Cálculo Diferencial e Integral II - Tagus Park **1o. Semestre 2009/2010**
2o. Teste **21/Novembro/2009**
Justifique as suas respostas

1. **(3 val.)** Calcule o comprimento de arco de $\mathbf{c}(t) = (t, \log t, 2\sqrt{2t})$ para $1 \le t \le 2$.

   Resposta:

$$||\mathbf{c}'(t)|| = ||(1, \frac{1}{t}, \sqrt{\frac{2}{t}})|| = \sqrt{1 + \frac{1}{t^2} + \frac{2}{t}} = \sqrt{\left(1 + \frac{1}{t}\right)^2} = 1 + \frac{1}{t} \qquad 1 \le t \le 2$$

   e portanto o comprimento pedido é:

$$\int_1^2 ||\vec{c}'(t)||dt = \int_1^2 \left(1 + \frac{1}{t}\right)dt = \Big[t + \ln t\Big]_1^2 = 2 + ln2 - 1 - \ln 1 = 1 + \ln 2$$

2. **(3 val.)** Calcule o gradiente de $f(x, y) = \arctan(x^2 + y^2)$. Calcule **explicitamente** o rotacional desse gradiente.

   Resposta:

$$\vec{grad}\, f(x, y) = \left(\frac{\partial}{\partial x} \arctan(x^2 + y^2), \frac{\partial}{\partial y} \arctan(x^2 + y^2)\right) =$$
$$= \left(\frac{2x}{1 + (x^2 + y^2)^2}, \frac{2y}{1 + (x^2 + y^2)^2}\right)$$

$$rot\left(\vec{grad}\, f\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{2x}{1+(x^2+y^2)^2} & \frac{2y}{1+(x^2+y^2)^2} & 0 \end{vmatrix} =$$
$$= 0\mathbf{i} + 0\mathbf{j} + \left(\frac{\partial}{\partial x}\frac{2y}{1 + (x^2 + y^2)^2} - \frac{\partial}{\partial y}\frac{2x}{1 + (x^2 + y^2)^2}\right)\mathbf{k} =$$
$$= \left(2y\frac{2(x^2 + y^2)2x}{\left(1 + (x^2 + y^2)^2\right)^2} - 2x\frac{2(x^2 + y^2)2y}{\left(1 + (x^2 + y^2)^2\right)^2}\right)\mathbf{k} =$$
$$= \left(\frac{2y2(x^2 + y^2)2x - 2x2(x^2 + y^2)2y}{\left(1 + (x^2 + y^2)^2\right)^2}\right)\mathbf{k} = 0\mathbf{k}$$

3. **(3 val.)** Calcule

$$\int_0^1 \int_0^{x^2} (x^2 + xy - y^2)dydx$$

   Esboce a região do plano XOY sobre a qual foi realizada esta integração.

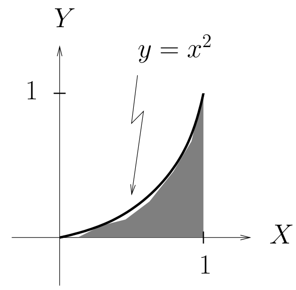

Figure 1: Esboço da região de integração

Resposta:

$$\int_0^1 \int_0^{x^2} (x^2 + xy - y^2) dy dx = \int_0^1 dx \int_0^{x^2} dy \, (x^2 + xy - y^2) =$$
$$= \int_0^1 dx \left[x^2 y + \frac{1}{2} x y^2 - \frac{1}{3} y^3 \right]_0^{x^2} = \int_0^1 dx \left[x^4 + \frac{1}{2} x^5 - \frac{1}{3} x^6 \right] =$$
$$= \left[\frac{1}{5} x^5 + \frac{1}{12} x^6 - \frac{1}{21} x^7 \right]_0^1 = \frac{1}{5} + \frac{1}{12} - \frac{1}{21}$$

4. **(3 val.)**Calcule
$$\int\int_W\int \sqrt{x^2 + y^2 + z^2} \, e^{-(x^2+y^2+z^2)} dx dy dz$$
onde W é o sólido entre as esferas dadas por $x^2 + y^2 + z^2 \leq a^2$ e $x^2 + y^2 + z^2 \leq b^2$, com os reais positivos $a < b$.

Resposta:

$$\int\int_W\int \sqrt{x^2 + y^2 + z^2} \, e^{-(x^2+y^2+z^2)} dx dy dz = \int_a^b d\rho \int_0^\pi d\phi \int_0^{2\pi} d\theta \, \rho^2 \sin\phi \, \rho e^{-\rho^2} = \cdots$$

$$\mathbf{P} \rho^2 \rho e^{-\rho^2} = \frac{-1}{2} \mathbf{P} \rho^2 (-2\rho) e^{-\rho^2} = \frac{-1}{2} \left[\rho^2 e^{-\rho^2} - \mathbf{P} 2\rho e^{-\rho^2} \right] = \frac{-1}{2} \left[\rho^2 e^{-\rho^2} + e^{-\rho^2} \right]$$

$$\cdots = \int_0^\pi d\phi \sin\phi \int_0^{2\pi} d\theta \int_a^b d\rho \rho^2 \rho e^{-\rho^2} = \Big[-\cos\phi \Big]_0^\pi 2\pi \frac{-1}{2} \left[\rho^2 e^{-\rho^2} + e^{-\rho^2} \right]_a^b =$$
$$= 2\pi \left[a^2 e^{-a^2} + e^{-a^2} - b^2 e^{-b^2} - e^{-b^2} \right]$$

5. **(3 val.)**Calcule o centro de massa da região entre $y = x^2$ e $y = x$ com a densidade $\delta(x, y) = x + y$

Resposta:

$$\int_0^1 dx \int_{x^2}^x dy\,(x+y) = \int_0^1 dx \left[xy + \frac{1}{2}y^2\right]_{x^2}^x = \int_0^1 dx \left[x^2 + \frac{1}{2}x^2 - x^3 - \frac{1}{2}x^4\right] =$$

$$= \left[\frac{1}{2}x^3 - \frac{1}{4}x^4 - \frac{1}{10}x^5\right]_0^1 = \frac{1}{2} - \frac{1}{4} - \frac{1}{10} = \frac{3}{20}$$

$$\int_0^1 dx \int_{x^2}^x dy\, x(x+y) = \int_0^1 dx \left[x^2 y + \frac{1}{2}xy^2\right]_{x^2}^x = \int_0^1 dx \left[x^3 + \frac{1}{2}x^3 - x^4 - \frac{1}{2}x^5\right] =$$

$$= \left[\frac{3}{8}x^4 - \frac{1}{5}x^5 - \frac{1}{12}x^6\right]_0^1 = \frac{3}{8} - \frac{1}{5} - \frac{1}{12} = \frac{45-24-10}{120} = \frac{11}{120}$$

$$\int_0^1 dx \int_{x^2}^x dy\, y(x+y) = \int_0^1 dx \left[\frac{1}{2}xy^2 + \frac{1}{3}y^3\right]_{x^2}^x = \int_0^1 dx \left[\frac{1}{2}x^3 + \frac{1}{3}x^3 - \frac{1}{2}x^5 - \frac{1}{3}x^6\right] =$$

$$= \left[\frac{5}{24}x^4 - \frac{1}{12}x^6 - \frac{1}{21}x^7\right]_0^1 = \frac{5}{24} - \frac{1}{12} - \frac{1}{21} = \frac{3}{24} - \frac{1}{21} = \frac{13}{168}$$

$$\bar{x} = \frac{\frac{11}{120}}{\frac{3}{20}} = \frac{11}{18} \qquad \bar{y} = \frac{\frac{13}{168}}{\frac{3}{20}} = \frac{130}{3 \cdot 84}$$

6. (3 **val.**)Determine a natureza do seguinte integral

$$\int_1^\infty \frac{3x^{15} + 2x^2 + \pi}{8x^{54} + 7x^{13} + e} dx$$

Resposta:

$$\frac{\frac{3x^{15}+2x^2+\pi}{8x^{54}+7x^{13}+e}}{\frac{1}{x^{39}}} = \frac{3x^{15} + 2x^2 + \pi}{x^{15}} \frac{x^{54}}{8x^{54} + 7x^{13} + e} = \left(3 + \frac{2}{x^{13}} + \frac{\pi}{x^{15}}\right) \frac{1}{8 + \frac{7}{x^{41}} + \frac{e}{x^{54}}}$$

$$\underset{x \mapsto \infty}{\longrightarrow} \frac{3}{8}$$

Como $\frac{3}{8} > 0$, pelo critério do limite, os integrais impróprios:

$$\int_1^\infty \frac{3x^{15} + 2x^2 + \pi}{8x^{54} + 7x^{13} + e} dx \qquad \int_1^\infty \frac{1}{x^{39}} dx$$

têm a mesma natureza.

Como $39 > 1$, então como vimos nas teóricas

$$\int_1^\infty \frac{1}{x^{39}} dx$$

converge. Então,

$$\int_1^\infty \frac{3x^{15} + 2x^2 + \pi}{8x^{54} + 7x^{13} + e} dx$$

tambem converge.

7. **(2 val.)**Sejam a e b números reais positivos. Calcule a área interior a uma elipse de semi-eixos a e b.

   Resposta:

$$\int_{-a}^{a} dx \int_{-b\sqrt{1-\left(\frac{x}{a}\right)^2}}^{b\sqrt{1-\left(\frac{x}{a}\right)^2}} dy = 2b \int_{-a}^{a} dx \sqrt{1-\left(\frac{x}{a}\right)^2} = \cdots$$

$$\frac{x}{a} = \sin t \qquad -\pi/2 \leq t \leq \pi/2 \qquad\qquad dx = a\cos t dt$$
$$x = -a \Rightarrow t = -\pi/2 \qquad\qquad x = a \Rightarrow t = \pi/2$$

$$\cdots = 2b \int_{-\pi/2}^{\pi/2} a\cos t dt \sqrt{1-\sin^2 t} = 2ab \int_{-\pi/2}^{\pi/2} \cos^2 t dt = \cdots$$

$$\cos 2t = \cos^2 t - \sin^2 t = \cos^2 t - 1 + \cos^2 t = 2\cos^2 t - 1 \Leftarrow \cos^2 t = \frac{1+\cos 2t}{2}$$

$$\cdots = 2ab \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2t}{2} tdt = 2ab \left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_{-\pi/2}^{\pi/2} =$$
$$= 2ab\frac{\pi/2}{2} + \frac{\sin 2\pi/2}{4} - \frac{-\pi/2}{2} - \frac{\sin 2(-\pi/2)}{4} = 2ab\pi/2 = \pi ab$$