Mathematical Relativity

June 12, 2018

MEFT and MMA

3 hours

Show your calculations

1. Consider a $3+1$ dimensional FLRW spacetime

$$ds^2 = -dt^2 + a(t)^2 h_{ij} dx^i dx^j$$

for a spherical universe with a positive cosmological constant $\Lambda = 3$ when $\alpha > \frac{\sqrt{3}}{9}$. Recall the Friedmann equations

$$\begin{cases} \frac{1}{2}\dot{a}^2 - \frac{\alpha}{a} - \frac{\Lambda}{6}a^2 = -\frac{k}{2}, \\ \frac{4}{3}\pi a^3 \rho = \alpha. \end{cases}$$

a) Construct the Penrose diagram for this spacetime.

b) Compute the second fundamental form of the vector field ∂_t and compute its expansion.

c) Verify directly Raychaudhuri's equality

$$X \cdot \theta = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}X^\mu X^\nu.$$

d) Characterize the region of spacetime where the strong energy condition is satisfied.

e) State and apply the Hawking Singularity Theorem to a suitable subset of this spacetime.

f) Suppose now $\alpha < \frac{\sqrt{3}}{9}$. Give initial data for the metric for which $K_{ij} = 0$ and check that the restriction equation

$$G_{00} + \Lambda g_{00} = 8\pi T_{00} \;\Leftrightarrow\; \bar{R} - 2\Lambda = 16\pi\rho$$

is satisfied. Note: you know the value of $\bar{R}$.

2. Let K be a Killing vector field.

a) Show that

$$\Box K_\mu = -R_{\mu\nu}K^\nu.$$

b) Use the previous equation to show that

$$\Box\phi = 0 \implies \Box(K \cdot \phi) = 0.$$

3. Compute the Komar mass of the Schwarzschild solution,

$$M = -\frac{1}{8\pi}\int_{\Sigma} *dK_{\flat}.$$

4. Suppose that (M, g) is a complete $3-$dimensional Riemannian manifold with a compact set C such that $M \backslash C$ is isometric to $3-$dimensional Euclidean space with a closed ball removed. Argue that if M is not isometric to $\mathbb{R}^3$ with the Euclidean metric then there must exist a point $p \in M$ such that $\bar{R}(p) < 0$, where $\bar{R}$ is the scalar curvature of (M, g).