 = -\frac{1}{\sqrt{3}}\left(\partial_\tau + \frac{\tau}{r}\partial_r + \frac{2}{r}\sqrt{1-\left(\frac{\tau}{2r}\right)^2}\partial_\theta\right),$$

and

$$E_2 = \sqrt{1-\left(\frac{\tau}{2r}\right)^2}\partial_r - \frac{\tau}{2r^2}\partial_\theta.$$

It will follow from the computations below that E_1 and E_2 are parallel along X.

f) Express $B_{\mu\nu}$ in the basis (X, E_1, E_2).

g) Compute the expansion θ, the deformation $\sigma_{\mu\nu}$ and the vorticity $\omega_{\mu\nu}$ using the coordinates corresponding to the basis (X, E_1, E_2).

h) Verify that

$$X \cdot \left(\frac{1}{r\sqrt{1-(\frac{\tau}{2r})^2}}\right) = 0.$$

i) Determine the integral curves (τ, r, θ) of X through $(0, r_0, \theta_0)$, parametrized in terms of the affine parameter u. Express τ and r in terms of

θ. Check that the lines which form the congruence are given by the intersection of the hyperboloids

$$x^2 + y^2 - \frac{\tau^2}{4} = r_0^2$$

with the planes

$$\cos\theta_0\, x + \sin\theta_0\, y = r_0.$$

j) Compute the change of basis matrix A such that

$$\begin{bmatrix} \partial_r \\ \partial_\theta \end{bmatrix} = A \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix}.$$

k) Verify that along the line of the congruence through $(0, r_0, \theta_0)$ we have

$$E_1 = -\frac{1}{\sqrt{3}}\left(\partial_\tau + 2\sin(\theta-\theta_0)\partial_r + 2\cos(\theta-\theta_0)\,\frac{1}{r}\partial_\theta\right),$$

and

$$E_2 = \cos(\theta-\theta_0)\partial_r - \sin(\theta-\theta_0)\,\frac{1}{r}\partial_\theta.$$

Express E_1 and E_2 in the basis $(\partial_\tau, \partial_x, \partial_y)$. What is B along the line?

l) Center your attention on the line L corresponding to $r_0 = 1$ and $\theta_0 = 0$. The vector fields $Y_1 = \partial_x$ and $Y_2 = \partial_\theta$ are Jacobi fields along L. Write Y_1 and Y_2 in the basis (X, E_1, E_2). Check directly that Y_1 and Y_2 satisfy the equation

$$\tfrac{d}{du} Y^\mu = B^\mu{}_\nu Y^\nu.$$

14. Apply Hawking's Singularity Theorem to the CDM model. Be careful to guarantee that the SEC is satisfied.

15. Consider $\mathbb{R}^3$ with the Minkowski metric written in polar coordinates as

$$g = -dt^2 + dr^2 + r^2\, d\theta^2.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be periodic with period 2π and

$$X = f(\theta)(\partial_t + \partial_r).$$

Consider the frame

$$\mathcal{V} = \left(\partial_t, X, \frac{1}{r}\partial_\theta\right).$$

a) Verify that X is null geodesic.

b) Compute the second fundamental form $B^\mu{}_\nu$ in the coordinates corresponding to $\mathcal{V}$ by calculating $\nabla_{\partial_t}X$ and $\nabla_{\frac{1}{r}\partial_\theta}X$.

c) Determine the integral curves (t, r, θ) of X through $(0, 1, \theta_0)$ in terms of the affine parameter u. Express r in terms of t.

d) The vector field $Y = \partial_\theta$ is a Jacobi field $\left(\partial_\theta = \partial_{\theta_0} - u\frac{f'(\theta_0)}{f(\theta_0)}\partial_u\right)$. Note however that Y does not commute with X. Correct the equation $\nabla_X Y^\mu = B^\mu{}_\nu Y^\nu$ to take this into account and verify the corrected equation directly.

e) Write the expression for the metric g in the frame $\mathcal{V}$. Compute the covector $X_\flat$.

f) Compute the second fundamental form $B_{\mu\nu}$ in the coordinates corresponding to $\mathcal{V}$ by calculating $\nabla_{\partial_t}X_\flat$ and $\nabla_{\frac{1}{r}\partial_\theta}X_\flat$. To check your answer, verify that $B_{\mu\nu} = g_{\mu\gamma}B^\gamma{}_\nu$.

16. Consider the metric

$$g = -dt^2 + a^2(t)(d\psi^2 + f^2(\psi)(d\theta^2 + \sin^2\theta d\varphi^2))$$

defined on M. When $M = \mathbb{R} \times S^3$, $f(\psi) = \sin\psi$; when $M = \mathbb{R} \times \mathbb{R}^3$, $f(\psi) = \psi$; when $M = \mathbb{R} \times H^3$, $f(\psi) = \sinh\psi$.

a) Using the geodesic Lagrangian obtained from the metric, compute the Christoffel symbols in the coordinates (t, ψ, θ, ϕ).

b) Verify that the null vector field

$$X = \frac{1}{a}\partial_t + \frac{1}{a^2}\partial_\psi$$

is geodesic.

c) Compute

$$\nabla_{\partial_\theta}X \qquad \text{and} \qquad \nabla_{\partial_\varphi}X.$$

Use these to obtain the expansion of the congruence of null geodesics tangent to X.

d) Let Σ be the sphere $t = t_0$ and $\psi = \psi_0$. Denote by ξ_r the flow of X. The 2-dimensional surface $\xi_r(\Sigma)$ is parametrized by

$$(\theta, \varphi) \mapsto (t(r), \psi(r), \theta, \varphi).$$

Here $t(r)$ is determined by

$$r = \int_{t_0}^{t} a(s)\,ds$$

and $\psi(r)$ satisfies

$$\psi = \psi_0 + \int_0^r \frac{1}{a^2(t(s))}\, ds.$$

Define

$$\tilde{A}(t, \psi) = a^2(t) f^2(\psi)$$

and

$$A(r) = \tilde{A}(t(r), \psi(r)).$$

Note that

$$\frac{dA}{dr} = X \cdot \tilde{A}.$$

Use the area element of $\xi_r(\Sigma)$, which is

$$A(r) \sin\theta \, d\theta \wedge d\varphi,$$

to confirm the result you obtained above for the expansion of the congruence of null geodesics.

17. Use ideas similar to those leading to the proof of Hawking's singularity theorem to prove Myers's Theorem: if (M, g) is a complete Riemannian manifold such that there exists an $\varepsilon > 0$ so that $R_{\mu\nu} X^\mu X^\nu \geq \varepsilon g_{\mu\nu} X^\mu X^\nu$, then M is compact. Can these ideas be used to prove a singularity theorem in Riemannian geometry?

18. Explain why Hawking's Singularity Theorem and explain why Penrose's Singularity Theorem do not apply to each of the following geodesically complete Lorentzian manifolds:

- **a)** Minkowski's spacetime;
- **b)** Einstein's spacetime;
- **c)** de Sitter's spacetime;
- **d)** Anti-de Sitter spacetime.

19. Check that Penrose's Singularity Theorem can be applied to a FLRW flat universe with $\alpha > 0$ and $\Lambda > 0$.

20. Write the wave equation for the FLRW models, $\Box\phi = \nabla^\mu \nabla_\mu \phi = 0$.

21. Compute the Komar mass of the Schwarzschild solution.

22. Let h be the spherically symmetric Riemannian metric defined in R^3 by

$$h = \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2),$$

where m is a smooth function in $[0, +\infty[$ such that $m(0) = m'(0) = m''(0) = 0$, $m(r) < \frac{r}{2}$ for all $r > 0$, there exist ε and M in $\mathbb{R}^+$ such that

$$\lim_{r\to+\infty} m(r) = M$$

and

$$m'(r) = O\Big(\frac{1}{r^{1+\varepsilon}}\Big) \quad \text{as } r \to \infty,$$

and

$$m''(r) = O\Big(\frac{1}{r^2}\Big) \quad \text{as } r \to \infty.$$

a) Check that in Cartesian coordinates we have

$$h_{ij} = \delta_{ij} + \frac{\frac{2m(r)}{r^3}}{1 - \frac{2m(r)}{r}} x^i x^j.$$

b) Check that h has ADM mass M.

c) Knowing that the scalar curvature of h is $\overline{R} = \frac{4m'(r)}{r^2}$, verify that h is asymptotically flat.

d) Relaxing the assumption on the second derivative of m to $m''(r) = O(r^\alpha)$ as $r \to \infty$, for what values of α is h is asymptotically flat?

e) Verify the rigidity statement of the Positive Mass Theorem for the metric h.

f) Assume now that m is constant, so that h defines a Riemannian metric only in the region where $r > 2m$. Compute the second fundamental form of the sphere $r = r_0$ and show that when $r_0 = 2m$ the sphere is a minimal surface. Check the Penrose inequality.

23. Consider the 3-dimensional Riemannian manifold M equal to the graph of the function $f : \mathbb{R}^3 \to \mathbb{R}$, with the metric induced by the Euclidean metric of $\mathbb{R}^4$. Let c be a constant and $S = S_c = \{(p, c) = (x, y, z, c) \in M : f(p) = c\}$ be a 2-dimensional surface such that $\nabla f(p) \neq 0$ for all $(p, c) \in M$. Suppose (u, v) are coordinates on S such that

$$\partial_u \cdot \partial_u = 1, \qquad \partial_v \cdot \partial_v = 1, \qquad \partial_u \cdot \partial_v = 0.$$

a) Check that the (unique up to sign) unit normal to S in TM is

$$X = \frac{\Big(\frac{\nabla f}{|\nabla f|}, |\nabla f|\Big)}{\sqrt{1 + |\nabla f|^2}}.$$

b) Let $\tilde{\nabla}$ denote the Levi-Civita connection of $\mathbb{R}^4$. Denote by

$$v_1 = \tilde{\nabla}_{\partial_u}(\nabla f, 0), \qquad v_2 = \tilde{\nabla}_{\partial_v}(\nabla f, 0),$$

and by

$$\alpha_1 = v_1 \cdot (\nabla f, 0), \qquad \alpha_2 = v_3 \cdot (\nabla f, 0).$$

Calculate $\tilde{\nabla}_{\partial_u} X$ and $\tilde{\nabla}_{\partial_v} X$ in terms of the v_i, α_i and ∇f. Simplify your answer and check directly that the vectors you obtained are orthogonal to X (which is obvious from $X \cdot X = 1$).

c) Let $\bar{\nabla}$ denote the Levi-Civita connection on M. Compute $\partial_u \cdot \bar{\nabla}_{\partial_u} X$, $\partial_v \cdot \bar{\nabla}_{\partial_u} X$, $\partial_u \cdot \bar{\nabla}_{\partial_v} X$ and $\partial_v \cdot \bar{\nabla}_{\partial_v} X$ in terms of $\partial_u \cdot v_i$, $\partial_v \cdot v_i$ and $|\nabla f|$. Write the second fundamental form of S in M.

d) Denote by ∇f the column vector

$$\nabla f = \begin{bmatrix} \partial_x f \\ \partial_y f \\ \partial_z f \end{bmatrix}.$$

Check that the metric on M is given by

$$g = I + \nabla f \, (\nabla f)^T$$

in the coordinates (x, y, z), i.e.

$$ds^2 = \begin{bmatrix} dx & dy & dz \end{bmatrix} g \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}.$$

e) Let (u, v, w) be coordinates on M, and l be a parametrization of M, such that u and v are as above, and let w be such that it vanishes on S and

$$\partial_w = \frac{\partial_x f \partial_x + \partial_y f \partial_y + \partial_z f \partial_z}{|\nabla f| \sqrt{1 + |\nabla f|^2}}.$$

Denote by

$$Dl = [\partial_u \ \partial_v \ \partial_w] = \begin{bmatrix} \partial_u l^1 & \partial_v l^1 & \partial_w l^1 \\ \partial_u l^2 & \partial_v l^2 & \partial_w l^2 \\ \partial_u l^3 & \partial_v l^3 & \partial_w l^3 \end{bmatrix}.$$

What is the matrix that represents the metric on M in the coordinates (u, v, w)? What does it reduce to over points that belong to S?

f) Use your answer to **e)** to justify that on points that belong to S

$$\frac{1}{2} L_{\partial_w} g \Big|_{TS} = \begin{bmatrix} du & dv & dw \end{bmatrix} \frac{1}{2} \partial_w [(Dl)^T Dl] \begin{bmatrix} du \\ dv \\ dw \end{bmatrix} \Bigg|_{TS}.$$

g) Recalling that $\bar{\Gamma}^k_{ij} = \omega^k(\bar{\nabla}_{X_i} X_j)$, use **f)** to check your answer to **c)**.

24. Consider the surfaces Σ_t obtained from the boundary $\partial K \equiv \Sigma_0$ of a compact convex set K contained in $\mathbb{R}^3$ by flowing a distance t along the unit normal. Let $A(t)$ be the area of Σ_t and $V(t)$ be the volume bounded by Σ_t. As we saw in class,

$$\dot{A}(0) = \int_{\partial K} \operatorname{tr} \kappa,$$

where κ is the second fundamental form of Σ.

a) Prove that $\ddot{A}(t) = 8\pi$. (Suggestion: use equalities (1) and (2) above, and the Gauss-Bonnet Theorem).

b) Deduce that

$$V(t) = \frac{4\pi}{3} t^3 + \frac{\dot{A}(0)}{2} t^2 + A(0)t + V(0).$$

c) Use the isoperimetric inequality, $A^3(t) \geq 36\pi V^2(t)$, to prove Minkowski's inequality

$$\dot{A}(0) \geq \sqrt{16\pi A(0)}.$$