# QUATERNIONIC ALGEBRAIC CYCLES AND REALITY

PEDRO F. DOS SANTOS AND PAULO LIMA-FILHO

Contents



*Date*: May 2001.
Partially supported by NSF.

## Introduction

In [Ati66] Atiyah developed a $K$-theory for spaces $(X, \sigma)$ with an involution $\sigma$, the *Real spaces* in his terminology. The construction uses the notion of a *Real bundle* $(E, \tau)$ over $(X, \sigma)$ which consists of a complex vector bundle $E$ over $X$, along with an anti-linear map $\tau : E \to E$ covering the involution $\sigma$ and satisfying $\tau^2 = 1$. The group $KR(X)$ is then defined as the Grothendieck group of the monoid of isomorphism classes of Real bundles over $(X, \sigma)$, and the resulting theory is called *$KR$-theory.*

In a similar fashion, J. Dupont developed in [Dup69] the *symplectic $K$-theory $KSp(X)$* for Real spaces $(X, \sigma)$. His construction is similar to Atiyah's, in that $KSp(X)$ is the Grothendieck group of the monoid of isomorphism classes of *symplectic bundles* over $(X, \sigma)$. In this context, a *symplectic bundle* $(E, \tau)$ over $(X, \sigma)$ consists of a complex vector bundle $E$ over $X$, along with an anti-linear map $\tau : E \to E$ covering the involution $\sigma$ and satisfying $\tau^2 = -1$. Subsequently, R. M. Seymour reintroduced this theory in [Sey73], where he called it *quaternionic $K$-theory* and denoted it by $KH(X)$. We adopt this terminology, for it avoids confusion with the non-equivariant notion of symplectic $K$-theory.

A clear and conceptual reason for the existence of these two competing theories arises when one tries to find their respective classifying spaces in the equivariant category. In fact, one can extend these theories to $RO(\mathbb{Z}/2)$-graded cohomology theories $KR^*$ and $KH^*$ in the sense of [Seg68] and [May77]. To this purpose, one constructs $\mathbb{Z}/2$-spaces $(\mathbb{Z} \times BU)_{\mathbb{C}}$ and $(\mathbb{Z} \times BU)_{\mathbb{H}}$ satisfying

$$[X_+, (\mathbb{Z} \times BU)_{\mathbb{C}}]_{\mathbb{Z}/2} \cong KR(X) \quad \text{and} \quad [X_+, (\mathbb{Z} \times BU)_{\mathbb{H}}]_{\mathbb{Z}/2} \cong KH(X);$$

cf. [LLFM98b] and Proposition 4.4, respectively. These spaces are shown to have the structure of $\mathbb{Z}/2$-equivariant infinite loop spaces in [LLFM98b, ] and Theorem 4.7, respectively, yielding the spectra classifying the desired equivariant cohomology theories.

In order to construct such classifying spaces, we first identify $\mathbb{Z}/2$ - the underlying group of the equivariant category - with the Galois group $Gal(\mathbb{C}/\mathbb{R})$. Recall that the Brauer group [Gro57] $Br(\mathbb{R})$ of $\mathbb{R}$ is also isomorphic to $\mathbb{Z}/2$. This will be shown to account for the two distinct $K$-theories in Section 4. The argument is roughly the following. Let $\mathbb{P}(\mathbb{C}^n)$ denote the projective space of complex 1-dimensional subspaces of $\mathbb{C}^n$. In the language os schemes, this is the set of complex-valued points of $\mathbb{P}^{n-1}$, endowed with the analytic topology. The Galois group $\mathbb{Z}/2 = Gal(\mathbb{C}/\mathbb{R})$ acts on $\mathbb{P}(\mathbb{C}^n)$ via complex conjugation. Similarly, let $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}\mathbf{j}$ denote the quaternions, and let $\mathbb{P}(\mathbb{H}^n)$ be the projective space of complex 1-dimensional subspaces of $\mathbb{H}^n$. We give $\mathbb{P}(\mathbb{H}^n)$ the $\mathbb{Z}/2$-action induced by multiplication by $\mathbf{j}$ on the left of $\mathbb{H}^n$. As a space, $\mathbb{P}(\mathbb{H}^n)$ is homeomorphic to $\mathbb{P}(\mathbb{C}^{2n})$, however the $\mathbb{Z}/2$-actions on $\mathbb{P}(\mathbb{C}^{2n})$ and $\mathbb{P}(\mathbb{H}^n)$ are quite distinct. In fact, these spaces are the complex-valued points of the two inequivalent Brauer-Severi schemes of rank $2n-1$ over $\mathbb{R}$, under the action of the Galois group. The aforementioned classifying spaces $(\mathbb{Z} \times BU)_{\mathbb{C}}$ and $(\mathbb{Z} \times BU)_{\mathbb{H}}$ are then constructed using the usual equivariant stabilization of the Grassmannians of complex linear subspaces of $\mathbb{P}(\mathbb{C}^n)$ and $\mathbb{P}(\mathbb{H}^n)$, respectively.

In order to develop a theory of characteristic classes for $KR^*$ and $KH^*$, one needs to introduce the appropriate equivariant cohomology theories. In [Kah87] B. Kahn defined characteristic classes for Real bundles, taking values in Galois-Grothendieck cohomology with coefficients in the $\mathbb{Z}/2$-modules $\mathbb{Z}(n)$. In the case of quaternionic bundles, Dupont poses in [Dup99] the question of which equivariant cohomology theory would be the natural target of characteristic classes, but the question was left unanswered.

In this paper we provide an answer to this question by constructing a cohomology theory $\mathfrak{Z}^*_{\mathbb{H}}$ in which the characteristic classes for quaternionic bundles take its values. Furthermore, we extend these characteristic classes to a natural transformation of $RO(\mathbb{Z}/2)$-graded cohomology theories $KH^* \to \mathfrak{Z}^*_{\mathbb{H}}$. A crucial aspect of our construction is the fact that *the two distinct Brauer-Severi varieties $\mathbb{P}(\mathbb{C}^n)$ and $\mathbb{P}(\mathbb{H}^n)$, used in the construction of the classifying spaces for $KR^*$ and $KH^*$, are also used to construct classifying spaces for the corresponding cohomology theories.* Under this approach, the classifying maps for the characteristic classes have a similar description in both cases.

In order to place our constructions under the proper perspective, let us describe how the characteristic classes for Real bundles were extended to $KR^*$ in [LLFM98b] and [dS00]. The main constructions go back to [LM91], [BLLF+93] and [LLFM96].

Let $\mathcal{Z}^q(\mathbb{P}(\mathbb{C}^n))$ denote the group of algebraic cycles of codimension $q$ in $\mathbb{P}(\mathbb{C}^n)$. This is an abelian topological group on which $Gal(\mathbb{C}/\mathbb{R})$ acts via topological automorphisms. See [LLFM98a] for details and additional references. The equivariant homotopy type of $\mathcal{Z}^q(\mathbb{P}(\mathbb{C}^n))$ was determined in [dS00], and it turns out to be a product of classifying spaces for equivariant coohomology with coefficients in the constant MacKey functor $\underline{\mathbb{Z}}$. More precisely, one has a canonical equivariant homotopy equivalence

$$\mathcal{Z}^q(\mathbb{P}(\mathbb{C}^n)) \cong \mathbb{Z} \times K(\mathbb{Z}(1), 2) \times K(\mathbb{Z}(2), 4) \times \cdots \times K(\mathbb{Z}(q), 2q).$$

The associated equivariant cohomology theory is $RO(\mathbb{Z}/2)$-graded (bigraded, in this case) and the resulting invariants arise naturally in $\mathbb{Z}/2$-homotopy theory. The functor represented by $K(\mathbb{Z}(q), 2q)$ is denoted $H^{2q,q}(-; \underline{\mathbb{Z}})$. See Section 1.3 for notation.

The situation in $KR$-theory follows a standard, albeit non-trivial, pattern. A canonical stabilization $\varinjlim Gr^q(\mathbb{C}^n)$ in the $\mathbb{Z}/2$-homotopy category, of Grassmannians as $Gal(\mathbb{C}/\mathbb{R})$-spaces, produces a classifying space $BU_{\mathbb{C}}$ for $KR$-theory. It is easy to see that the Bredon cohomology of $BU_{\mathbb{C}}$ with coefficients in $\underline{\mathbb{Z}}$ is a polynomial ring over $\mathbb{Z}$ on certain characteristic classes, the equivariant Chern classes for $KR$-bundles. Furthermore, the inclusion of $Gr^q(\mathbb{C}^n) \hookrightarrow \mathcal{Z}^q(\mathbb{P}(\mathbb{C}^n))$ stabilizes to give a map of equivariant infinite loop spaces

$$c : BU_{\mathbb{C}} \to \mathcal{Z} \cong \prod_p K(\mathbb{Z}(p), 2p),$$

which classifies the total Chern class.

In this paper we provide the quaternionic counterpart of the constructions in $KR^*$-theory described above. This turns out to be a more subtle issue, and our answer was inspired by the following observation, made in [Ati66] and [Dup69]. If $X$ be a Real space, then one has an isomorphism:

$$KR(X \times \mathbb{P}(\mathbb{H})) \cong KR(X) \oplus KH(X). \tag{1}$$

The first task is to determine the $\mathbb{Z}/2$-homotopy type of $\mathcal{Z}^q(\mathbb{P}(\mathbb{H}^n))$, under the $\mathbb{Z}/2$-action induced by $\mathbf{j}$, as explained above. This is the action induced by the $Gal(\mathbb{C}/\mathbb{R})$ action on the complex points $\mathbb{P}(\mathbb{H}^n)$ of the Brauer-Severi variety of rank $n-1$ over $\mathbb{R}$. This problem was first considered in [LLFM98b], where it was proved that quaternionic suspension, $\Sigma_{\mathbb{H}} : \mathcal{Z}^q(\mathbb{P}(\mathbb{H}^n)) \to \mathcal{Z}^q(\mathbb{P}(\mathbb{H}^{n+1}))$, is a $\mathbb{Z}/2$-homotopy equivalence. This reduced the problem to computing the $\mathbb{Z}/2$-homotopy type of cycle spaces of dimensions 0 and 1. Their homotopy type is quite distinct, and this somehow reflects the sharp difference between quaternionic bundles of even and odd complex rank. Later, in [LLFM98c] the homotopy type of the space of cycles $\mathcal{Z}^q(\mathbb{P}(\mathbb{H}^n))^{\mathbb{Z}/2}$ was computed using suspension to a real bundle other than $\mathcal{O}(1)$ (which corresponds to complex suspension).

Based on the techniques of [LLFM98c], and on the $\mathbb{Z}/2$-equivariant perspective of [dS99a], we first establish the following splitting result.

**Theorem 3.4.** *For $k < n$ there are canonical equivariant homotopy equivalences:*

$$\mathcal{Z}^{2k-1}(\mathbb{P}(\mathbb{H}^n)) \cong \prod_{j=1}^{k} F\left(\mathbb{P}(\mathbb{H})_+,\ K(\mathbb{Z}(2j-1),\ 4j-2)\ \right) \tag{2}$$

*and*

$$\mathcal{Z}^{2k}(\mathbb{P}(\mathbb{H}^n)) \cong \prod_{j=1}^{k} F\left(\mathbb{P}(\mathbb{H})_+,\ K(\mathbb{Z}(2j),\ 4j)\ \right) \tag{3}$$

*where $F(-,-)$ denotes based maps.*

Note that the spaces $F\left(\mathbb{P}(\mathbb{H})_+,\ K(\mathbb{Z}(q), 2q)\right)$ are classifying spaces for the cohomology functors $H^{2q,q}(-\times\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$. This result completely determines the $\mathbb{Z}/2$-homotopy type of $\mathcal{Z}^m(\mathbb{P}(\mathbb{H}^n))$.

We then apply a suitable stabilization procedure using the spaces $\mathcal{Z}^q(\mathbb{P}(\mathbb{H}^n))$. The resulting space $\mathcal{Z}_{\mathbb{H}}$ has the property that all of its connected components are products of classifying spaces for the functors $H^{2*,*}(-\times\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$, according to the splitting of Theorem 3.4.

**Theorem 3.9.** *The space $\mathcal{Z}_{\mathbb{H}}$ is written as a disjoint union of connected spaces*

$$\mathcal{Z}_{\mathbb{H}} = \coprod_{j=-\infty}^{\infty} \mathcal{Z}_{\mathbb{H}}^{j},$$

*where the equivariant homotopy type of $\mathcal{Z}_{\mathbb{H}}^{j}$ is totally determined by*

$$\mathcal{Z}_{\mathbb{H}}^{j} \cong \begin{cases} \prod_{k=1}^{\infty} F\left(\mathbb{P}(\mathbb{H})_+,\ K(\mathbb{Z}(2k-1),\ 4k-2)\ \right) & \text{, if } j \text{ is odd} \\ \prod_{k=1}^{\infty} F\left(\mathbb{P}(\mathbb{H})_+,\ K(\mathbb{Z}(2k),\ 4k)\ \right) & \text{, if } j \text{ is even.} \end{cases} \tag{4}$$

Using standard results in equivariant homotopy theory (cf. [CW91]), we prove that the complex join pairing on algebraic cycles induces an equivariant infinite loop space structure on $\mathcal{Z}_{\mathbb{H}}$. We denote by $\mathfrak{Z}_{\mathbb{H}}^{*}$ the resulting equivariant cohomology theory. Note that for a compact $\mathbb{Z}/2$-space $X$, one has an identification

$$\mathfrak{Z}_{\mathbb{H}}^{0}(X) = [X, \mathcal{Z}_{\mathbb{H}}]_{\mathbb{Z}/2} = \bigoplus_{j\in\mathbb{Z}} \left[X, \mathcal{Z}_{\mathbb{H}}^{j}\right]_{\mathbb{Z}/2} = \bigoplus_{j\in\mathbb{Z}} \prod_{r\geq 1} H^{4r-2\epsilon(j),\ 2r-\epsilon(j)}(X\times\mathbb{P}(\mathbb{H}), \underline{\mathbb{Z}}), \tag{5}$$

where $\epsilon(j)$ is 0 if $j$ is even and 1 if $j$ is odd. The group structure on $\mathfrak{Z}_{\mathbb{H}}^{0}(X)$, coming from the $H$-space structure on $\mathcal{Z}_{\mathbb{H}}$ induced by the algebraic join of cycles, has the following description.

**Proposition 5.10.** *Let $X$ be a $\mathbb{Z}/2$-space, and let $a\cdot b$ denote the product of elements $a, b$ in $\mathfrak{Z}_{\mathbb{H}}^{0}(X)$. Consider $\mathfrak{Z}_{\mathbb{H}}^{0}(X)$ included in*

$$\bigoplus_{j\in\mathbb{Z}} \prod_{r,s\geq 1} H^{r,s}(X\times\mathbb{P}(\mathbb{H}), \underline{\mathbb{Z}}),$$

*as in (60). Then, under this inclusion we have,*

$$a\cdot b = a\cup b + pr^*(a/z)\cup pr^*(b/z),$$

*where $\mathbf{z}\in H_{2,1}(\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$ is the fundamental class $\mathbb{P}(\mathbb{H})$, $-/z$ denotes slant product with $z$ and $pr$ is the projection onto the first factor in the product $X\times\mathbb{P}(\mathbb{H})$.*

A similar stabilization procedure is then applied to the Grassmannians $G^q(\mathbb{H}^n)$ and the result is an equivariant infinite loop space $(\mathbb{Z}\times BU)_{\mathbb{H}} = \amalg_{j\in\mathbb{Z}}\ BU^j_{\mathbb{H}}$ which classifies $KH^*$, as described above. The inclusion $\mathrm{Gr}^q(\mathbb{H}^n)\subset \mathcal{Z}^q(\mathbb{P}(\mathbb{H}^n))$ induces a total Chern class map $c_{\mathbb{H}} : (\mathbb{Z}\times BU)_{\mathbb{H}} \to \mathcal{Z}_{\mathbb{H}}$, which turns out to be a map of equivariant infinite loop spaces; cf. (48).

In order to understand this Chern class map, we compute the equivariant cohomology group $\mathfrak{Z}^0_{\mathbb{H}}((\mathbb{Z}\times BU)_{\mathbb{H}}) = [\ (\mathbb{Z}\times BU)_{\mathbb{H}},\ \mathcal{Z}_{\mathbb{H}}\ ]_{\mathbb{Z}/2}$. It follows from Proposition 5.10 that we first need to compute the cohomology ring $H^{*,*}(BU^j_{\mathbb{H}}\times \mathbb{P}(\mathbb{H}), \underline{\mathbb{Z}})$.

If $E$ is a *Real* bundle over a $\mathbb{Z}/2$-space $X$, denote by $\widetilde{c}_k(E)\in H^{2k,k}(X;\underline{\mathbb{Z}})$ its $k$-th equivariant Chern class, as described in [dS00]. Let $\xi^{2n}$ be the universal quotient bundle over $BU^{2n}_{\mathbb{H}}$, and observe that $\xi^{2n}\otimes\mathcal{O}(1)$ is a real bundle over $BU^{2n}_{\mathbb{H}}\times\mathbb{P}(\mathbb{H})$. Define classes $\mathbf{d}_k \in H^{2k,k}(\mathrm{BU}^{2n}_{\mathbb{H}}\times\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$ using the formulas

$$d_{2n-(2i+\delta)} := \widetilde{c}_{2n-(2i*\delta)}(\xi^{2n}\otimes\mathcal{O}(1)) - i\ \widetilde{c}_{2n-(2i*\delta)-1}(\xi^{2n}\otimes\mathcal{O}(1))\ \mathbf{x},$$

for $0\le\delta\le 1$ and $0\le 2i+\delta\le 2n$, and where $\mathbf{x}\in H^{2,1}(\mathbb{P}(\mathbb{H}),\underline{\mathbb{Z}})\cong\mathbb{Z}$ is the canonical generator; cf. Section 1.8.

**Theorem 5.5.** *Let $\mathbf{d}_k$ be the classes defined above. Then we have ring isomorphisms*

$$H^{*,*}(BU^{ev}_{\mathbb{H}}\ \times\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}}) \cong H^{*,*}(BU^{odd}_{\mathbb{H}}\ \times\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}}) \cong H^{*,*}(\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})\,[\mathbf{d}_1,\mathbf{d}_2,\ldots,\mathbf{d}_k,\ldots].$$

The cohomology ring $H^{*,*}(\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$ is computed in Section 1.8.

The total Chern class map $c_{\mathbb{H}}\ :\ \mathrm{BU}_{\mathbb{H}} = \amalg_{j\in\mathbb{Z}}BU^j_{\mathbb{H}}\to\mathcal{Z}_{\mathbb{H}} = \amalg_{j\in\mathbb{Z}}\mathcal{Z}^j_{\mathbb{H}}$ sends the component $BU^j_{\mathbb{H}}$ to the component $\mathcal{Z}^j_{\mathbb{H}}$. Its equivariant homotopy type is determined by the following result.

**Theorem 5.9.** *The equivariant cohomology classes determined by total quaternionic Chern class map $c_{\mathbb{H}}$ and the splitting* (41) *of Theorem 3.9 are given by*

$$\mathbf{1}+\mathbf{d}_2+\mathbf{d}_4+\cdots+\mathbf{d}_{2n}+\cdots \qquad \textit{on } BU^{ev}_{\mathbb{H}} \tag{6}$$

$$\mathbf{d}_1+\mathbf{d}_3+\cdots+\mathbf{d}_{2n+1}+\cdots \qquad \textit{on } BU^{odd}_{\mathbb{H}} \tag{7}$$

In [dSLF01] we work in the category of real algebraic varieties, addressing the issue of replacing continuous maps by morphisms of algebraic varieties. The $K$-theoretic constructions yield a semi-topological *quaternionic $K$-theory for real varieties*. This is related to Friedlander-Walker's semi-topological $K$-theory for real varieties [FW01] in the same way as Seymour's $KH^*$ is related to Atiyah's $KR^*$. Furthermore, we establish relations to the algebraic $K$-theory of real varieties and to Quillen's computation of the $K$-theory of Brauer-Severi varieties in [Qui73]. In the level of "morphism spaces" into algebraic cycles, the splittings of Theorem 3.4 still hold. Using them we introduce the the *quaternionic morphic cohomology* for real varieties, and discuss Chern classes for the quaternionic $K$-theory of real varieties.

This paper is organized as follows: in Section 1 we introduce the necessary background from $\mathbb{Z}/2$-homotopy theory needed to state our results. In Section 2 we establish a canonical splitting for the space of zero cycles on $\mathbb{P}(\mathbb{H}^n)$. In Section 3 we compute the$\mathbb{Z}/2$-homotopy type of $\mathcal{Z}^q(\mathbb{P}(\mathbb{H}^n))$ and define the infinite loop space of stabilized cycles $\mathcal{Z}_{\mathbb{H}}$. In Section 4 we apply the same stabilization procedure to the Grassmannians $G^q(\mathbb{H}^n)$, obtaining an equivariant infinite loop space $(\mathbb{Z}\times BU)_{\mathbb{H}}$. We show that $(\mathbb{Z}\times BU)_{\mathbb{H}}$ classifies Dupont's quaternionic K-theory. Section 5 is dedicated to computations involving the characteristic classes for quaternionic bundles defined in Section 4; a projective bundle formula is proved and the characteristic classes for the universal quaternionic bundle over $(\mathbb{Z} \times BU)_{\mathbb{H}}$ are computed.

**Acknowledgement.** The authors would like to thank H. Blaine Lawson, Jr., for fruitful conversations during the elaboration of this work. The first author thanks Texas A&M University and the second author thanks the Instituto Técnico Superior (Lisbon) for their respective hospitality during the ellaboration of this work.

## 1. Preliminary results from equivariant homotopy theory

In this section we review the definitions and results from equivariant homotopy theory needed for the purposes of this paper. Throughout this section $G$ will be an arbitrary finite group, and later on we will specialize to the case $G = \mathbb{Z}/2$.

**Notation 1.1.** If $V$ is a representation of $G$, $S^V$ denotes the one point compactification of $V$ and, for a based $G$-space $X$, $\Omega^V X$ denotes the space of based maps $F(S^V, X)$. The space $F(S^V, X)$ is equipped with the its standard $G$-space structure. The set of equivariant homotopy classes $[S^V, X]_G$ is denoted by $\pi_V(X)$. Given a $G$-space $X$, we denote by $X_+$ the pointed $G$-space $X \cup \{+\}$, where $+$ is a point fixed by $G$.

1.1. **Coefficient systems and Mackey functors.** Let $\mathcal{F}_G$ be category of finite $G$-sets and $G$-maps. The coefficients for ordinary equivariant (co)homology are (contravariant) covariant functors from $\mathcal{F}_G$ to the category $\mathcal{A}b$ of abelian groups which send disjoint unions to direct sums.

Given a contravariant coefficient system $M$ there are Bredon cohomology groups $H^*(-; M)$ with coefficients in $M$. They satisfy $G$-homotopy invariance and the suspension axiom, and they are classical cohomology theories in the sense that they satisfy the dimension axiom $H^0(pt; M) = M$ and $H^n(pt; M) = 0$, for $n > 0$.

There are certain coefficient systems – called Mackey functors – for which Bredon cohomology can be extented to an $RO(G)$-graded theory. A Mackey functor $M$ is a pair $(M_*, M^*)$ of functors $M_* : \mathcal{F}_G \to \mathcal{A}b$ and $M^* : \mathcal{F}_G^{op} \to \mathcal{A}b$ with the same value on objects and which transform each pull-back diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{k} & D \end{array}$$

in $\mathcal{F}_G$ into a commutative diagram in $\mathcal{A}b$

$$\begin{array}{ccc} M(A) & \xrightarrow{M_*(f)} & M(B) \\ {\scriptstyle M^*(g)}\uparrow & & \uparrow{\scriptstyle M^*(h)} \\ M(C) & \xrightarrow{M_*(k)} & M(D) \end{array}$$

**Example 1.2.** In this paper we are interested in the case where $M = \underline{\mathbb{Z}}$ is the Mackey functor constant at $\mathbb{Z}$. This Mackey functor is uniquely determined by the following conditions; cf. [May86, Prop. 9.10].

(i) $\underline{\mathbb{Z}}(G/H) = \mathbb{Z}$, for $H \leq G$;

(ii) if $K \leq H$, the value of the contravariant functor $\underline{\mathbb{Z}}^*$ on the projection $\rho : G/K \to G/H$ is the identity.

The $RO(G)$-graded cohomology groups with coefficients in a Mackey functor $M$ are denoted $H^*(-;M)$ and the corresponding reduced cohomology groups are denoted $\widetilde{H}^*(-;M)$. For each real orthogonal representation $V$ there is a classifying space $K(M,V)$ such that, for any $G$-space $X$,

$$H^V(X;M) \cong [X_+, K(M,V)]_G.$$

The spaces $K(M,V)$ fit together to give an equivariant Eilenberg-Mac Lane spectrum $HM$, i.e., given $G$-representations $V$, $W$, there is a $G$-homotopy equivalence $K(M,V) \cong \Omega^W K(M,V+W)$ satisfying various compatibility properties; cf. [May96]. This implies that $H^*(-;M)$ satisfies the suspension axiom in the direction of any representation:

$$\widetilde{H}^{V+W}(S^V \wedge X; M) \cong \widetilde{H}^W(X;M).$$

1.2. **Dold-Thom theorem.** Our interest on the Mackey functor $\underline{\mathbb{Z}}$ lies in the fact that just as the spaces of zero cycles (of degree zero) on the sphere $S^n$ is a model for the non-equivariant Eilenberg-MacLane space $K(\mathbb{Z},n)$, zero cycles on a representation sphere $S^V$ provide a model for $K(\underline{\mathbb{Z}},V)$. This is a consequence of the following equivariant version of the classical Dold-Thom theorem.

**Notation 1.3.** Let $X$ be a $G$-space. The topological group of zero cycles on $X$ is denoted by $\mathcal{Z}_0(X)$. Its elements are formal sums $\sum_i n_i x_i$, with $n_i \in \mathbb{Z}$ and $x_i \in X$. There is an augmentation homomorphism $deg : \mathcal{Z}_0(X) \to \mathbb{Z}$, whose kernel we denote by $\mathcal{Z}_0(X)_\circ$. Note that $\mathcal{Z}_0(X)$ is isomorphic to $\mathcal{Z}_0(X_+)_\circ$.

**Theorem 1.4.** [dS99b] *Let $G$ be a finite group, let $X$ be a based $G$-CW-complex and let $V$ be a finite dimensional $G$-representation, then there is a natural equivalence*

$$\pi_V AG(X) \cong \widetilde{H}^G_V(X;\underline{\mathbb{Z}}).$$

*In particular, $AG(S^V)$ is a $K(\underline{\mathbb{Z}},V)$ space.*

1.3. **Motivic notation.** From now on we restrict ourselves to the case of $G = \mathbb{Z}/2$. We will use motivic notation for $\mathbb{Z}/2$-equivariant cohomology, for it is compatible with the invariants used in algebraic geometry.

**Notation 1.5.** Let $\mathbf{s}$ be the one dimensional real sign representation of $\mathbb{Z}/2$ and let $\mathbf{1}$ stand for the one dimensional trivial representation. Then $RO(\mathbb{Z}/2) = \mathbb{Z}\cdot\mathbf{1} + \mathbb{Z}\cdot\mathbf{s}$. With $p \geq q$, we write

(1) $\mathbb{R}^{p,q}$ for $(p-q)\cdot\mathbf{1} \oplus q\cdot\mathbf{s}$;
(2) $S^{p,q}$ for $\mathbb{R}^{p,q} \cup \{\infty\}$;
(3) $H^{p,q}(-;\underline{\mathbb{Z}})$ for $H^{\mathbb{R}^{p,q}}(-;\underline{\mathbb{Z}})$;
(4) $K(\mathbb{Z}(q),p)$ for $K(\underline{\mathbb{Z}},\mathbb{R}^{p,q})$.

1.4. **Real bundles and equivariant Chern classes.** Recall that a real bundle over a $\mathbb{Z}/2$-space $X$, in the sense of [Ati66], is a complex bundle $\xi \to X$ with a bundle map $\tau : \xi \to \xi$ which is an anti-linear involution that covers the involution on $X$. Atiyah's $KR$-theory of $X$ is defined [Ati66] as the Grothendieck group of isomorphism classes of real bundles over $X$.

It turns out that $KR(-)$ is a $\mathbb{Z}/2$-equivariant cohomology theory in the sense that it is represented by a $\mathbb{Z}/2$-spectrum as defined above. In fact, $\mathbb{Z}\times BU$ has a natural $\mathbb{Z}/2$-action induced by complex conjugation, and we denote this $\mathbb{Z}/2$-space by $(\mathbb{Z} \times BU)_{\mathbb{C}}$. This is the classifying space for $KR$-theory, *i.e.*, for a $\mathbb{Z}/2$-space $X$,

$$KR(X) \cong [X_+, (\mathbb{Z} \times BU)_{\mathbb{C}}]_{\mathbb{Z}/2}.$$

In [Ati66] it is proved that the usual periodicity of $\mathbb{Z} \times BU$ is actually a $(2,1)$ periodicity, *i.e.*, there is an equivariant homotopy equivalence

$$\mathbb{Z} \times BU \cong \Omega^{2,1}(\mathbb{Z} \times BU).$$

Hence $(\mathbb{Z} \times BU)_{\mathbb{C}}$ is the zero-th space of a periodic $\mathbb{Z}/2$-spectrum and there are groups $KR^{p,q}(-)$, for $p, q \in \mathbb{Z}$, satisfying suspension, exact sequences and $\mathbb{Z}/2$-homotopy invariance. One can define Chern classes for real bundles with values in $H^{*,*}(-;\underline{\mathbb{Z}})$ as usual, by pulling back certain classes from $H^{*,*}(BU_{\mathbb{C}};\underline{\mathbb{Z}})$.

**Theorem 1.6.** *There exist unique classes $\widetilde{c_n} \in H^{2n,n}(BU_{\mathbb{C}};\underline{\mathbb{Z}})$ whose image under the forgetful map to singular cohomology is the $n$-Chern class $c_n \in H^{2n}(BU_{\mathbb{C}};\mathbb{Z})$. Furthermore, we have the following ring isomorphism*

$$H^{*,*}(BU_{\mathbb{C}};\underline{\mathbb{Z}}) \cong R[\widetilde{c_1}, \ldots, \widetilde{c_n}, \ldots],$$

*where $R$ is the cohomology ring of a point, $H^{*,*}(pt;\underline{\mathbb{Z}})$.*

*Proof.* This follows from the fact that $BU_{\mathbb{C}}$ has an equivariant cell decomposition given by the Schubert decomposition. $\square$

**Definition 1.7.** For a virtual real bundle $\xi$ over $X$, with a classifying map $f : X \to BU_{\mathbb{C}}$ the $n$-th *equivariant Chern class* is

$$f^*(\widetilde{c_n}) \in H^{2n,n}(X;\underline{\mathbb{Z}}).$$

The equivariant Chern classes satisfy the Whitney sum and projective bundle formulas.

1.5. **Thom isomorphism for real bundles.** As in non-equivariant homotopy theory, the existence of a Thom-isomorphism for some cohomology theory is directly related to the existence of orientations in that theory. We will see that real bundles are $\mathrm{H}\underline{\mathbb{Z}}$-orientable and hence there is a corresponding Thom isomorphism theorem for real bundles.

**Definition 1.8** ([May96])**.** Let $G$ be a finite group. Let $\xi \xrightarrow{p} X$ be an $n$-plane $G$-bundle over a $G$-space $X$. An $\mathrm{H}\underline{\mathbb{Z}}$-orientation of $\xi$ is an element $\mu$ of $\mathrm{H}\underline{\mathbb{Z}}^{\alpha}(\mathrm{T}(X))$ for some $\alpha$ of virtual dimension $n$, such that, for each inclusion $i : G/H \to X$ the restriction $i^*\mu$ to $\mathrm{T}(i^*\xi)$ is a generator of the free $\mathrm{H}\underline{\mathbb{Z}}^*(S^0)$-module $\mathrm{H}\underline{\mathbb{Z}}^*_H(\mathrm{T}(i^*\xi))$.

**Proposition 1.9.** *Let $X$ be a real space and let $\xi \xrightarrow{p} X$ be a real bundle. Then $\xi$ is $\mathrm{H}\underline{\mathbb{Z}}$-orientable.*

*Proof.* It suffices to consider the case where $\xi$ is a real line bundle. In this case we observe that $\mathrm{T}(\xi) = \mathbb{P}(\xi \oplus \mathbb{C})/\mathbb{P}(\xi)$ and set $\mu = c_1(\lambda)$ where $\lambda \to \mathbb{P}(\xi \oplus \mathbb{C})$ is the tautological line bundle and $c_1$ denotes the real first Chern class. Since $\lambda|_{\mathbb{P}(\xi)}$ is trivial, then $\mu$ descends to a class in the cohomology of $\mathrm{T}(\xi)$, which is also denote by $\mu$. Consider an equivariant map $i : (\mathbb{Z}/2)/H \to X$. There are two possible cases.

(i) $H = \mathbb{Z}/2$: In this case $\mathrm{T}(i^*\xi) = \mathbb{P}(\mathbb{C}^2)$ with the $\mathbb{Z}/2$-action given by complex conjugation. Then $c_1(\lambda)|_{\mathbb{P}(\mathbb{C}^2)}$ is the first Chern class of the tautological bundle over $\mathbb{P}(\mathbb{C}^2)$, which is a generator for $H^{*,*}(\mathbb{P}(\mathbb{C}^2), \underline{\mathbb{Z}})$ over $\mathrm{H}\underline{\mathbb{Z}}^*(S^0) = H^{*,*}(pt, \underline{\mathbb{Z}})$.

(ii) $H = \{0\}$. In this case $\mathrm{T}(\xi) = \mathbb{Z}/2_+ \wedge S^{2,0}$. We have $H^{*,*}(\mathbb{Z}/2_+ \wedge S^{2,0}, \underline{\mathbb{Z}}) \cong H^*(S^2;\mathbb{Z})$ and $KR(\mathbb{Z}/2_+ \wedge S^{2,0}) \cong K(S^2)$. It is easy to see that, under these isomorphisms, $c_1(\lambda)$ is the usual Chern class and hence it generates $H^{*,*}(\mathbb{Z}/2_+ \wedge S^{2,0}, \underline{\mathbb{Z}})$.

□

**Definition 1.10.** Given a vector bundle $\xi \to X$ over $X$, the projection $p : \mathbb{P}(\xi \oplus 1) \to X$ along with the quotient map $q : \mathbb{P}(\xi \oplus 1) \to \mathrm{T}(\xi) = \mathbb{P}(\xi \oplus 1)/\mathbb{P}(\xi)$ induce a map $\Delta : \mathrm{T}(\xi) \to \mathrm{T}(\xi) \wedge X_+$ called the *Thom diagonal* of $\xi$.

**Proposition 1.11** (Thom isomorphism for real bundles). *Let $\xi \xrightarrow{p} X$ be a real $n$-bundle over a real space $X$ and let $\mu \in H^{2n,n}(\mathrm{T}(\xi), \underline{\mathbb{Z}})$ be an orientation for $\xi$. Then*

$$\cup\mu : H^{p,q}(X_+, \underline{\mathbb{Z}}) \longrightarrow \widetilde{H}^{2n+p,n+q}(\mathrm{T}(\xi), \underline{\mathbb{Z}})$$

*is an isomorphism for all $p, q$. Furthermore, there is an equivariant homotopy equivalence*

$$\phi_\mu : \mathcal{Z}_0(\mathrm{T}(\xi))_o \xrightarrow{\cong} \mathcal{Z}_0\left(X_+ \wedge S^{2n,n}\right)_o$$

*which induces the Thom isomorphism in homology*

$$\widetilde{H}_{p,q}(\mathrm{T}(\xi), \underline{\mathbb{Z}}) \cong H_{p-2n,q-n}(X, \underline{\mathbb{Z}}).$$

*Proof.* The existence of the Thom isomorphism is an immediate consequence of Proposition 1.9; see [May96]. We proceed to construct an explicit map $\phi_\mu$ at the classifying space level which induces this isomorphism. Let $f_\mu : \mathrm{T}(\xi) \to \mathcal{Z}_0\left(S^{2n,n}\right)_{\mathrm{o}}$ be a (based) classifying map for the orientation class $\mu$, hence $f_\mu(\infty) = 0$. Consider the composition $\mathrm{T}(\xi) \xrightarrow{\Delta} \mathrm{T}(\xi) \wedge X_+ \xrightarrow{f_\mu \wedge id} \mathcal{Z}_0\left(S^{2n,n}\right)_{\mathrm{o}} \wedge X_+ \to \mathcal{Z}_0\left(S^{2n,n} \wedge X_+\right)_{\mathrm{o}}$, where $\Delta$ is the Thom diagonal, and the last map comes from the structure of "functor with smash products" (FSP) for $\mathcal{Z}_0(-)$. This composition induces a function $\phi_\mu : \mathcal{Z}_0(\mathrm{T}(\xi))_{\mathrm{o}} \to \mathcal{Z}_0\left(S^{2n,n} \wedge X_+\right)_{\mathrm{o}}$.

We claim that $\phi_\mu$ induces the Thom isomorphism in homology. Indeed, $\mathcal{Z}_0\left(S^{p,q}\right)_{\mathrm{o}}$ is a $K(\mathbb{Z}(q), p)$-space and so a model for $\mathrm{H}\underline{\mathbb{Z}}$ is given by $(p, q) \mapsto \mathcal{Z}_0\left(S^{p,q}\right)_{\mathrm{o}}$. Moreover $\wedge$ induces a pairing

$$K(\mathbb{Z}(q), p) \wedge K(\mathbb{Z}(q'), p') \longrightarrow K(\mathbb{Z}(q + q'), p + p')$$

which gives the usual ring spectrum structure on $\mathrm{H}\underline{\mathbb{Z}}$, cf. [Dug01] and [dS99b]. It follows that the map in homology represented by $\phi_\mu$ is

$$\widetilde{H}_{p,q}(\mathrm{T}(\xi), \underline{\mathbb{Z}}) \ni a \xrightarrow{\phi_{\mu_*}} p_*(\mu \cap a) \in H_{p-2n,q-n}(X, \underline{\mathbb{Z}}),$$

the usual definition of the Thom isomorphism. The proof that $\phi_{\mu*}$ is in fact an isomorphism goes exactly as in the non-equivariant case. Using the five lemma and the Mayer-Vietoris sequence it is possible to reduce to the case where $\xi \xrightarrow{p} X$ is the trivial bundle, in which case $\mathrm{T}(\xi) \cong X_+ \wedge S^{2n,n}$ and it is clear that $\phi_{\mu_*}$ is an isomorphism. □

1.6. **Poincaré duality.** A smooth manifold $X$ is called a *Real $n$-manifold* if it has the structure of a Real space $(X, \sigma)$ whose tangent bundle becomes a Real $n$-bundle over $(X, \sigma)$ under the action induced by $d\sigma$.

**Proposition 1.12.** *Let $X$ be a connected Real manifold of dimension $n$. Then, for each $k \geq 0$, there is an equivariant homotopy equivalence*

$$\mathcal{P} : F(X_+; \mathcal{Z}_0\left(S^{2(n+k),n+k}\right)_o) \xrightarrow{\cong} \mathcal{Z}_0\left(S^{2k,k} \wedge X_+\right)_o, \tag{8}$$

*which on passage to homotopy groups induces the Poincaré duality isomorphism*

$$H^{2n-p,n-q}(X, \underline{\mathbb{Z}}) \cong H_{p,q}(X, \underline{\mathbb{Z}}). \tag{9}$$

*Proof.* By Proposition 1.11 the tangent bundle of $X$ has an orientation, which is a cohomology class in dimension $(2n, n)$. It follows [CW92] that $X$ satisfies Poincaré duality as in (9) and the duality isomorphism is given by cap product with the fundamental class $\mathbf{z} \in H_{2n,n}(X, \underline{\mathbb{Z}})$ – which corresponds to $1 \in H^{0,0}(X, \underline{\mathbb{Z}}) = H^0(X_{\mathbb{Z}/2}; \mathbb{Z})$; cf. [CW92].

We now define a homotopy equivalence at the classifying space level realizing the Poincaré duality isomorphism. Let $r = n + k$ and let $D$ denote the composition

$$\begin{aligned} \mathcal{Z}_0(X) \wedge F(X_+, \mathcal{Z}_0\left(S^{2r,r}\right)_o) &\xrightarrow{\Delta\wedge\mathrm{id}} \mathcal{Z}_0(X) \wedge \mathcal{Z}_0(X) \wedge F(X_+, \mathcal{Z}_0\left(S^{2r,r}\right)_o) \\ &\xrightarrow{\mathrm{id}\wedge\epsilon} \mathcal{Z}_0(X) \wedge \mathcal{Z}_0\left(S^{2r,r}\right)_o \to \mathcal{Z}_0\left(S^{2r,r} \wedge X_+\right)_o. \end{aligned}$$

where $\Delta$ is the diagonal map, $\varepsilon$ is the group homomorphism induced by the evaluation map $X \wedge F(X_+, \mathcal{Z}_0\left(S^{2r,r}\right)_o) \to \mathcal{Z}_0\left(S^{2r,r}\right)_o$, and the last arrow comes from the structure of "functor with smash products" (FSP) for $\mathcal{Z}_0(-)$. Composing $D$ with a classifying map $S^{2r,r} \to \mathcal{Z}_0(X)$ for the fundamental class $\mathbf{z}$ we obtain a map $S^{2n,n} \wedge F(X_+, \mathcal{Z}_0\left(S^{2r,r}\right)_o) \to \mathcal{Z}_0\left(S^{2r,r} \wedge X_+\right)_o$, with adjoint $F(X_+, \mathcal{Z}_0\left(S^{2r,r}\right)_o) \to \Omega^{2n,n}\mathcal{Z}_0\left(S^{2r,r} \wedge X_+\right)_o$. Composing with the natural equivalence $\Omega^{2n,n}\mathcal{Z}_0\left(S^{2r,r} \wedge X_+\right)_o \cong \mathcal{Z}_0\left(S^{2k,k} \wedge X_+\right)_o$, yields a map

$$\mathcal{P} : F(X_+, \mathcal{Z}_0\left(S^{2(n+k),n+k}\right)_o) \to \mathcal{Z}_0\left(S^{2k,k} \wedge X_+\right)_o$$

which induces the cap product with $\mathbf{z}$. Hence $\mathcal{P}$ is an equivariant homotopy equivalence. ☐

1.7. **Relation with group cohomology and Galois-Grothendieck cohomology.** Cohomology theories like the one represented by $\mathrm{H}\underline{\mathbb{Z}}$ are not what algebraic geometers usually mean by equivariant cohomology. However, it is well known that for spaces which are free (under the action of a finite group), the invariants given by $H^{*,*}(-; \underline{\mathbb{Z}})$ are closely related to the equivariant cohomology theories used in algebraic geometry. The goal of this section is to describe the relation between the theory $H^{*,*}(-; \underline{\mathbb{Z}})$, and Galois-Grothendieck cohomology, in the case of $\mathbb{Z}/2$-actions.

Let $G$ be a finite group. Recall that, in most geometrical contexts, the Borel cohomology of a $G$-space $X$ (with coefficients in a ring $R$) is just the ordinary cohomology of the Borel construction $X_{hG} := X \times_G EG$. We denote these groups by $\hat{H}^*_G(X; R)$. Galois-Grothendieck cohomology can be thought of as Borel cohomology with twisted coefficients: if $\mathcal{F}$ is a $G$-sheaf over $X$, then $\mathcal{F}\times_G EG$ is sheaf over $X_{hG}$ and we can define $\hat{H}^*_G(X; \mathcal{F}) := H^*(X_{hG}; \mathcal{F}\times_G EG)$. These are the Galois-Grothendieck cohomology groups of $X$ with coefficients in $\mathcal{F}$. Given any $G$-spectrum $k_G$ there is also a Borel type cohomology theory associated with $k_G$, defined as the cohomology represented by the $G$-spectrum $F(EG_+, k_G)$.

We now specialize to the case $G = \mathbb{Z}/2$, and $k_G = \mathrm{H}\underline{\mathbb{Z}}$. Consider the $\mathbb{Z}/2$-sheaves $\mathbb{Z}(n)$, which denote the constant sheaf $\mathbb{Z}$ consider as a $\mathbb{Z}/2$-sheaf with the $\mathbb{Z}/2$-action of multiplication by $(-1)^n$. The following proposition relates Galois-Grothendieck cohomology with coefficients in the sheaves $\mathbb{Z}(n)$ to the cohomology represented by $F(E\mathbb{Z}/2_+, \mathrm{H}\underline{\mathbb{Z}})$.

**Proposition 1.13.** *Let $p, q$ be non-negative integers such that $p \geq q$. There are natural isomorphisms*

$$\Psi : H^{p,q}(- \times E\mathbb{Z}/2_+; \underline{\mathbb{Z}}) \to \hat{H}^p_{\mathbb{Z}/2}(-; \mathbb{Z}(q)), \tag{10}$$

*which assemble into a ring homomorphism*

$$\Psi : H^{*,*}(- \times E\mathbb{Z}/2_+; \underline{\mathbb{Z}}) \to \hat{H}^*_{\mathbb{Z}/2}(-; \mathbb{Z}(*)).$$

*In particular, if $X$ is a free $\mathbb{Z}/2$-space, $H^{*,*}(X; \underline{\mathbb{Z}})$ is periodic with period $(0, 2)$, and the periodicity isomorphism is given by multiplication with a generator of $H^{0,2}(E\mathbb{Z}/2; \underline{\mathbb{Z}}) \cong \mathbb{Z}$.*

*Proof.* The existence of periodicity in the equivariant cohomology spaces with free actions is a result of Waner [Wan86]. Our proof uses $\mathcal{Z}_0\left(S^{p,q}\right)_{\mathrm{o}}$ as a model for the classifying space $K(\underline{\mathbb{Z}}, \mathbb{R}^{p,q})$. We start by observing that the cohomology groups $\hat{H}^p(X; \mathbb{Z}(q))$ can be expressed as Bredon cohomology groups of $X \times E\mathbb{Z}/2$. In fact, we have $\hat{H}^p_{\mathbb{Z}/2}(X; \mathbb{Z}(q)) \cong H^p(X \times E\mathbb{Z}/2; \underline{\mathbb{Z}}(q))$, where right-hand side denotes Bredon cohomology and $\underline{\mathbb{Z}}(q)$ is the coefficient system determined by the $\mathbb{Z}/2$-module $\mathbb{Z}(q)$. Hence, $\hat{H}^p_{\mathbb{Z}/2}(X; \mathbb{Z}(q)) \cong [X \times E\mathbb{Z}/2_+, \mathcal{Z}_0\left(S^p\right)_{\mathrm{o}} \otimes \mathbb{Z}(q)]_{\mathbb{Z}/2}$, where $\mathcal{Z}_0\left(S^p\right)_{\mathrm{o}} \otimes \mathbb{Z}(q)$ is $\mathcal{Z}_0\left(S^p\right)_{\mathrm{o}}$ with the $\mathbb{Z}/2$-action of multiplication by $(-1)^q$.

A direct computation shows that there is a class $\alpha_{p,q}$ in $\hat{H}^p_{\mathbb{Z}/2}(S^{p,q}; \mathbb{Z}(q))$, whose image under the forgetful map to singular cohomology is the fundamental class of $S^{p,q}$. This implies that there is an equivariant map $\mathcal{Z}_0\left(S^{p,q}\right)_{\mathrm{o}} \times E\mathbb{Z}/2 \to \mathcal{Z}_0\left(S^p\right)_{\mathrm{o}} \otimes \mathbb{Z}(q)$, which is a non-equivariant homotopy equivalence. Composition with this map induces an equivariant homotopy equivalence

$$F(E\mathbb{Z}/2_+, \mathcal{Z}_0\left(S^{p,q}\right)_{\mathrm{o}} \times E\mathbb{Z}/2) \cong F(E\mathbb{Z}/2_+, \mathcal{Z}_0\left(S^p\right)_{\mathrm{o}} \otimes \mathbb{Z}(q)).$$

Composing with the map $E\mathbb{Z}/2 \to *$ gives an equivariant homotopy equivalence

$$F(E\mathbb{Z}/2_+, \mathcal{Z}_0\left(S^{p,q}\right)_{\mathrm{o}} \times E\mathbb{Z}/2) \cong F(E\mathbb{Z}/2_+, \mathcal{Z}_0\left(S^{p,q}\right))_{\mathrm{o}},$$

which induces the isomorphism $\Psi$ in (10).

The pairing $\mathbb{Z}(q) \otimes \mathbb{Z}(q') \to \mathbb{Z}(q + q')$ induces a pairing

$$\hat{H}^p_{\mathbb{Z}/2}(-; \mathbb{Z}(q)) \otimes \hat{H}^p_{\mathbb{Z}/2}(-; \mathbb{Z}(q')) \to \hat{H}^p_{\mathbb{Z}/2}(-; \mathbb{Z}(q + q')),$$

which is easily seen to correspond, under the isomorphism (10), to the cup product in $H^{*,*}(- \times E\mathbb{Z}/2_+; \underline{\mathbb{Z}})$ (because $\alpha_{p,q} \cup \alpha_{p',q'} = \alpha_{p+p',q+q'}$). Let $t$ be a generator of $H^{0,2}(E\mathbb{Z}/2; \underline{\mathbb{Z}}) \cong \mathbb{Z}$ and let $X$ be a free $\mathbb{Z}/2$-space. Since $X$ is $\mathbb{Z}/2$-homotopy equivalent to $X \times E\mathbb{Z}/2$, the cohomology of $X$ is a module over $H^{*,*}(E\mathbb{Z}/2; \underline{\mathbb{Z}})$. The compatibility of $\Psi$ with products shows that the following diagram commutes

$$\begin{array}{ccc} H^{p,q+2}(X; \underline{\mathbb{Z}}) & \xrightarrow{\Psi} & \hat{H}^p_{\mathbb{Z}/2}(X; \mathbb{Z}(q+2)) \\ {\scriptstyle t\cup -}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ H^{p,q}(X; \underline{\mathbb{Z}}) & \xrightarrow{\Psi} & \hat{H}^p_{\mathbb{Z}/2}(X; \mathbb{Z}(q)) \end{array},$$

and hence $t \cup -$ is an isomorphism. □

**Notation 1.14.** As observed above, if $X$ is a free $\mathbb{Z}/2$, there is a natural homomorphism

$$H^{*,*}(X \times E\mathbb{Z}; \underline{\mathbb{Z}}) \to H^{*,*}(X; \underline{\mathbb{Z}})$$

making $H^{*,*}(X;\underline{\mathbb{Z}})$ into a module over $H^{*,*}(E\mathbb{Z}/2;\underline{\mathbb{Z}})$. From now on we will use $t$ to denote both a generator of $H^{0,2}(E\mathbb{Z}/2;\underline{\mathbb{Z}}) \cong \mathbb{Z}$ and its image under the homomorphism above. Multiplication by $t$ induces the $(0,2)$ periodicity in $H^{*,*}(X;\underline{\mathbb{Z}})$.

**Corollary 1.15.** *Let $X$ be a space with a free $\mathbb{Z}/2$-action. There is an $E_2$ spectral sequence*

$$H^s(B\mathbb{Z}/2;\mathcal{H}^t(X;\mathbb{Z}(q))) \Longrightarrow H^{s+t,q}(X;\underline{\mathbb{Z}})$$

*Proof.* It is shown in [Gro57] that there is an $E_2$ term spectral sequence

$$H^s(B\mathbb{Z}/2;\mathcal{H}^t(X;\mathbb{Z}(q))) \Longrightarrow \hat{H}^{s+t}_{\mathbb{Z}/2}(X;\underline{\mathbb{Z}}(q)).$$

□

1.8. **The $RO(\mathbb{Z}/2)$-graded cohomology of the Brauer-Severi curve $\mathbb{P}(\mathbb{H})$.** Let $\mathbb{P}(\mathbb{H})$ denote the complex points of the Brauer-Severi variety associated with the real algebra $\mathbb{H}$. Here we describe the cohomology ring $H^{*,*}(\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$. The computations follow directly from the results above.

We start by computing $H^{*,q}(\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$, for $q = 0, 1$. For $q = 0$, we have

$$H^{*,0}(\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}}) = H^*(\mathbb{P}(\mathbb{H})_{\mathbb{Z}/2};\mathbb{Z}) = H^*(\mathbb{RP}^2;\mathbb{Z}).$$

For $q = 1$, the spectral sequence (1.15) gives

$$H^{p,1}(\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}}) \cong \begin{cases} \mathbb{Z}/2 & p = 1 \\ \mathbb{Z} & p = 2 \\ 0 & \text{otherwise} \end{cases}$$

Moreover, $H^{1,1}(\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$ and $H^{2,0}(\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$ are generated by the image of the homomorphism

$$H^{*,*}(E\mathbb{Z}/2;\underline{\mathbb{Z}}) \to H^{*,*}(\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}}).$$

Let $\epsilon$ denote the generator of $H^{1,1}(\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$ and let $\epsilon'$ be its image in $H^1(\mathbb{P}(\mathbb{H});\mathbb{Z}(1))$ under the ring homomorphism $\Psi$ of (10). One can check that $\epsilon'^2 \neq 0$ hence $\epsilon^2$ is the generator of $H^{2,2}(\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$.

The group $H^{2,1}(\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$ is generated by the fundamental class of $\mathbb{P}(\mathbb{H})$, which we denote by $\mathbf{x}$. Note that $\mathbf{x}^2 = 0$, since, by $(0,2)$ periodicity,

$$\mathbf{x}^2 \in H^{4,2}(\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}}) \cong H^4(\mathbb{RP}^2;\mathbb{Z}) = 0.$$

The same argument shows that $\mathbf{x}\epsilon = \mathbf{x}^3 = 0$.

**Remark 1.16.** The generator $\mathbf{x}$ is given by the first Chern class $\tilde{c}_1(\mathcal{O}(2))$ of the Real bundle $\mathcal{O}(2)$ over $\mathbb{P}(\mathbb{H})$.

Putting all these facts together and using that fact that $t \cup -$ is an isomorphism, we obtain a ring isomorphism

$$H^{*,*}(\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}}) \cong \mathbb{Z}[\epsilon, \mathbf{x}, t, t^{-1}]/(2\epsilon, \epsilon^3, \mathbf{x}\epsilon, \mathbf{x}^2), \tag{11}$$

where $\epsilon, t$ and $\mathbf{x}$ have degrees $(1,1), (0,2)$ and $(2,1)$, respectively.

## 2. The $\mathbb{Z}/2$-homotopy type of zero-cycles

A structure on a complex vector space $V$ is a complex anti-linear map $\mathbf{j} : V \to V$ that satisfies either $\mathbf{j}^2 = I$ or $\mathbf{j}^2 = -I$. In the first case, $\mathbf{j}$ is called a *real structure* and the pair $(V, \mathbf{j})$ is a *real vector space*, and in the latter case, $\mathbf{j}$ is called a *quaternionic structure* and the pair $(V, \mathbf{j})$ is a *quaternionic vector space*. A morphism $f : (V, \mathbf{j}_V) \to (W, \mathbf{j}_W)$ of vector spaces with structure is a complex linear map from $V$ to $W$ commuting with the respective structures.

Any real vector space $(V, \mathbf{j})$ of complex dimensin $n$ is isomorphic as a vector space with structure to $(\mathbb{C}^n, \mathbf{j}_o)$, where $\mathbf{j}_o$ denotes the usual complex conjugation on $\mathbb{C}^n$. Similarly, any quaternionic vector space $(V, \mathbf{j})$ of complex dimension $2n$ is isomorphic to $(\mathbb{H}^n, \mathbf{j}_o)$, where $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}\jmath$ denotes the algebra of the quaternions and $\mathbf{j}_o$ is induced by multiplication by $\jmath$ on the left of $\mathbb{H}^n$.

**Remark 2.1.** It is clear that if $(V, \mathbf{j})$ is a quaternionic vector space, then $\mathbf{j}$ naturally induces a structure $\mathbf{j}_d$ on the symmetric power $\mathrm{Sym}_d(V)$ which is *real* if $d$ is even and *quaternionic* if $d$ is odd. In particular, one has

$$(\mathrm{Sym}_{2k}(\mathbb{H}), \mathbf{j}_{2k}) \cong (\mathbb{C}^{2k+1}, \mathbf{j}_o) \quad \text{and} \quad (\mathrm{Sym}_{2k+1}(\mathbb{H}), \mathbf{j}_{2k+1}) \cong (\mathbb{H}^{k+1}, \mathbf{j}_o). \tag{12}$$

Let $\mathbb{H}^\vee$ denote the complex dual of $\mathbb{H}$ and let $\mathbb{P}(\mathbb{H})$ denote the projective space of 1-dimensional complex subspaces of $\mathbb{H}$, which we identify with the complex subspaces of codimension 1 in $\mathbb{H}^\vee$. In particular, if $f : \mathbb{H}^\vee \to \mathbb{C}$ is a non-zero linear functional then we denote its zero locus by $[f] \in \mathbb{P}(\mathbb{H})$.

The $d$-fold symmetric product $\mathrm{SP}_d(\mathbb{P}(\mathbb{H}))$ inherits an anti-holomorphic involution $\sigma : \mathrm{SP}_d(\mathbb{P}(\mathbb{H})) \to \mathrm{SP}_d(\mathbb{P}(\mathbb{H}))$ induced by $\jmath$, hence it becomes a *Real space* in the sense of Atiyah [Ati66]. If one denotes by $[f_1] \cdots [f_d]$ an element in $\mathrm{SP}_d(\mathbb{P}(\mathbb{H}))$, then the map sending $[f_1] \cdots [f_d]$ to $[f_1 \cdots f_d]$ induces the classical isomorphism between $\mathrm{SP}_d(\mathbb{P}(\mathbb{H}))$ and $\mathbb{P}\big(\mathrm{Sym}_d(\mathbb{H})\big)$. This is an isomorphism of *Real spaces.*

The canonical inclusion $i_{n,q} : \mathbb{P}\big(\mathbb{H}^n\big) \hookrightarrow \mathbb{P}\big(\mathbb{H}^q\big)$, for $n < q$, given by setting the last coordinates zero can be described (up to linear isomorphism) in terms of symmetric products as follows. Let $[f_o] \in \mathbb{P}(\mathbb{H})$ be some point and let $[f_o^\sigma]$ denote its image under the quaternionic involution $\sigma$. Then define

$$\begin{aligned} i_{n,q} : \mathbb{P}\big(\mathbb{H}^n\big) = \mathrm{SP}_{2n-1}(\mathbb{P}(\mathbb{H})) &\to \mathbb{P}\big(\mathbb{H}^q\big) = \mathrm{SP}_{2q-1}(\mathbb{P}(\mathbb{H})) \\ [f_1] \cdots [f_{2n-1}] &\to ([f_o][f_o^\sigma])^{q-n} \cdot [f_1] \cdots [f_{2n-1}]. \end{aligned} \tag{13}$$

Note that $i_{n,q}$ is a morphism of Real spaces.

Following [LLFM98c] we define, for $a < b$, the map

$$\begin{aligned} r_{b,a} : \mathrm{SP}_b(\mathbb{P}(\mathbb{H})) &\to \mathrm{SP}_{\binom{b}{a}}(\mathrm{SP}_a(\mathbb{P}(\mathbb{H}))) \\ [f_1] \cdots [f_b] &\mapsto \sum_{|I|=a} [f_{i_1}] \cdots [f_{i_a}], \end{aligned} \tag{14}$$

where the sum runs over all multi-indexes $I = \{1 \le i_1 < \cdots < i_a \le b\}$. Notice that although $a$ and $b$ can be even or odd, the maps $r_{b,a}$ are always morphisms of Real spaces.

To simplify notation, denote

$$M_n = \mathcal{Z}_0\left(\mathrm{SP}_{2n-1}(\mathbb{P}(\mathbb{H}))\right) = \mathcal{Z}_0\left(\mathbb{P}\big(\mathbb{H}^n\big)\right), \tag{15}$$

and let

$$\Psi_n : M_n \to Q_n := M_n / M_{n-1} \tag{16}$$

denote the quotient map. We adopt the convention that $M_{-1} = \{0\}$.

Now, use the morphisms $r_{b,a}$ to construct maps

$$R_{q,n} : M_q \to M_n \tag{17}$$

as follows. Given $\tau \in \mathbb{P}(\mathbb{H}^q)$, define for $n < q$,

$$R_{q,n}(\tau) = r_{2q-1,2n-1}(\tau) + \sum_{j=1}^{2n-1} (-1)^j \binom{q-n+j-1}{j} \left([f_o]^j + [f_o^\sigma]^j\right) \; r_{2q-1,2n-1-j}(\tau). \tag{18}$$

The map $R_{q,n}$ can then be extended by linearity to arbitrary 0-cycles on $\mathbb{P}(\mathbb{H}^q)$. Finally, for $n < q$, define $q_{q,n} : M_q \to Q_n$ as the composition $q_{q,n} = \Psi_n \circ R_{q,n}$ and let $q_{n,n} = \Psi_n$.

**Proposition 2.2.** *Let $\{(M_n, Q_n, q_{q,n}, i_{q,n}) \mid q_{q,n} : M_q \to Q_n,\ i_{n,q} : M_n \hookrightarrow M_q,\ 0 \le n \le q\}$ be the collection of groups and maps defined above. Then the following assertions hold:*

- **a:** *The maps $i_{n,q}$ and $q_{q,n}$ are equivariant homomorphisms for the $\mathbb{Z}/2$ actions on $M_n$ and $Q_n$ induced by the quaternionic structure on $\mathbb{H}^n$.*
- **b:** *The sequence $M_{n-1} \xrightarrow{i_{n-1,n}} M_n \xrightarrow{q_{n,n}} Q_n$ is an equivariant principal fibration, for all $n$.*
- **c:** *The following diagram commutes:*

$$\begin{array}{ccc} M_n & \xrightarrow[i_{n,q}]{} & M_q \\ {\scriptstyle q_{n,n}}\downarrow & & \downarrow{\scriptstyle q_{q,n}} \\ Q_n & \xrightarrow[id]{} & Q_n. \end{array} \tag{19}$$

*Proof.* The first assertion is evident from the definitions.

Now, observe that the $\mathbb{Z}/2$ involution on $\mathbb{P}(\mathbb{H}^n)$ induced by the quaternionic structure on $\mathbb{H}^n$ is a real analytic involution. Therefore, the pair $(\mathbb{P}(\mathbb{H}^n), \mathbb{P}(\mathbb{H}^{n-1}))$ becomes a $\mathbb{Z}/2$-simplicial pair after a suitable equivariant triangulation. Therefore, the second assertion follows from [LF97, Thm. 2.7]

In order to prove the last assertion, consider elements $x_1, \ldots, x_{2q-1} \in \mathbb{P}(\mathbb{H})$ as free variables and let $S$ denote the polynomial ring $S := \mathbb{Z}[x_1, \ldots, x_{2q-1}]$. It is then clear that $r_{2q-1,2n-1}(x_1 \cdots x_{2q-1}) \in \mathrm{SP}_{\binom{2q-1}{2n-1}}(\mathrm{SP}_{2n-1}(\mathbb{P}(\mathbb{H})))$ can be seen as the coefficient of $t^{2n-1}$ in the polynomial

$$P_t^{2q-1}(x_1, \ldots, x_{2q-1}) := \prod_{i=1}^{2q-1} (1 + x_i t) \;\in\; S[t]. \tag{20}$$

In particular,

$$r_{2q-1,2n-1-j} \circ i_{n,q}(x_1 \cdots x_{2n-1}) = r_{2q-1,2n-1-j}(x_1 \cdots x_{2n-1} \; (f_o f_o^\sigma)^{n-q})$$

is the coefficient of $t^{2n-1-j}$ in the polynomial $P_t^{2n-1}(x_1, \ldots, x_{2n-1}) \; (1 + f_o t)^{q-n} (1 + f_o^\sigma t)^{q-n}$.

Using the above observations, and the definition of $R_{q,n}$, one concludes that $R_{q,n}(i_{n,q}(x_1\cdots x_{2n-1}))$ is the coefficient of $t^{2n-1}$ in

$$
\begin{aligned}
(21)\quad & \left(\prod_{i=1}^{2n-1}(1+x_it)\right)(1+f_ot)^{q-n}(1+f_o^\sigma t)^{q-n}\\
&+\sum_{j=1}^{\infty}(-1)^j\binom{q-n+j-1}{j}\{(f_o)^j+(f_o^\sigma)^j\}\,t^j\left(\prod_{i=1}^{2n-1}(1+x_it)\right)(1+f_ot)^{q-n}(1+f_o^\sigma t)^{q-n}\\
&=\left(\prod_{i=1}^{2n-1}(1+x_it)\right)\cdot\Bigg\{(1+f_ot)^{q-n}(1+f_o^\sigma t)^{q-n}\\
&+\sum_{j=1}^{\infty}(-1)^j\binom{q-n+j-1}{j}\{(f_o)^j+(f_o^\sigma)^j\}\,t^j\,(1+f_ot)^{q-n}(1+f_o^\sigma t)^{q-n}\Bigg\}
\end{aligned}
$$

seen as an element in the ring $\mathbb{Z}[x_1,\ldots,x_{2n-1}][\![f_o,f_o^\sigma,t]\!]$ of formal power series in the variables $f_o,f_o^\sigma,t$ with coefficients in $\mathbb{Z}[x_1,\ldots x_{2n-1}]$.

We now observe that the inverse of $(1+y)^N$ in $\mathbb{Z}[\![y]\!]$ is given by $\sum_{j=0}^\infty(-1)^j\binom{N+j-1}{j}\,y^j$ and hence, the element

$$
(1+f_ot)^{q-n}(1+f_o^\sigma t)^{q-n}+\sum_{j=1}^{\infty}(-1)^j\binom{q-n+j-1}{j}\{(f_o)^j+(f_o^\sigma)^j\}\,t^j\,(1+f_ot)^{q-n}(1+f_o^\sigma t)^{q-n}
$$

in $\mathbb{Z}[\![f_o,f_o^\sigma,t]\!]$ is seen to be equal to

$$
\begin{aligned}
(22)\quad & (1+f_ot)^{q-n}(1+f_o^\sigma t)^{q-n}\left\{(1+f_ot)^{n-q}+(1+f_o^\sigma t)^{n-q}-1\right\}\\
&=(1+f_o^\sigma t)^{q-n}+(1+f_ot)^{q-n}-(1+f_ot)^{q-n}(1+f_o^\sigma t)^{q-n}.
\end{aligned}
$$

If one writes $(1+yt)^N=1+y\alpha_t^N(y)$, then the right hand side of the formula (22) above can can be written as

$$
\begin{aligned}
&(1+f_o^\sigma t)^{q-n}+(1+f_ot)^{q-n}-(1+f_ot)^{q-n}(1+f_o^\sigma t)^{q-n}\\
=&(1+f_o^\sigma t)^{q-n}+(1+f_ot)^{q-n}-(1+f_o^\sigma\ \alpha_t^{q-n}(f_o^\sigma))(1+f_o\ \alpha_t^{q-n}(f_o))\\
=&(1+f_o^\sigma t)^{q-n}+(1+f_ot)^{q-n}\\
&-\left\{1+f_o^\sigma\ \alpha_t^{q-n}(f_o^\sigma)+f_o\ \alpha_t^{q-n}(f_o))+f_o^\sigma f_o\ \ \alpha_t^{q-n}(f_o^\sigma)\alpha_t^{q-n}(f_o)\right\}\\
=&(1+f_o^\sigma t)^{q-n}+(1+f_ot)^{q-n}\\
&-\left\{(1+f_o^\sigma t)^{q-n}+(1+f_ot)^{q-n}-1+f_o^\sigma f_o\ \ \alpha_t^{q-n}(f_o^\sigma)\alpha_t^{q-n}(f_o)\right\}\\
=&1-f_o^\sigma f_o\left\{\alpha_t^{q-n}(f_o^\sigma)\alpha_t^{q-n}(f_o)\right\}.
\end{aligned}
$$

We now apply the latter identity to equation (21) and obtain

$$
R_{q,n}\left(i_{n,q}(x_1\cdots x_{2n-1})\right)=x_1\cdots x_{2n-1}-f_o^\sigma f_o(\star),
$$

for some element $\star$. Therefore,

$$q_{q,n} \circ i_{n,q}(x_1 \cdots x_{2n-1}) = \Psi_n\left(R_{q,n}(i_{n,q}(x_1 \cdots x_{2n-1}))\right) = \Psi_n\left(x_1 \cdots x_{2n-1}\right),$$

and this shows that the diagram (19) commutes, concluding the proof. □

**Corollary 2.3.** *Let $X$ be a based $\mathbb{Z}/2$-space. The maps $q_{n,j} : \mathcal{Z}_0\left(\mathbb{P}(\mathbb{H}^n)\right) \to \mathcal{Z}_0\left(\mathbb{P}(\mathbb{H}^j)\right)/\mathcal{Z}_0\left(\mathbb{P}(\mathbb{H}^{j-1})\right)$ induce an equivariant homotopy equivalence*

$$\mathcal{Z}_0\Big(\ X \wedge \mathbb{P}(\mathbb{H}^n)_+\ \Big) \xrightarrow{\tilde{q}_{n,n}\times\tilde{q}_{n,n-1}\times\cdots\times\tilde{q}_{n,1}} \prod_{j=1}^{n} \mathcal{Z}_0\Big(\ X \wedge \{\mathbb{P}(\mathbb{H}^j)/\mathbb{P}(\mathbb{H}^{j-1})\}_+\ \Big)_o.$$

*Proof.* Define $M_j = \mathcal{Z}_0\Big(\ X \wedge \mathbb{P}(\mathbb{H}^j)_+\ \Big)$ and $Q_j = \mathcal{Z}_0\Big(\ X \wedge \{\mathbb{P}(\mathbb{H}^j)/\mathbb{P}(\mathbb{H}^{j-1})\}_+\ \Big)_o$ and observe that one has an equivariant isomorphism $M_j/M_{j-1} \cong Q_j$. Let $\Psi_j : M_j \to Q_j$ denote the projection and let $\tilde{i}_{n,q} : M_n \hookrightarrow M_q$ be the canonical inclusion induced by the inclusion of spaces when $n < q$.

The maps $r_{2q-1,2n-1}$, described in (14), induce maps

$$\tilde{r}_{2q-1,2n-1} : X \wedge \mathbb{P}(\mathbb{H}^q)_+ \to \mathrm{SP}_{\binom{2q-1}{2n-1}}(X \wedge \mathbb{P}(\mathbb{H}^n)_+) \tag{23}$$

defined as the composition

$$X \wedge \mathbb{P}(\mathbb{H}^q)_+ \xrightarrow{id\wedge r_{2q-1,2n-1}} X \wedge \Big\{\mathrm{SP}_{\binom{2q-1}{2n-1}}(\mathbb{P}(\mathbb{H}^n))\Big\}_+ \xrightarrow{\wedge} \mathrm{SP}_{\binom{2q-1}{2n-1}}(X \wedge \mathbb{P}(\mathbb{H}^n)_+),$$

where the latter is the natural structural map when we see $\mathrm{SP}_*(-)$ as a functor with smash products. Finally, define $\tilde{q}_{q,n} := \Psi_n \circ \tilde{r}_{2q-1,2n-1}$.

It is immediate from the definitions that all the assertions in Proposition 2.2 hold for the new collection $(M_n, Q_n, \tilde{q}_{q,n}, \tilde{i}_{n,q})$ above. These assertions guarantee that the spaces and maps involved, along with their restrictions to fixed point sets, satisfy the hypothesis of [FL92, Prop. 2.13]. The corollary then follows. □

In order to fully understand the equivariant homotopy type of $\mathcal{Z}_0\left(\mathbb{P}(\mathbb{H}^n)\right)$ we are reduced to understanding $\mathcal{Z}_0\left(\mathbb{P}(\mathbb{H}^j)/\mathbb{P}(\mathbb{H}^{j-1})\right)_o$. However, $\mathbb{P}(\mathbb{H}^j)/\mathbb{P}(\mathbb{H}^{j-1}) = \mathrm{T}\left(\mathcal{O}_{\mathbb{P}(\mathbb{H})}(1) \otimes \mathbb{H}^{j-1}\right)$ is the Thom space of the Real bundle $\mathcal{O}_{\mathbb{P}(\mathbb{H})}(1) \otimes \mathbb{H}^{j-1}$. It follows from Propositions 1.11 and 1.12 that the Thom class of $\mathcal{O}_{\mathbb{P}(\mathbb{H})}(1)$, along with Poincaré duality, determines a unique equivariant homotopy equivalence

$$\mathcal{Z}_0\left(\mathbb{P}(\mathbb{H}^j)/\mathbb{P}(\mathbb{H}^{j-1})\right)_o \cong \mathcal{Z}_0\left(\mathrm{T}\left(\mathcal{O}_{\mathbb{P}(\mathbb{H})}(1) \otimes \mathbb{H}^{j-1}\right)\right)_o \cong F(\mathbb{P}(\mathbb{H})_+, \mathcal{Z}_0\left(S^{4j-2,2j-1}\right)_o). \tag{24}$$

This proves the following result:

**Corollary 2.4.** *There is a canonical equivariant homotopy equivalence*

$$\mathcal{Z}_0\left(\mathbb{P}(\mathbb{H}^n)\right) \cong \prod_{j=1}^{n} F\left(\mathbb{P}(\mathbb{H})_+,\ \mathcal{Z}_0\left(S^{4j-1,2j-1}\right)_o\right) \cong \prod_{j=1}^{n} F\left(\mathbb{P}(\mathbb{H})_+,\ K(\mathbb{Z}(2j-1), 4j-2)\right).$$

The last equivalence follows from the equivariant Dold-Thom theorem proven in [dS99a]; cf. (1.4).

## 3. Quaternionic algebraic cycles and the join pairing

In this section we study the equivariant topology of groups of algebraic cycles under quaternionic involution, and construct stabilizations of such objects that yield equivariant $\mathbb{Z}/2$-spectra.

3.1. **Equivariant homotopy type of algebraic cycles.** Let $(V, \mathbf{j})$ be a quaternionic vector space. An *algebraic cycle* on $\mathbb{P}(V)$ of codimension $q$ is a finite linear combination $\sum_{i=0}^{k} m_i A_i$, where $m_i \in \mathbb{Z}$ and $A_i \subset \mathbb{P}(V)$ is an irreducible subvariety of codimension $q$ in $\mathbb{P}(V)$. The following properties hold.

**Facts 3.1.**

**a:** The collection of algebraic cycles of codimension $q$ in $\mathbb{P}(V)$ forms an abelian topological group $\mathcal{Z}^q(\mathbb{P}(V))$ under addition of cycles; cf. [LF94].

**b:** The quaternionic structure $\mathbf{j}$ on $V$ induces a continuous involution $\mathbf{j}_* : \mathcal{Z}^q(\mathbb{P}(V)) \to \mathcal{Z}^q(\mathbb{P}(V))$ which is also a group homomorphism; cf. [LLFM98c]. This gives an action of $\mathbb{Z}/2$ on $\mathcal{Z}^q(\mathbb{P}(V))$ via group automorphisms. We reserve the word *equivariant* in the present context to mean $\mathbb{Z}/2$-equivariant under this quaternionic action.

**c:** There is a continuous degree homomorphism $\deg : \mathcal{Z}^q(\mathbb{P}(V)) \to \mathbb{Z}$ which assigns to a cycle $\sum_i n_i A_i$ the integer $\sum_i n_i \deg(A_i)$, where $\deg(A_i)$ is the degree of $A_i$ as a subvariety of $\mathbb{P}(V)$. For each $d \in \mathbb{Z}$, denote $\mathcal{Z}^q(\mathbb{P}(V))_d := \deg^{-1}(d)$ the subspace of cycles of degree $d$. Each $\mathcal{Z}^q(\mathbb{P}(V))_d$ is a connected component of $\mathcal{Z}^q(\mathbb{P}(V))$.

Given two quaternionic vector spaces $(V, \mathbf{j}_V)$ and $(W, \mathbf{j}_W)$, one has an equivariant external *join pairing*

$$\# : \mathcal{Z}^q(\mathbb{P}(V)) \times \mathcal{Z}^{q'}(\mathbb{P}(W)) \to \mathcal{Z}^{q+q'}(\mathbb{P}(V \oplus W)) \tag{25}$$

given by the ruled join of cycles. Roughly speaking, $\#$ is the bilinear extension of the following operation. Given an irreducible subvariety $A \subset \mathbb{P}(V)$ of codimension $q$, and an irreducible subvariety $B \subset \mathbb{P}(W)$ of codimension $q'$, let $A\#B$ be the irreducible subvariety of $\mathbb{P}(V \oplus W)$ obtained by taking the union of all projective lines in $\mathbb{P}(V \oplus W)$ joining points in $A$ to points in $B$, after taking the embeddings $A \subset \mathbb{P}(V) = \mathbb{P}(V \oplus 0) \subset \mathbb{P}(V \oplus W)$ and $B \subset \mathbb{P}(W) = \mathbb{P}(0 \oplus W) \subset \mathbb{P}(V \oplus W)$. We refer the reader to [LLFM96] for more details.

**Remark 3.2.** The degree of cycles is additive with respect to addition of cycles and multiplicative with respect to the join. In other words, given cycles $\sigma_1, \sigma_2 \in \mathcal{Z}^q(\mathbb{P}(V))$ and $\tau_1 \in \mathcal{Z}^q(\mathbb{P}(W))$, then

$$\deg(\sigma_1 + \sigma_2) = \deg \sigma_1 + \deg \sigma_2 \quad \text{and} \quad \deg(\sigma_1 \# \tau_1) = \deg \sigma_1 \deg \tau_1.$$

If one thinks of $\mathbb{P}(\mathbb{H})$ as an element in $\mathcal{Z}^0(\mathbb{P}(\mathbb{H}))$, one can use the join to define the quaternionic suspension map:

$$\begin{aligned} \not{\Sigma}_{\mathbb{H}} : \mathcal{Z}^q(\mathbb{P}(V)) &\to \mathcal{Z}^q(\mathbb{P}(V \oplus \mathbb{H})) \\ c &\mapsto c\#\mathbb{P}(\mathbb{H}). \end{aligned} \tag{26}$$

**Remark 3.3.**
(1) This definition parallels the construction of the complex suspension map $\not{\Sigma}_{\mathbb{C}} : \mathcal{Z}^q(\mathbb{P}(V)) \to \mathcal{Z}^q(\mathbb{P}(V \oplus \mathbb{C}))$ for a real vector space $(V, \mathbf{j})$, whose (non-equivariant) homotopy properties were first studied in [Law89]. Equivariant properties for this map, with respect to the complex conjugation involution, were studied in [Lam90], [LLFM98a], [LLFM98b], [dS99a], [Mos98].
(2) Another useful description of the suspension map is the following. Consider a *surjection* $f : V \to W$ of quaternionic vector spaces, and let $C \subset V$ be a (quaternionic) complement to the kernel $K := \ker f$, so that $V$ is the internal direct sum $K \oplus C$ with $C \cong W$. The

latter isomorphism induces a homomorphism $\mathcal{Z}^q(\mathbb{P}(W)) \cong \mathcal{Z}^q(\mathbb{P}(C)) \xrightarrow{\Sigma_K} \mathcal{Z}^q(\mathbb{P}(C \oplus K)) \cong \mathcal{Z}^q(\mathbb{P}(V))$, which is an equivariant homotopy equivalence. One can easily verify that the resulting "pull-back map" $f^* : \mathcal{Z}^q(\mathbb{P}(W)) \to \mathcal{Z}^q(\mathbb{P}(V))$ is independent of the choice of the complement $C$.

We now proceed to determine the equivariant homotopy type of spaces of algebraic cycles $\mathcal{Z}^q(\mathbb{P}(\mathbb{H}^n))$ of arbitrary codimension $q$ on $\mathbb{P}(\mathbb{H}^n)$. It is shown in [LLFM98a] that, given a quaternionic vector space $(V, \mathbf{j})$, the suspension homomorphism (26) $\Sigma_{\mathbb{H}} : \mathcal{Z}^q(\mathbb{P}(V)) \to \mathcal{Z}^q(\mathbb{P}(V \oplus \mathbb{H}))$ gives an equivariant homotopy equivalence. In particular, for $k < n$, one obtains equivariant homotopy equivalences:

$$\Sigma_{\mathbb{H}}^{n-k} : \mathcal{Z}^{2k-1}(\mathbb{P}(\mathbb{H}^k)) = \mathcal{Z}_0\left(\mathbb{P}(\mathbb{H}^k)\right) \to \mathcal{Z}^{2k-1}(\mathbb{P}(\mathbb{H}^n)) \tag{27}$$

and

$$\Sigma_{\mathbb{H}}^{n-k-1} : \mathcal{Z}^{2k}(\mathbb{P}(\mathbb{H}^{k+1})) = \mathcal{Z}_1(\mathbb{P}(\mathbb{H}^{k+1}) \to \mathcal{Z}^{2k}(\mathbb{P}(\mathbb{H}^n)) \tag{28}$$

Now, let $(V, \mathbf{j})$ be a quaternionic vector space, and recall that $\mathrm{Sym}_2(V)$ has a natural structure of a real vector space; cf. Remark 2.1. It follows that the image of $\mathbb{P}(V)$ under the Veronese embedding $\nu_2 : \mathbb{P}(V) \hookrightarrow \mathbb{P}(\mathrm{Sym}_2(V))$ given by $\mathcal{O}_{\mathbb{P}(V)}(2)$ becomes a real subvariety of $\mathbb{P}(\mathrm{Sym}_2(V))$. Define

$$Q(V) := \mathrm{T}\left(\mathcal{O}_{\mathbb{P}(V)}(2)\right). \tag{29}$$

It is clear that $Q(V)$ can be identified with the complex suspension

$$\Sigma_{\mathbb{C}}(\nu_2(\mathbb{P}(V))) = \nu_2(\mathbb{P}(V))\#p_\infty \subset \mathbb{P}(\mathrm{Sym}_2(V) \oplus \mathbb{C}),$$

where $p_\infty = \mathbb{P}(0 \oplus C) \in \mathbb{P}(\mathrm{Sym}_2(V) \oplus \mathbb{C})$.

It is shown in [LLFM98c, Prop. 6.1] that the complex suspension $\Sigma : \mathcal{Z}^q(\mathbb{P}(V)) \to \mathcal{Z}^q(Q(V))$ composed with the Veronese embedding $\mathbb{P}(V) \hookrightarrow \mathbb{P}(\mathrm{Sym}_2(V))$ induces an equivariant homotopy equivalence $\Sigma : \mathcal{Z}^q(\mathbb{P}(V)) \to \mathcal{Z}^q(Q(V))$. This fact, together with the equivalence $\mathcal{Z}^q(Q(\mathbb{H}^{k+1})) \cong \mathcal{Z}^q(Q(\mathbb{H}^k))$ proven in [LLFM98c] gives an equivalence

$$\mathcal{Z}^{2k}(\mathbb{P}(\mathbb{H}^{k+1})) \cong \mathcal{Z}_0\left(Q(\mathbb{H}^k)\right). \tag{30}$$

These observations imply the following result.

**Theorem 3.4.** *For $k < n$ there are canonical equivariant homotopy equivalences:*

$$\mathcal{Z}^{2k-1}(\mathbb{P}(\mathbb{H}^n)) \cong \prod_{j=1}^{k} F\left(\mathbb{P}(\mathbb{H})_+,\ K(\mathbb{Z}(2j-1),\ 4j-2)\ \right) \tag{31}$$

*and*

$$\mathcal{Z}^{2k}(\mathbb{P}(\mathbb{H}^n)) \cong \prod_{j=1}^{k} F\left(\mathbb{P}(\mathbb{H})_+,\ K(\mathbb{Z}(2j),\ 4j)\ \right) \tag{32}$$

*Proof.* The first equivalence follows from the equivalence $\mathcal{Z}^{2k-1}(\mathbb{P}(\mathbb{H}^n)) \cong \mathcal{Z}_0\left(\mathbb{P}(\mathbb{H}^k)\right)$ given by (27) and Corollary 2.4. To prove the second one, first consider the equivalences

$$\mathcal{Z}^{2k}(\mathbb{P}(\mathbb{H}^n)) \underset{(28)}{\cong} \mathcal{Z}^{2k}(\mathbb{P}(\mathbb{H}^{k+1})) = \mathcal{Z}_1(\mathbb{P}(\mathbb{H}^{k+1})) \underset{(30)}{\cong} \mathcal{Z}_0\left(Q(\mathbb{H}^k)\right) \tag{33}$$

and

$$\mathcal{Z}_0\left(Q(\mathbb{H}^k)\right) = \mathcal{Z}_0\left(\mathrm{T}\left(\mathcal{O}_{\mathbb{P}\left(\mathbb{H}^k\right)}(2)\right)\right) \cong \mathcal{Z}_0\left(S^{2,1} \wedge \mathbb{P}(\mathbb{H}^k)_+\right); \tag{34}$$

cf. (27), (28), (30) and Propositions 1.11 and 1.12.

Using Corollary 2.3 one obtains

$$\begin{aligned} \mathcal{Z}_0\left(S^{2,1} \wedge \mathbb{P}(\mathbb{H}^k)_+\right) &\cong \prod_{j=1}^{n} \mathcal{Z}_0\left(\ S^{2,1} \wedge \{\mathbb{P}(\mathbb{H}^j)/\mathbb{P}(\mathbb{H}^{j-1})\}_+\ \right)_{\mathrm{o}} \\ &= \prod_{j=1}^{n} \mathcal{Z}_0\left(\ S^{2,1} \wedge \{\mathrm{T}\left(\mathcal{O}_{\mathbb{P}(\mathbb{H})}(1) \otimes \mathbb{H}^{j-1}\right)\}_+\right)_{\mathrm{o}} \\ &= \prod_{j=1}^{n} \mathcal{Z}_0\left(\ \mathrm{T}\left(\{\mathcal{O}_{\mathbb{P}(\mathbb{H})}(1) \otimes \mathbb{H}^{j-1}\} \oplus \mathbf{1}\right)\right)_{\mathrm{o}}. \end{aligned} \tag{35}$$

Finally, the canonical equivalence $\mathcal{Z}_0\left(\ \mathrm{T}\left(\{\mathcal{O}_{\mathbb{P}(\mathbb{H})}(1) \otimes \mathbb{H}^{j-1}\} \oplus \mathbf{1}\right)\right)_{\mathrm{o}} \cong F\left(\mathbb{P}(\mathbb{H})_+,\ \mathcal{Z}_0\left(S^{4j,2j}\right)_{\mathrm{o}}\right)$ established in Propositions 1.11 and 1.12, along with (31), (32) and (33), proves the second assertion of the proposition. □

3.2. **Stabilizations of cycle spaces.** Here we use the group of algebraic cycles, with the quaternionic $\mathbb{Z}/2$ action defined above, to construct equivariant infinite loop spaces.

Consider a real vector space $(V, \sigma)$, of complex dimension $v$, and let $V^*$ denote its complex dual. Denote by $(V_{\mathbb{H}}, \mathbf{j})$ the quaternionic vector space $V_{\mathbb{H}} := V \otimes_{\mathbb{C}} \mathbb{H}$ with quaternionic structure induced by multiplication by $j$ on the right, and define

$$\mathcal{Z}(V) := \coprod_{j=-2v}^{2v} \mathcal{Z}^{2v+j}(V_{\mathbb{H}}^* \oplus V_{\mathbb{H}})_1, \tag{36}$$

where $\mathcal{Z}^{2v+j}(\mathbb{P}(V_{\mathbb{H}}^* \oplus V_{\mathbb{H}}))_1$ denotes the spaces of algebraic cycles of codimension $2v+j$ and degree 1 in $\mathbb{P}(V_{\mathbb{H}}^* \oplus V_{\mathbb{H}})$. Here we see $\mathcal{Z}(V)$ as a $\mathbb{Z}/2$-space under the quaternionic action induced by $j$ on $\mathbb{P}(V_{\mathbb{H}}^* \oplus V_{\mathbb{H}})$. The spaces $\mathcal{Z}(V)$ have a natural basepoint $\mathbf{1}_V := \mathbb{P}(V_{\mathbb{H}}^* \oplus \{0\}) \in \mathcal{Z}^{2v}(V_{\mathbb{H}}^* \oplus V_{\mathbb{H}})_1 \subset \mathcal{Z}(V)$. Define $\mathcal{Z}(0)$ to be the one-point set $\{\mathbf{1}_0\}$.

Given an inclusion of real vector spaces $i : (V, \sigma) \mapsto (W, \sigma')$, let $i^* : W^* \to V^*$ denote the adjoint surjection. The inclusion $i$ induces maps

$$(id \oplus i)_* \ : \ \mathcal{Z}^{2v+j}(\mathbb{P}(V_{\mathbb{H}}^* \oplus V_{\mathbb{H}}))_1 \to \mathcal{Z}^{2w+j}(\mathbb{P}(V_{\mathbb{H}}^* \oplus W_{\mathbb{H}}))_1,$$

given by the inclusion of cycles, and the surjection $i^*$ induces maps

$$(i^* \oplus id)^* \ : \ \mathcal{Z}^{2w+j}(\mathbb{P}(V_{\mathbb{H}}^* \oplus W_{\mathbb{H}}))_1 \to \mathcal{Z}^{2w+j}(\mathbb{P}(W_{\mathbb{H}}^* \oplus W_{\mathbb{H}}))_1,$$

given by the appropriate pull-back of cycles; cf. Remark 3.3. Define

$$i_\sharp \ : \ \coprod_{j=-2v}^{2v} \mathcal{Z}^{2v+j}(\mathbb{P}\big(V_{\mathbb{H}}^* \oplus V_{\mathbb{H}}\big))_1 \to \coprod_{j=-2w}^{2w} \mathcal{Z}^{2w+j}(\mathbb{P}\big(W_{\mathbb{H}}^* \oplus W_{\mathbb{H}}\big))_1 \tag{37}$$

as the composition $(i^* \oplus id)^* \circ (id \oplus i)_*$. This makes $\mathcal{Z}(-)$ into a functor from the category of finite dimensional real vector spaces and linear monomorphisms, to the category of pointed $\mathbb{Z}/2$-spaces.

The functor $\mathcal{Z}(-)$ comes with an additional structure, an equivariant "Whitney sum" pairing, defined as follows. Given real vector spaces $(V, \sigma)$ and $(W, \sigma')$, define

$$\#_{V,W} \ : \ \mathcal{Z}(V) \times \mathcal{Z}(W) \to \mathcal{Z}(V \oplus W) \tag{38}$$

as the map whose restriction to the components is the composition

$$\begin{aligned} \mathcal{Z}^{2v+j}(\mathbb{P}\big(V_{\mathbb{H}}^* \oplus V_{\mathbb{H}}\big))_1 \times \mathcal{Z}^{2w+k}(\mathbb{P}\big(W_{\mathbb{H}}^* \oplus W_{\mathbb{H}}\big))_1 \xrightarrow{\#} & \\ \mathcal{Z}^{2(w+v)+j+k}(\mathbb{P}\big(V_{\mathbb{H}}^* \oplus V_{\mathbb{H}} \oplus W_{\mathbb{H}}^* \oplus W_{\mathbb{H}}\big))_1 & \\ \xrightarrow{\tau} \mathcal{Z}^{2(w+v)+j+k}(\mathbb{P}\big((V \oplus W)_{\mathbb{H}}^* \oplus (V \oplus W)_{\mathbb{H}}\big))_1, & \end{aligned} \tag{39}$$

where $\#$ is the join pairing (38). Here $\tau$ denotes the map on cycles induced by the composition of linear isomorphisms $V_{\mathbb{H}}^* \oplus V_{\mathbb{H}} \oplus W_{\mathbb{H}}^* \oplus W_{\mathbb{H}} \to (V_{\mathbb{H}}^* \oplus W_{\mathbb{H}}^*) \oplus (V_{\mathbb{H}} \oplus W_{\mathbb{H}}) \to (V \oplus W)_{\mathbb{H}}^* \oplus (V \oplus W)_{\mathbb{H}}$, where the former map is a shuffle isomorphism and the latter is the usual natural identification. Define $\#_{0,W}$ to be the identity map, after the natural identification $\mathcal{Z}(\mathbb{P}\big(0 \oplus V\big)) \equiv \mathcal{Z}(\mathbb{P}\big(V\big))$.

**Proposition 3.5.** *The assignments $V \mapsto \mathcal{Z}(V)$ along with the pairings $\#_{V,W}$ give $\mathcal{Z}(-)$ the structure of an equivariant $(\mathbb{Z}/2)\mathcal{I}^*$-functor, in the language of* [May77]. *See also* [LLFM96].

*Proof.* This amounts to checking various coherence properties, and proceeds exactly as in the non-equivariant case done in [BLLF+93], or in the equivariant study of real algebraic cycles, done in [LLFM98b]. ☐

Now, consider $(\mathbb{C}^\infty, \sigma)$ as a real vector space under complex conjugation $\sigma$, and observe that it is a complete $\mathbb{Z}/2$-universe; cf. [May96]. In other words, it contains infinitely countably many copies of each irreducible representation of $\mathbb{Z}/2$. Define $\mathcal{Z}_{\mathbb{H}}$ as the colimit

$$\mathcal{Z}_{\mathbb{H}} := \varinjlim_{V \subset \mathbb{C}^\infty} \mathcal{Z}(V), \tag{40}$$

where $V$ runs over all real subspaces of $\mathbb{C}^\infty$.

**Theorem 3.6.** *The space $\mathcal{Z}_{\mathbb{H}}$ is an equivariant $\mathbb{Z}/2$-infinite loop space. In other words, for each real $\mathbb{Z}/2$-module $V$ there is a $\mathbb{Z}/2$-space $\mathcal{Z}_{\mathbb{H},V}$ along with coherent equivariant homotopy equivalences*

$$\mathcal{Z}_{\mathbb{H}} = \mathcal{Z}_{\mathbb{H},0} \cong \Omega^V \mathcal{Z}_{\mathbb{H},V}.$$

**Remark 3.7.** Given any based $\mathbb{Z}/2$-space $X$ and $\mathbb{Z}/2$-module $V$, the space $\Omega^V X$ of $V$-fold loops in $X$ is the space $F(S^V, X)$ of based maps from the one-point compactification $S^V$ of $V$ into $X$, with its usual topology and the usul $\mathbb{Z}/2$ action on function spaces. The coherence properties mentioned in the statement above are the ones that define an equivariant infinite loop space; cf. [May96].

**Definition 3.8.** The infinite loop space structure on $\mathcal{Z}_{\mathbb{H}}$ determines an equivariant spectrum (cf. [May96]) denoted $\mathfrak{Z}_{\mathbb{H}}$, satisfying $\mathfrak{Z}_{\mathbb{H}}(0) \cong \mathcal{Z}_{\mathbb{H}}$.

Before proving the theorem, let us analyze the space $\mathcal{Z}_{\mathbb{H}}$ in detail. Since the increasing coordinate flag $\{0\} \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^\infty$ is cofinal among the finite dimensional subspaces of $\mathbb{C}^\infty$, one observes that

$$\mathcal{Z}_{\mathbb{H}} = \varinjlim_n \mathcal{Z}(\mathbb{C}^n) = \varinjlim_n \coprod_{j=-2n}^{2n} \mathcal{Z}^{2n+j}(\mathbb{P}(\mathbb{H}^{n*} \oplus \mathbb{H}^n))_1.$$

Here we are making the usual identification $\mathbb{C}^n_{\mathbb{H}} = \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{H} \equiv \mathbb{H}^n$. The maps $i_\sharp$ in the colimit above are compositions of algebraic suspensions and coordinatewise inclusions. Therefore, they are compatible with the splittings given in Theorem 3.4.

**Theorem 3.9.** *The space $\mathcal{Z}_{\mathbb{H}}$ is written as a disjoint union of connected spaces*

$$\mathcal{Z}_{\mathbb{H}} = \coprod_{j=-\infty}^{\infty} \mathcal{Z}^j_{\mathbb{H}},$$

*where the equivariant homotopy type of $\mathcal{Z}^j_{\mathbb{H}}$ is totally determined by*

$$\mathcal{Z}^j_{\mathbb{H}} \cong \begin{cases} \prod_{j=1}^\infty \ F\left(\mathbb{P}(\mathbb{H})_+,\ K(\mathbb{Z}(2j-1),\ 4j-2)\ \right) & \text{, if } j \text{ is odd} \\ \prod_{j=1}^\infty \ F\left(\mathbb{P}(\mathbb{H})_+,\ K(\mathbb{Z}(2j),\ 4j)\ \right) & \text{, if } j \text{ is even.} \end{cases} \tag{41}$$

*Proof.* It is evident that if one defines

$$\mathcal{Z}^j_{\mathbb{H}} := \varinjlim_n \mathcal{Z}^{j+2n}(\mathbb{P}(\mathbb{H}^{n*} \oplus \mathbb{H}^n)), \tag{42}$$

then $\mathcal{Z}_{\mathbb{H}} = \coprod_{j=-\infty}^\infty \mathcal{Z}^j_{\mathbb{H}}$. The result now follows from the remark preceding the Proposition together with Theorem 3.4. □

**Remark 3.10.** Note the the equivariant homotopy type of $\mathcal{Z}^j_{\mathbb{H}}$ is completely determined by the parity of $j$. Furthermore, a canonical inclusion of coordinate hyperplanes gives immediately an equivariant homotopy equivalence $\mathcal{Z}^j_{\mathbb{H}} \cong \mathcal{Z}^{j+2}_{\mathbb{H}}$, for all $j$. For that reason we establish the notation

$$\mathcal{Z}^{\mathrm{ev}}_{\mathbb{H}} := \text{ the equivariant homotopy type of } \mathcal{Z}^j_{\mathbb{H}}, \text{ for } j \text{ even,}$$

and

$$\mathcal{Z}^{\mathrm{odd}}_{\mathbb{H}} := \text{ the equivariant homotopy type of } \mathcal{Z}^j_{\mathbb{H}}, \text{ for } j \text{ odd.}$$

We now proceed to prove Theorem 3.6.

*Proof.* The same arguments of [BLLF+93], [LLFM96], or [LLFM98b], together with Proposition 3.5, imply that the join operation induces an action of the equivariant $\mathbb{Z}/2$-linear isometries operad on $\mathcal{Z}_{\mathbb{H}}$. In particular, both $\mathcal{Z}_{\mathbb{H}}$ and its fixed point set $\mathcal{Z}^{\mathbb{Z}/2}_{\mathbb{H}}$ have an induced action of the usual linear isometries operad.

It follows from (38) that the $H$-space structure on $\mathcal{Z}_{\mathbb{H}}$ given by the join induces a group isomorphism $\pi_0(\mathcal{Z}_{\mathbb{H}}) \cong \mathbb{Z}$. Furthermore, it is easy to see (cf. [LLFM98c]) that $\left\{\mathcal{Z}^{j+2n}(\mathbb{P}(\mathbb{H}^{n*} \oplus \mathbb{H}^n))_1\right\}^{\mathbb{Z}/2}$ is empty if $j$ is odd, and non-empty and connected if $j$ is even. This implies, after passage to the colimit, that

$$\pi_0(\mathcal{Z}^{\mathbb{Z}/2}_{\mathbb{H}}) \cong 2\mathbb{Z} \subset \mathbb{Z} \cong \pi_0(\mathcal{Z}_{\mathbb{H}}).$$

Therefore, the $H$-space $\mathcal{Z}_{\mathbb{H}}$ is $\mathbb{Z}/2$-group complete, in the language of [CW91]. It follows from the equivariant "recognition principle" in [CW91] that $\mathcal{Z}_{\mathbb{H}}$ is an equivariant infinite loop space; see also [dSLF01]. □

**Remark 3.11.** We must point out that the construction of our $\mathcal{I}^*$-functor $\mathcal{Z}(-)$, in Proposition 3.5, only uses embeddings $V \hookrightarrow W$, not *isometric* embeddings of hermitian vector spaces. As a consequence, we can replace the linear isometries operad $\mathcal{L}$ by the *linear embeddings operad*

$$\{ \ \mathcal{E}(n) := \mathrm{Emb}(\mathbb{C}^\infty \oplus \cdots \oplus \mathbb{C}^\infty \ , \ \mathbb{C}^\infty) \mid n \in \mathbb{N} \ \}.$$

This operad has the advantage of being algebraic in nature and is more suitable for possible "motivic" generalizations and various other applications to algebraic geometry.

## 4. Quaternionic $K$-theory

Given a $\mathbb{Z}/2$-space $X$, one can use its equivariant structure to study two classes of complex bundles over $X$, namely the real and quaternionic bundles. This study yields two distinct, albeit related, equivariant theories: the *Real $K$-theory* studied by Atiyah in [Ati66] and the (equivariant) quaternionic $K$-theory studied by Dupont in [Dup69].

Let us recall the basic definitions.

**Definition 4.1.** Let $(X, \sigma)$ be a $\mathbb{Z}/2$-space and let $p : E \to X$ be a complex vector bundle over $X$. Let $\tau : E \to E$ be a continuous map covering $\sigma$, i.e. $p \circ \tau = \sigma \circ p$, and such that for any $x \in X$ the resulting map $\tau : E_x \to E_{\sigma x}$ is anti-linear.

**a:** If $\tau^2 = \mathrm{id}$ then $(E, \tau)$ is a *real bundle*;

**b:** If $\tau^2 = -\,\mathrm{id}$ then $(E, \tau)$ is a *quaternionic bundle*.

The dimension of a real or quaternionic bundle is defined as its complex dimension, and morphisms between such bundles are bundle morphisms that commute with the structure maps $\tau$.

The isomorphism classes real bundles form a monoid under Whitney sum, whose Grothendieck group is called the *real $K$-theory* $KR^0(X)$ of $X$. Similarly, the Grothendieck group of isomorphism classes of quaternionic bundles gives the *quaternionic $K$-theory* groups $KH^0(X)$ of $X$.

In [LLFM98b] the connective version of $KR$-theory is studied from the equivariant point of view, along with a suitable theory of equivariant Chern classes. In this section and the next, we provide quaternionic analogues, along with a suitable theory of equivariant Chern classes and their equivariant deloopings. We must point out that in [Dup99], Dupont conjectures the existence of an appropriate theory of Chern classes for quaternionic bundles. Here we provide a quite natural answer to his question.

### 4.1. **Classifying spaces and equivariant quaternionic $K$-theory spectrum.**

In this section we describe a classifying space for quaternionic $K$-theory in the equivariant category, and at the same time we prove the existence of equivariant deloopings of this space.

Given a real vector space $(V, \sigma)$ of complex dimension $v$, we follow (36) and define

$$\mathrm{Gr}(V) := \coprod_{j=-2v}^{2v} \mathrm{Gr}^{2v+j}(V^*_{\mathbb{H}} \oplus V_{\mathbb{H}}). \tag{43}$$

We make the assignment $V \mapsto \mathrm{Gr}(V)$ functorial as follows. Consider an inclusion $i : W \hookrightarrow V$ of real vector spaces of dimensions $w$ and $v$, respectively. As in the previous section, we observe that $i$ induces an inclusion $(id \oplus i)_* \ : \ \mathrm{Gr}^{2w+j}(W^*_{\mathbb{H}} \oplus W_{\mathbb{H}}) \to \mathrm{Gr}^{2v+j}(W^*_{\mathbb{H}} \oplus V_{\mathbb{H}})$ given by the inclusion of linear subspaces, and that the surjection $i^* : V^* \to W^*$ induces another inclusion $(i^* \oplus id)^{-1} \ :$

$\mathrm{Gr}^{2v+j}(W_{\mathbb{H}}^{*}\oplus V_{\mathbb{H}})\to \mathrm{Gr}^{2v+j}(V_{\mathbb{H}}^{*}\oplus V_{\mathbb{H}})$, given by taking inverse images. Let $i_{\sharp}:\mathrm{Gr}(W)\to \mathrm{Gr}(V)$ be the map given, on each connected component, by the composition

$$i_{\sharp} := (i^{*}\oplus id)^{-1}\circ (id\oplus i)_{*}. \tag{44}$$

Given any space $X$, and any real vector space $(V,\sigma)$, we denote by $\underline{V}_{\mathbb{H}}$ the quaternionic vector bundle $X\times V_{\mathbb{H}}$ over $X$. Now, let $\xi_V^j$ denote the universal quotient bundle over $\mathrm{Gr}^{2v+j}(V_{\mathbb{H}}^{*}\oplus V_{\mathbb{H}})$. The proof of the following result is standard.

**Lemma 4.2.** *The map on Grassmannians $i_{\sharp}: Gr^{2w+j}(W_{\mathbb{H}}^{*}\oplus W_{\mathbb{H}})\to Gr^{2v+j}(V_{\mathbb{H}}^{*}\oplus V_{\mathbb{H}})$, induced by an inclusion $i:W\hookrightarrow V$, satisfies*

$$(i_{\sharp})^{*}\xi_V^j = \xi_W^j\oplus \underline{(V/W)}_{\mathbb{H}}.$$

**Definition 4.3.** Define the $(\mathbb{Z}/2)$-space $(\mathbb{Z}\times BU)_{\mathbb{H}}$ as the colimit

$$(\mathbb{Z}\times BU)_{\mathbb{H}} := \varinjlim_{V\subset\mathbb{C}^\infty} \mathrm{Gr}(V). \tag{45}$$

Observe that, in the same fashion as $\mathcal{Z}_{\mathbb{H}}$ (cf. Theorem 3.9), the space $(\mathbb{Z}\times BU)_{\mathbb{H}}$ can be written as a disjoint union $\coprod_{j=-\infty}^{\infty}\mathrm{BU}_{\mathbb{H}}^{j}$ of $(\mathbb{Z}/2)$-spaces $\mathrm{BU}_{\mathbb{H}}^{j}$ defined as $\mathrm{BU}_{\mathbb{H}}^{j} := \varinjlim_{V\subset\mathbb{C}^\infty} \mathrm{Gr}^{2v+j}(V_{\mathbb{H}}^{*}\oplus V_{\mathbb{H}})$. Furthermore, the coordinate-wise inclusion $\mathbb{C}^\infty\hookrightarrow\mathbb{C}^\infty$, given by setting the first coordinate zero, induces equivariant homotopy equivalences

$$\mathrm{BU}_{\mathbb{H}}^{j}\cong \mathrm{BU}_{\mathbb{H}}^{j+2}\cong\cdots\cong \mathrm{BU}_{\mathbb{H}}^{j+2r}\cong\cdots. \tag{46}$$

Hence all connected components of $(\mathbb{Z}\times BU)_{\mathbb{H}}$ are either equivalent to $\mathrm{BU}_{\mathbb{H}}^{0}$ or to $\mathrm{BU}_{\mathbb{H}}^{1}$, and we denote by $\mathrm{BU}_{\mathbb{H}}^{ev}$ and $\mathrm{BU}_{\mathbb{H}}^{odd}$ their respective equivariant homotopy types.

**Proposition 4.4.** *The space $(\mathbb{Z}\times BU)_{\mathbb{H}}$ classifies quaternionic $K$-theory. In other words, given a compact $(\mathbb{Z}/2)$-space $X$ one has a natural isomorphism*

$$[X_+,\ (\mathbb{Z}\times BU)_{\mathbb{H}}\ ]_{\mathbb{Z}/2}\xrightarrow{\cong} KH^0(X).$$

*Proof.* The proof follows standard arguments, as in [Seg68], and we only outline the details which are particular to this case.

Let $(X,\sigma)$ be a compact $(\mathbb{Z}/2)$-space. Given an equivariant map $f:X\to(\mathbb{Z}\times BU)_{\mathbb{H}}$, one can find a real subspace $W\subset\mathbb{C}^\infty$, $\dim W=w$, so that the map $f$ factors as a composition

$$X\xrightarrow{f_W}\mathrm{Gr}(W)\xrightarrow{i_W}\amalg_{j=-2w}^{2w}\mathrm{BU}_{\mathbb{H}}^{j}\ \subset\ (\mathbb{Z}\times BU)_{\mathbb{H}},$$

where the $i_W$'s are the natural maps from the directed system defining $(\mathbb{Z}\times BU)_{\mathbb{H}}$.

Now, assign to $f$ the isomorphism class of the virtual bundle $f_W^{*}(\xi_W)-\underline{W}_{\mathbb{H}}$, where $\xi_W$ is the bundle over $\mathrm{Gr}(W)=\amalg_{j=-2w}^{2w}\ \mathrm{Gr}^{2w+j}(W_{\mathbb{H}}^{*}\oplus W_{\mathbb{H}})$ whose restriction to the component $\mathrm{Gr}^{2w+j}(W_{\mathbb{H}}^{*}\oplus W_{\mathbb{H}})$ is the universal quotient bundle $\xi_W^j$. If $i:W\hookrightarrow V\subset\mathbb{C}^\infty$ is an inclusion, then it follows from the construction of $\mathrm{BU}_{\mathbb{H}}^{j}$ that $f_V=i_{\sharp}\circ f_W$, and hence one has equalities in $KH^0(X)$:

$$\begin{aligned} f_V^{*}(\xi_V)\ -\ \underline{V}_{\mathbb{H}} &= f_W^{*}(i_{\sharp}^{*}(\xi_V))\ -\ \underline{V}_{\mathbb{H}} = \left(f_W^{*}(\xi_W)\oplus\underline{(V/W)}_{\mathbb{H}}\right)\ -\ \underline{V}_{\mathbb{H}}\\ &= f_W^{*}(\xi_W)+\underline{(V/W)}_{\mathbb{H}}\ -\ \underline{V}_{\mathbb{H}} = f_W^{*}(\xi_W)\ -\ \underline{W}_{\mathbb{H}}, \end{aligned}$$

where the second equality comes from Lemma 4.2. This shows that the element in $KH^0(X)$ thus obtained is independent of the factorization through a finite dimensional Grassmannian. Standard arguments, e.g. [Seg68], show that this assignment only depends on the equivariant homotopy class of $f$ and that the resulting map $\Psi : [X_+,\ (\mathbb{Z}\times BU)_{\mathbb{H}}\ ]_{\mathbb{Z}/2} \to KH^0(X)$ is injective.

In order to prove surjectivity, let $(E,\tau)$ be a quaternionic bundle over $(X,\sigma)$. One can find sections $s_i : X \to E, i = 1,\dots,k$ generating $E$, i.e., for each $x\in X$ the map $\varphi_x : \mathbb{C}^k \to E_x$ sending $(\lambda_1,\dots,\lambda_k)$ to $\sum_{i=1}^k \lambda_i\ s_i(x)$ is surjective. Now, define

$$\phi : \mathbb{H}^n \times X \to E$$

by sending $(a_1+b_1 j,\dots,a_k+b_k j;\ x)$ to $\sum_{i=1}^k\ a_i s_i(x)\ +\sum_{i=1}^k\ b_i\tau(s_i(\sigma x)) \in E_x$. It is clear that $\phi$ is onto, by construction, and that $\phi$ is a map of quaternionic bundles for the diagonal quaternionic structure on $\mathbb{H}^n\times X$ given by right multiplication by $j$ on the first factor, and by $\sigma$ on the second factor. It follows that the map $f : X \to \mathrm{Gr}^{\dim E}(\mathbb{H}^k)$ defined as $f(x) = \ker(\phi_x)$ is equivariant and satisfies $f^*Q \cong E$, where $Q$ is the universal quotient bundle over $\mathrm{Gr}^{\dim E}(\mathbb{H}^k)$. A little manipulation with $f$ and the directed system giving $(\mathbb{Z}\times BU)_{\mathbb{H}}$ then shows that $\Psi$ is onto. □

**Definition 4.5.** For each $j\in\mathbb{Z}$, we denote by $\xi^j$ the virtual universal quotient bundle over $\mathrm{BU}^j_{\mathbb{H}}$, of virtual dimension $j$, whose restriction to $\mathrm{Gr}^{2v+j}(V_{\mathbb{H}}^*\oplus V_{\mathbb{H}})$ is $\xi^j_V - V_{\mathbb{H}}$.

One must notice that the construction of $\mathrm{Gr}(V)$ here parallels (in fact it precedes) that of $\mathcal{Z}(V)$, given in (36). Furthermore, given real vector spaces $(V,\sigma)$, $(W,\sigma')$ one can define

$$\oplus_{V,W}\ :\ \mathrm{Gr}(V)\times\mathrm{Gr}(W)\to\mathrm{Gr}(V\oplus W) \tag{47}$$

by sending $L\subset V_{\mathbb{H}}^*\oplus V_{\mathbb{H}}$ and $L'\subset W_{\mathbb{H}}^*\oplus W_{\mathbb{H}}$ to $\tau(L\oplus L)$, where $\tau$ is the shuffle map which switches coordinates from $(V_{\mathbb{H}}^*\oplus V_{\mathbb{H}})\ \oplus\ (W_{\mathbb{H}}^*\oplus W_{\mathbb{H}})$ to $(V_{\mathbb{H}}\oplus W_{\mathbb{H}})^*\ \oplus\ (V_{\mathbb{H}}\oplus W_{\mathbb{H}})$. The following proposition is analogous to Proposition 3.5 and is proven in a similar fashion.

**Proposition 4.6.** *The assignments* $V\mapsto\mathrm{Gr}(V)$ *along with the pairings* $\oplus_{V,W}$ *give* $\mathrm{Gr}(-)$ *the structure of an equivariant* $(\mathbb{Z}/2)\mathcal{I}^*$*-functor, in the language of* [May77]*. See also* [LLFM96].

The proof of the following result is identical to the proof of Theorem 3.6.

**Theorem 4.7.** *The direct sum operation induces an equivariant* $\mathbb{Z}/2$*-infinite loop space structure on the space* $(\mathbb{Z}\times BU)_{\mathbb{H}}$.

**Definition 4.8.** The infinite loop space structure on $(\mathbb{Z}\times BU)_{\mathbb{H}}$ determines an equivariant spectrum (cf. [May96]) denoted $\mathfrak{Ksp}$, satisfying $\mathfrak{Ksp}(0)\cong(\mathbb{Z}\times BU)_{\mathbb{H}}$. This is the connective *quaternionic* $K$*-theory spectrum.*

**Remark 4.9.** The same construction with $V_{\mathbb{C}}$ replacing $V_{\mathbb{H}}$, and with the complex conjugation action on $V_{\mathbb{C}}$, would give $(\mathbb{Z}\times BU)_{\mathbb{C}}$ along with the equivariant infinite loop space structure classifying KR()-theory.

Let $(V,\sigma)$ be a real vector space. An important feature of our constructions is the fact that a complex linear subspace $L$ of codimension $2v+j$ in $V_{\mathbb{H}}^*\oplus V_{\mathbb{H}}$ is also an irreducible subvariety of codimension $2v+j$ and degree 1 in $\mathbb{P}\big(V_{\mathbb{H}}^*\oplus V_{\mathbb{H}}\big)$. Furthermore, the external direct sum of two such subspaces corresponds to their algebraic join when seen as projective subvarieties. See [BLLF$^+$93] for more details. This gives a natural transformation $c:\mathcal{Z}(-)\to\mathrm{Gr}(-)$ of $(\mathbb{Z}/2)\mathcal{I}_*$-functors.

Standard arguments, such as in [BLLF+93], show that the resulting map of colimits

$$c_{\mathbb{H}} \; : \; (\mathbb{Z}\times BU)_{\mathbb{H}} \to \mathcal{Z}_{\mathbb{H}} \tag{48}$$

is a map of equivariant infinite loop spaces. In other words, it induces a map of equivariant spectra $c_{\mathbb{H}} \; : \; \mathfrak{Ksp} \to \mathfrak{Z}_{\mathbb{H}}$.

**Definition 4.10.** The map $c_{\mathbb{H}} \; : \; \mathfrak{Ksp} \to \mathfrak{Z}_{\mathbb{H}}$ is called the *total quaternionic Chern class map.*

In order to analyze the quaternionic Chern class map in the level of classifying spaces, we first need to understand the equivariant cohomology of $(\mathbb{Z}\times BU)_{\mathbb{H}} \times \mathbb{P}(\mathbb{H})$. This computation and the subsequent analysis form the content of our next section.

## 5. Characteristic Classes

In this section we introduce characteristic classes for quaternionic bundles and establish their relation to the total quaternionic Chern class map $c_{\mathbb{H}} \; : \; (\mathbb{Z}\times BU)_{\mathbb{H}} \to \mathcal{Z}_{\mathbb{H}}$, in the level of classifying spaces.

The characteristic classes $d_k(E) \in H^{2k,k}(X\times\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$, associated to a quaternionic bundle $E\to X$ are defined as follows.

**Definition 5.1.** Let $E$ be a rank $e$ quaternionic bundle over a $\mathbb{Z}/2$-space $X$. For $\delta = 0, 1$, and $i$ satisfying $0 \le 2i+\delta \le e$, define

$$d_{e-(2i+\delta)}(E) := \widetilde{c}_{e-(2i+\delta)}(E\otimes\mathcal{O}(1)) - (i+\delta)\; \widetilde{c}_{e-(2i+\delta)-1}(E\otimes\mathcal{O}(1))\;\mathbf{x},$$

where $c_{-1}(-) = 0$.

Note that, since $X\times\mathbb{P}(\mathbb{H})$ is a free $\mathbb{Z}/2$-space, the characteristic classes $\tilde{c}_k(E\otimes\mathcal{O}(1))$ can also be defined with values in Galois-Grothendieck cohomology as in [Kah87], according to the discussion in Section 1.7.

In order to understand the meaning of this definition, and its relation to the total Chern class map, we first compute the equivariant cohomology of $(\mathbb{Z}\times BU)_{\mathbb{H}}\times\mathbb{P}(\mathbb{H})$ in the next section.

### 5.1. **Cohomology of** $(\mathbb{Z}\times BU)_{\mathbb{H}}$.

We start by observing that all the components of $(\mathbb{Z}\times BU)_{\mathbb{H}}\times\mathbb{P}(\mathbb{H})$ have the same equivariant homotopy type, hence it suffices to compute $H^{*,*}(\mathrm{BU}^{ev}_{\mathbb{H}}\times\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$. In fact, an equivariant homotopy equivalence $\Psi \; : \; \mathrm{BU}^{ev}_{\mathbb{H}}\times\mathbb{P}(\mathbb{H}) \to \mathrm{BU}^{odd}_{\mathbb{H}}\times\mathbb{P}(\mathbb{H})$ can be constructed as follows. Given a real vector space $(V,\sigma)$, let

$$\psi_V^{j,j+1} \; : \; \mathrm{Gr}^{2v+j}(V^*_{\mathbb{H}}\oplus V_{\mathbb{H}})\times\mathbb{P}(\mathbb{H}) \to \mathrm{Gr}^{2v+j+1}((V\oplus\mathbb{C})^*_{\mathbb{H}}\oplus(V\oplus\mathbb{C})_{\mathbb{H}})$$

denote the composition

$$\begin{aligned}\mathrm{Gr}^{2v+j}(V^*_{\mathbb{H}}\oplus V_{\mathbb{H}})\times\mathbb{P}(\mathbb{H}) \xrightarrow{id\times\iota} \mathrm{Gr}^{2v+j}(V^*_{\mathbb{H}}\oplus V_{\mathbb{H}})\times\mathrm{Gr}^1(\mathbb{H}^*\oplus\mathbb{H}) \xrightarrow{\oplus_{V,\mathbb{C}}} \\ \mathrm{Gr}^{2v+j+1}((V\oplus\mathbb{C})^*_{\mathbb{H}}\oplus(V\oplus\mathbb{C})_{\mathbb{H}}),\end{aligned} \tag{49}$$

where $\iota : \mathbb{P}(\mathbb{H}) \to \mathrm{Gr}^1(\mathbb{H}^*\oplus\mathbb{H})$ is the linear embedding sending $L$ to $L\oplus\mathbb{H}$, and $\oplus_{V,\mathbb{C}}$ is the Whitney sum map (47) after one identifies $\mathbb{H}\equiv\mathbb{C}_{\mathbb{H}}$.

It is easy to see that

$$(\psi_V^{j,j+1})^*(\xi^{j+1}_{V\oplus\mathbb{C}}) = \pi_1^*(\xi^j_V)\oplus\underline{\mathbb{H}}\oplus\pi_2^*(\mathcal{O}(1)),$$

where the $\pi_i$'s are the projections from $\mathrm{Gr}^{2v+j}(V^*_{\mathbb{H}}\oplus V_{\mathbb{H}})\times\mathbb{P}(\mathbb{H})$ to the respective factors, $\xi^j_V$ is the universal quotient bundle over the Grassmannian, and $\mathcal{O}(1)$ is the hyperplane bundle over $\mathbb{P}(\mathbb{H})$.

Note that $\mathcal{O}(1)$ is also a quaternionic bundle. Furthermore, the maps $\psi_V^{j,j+1}$ assemble to give a morphism of directed systems inducing a $(\mathbb{Z}/2)$-equivariant map $\psi^{j,j+1} : \mathrm{BU}_{\mathbb{H}}^{j} \times \mathbb{P}(\mathbb{H}) \to \mathrm{BU}_{\mathbb{H}}^{j+1}$ having the property that $(\psi^{j,j+1})^*(\xi^{j+1}) = \xi^j \times \mathcal{O}(1)$. Here we denote $\pi_1^*(\xi^j) \oplus \pi_2^*(\mathcal{O}(1))$ as a product $\xi^j \times \mathcal{O}(1)$.

It follows that one obtains a $\mathbb{Z}/2$-map $\psi : \mathrm{BU}_{\mathbb{H}}^{ev} \times \mathbb{P}(\mathbb{H}) \to \mathrm{BU}_{\mathbb{H}}^{odd}$. Now, define

$$\Psi := \psi \times \mathrm{id} : \mathrm{BU}_{\mathbb{H}}^{ev} \times \mathbb{P}(\mathbb{H}) \to \mathrm{BU}_{\mathbb{H}}^{odd} \times \mathbb{P}(\mathbb{H}), \tag{50}$$

and observe that $\Psi$ is a $\mathbb{Z}/2$-map which is a non-equivariant homotopy equivalence. Since $\mathrm{BU}_{\mathbb{H}}^{ev} \times \mathbb{P}(\mathbb{H})$ and $\mathrm{BU}_{\mathbb{H}}^{odd} \times \mathbb{P}(\mathbb{H})$ are free $(\mathbb{Z}/2)$-spaces, it then follows that $\Psi$ is an equivariant homotopy equivalence.

**Remark 5.2.** The same procedure used to construct $\Psi$ can be applied to produce an equivariant homotopy equivalence $\Phi : \mathcal{Z}_{\mathbb{H}}^{\mathrm{ev}} \times \mathbb{P}(\mathbb{H}) \to \mathcal{Z}_{\mathbb{H}}^{\mathrm{odd}} \times \mathbb{P}(\mathbb{H})$. Indeed, given a real vector space $(V, \sigma)$, let

$$\phi_V^{j,j+1} \;:\; \mathcal{Z}^{2v+j}(V_{\mathbb{H}}^* \oplus V_{\mathbb{H}}) \times \mathbb{P}(\mathbb{H}) \to \mathcal{Z}^{2v+j+1}((V \oplus \mathbb{C})_{\mathbb{H}}^* \oplus (V \oplus \mathbb{C})_{\mathbb{H}})$$

denote the composition

$$\begin{aligned} \mathcal{Z}^{2v+j}(V_{\mathbb{H}}^* \oplus V_{\mathbb{H}}) \times \mathbb{P}(\mathbb{H}) \xrightarrow{id \times \iota} \mathcal{Z}^{2v+j}(V_{\mathbb{H}}^* \oplus V_{\mathbb{H}}) \times \mathcal{Z}^1(\mathbb{H}^* \oplus \mathbb{H}) \xrightarrow{\#_{V,\mathbb{C}}} \\ \mathcal{Z}^{2v+j+1}((V \oplus \mathbb{C})_{\mathbb{H}}^* \oplus (V \oplus \mathbb{C})_{\mathbb{H}}), \end{aligned} \tag{51}$$

As before, one can check that the maps $\phi_V^{j,j+1}$ assemble to give a morphism of directed systems inducing a $(\mathbb{Z}/2)$-equivariant map $\phi^{j,j+1} : \mathcal{Z}_{\mathbb{H}}^{j} \times \mathbb{P}(\mathbb{H}) \to \mathcal{Z}_{\mathbb{H}}^{j+1}$. Finally, define

$$\Phi := \phi \times \mathrm{id} : \mathcal{Z}_{\mathbb{H}}^{\mathrm{ev}} \times \mathbb{P}(\mathbb{H}) \to \mathcal{Z}_{\mathbb{H}}^{\mathrm{odd}} \times \mathbb{P}(\mathbb{H}).$$

Non-equivariantly, one can fix a point $t \in \mathbb{P}(\mathbb{H})$, and observe that the map $\Phi(-, t)$ is just suspension to $t$. It follows that $\Phi$ is a non-equivariant homotopy equivalence. Since $\mathcal{Z}_{\mathbb{H}}^{\mathrm{ev}} \times \mathbb{P}(\mathbb{H})$ and $\mathcal{Z}_{\mathbb{H}}^{\mathrm{odd}} \times \mathbb{P}(\mathbb{H})$ are free $(\mathbb{Z}/2)$-spaces, it then follows that $\Phi$ is also an equivariant homotopy equivalence.

Before we compute the equivariant cohomology of $\mathrm{BU}_{\mathbb{H}}^{ev} \times \mathbb{P}(\mathbb{H})$ some notation is needed.

**Notation 5.3.** Given a real bundle $E$ over a $\mathbb{Z}/2$-space $X$, its $k$-th equivariant Chern class is denoted by $\widetilde{c}_k(E) \in H^{2k,k}(X; \underline{\mathbb{Z}})$. The generator of $H^{2,1}(\mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}}) \cong \mathbb{Z}$ is denoted by $x$, as before. We denote the line bundle $\mathcal{O}(m)$ over $\mathbb{P}(\mathbb{H}^n)$ by $\mathcal{O}_n(m)$. In the case $n = 1$ we write $\mathcal{O}(m)$ instead of $\mathcal{O}_1(m)$. Notice that $\mathcal{O}_n(m)$ is real, if $m$ is even, and quaternionic if $m$ is odd. Also, note that $\mathbf{x} = \widetilde{c}_1(\mathcal{O}(2))$. The fact that a tensor product of two quaternionic bundles is a real bundle will also be used throughout.

**Remark 5.4.** Let $\xi_V^j$ denote the universal quotient bundle over $\mathrm{Gr}^{2v+j}(V_{\mathbb{H}}^* \oplus V_{\mathbb{H}})$, cf. Lemma 4.2, and let $i_\sharp : \mathrm{Gr}^{2w+j}(W_{\mathbb{H}}^* \oplus W_{\mathbb{H}}) \to \mathrm{Gr}^{2v+j}(V_{\mathbb{H}}^* \oplus V_{\mathbb{H}})$ be the maps defined in (44). It is easy to see that one has $i_\sharp^*(d_k(\xi_V^j)) = d_k(\xi_W^j)$. Hence, these classes yield elements $\mathbf{d}_k \in H^{2k,k}(\mathrm{BU}_{\mathbb{H}}^{j} \times \mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}})$ that are compatible with the equivalences $\mathrm{BU}_{\mathbb{H}}^{j} \cong \mathrm{BU}_{\mathbb{H}}^{j+2}$, described in (46). It follows that we obtain well-defined classes in $H^{2k,k}(\mathrm{BU}_{\mathbb{H}}^{ev} \times \mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}})$. Using the homotopy equivalence $\Psi : \mathrm{BU}_{\mathbb{H}}^{ev} \times \mathbb{P}(\mathbb{H}) \to \mathrm{BU}_{\mathbb{H}}^{odd} \times \mathbb{P}(\mathbb{H})$, we define the corresponding classes in $\mathrm{BU}_{\mathbb{H}}^{ev} \times \mathbb{P}(\mathbb{H})$, and also denote these classes by $\mathbf{d}_k$. The following facts are easily verified:

**a:** Let $f : X \to (\mathbb{Z} \times BU)_{\mathbb{H}}$ be a classifying map for a quaternionic bundle $E$ over $X$. Then:

$$d_k(E) := (f \times \mathrm{id})^*(\mathbf{d}_k).$$

**b:** Let $E \to X$ be a quaternionic bundle of rank $e$, and let $\rho$ denote the forgetful functor from equivariant cohomology to singular cohomology. Then, denoting the fundamental cohomology class of $\mathbb{P}(\mathbb{H})$ by $\beta$, we have

$$\rho(d_k(E)) = \begin{cases} c_k(E) \times 1 + c_{k-1}(E) \times \beta & , \quad k \equiv e \mod 2 \\ c_k(E) \times 1 & , \quad k \not\equiv e \mod 2. \end{cases}$$

In particular, considering the universal bundles $\xi_k^j$ over $\mathrm{Gr}^{2k+j}(\mathbb{H}^{k^*} \oplus \mathbb{H}^k)$, we see that the classes $\rho(\mathbf{d}_k)$ generate the singular cohomology of $(\mathbb{Z} \times BU)_{\mathbb{H}} \times \mathbb{P}(\mathbb{H})$ over $H^*(\mathbb{P}(\mathbb{H}); \mathbb{Z})$.

We now compute the equivariant cohomology of $(\mathbb{Z} \times BU)_{\mathbb{H}}$.

**Theorem 5.5.** *Let $\mathbf{d}_k$ be the classes defined in* (5.1)*. Then we have the following ring isomorphism*

$$H^{*,*}(BU^{ev}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}}) \cong H^{*,*}(\mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}})\, [\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_k, \ldots]$$

*Proof.* Since $\mathrm{BU}^{ev}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H})$ is free, it suffices to compute the Galois-Grothendieck cohomology groups with $\mathbb{Z}(n)$ coefficients. For this we use the spectral sequence (cf. Section 1.7)

$$E_2^{r,s}(n) := H^r(B\mathbb{Z}/2; \mathcal{H}(\mathrm{BU}^{ev}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H}); \mathbb{Z}(n))) \Rightarrow \hat{H}^{r+s}_{\mathbb{Z}/2}(\mathrm{BU}^{ev}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H}); \mathbb{Z}(n)),$$

and the pairing of spectral sequences $E_r^{p,q}(n) \otimes E_r^{p',q'}(n') \to E_r^{p+p',q+q'}(n+n')$, induced by the cup product. This pairing makes $E_r^{*,*}(*)$ into a spectral sequence of $\mathbb{Z}^2 \times \mathbb{Z}/2$-graded rings.

Set

$$F_2^{p,q}(n) := H^r(B\mathbb{Z}/2; \mathcal{H}(\mathbb{P}(\mathbb{H}); \mathbb{Z}(n))),$$

and note that $E_2^{*,*}(*)$ is a module over $F_2^{*,*}(*)$. Corresponding to each of the classes $\mathbf{d}_k$ there are elements in $E_2^{0,2k}(k)$ which are universal cycles. These cycles are denoted $\widetilde{d}_k$. The correspondence $\mathbf{d}_k \mapsto \widetilde{d}_k$ defines a ring homomorphism $\Theta : F_2^{*,*}(*)\ [\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n, \ldots] \to E_2^{*,*}(*)$.

Observe that the action of $\mathbf{j}_*$ in $H^{2q}(\mathrm{BU}^{ev}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H}); \mathbb{Z})$ is multiplication by $(-1)^q$, hence

$$E^{0,q}(n) = \begin{cases} H^q(\mathrm{BU}^{ev}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H}); \mathbb{Z}) & , q = 2q'\,, n = q' \mod 2 \\ 0 & , \text{ otherwise} \end{cases}$$

As noted in remark 5.4 , the image of the $\mathbf{d}_k$'s, under the forgetful functor to singular cohomology, generates the cohomology of $\mathrm{BU}^{ev}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H})$ over $H^*(\mathbb{P}(\mathbb{H}); \mathbb{Z})$. This implies that $\Theta^{0,2q}(q)$ is an isomorphism. Since, $E^{p,0}(n) = H^p(B\mathbb{Z}/2; \mathbb{Z}(n))$, we see that $\Theta^{p,0}(n)$ is also an isomorphism. By Zeeman's comparison theorem, it follows that

$$E_\infty^{*,*}(*) \cong F_\infty^{*,*}(*)[\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n, \ldots].$$

A standard argument can be used to show that there is actually a ring isomorphism

$$\hat{H}^*_{\mathbb{Z}/2}(\mathrm{BU}^{ev}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H}); \mathbb{Z}(*)) \cong \hat{H}^*_{\mathbb{Z}/2}(\mathbb{P}(\mathbb{H}); \mathbb{Z}(*))\, [\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_k, \ldots].$$

By Proposition 1.13, we conclude that

$$H^{*,*}(\mathrm{BU}^{ev}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}}) \cong H^{*,*}(\mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}})\ [\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_k, \ldots].$$

□

The following result will be used subsequently.

**Lemma 5.6.** *Let $E \to X$ be a quaternionic bundle of dimension $e$, and let $\mathbf{z} \in H_{2,1}(\mathbb{P}(\mathbb{H});\mathbb{Z})$ be the fundamental homology class. Then we have*

$$pr^*(d_{e-2k}(E)/\mathbf{z}) = d_{e-2k-1}(E), \qquad k = 0, \dots, \lfloor \frac{e-2}{2} \rfloor,$$

*where $-/\mathbf{z}$ denotes slant product with $\mathbf{z}$ and $pr$ is projection to the first factor of $X \times \mathbb{P}(\mathbb{H})$.*

*Proof.* We will prove the case where $e$ is even. The other case is proven similarly. Recall from Theorem 5.5 that the equivariant cohomology of $\mathrm{BU}^{ev}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H})$ is generated over $H^{*,*}(\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$ by the classes $\mathbf{d}_k$, $k \geq 1$. The result will follow once we prove the identity

$$pr^*(\mathbf{d}_{2k}/\mathbf{z}) = \mathbf{d}_{2k-1},$$

where $pr : \mathrm{BU}^{ev}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H})$ is the projection onto the first factor. Observe that, by construction, the restriction of $\mathbf{d}_{2k}$ to $\mathrm{Gr}^{2k-2}(\mathbb{H}^{r*} \oplus \mathbb{H}^s)$ is zero. Hence, by Theorem 5.5, $pr^*(\mathbf{d}_{2k}/\mathbf{z})$ must be a multiple of $\mathbf{d}_{2k-1}$. Since

$$\rho(pr^*(\mathbf{d}_{2k}/\mathbf{z})) = \rho(\mathbf{d}_{2k-1}) = c_{2k-1} \times 1,$$

it follows that $pr^*(\mathbf{d}_{2k}/\mathbf{z}) = \mathbf{d}_{2k-1}$. □

5.2. **Projective bundle formula.** In the previous section we defined characteristic classes, $d_k(E)$, for a quaternionic bundle $E$ over $X$, with values in the cohomology theory $H^{*,*}(X \times \mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$. By analogy with Chern classes, it is natural to look for a relation between the classes $d_k(E)$ and the structure of $H^{*,*}(\mathbb{P}(E) \times \mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$, as module over $H^{*,*}(X \times \mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$. The next result addresses this question.

**Proposition 5.7.** *Let $\mathbb{P}(E) \xrightarrow{p} X$ be the projectivization of a rank $e$ quaternionic bundle $E \xrightarrow{p} X$. Let $\xi$ in $KH(\mathbb{P}(E))$ be the universal quotient bundle over $\mathbb{P}(E)$. Then $H^{*,*}(\mathbb{P}(E) \times \mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$ is a free $H^{*,*}(X \times \mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$-module generated by $d_k(\xi)$, $k = 0, \dots, e-1$. Moreover, the classes $d_0(\xi), \dots, d_e(\xi)$ satisfy the following relation*

$$\sum_{k=0}^{e} (-1)^k d_k(\xi) d_{e-k}(p^*E) = 0. \tag{52}$$

*Proof.* Let $p_1 : X \times \mathbb{P}(\mathbb{H}) \to X$ and $p_2 : X \times \mathbb{P}(\mathbb{H}) \to \mathbb{P}(\mathbb{H})$ be the projections. Then $G := p_1^*(E) \otimes p_2^*(\mathcal{O}_{\mathbb{P}(\mathbb{H})}(1))$ is a real bundle, and there is an equivariant homeomorphism

$$\mathbb{P}(G) = \mathbb{P}(p_1^*(E) \otimes p_2^*(\mathcal{O}_{\mathbb{P}(\mathbb{H})}(1))) \cong \mathbb{P}(p_1^*(E)) \cong \mathbb{P}(E) \times \mathbb{P}(\mathbb{H}).$$

It follows from the projective bundle formula for real bundles that, if $\mathcal{O}_G(1)$ is the dual of the tautological line bundle over $\mathbb{P}(E)$ and $t = \tilde{c}_1(\mathcal{O}_G(1))$, then $H^{*,*}(\mathbb{P}(E) \times \mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$ is a free $H^{*,*}(X \times \mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$-module generated by $1, t, \dots, t^{e-1}$. Moreover, the classes $\tilde{c}_0(p^*G), \dots, \tilde{c}_e(p^*G)$ satisfy the relation

$$\sum_{\substack{0 \leq \delta \leq 1 \\ 0 \leq 2i+\delta \leq e}} (-1)^{2i+\delta}\, t^{2i+\delta}\, \tilde{c}_{e-2i+\delta}(q^*G) = 0, \tag{53}$$

where $q : \mathbb{P}(G) \to X \times \mathbb{P}(\mathbb{H})$ is the bundle projection.

A simple computation shows that the classes $t^k$ can be expressed in terms of the classes $d_k(\xi)$. In fact, denoting the first Chern class of the real bundle $\mathcal{O}(2)$ by $\mathbf{x}$ as before, we have

$$t^{2i+\delta} = d_{2i+\delta}(\xi) + i\, d_{2i+\delta-1}(\xi)\mathbf{x}, \tag{54}$$

for $0 \le \delta \le 1$ and $0 \le 2i+\delta \le e$.

Hence $H^{*,*}(\mathbb{P}(E)\times\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$ is also generated by $1, d_1(\xi),\dots,d_e(\xi)$. It remains to show that (52) holds. Now,

$$d_{e-(2i-\delta)}(p^*E) = c_{e-(2i-\delta)}(q^*G) - i\ c_{e-(2i-\delta)-1}(q^*G)\mathbf{x},$$

for $0 \le \delta \le 1$ and $0 \le 2i+\delta \le e$ (where we've set $c_{-1}(G) := 0$). Hence from (53), (54) and the equality $\mathbf{x}^2 = 0$, we get

$$\begin{aligned}
(55)\quad \sum_{k=0}^{e} (-1)^k\ d_k\ (\xi)\ d_{e-k}(p^*E) \quad &= \sum_{\substack{0\le\delta\le 1\\ 0\le 2i+\delta\le e}} (-1)^{2i+\delta}\ d_{2i+\delta}\ (\xi)\ d_{e-(2i+\delta)}(p^*E)\\
= \sum_{\substack{0\le\delta\le 1\\ 0\le 2i+\delta\le e}} (-1)^{2i+\delta} &\left(t^{2i+\delta} - i\ t^{2i+\delta-1}\ \mathbf{x}\right)\left(c_{e-(2i+\delta)}(q^*G) - i\ c_{e-(2i+\delta)-1}(q^*G)\ \mathbf{x}\right)\\
= \sum_{\substack{0\le\delta\le 1\\ 0\le 2i+\delta\le e}} (-1)^{2i+\delta-1}\ i\ &\left(t^{2i+\delta}\ c_{e-(2i+\delta)-1}(q^*G) - \ t^{2i+\delta-1}c_{e-(2i+\delta)}(q^*G)\ \mathbf{x}\right) = 0.
\end{aligned}$$

□

5.3. **The quaternionic total Chern class map.** Recall from Theorem 3.9 that the space $\mathcal{Z}_{\mathbb{H}}$ splits equivariantly as a product of classifying spaces for the functors $H^{*,*}(-\times\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$. Given any equivariant map, $X \to \mathcal{Z}_{\mathbb{H}}$, such a splitting determines a set of classes in $H^{*,*}(X\times\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$.

We have seen in Definition 4.10 that the inclusion of linear spaces in the space of all algebraic cycles induces an equivariant map $c_{\mathbb{H}} : (\mathbb{Z}\times BU)_{\mathbb{H}} \to \mathcal{Z}_{\mathbb{H}}$. In this section we will compute the cohomology classes determined by the map $c_{\mathbb{H}}$ and the splitting (41) of Theorem 3.9.

**Proposition 5.8.** *Let $\xi^n \in KH(\mathbb{P}(\mathbb{H}^n))$ be the universal quotient bundle over $\mathbb{P}(\mathbb{H})$, and let*

$$(56)\qquad \psi_n : \mathcal{Z}_0\left(\mathbb{P}(\mathbb{H}^n)\right) \to \prod_{i=1}^{n} F\left(\mathbb{P}(\mathbb{H})_+,\ K(\mathbb{Z}(2i-1), 4i-2)\right)$$

*be the equivariant homotopy equivalence of corollary 2.4, and let $j_n : \mathbb{P}(\mathbb{H}^n) \hookrightarrow \mathcal{Z}_0\left(\mathbb{P}(\mathbb{H}^n)\right)$ denote the natural inclusion. Then the composition*

$$\phi_n := \psi_n \circ j_n : \mathbb{P}(\mathbb{H}^n) \to \prod_{i=1}^{n} F\left(\mathbb{P}(\mathbb{H})_+,\ K(\mathbb{Z}(2i-1), 4i-2)\right)$$

*classifies $d_1(\xi^n),\ d_3(\xi^n),\dots,\ d_{2n-1}(\xi^n)$.*

*Proof.* For $n=1$, we need to identify the element $\alpha$ of $H^{2,1}(\mathbb{P}(\mathbb{H})\times\mathbb{P}(\mathbb{H});\underline{\mathbb{Z}})$ classified by the composition $\mathbb{P}(\mathbb{H}) \xrightarrow{j} \mathcal{Z}_0\left(\mathbb{P}(\mathbb{H})\right) \xrightarrow{\psi} F(\mathbb{P}(\mathbb{H})_+, K(\mathbb{Z}(1),2)$. Since $\psi$ is the map that realizes Poincaré duality, $\alpha$ is the Poincaré dual of the diagonal $\Delta \subset \mathbb{P}(\mathbb{H})\times\mathbb{P}(\mathbb{H})$. From the projective bundle formula (Proposition 5.7) it is easy to see that

$$\alpha = \mathcal{P}(\Delta) = c_1\left(pr_1^*\mathcal{O}(1)\otimes pr_2^*\mathcal{O}(1)\right) = d_1(\xi^1).$$

Assume that the proposition holds for $k<n$, and note that Proposition 2.2**(c)** implies that $\phi_n$ restricts to $\phi_k$ on $\mathbb{P}(\mathbb{H}^k)$, for $k<n$. The projective bundle formula shows that $d_{2i-1}(\xi^n)$ is the only

class whose restriction to $\mathbb{P}(\mathbb{H}^k)$ is $d_{2i-1}(\xi^k)$, for all $i = 1, \ldots, k$. Hence it suffices to show that $p_n \circ \phi_n$ classifies $d_n(\xi^n)$, where $p_n$ denotes the projection onto the last factor in (56).

Observe that $p_n \circ \phi_n$ is given by the composition

$$\mathbb{P}(\mathbb{H}^n)_+ \xrightarrow{\pi} \mathbb{P}(\mathbb{H}^n)/\mathbb{P}(\mathbb{H}^{n-1}) \xrightarrow{j} \mathcal{Z}_0\left(\mathbb{P}(\mathbb{H}^n)/\mathbb{P}(\mathbb{H}^{n-1})\right)_{\mathrm{o}} \xrightarrow{\theta} F(\mathbb{P}(\mathbb{H})_+, K(\mathbb{Z}(2n-1), 4n-2)),$$

where $\pi$ is the projection, $j$ is the natural inclusion and $\theta$ is the equivalence given in (8). The adjoint map $(\theta \circ j)^\vee : \frac{\mathbb{P}(\mathbb{H}^n)\times\mathbb{P}(\mathbb{H})}{\mathbb{P}(\mathbb{H}^{n-1})\times\mathbb{P}(\mathbb{H})} \to K(\mathbb{Z}(2n-1), 4n-2)$, can be described as the composition

$$\begin{aligned}
(57)\quad & \mathrm{T}\left(\mathcal{O}(1) \otimes \mathbb{H}^{n-1}\right) \wedge \mathbb{P}(\mathbb{H})_+ \\
& \xrightarrow{\Delta \wedge id} \left(\mathrm{T}\left(\mathcal{O}(1) \otimes \mathbb{H}^{n-1}\right) \wedge \mathbb{P}(\mathbb{H})_+\right) \wedge \mathbb{P}(\mathbb{H})_+ = \mathrm{T}\left(\mathcal{O}(1) \otimes \mathbb{H}^{n-1}\right) \wedge (\mathbb{P}(\mathbb{H}) \wedge \mathbb{P}(\mathbb{H}))_+ \\
& \xrightarrow{f_\mu \wedge d_1} \mathcal{Z}_0\left(S^{4(n-1),2(n-1)}\right)_{\mathrm{o}} \wedge \mathcal{Z}_0\left(S^{2,1}\right)_{\mathrm{o}} \to \mathcal{Z}_0\left(S^{4n-2,2n-1}\right)_{\mathrm{o}} = K(\mathbb{Z}(2n-1), 4n-2),
\end{aligned}$$

where $f_\mu$ classifies the Thom class of $\mathcal{O}(1) \otimes \mathbb{H}^{n-1}$. One can easily check that the pull-back of $[f_\mu]$ under the projection $\pi : \mathbb{P}(\mathbb{H}^n) \to \mathrm{T}\left(\mathcal{O}(1) \otimes \mathbb{H}^{n-1}\right)$ coincides with $d_{2n-2}(\xi^n)$. Therefore, $[p_n \circ \phi_n] = d_{2n-2}(\xi^n) d_1(\xi^n) = d_{2n-1}(\xi^n)$, and this completes the proof of the Proposition. ☐

We can now compute the classes determined by total quaternionic Chern class map $c_{\mathbb{H}}$

**Theorem 5.9.** *The equivariant cohomology classes determined by total quaternionic Chern class map* $c_{\mathbb{H}} : (\mathbb{Z} \times BU)_{\mathbb{H}} \to \mathcal{Z}_{\mathbb{H}}$ *and the splitting* (41) *of Theorem 3.9 are*

$$\mathbf{1} + \mathbf{d}_2 + \mathbf{d}_4 + \cdots + \mathbf{d}_{2n} + \cdots \quad \text{on } BU_{\mathbb{H}}^{ev} \tag{58}$$

$$\mathbf{d}_1 + \mathbf{d}_3 + \cdots + \mathbf{d}_{2n+1} + \cdots \quad \text{on } BU_{\mathbb{H}}^{odd} \tag{59}$$

*Proof.* Recall that there are natural equivalences $\mathrm{BU}_{\mathbb{H}}^j \cong \mathrm{BU}_{\mathbb{H}}^{j+2}$ and $\mathcal{Z}_{\mathbb{H}}^j \cong \mathcal{Z}_{\mathbb{H}}^{j+2}$ and, moreover, the map $c_{\mathbb{H}} : (\mathbb{Z} \times BU)_{\mathbb{H}} \to \mathcal{Z}_{\mathbb{H}}$ is compatible with these equivalences. Thus $c_{\mathbb{H}}$ induces maps

$$c_{\mathbb{H}}^{ev} : \mathrm{BU}_{\mathbb{H}}^{ev} \to \mathcal{Z}_{\mathbb{H}}^{\mathrm{ev}} \quad \text{and} \quad c_{\mathbb{H}}^{odd} : \mathrm{BU}_{\mathbb{H}}^{odd} \to \mathcal{Z}_{\mathbb{H}}^{\mathrm{odd}}$$

and it suffices to compute the equivariant cohomology classes they classify.

The map $c_{\mathbb{H}}^{odd} : \mathrm{BU}_{\mathbb{H}}^{odd} \to \mathcal{Z}_{\mathbb{H}}^{\mathrm{odd}}$ classifies an element $D_1 + D_3 + \cdots + D_{2n-1} + \cdots$ with

$$D_{2i-1} \in H^{4i-2,2i-1}(\mathrm{BU}_{\mathbb{H}}^{odd} \times \mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}}) \quad , i \geq 1.$$

Note that by construction, we have $c_{\mathbb{H}}(\mathrm{Gr}^{2q-3}(\mathbb{H}^{k*} \oplus \mathbb{H}^k)) \subset \mathcal{Z}_{\mathbb{H}}^{2q-3}(\mathbb{H}^{k*} \oplus \mathbb{H}^k)$ hence

$$(D_{2q-1})_{|\mathrm{Gr}^{2q-3}(\mathbb{H}^{k*} \oplus \mathbb{H}^k)} = 0, \quad k \geq 0.$$

It follows from Theorem 5.5 that there are constants $\lambda_1, \lambda_2$ such that

$$D_{2q-1} = \lambda_1 \mathbf{d}_{2q-1} + \lambda_2\, \mathbf{x} \cdot \mathbf{d}_{2q},$$

where $\mathbf{x} \in H^{2,1}(\mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}})$ is the fundamental class of $\mathbb{P}(\mathbb{H})$. To compute $\lambda_1, \lambda_2$ we observe that, by Proposition 5.8, the restriction of $D_{2q-1}$ to $\mathbb{P}(\mathbb{H}^q)$ is $d_{2q-1}(\xi^q)$, where $\xi^q$ is the universal quotient bundle over $\mathbb{P}(\mathbb{H}^q)$. Since the inclusion

$$\mathbb{P}(\mathbb{H}^q) \subset \mathrm{Gr}^{2q-1}(\mathbb{H}^{q*})\mathrm{Gr}^{2q-1}(\mathbb{H}^{q*}) \subset \mathrm{Gr}^{2q-1}(\mathbb{H}^{q*} \oplus \mathbb{H}^q) \subset \mathrm{BU}_{\mathbb{H}}^{2q-1}$$

classifies $\xi^q$, it follows that ${D_{2q-1}}_{|\mathbb{P}(\mathbb{H}^q)} = d_{2q-1}(\xi^q)$. Thus $\lambda_1 = 1, \lambda_2 = 0$ and $D_{2q-1} = \mathbf{d}_{2q-1}$.

Now, consider the map $c^{ev}_{\mathbb{H}} : \mathrm{BU}^{ev}_{\mathbb{H}} \to \mathcal{Z}^{\mathrm{ev}}_{\mathbb{H}}$. It classifies an element $1 + D_2 + D_4 + \cdots + D_{2n} + \cdots$ with $D_{2i} \in H^{4i,2i}(\mathrm{BU}^{ev}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}})$ $i \geq 1$. Once again, we observe that $c_{\mathbb{H}}(\mathrm{Gr}^{2q-2}(\mathbb{H}^{k^*} \oplus \mathbb{H}^k)) \subset \mathcal{Z}^{2q-2}_{\mathbb{H}}(\mathbb{H}^{k^*} \oplus \mathbb{H}^k)$, implying

$$(D_{2q-2})_{|\mathrm{Gr}^{2q-2}(\mathbb{H}^{k^*} \oplus \mathbb{H}^k)} = 0, \quad k \geq 0.$$

As before, we conclude that there are constants $\lambda_1, \lambda_2$ such that $D_{2q} = \lambda_1 \mathbf{d}_{2q} + \lambda_2\ \mathbf{x} \cdot \mathbf{d}_{2q-1}$. To compute $\lambda_1, \lambda_2$ it suffices to compute the image of $D_{2q}$ under the forgetful map to singular cohomology. In [LM88] it is shown that, non-equivariantly, $\mathcal{Z}_1(\mathbb{P}(\mathbb{H}^n)_1 \cong K(\mathbb{Z}, 2) \times \cdots \times K(\mathbb{Z}, 4n - 2)$, and that under this equivalence $c_{\mathbb{H}}$ classifies the total Chern class. One can show that the decomposition (34) is compatible with this non-equivariant splitting. It follows that $\rho(D_{2k}) = c_k + c_{k-1}\beta$, hence $\lambda_1 = 1$ and $\lambda_2 = 0$. □

5.4. **The group struture on $\mathfrak{Z}^0_{\mathbb{H}}(X)$.** In this section we compute the group structure induced by the algebraic join of cycles on $\mathcal{Z}_{\mathbb{H}}$. Recall from Proposition 3.5 that the algebraic join # induces a pairing

$$\# : \mathcal{Z}_{\mathbb{H}} \times \mathcal{Z}_{\mathbb{H}} \to \mathcal{Z}_{\mathbb{H}}$$

satisfying

$$\# : \mathcal{Z}^j_{\mathbb{H}} \times \mathcal{Z}^j_{\mathbb{H}} \to \mathcal{Z}^{j+j'}_{\mathbb{H}}.$$

For a $\mathbb{Z}/2$-space $X$, one has an identification

$$\mathfrak{Z}^0_{\mathbb{H}}(X) = [X, \mathcal{Z}_{\mathbb{H}}]_{\mathbb{Z}/2} = \bigoplus_{j \in \mathbb{Z}} \left[X, \mathcal{Z}^j_{\mathbb{H}}\right]_{\mathbb{Z}/2} = \bigoplus_{j \in \mathbb{Z}} \prod_{r \geq 1} H^{4r-2\epsilon(j),\ 2r-\epsilon(j)}(X \times \mathbb{P}(\mathbb{H}), \underline{\mathbb{Z}}), \tag{60}$$

where $\epsilon(j)$ is 0 if $j$ is even and 1 if $J$ is odd. Given the splitting above one might conjecture that the group structure induced by # is induced by the cup on $H^{*,*}(X \times \mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}})$, however this is not the case as we will show. From (60) it follows that the group structure on $\mathfrak{Z}^0_{\mathbb{H}}(X)$ is completely determined by the cohomology class represented by the map # under the equivalences

$$[\mathcal{Z}^j_{\mathbb{H}} \times \mathcal{Z}^{j'}_{\mathbb{H}}, \mathcal{Z}^{j+j'}_{\mathbb{H}}]_{\mathbb{Z}/2} \cong \prod_{r \geq 1} H^{4r-2\epsilon(j+j'),\ 2r-\epsilon(j+j')}(\mathcal{Z}^j_{\mathbb{H}} \times \mathcal{Z}^{j'}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H}), \underline{\mathbb{Z}}).$$

Also recall that $\mathrm{BU}^j_{\mathbb{H}}$ maps to $\mathcal{Z}^j_{\mathbb{H}}$ by $c_{\mathbb{H}}$ and that the following diagram commutes

$$\begin{tikzcd}
\mathrm{BU}^j_{\mathbb{H}} \times \mathrm{BU}^{j'}_{\mathbb{H}} \arrow[r, "\oplus"] \arrow[d, "c_{\mathbb{H}} \times c_{\mathbb{H}}"'] & \mathrm{BU}^{j+j'}_{\mathbb{H}} \arrow[d, "c_{\mathbb{H}}"] \\
\mathcal{Z}^j_{\mathbb{H}} \times \mathcal{Z}^{j'}_{\mathbb{H}} \arrow[r, "\#"] & \mathcal{Z}^{j+j'}_{\mathbb{H}}
\end{tikzcd}$$

We claim that the maps $c_{\mathbb{H}}$ above induce injective maps

$$H^{*,*}(\mathcal{Z}^j_{\mathbb{H}} \times \mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}}) \to H^{*,*}(\mathcal{Z}_0 \left(\mathrm{BU}^j_{\mathbb{H}}\right)_1 \times \mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}})$$

and hence the pairing on $\mathfrak{Z}^0_{\mathbb{H}}(-)$ is completely determined by the formula for the quaternionic Chern class of a Whitney sum.

Let us start with the case $j = 1$. Let $\iota : \mathbb{P}(\mathbb{H}^\infty) \to \mathrm{BU}^1_{\mathbb{H}}$ be the map induced by the inclusions $\iota_n : \mathbb{P}(\mathbb{H}^n) \to \mathrm{Gr}^1(\mathbb{H}^{n*} \oplus \mathbb{H}^n)$ that send $L$ to $L \oplus \mathbb{H}^n$. Composing with $c_{\mathbb{H}}$ gives a map from $\mathbb{P}(\mathbb{H}^\infty)$ to $\mathcal{Z}^1_{\mathbb{H}}$. The linear extension $\mathcal{Z}_0(\mathbb{P}(\mathbb{H}^n)) \to \mathcal{Z}^1_{\mathbb{H}}$ of this composition is, by the quaternionic suspension theorem, an equivariant homotopy equivalence when restricted to the component of 1. This map factors as $\mathcal{Z}_0(\mathbb{P}(\mathbb{H}^n))_1 \xrightarrow{\tilde{\iota}} \mathcal{Z}_0\left(\mathrm{BU}^1_{\mathbb{H}}\right)_1 \xrightarrow{\widetilde{c_{\mathbb{H}}}} \mathcal{Z}^1_{\mathbb{H}}$, where $\tilde{\iota}$ and $\widetilde{c_{\mathbb{H}}}$ denote the linear extension of $\iota$ and $c_{\mathbb{H}}$, respectively. It follows that, for any cohomology theory $\mathfrak{H}^*$, the map $\mathfrak{H}(\widetilde{c_{\mathbb{H}}}) : \mathfrak{H}^*(\mathcal{Z}^1_{\mathbb{H}}) \to \mathfrak{H}^*(\mathcal{Z}_0\left(\mathrm{BU}^1_{\mathbb{H}}\right)_1)$ is injective. For $1 \neq j$ odd it suffices to observe that $\widetilde{c_{\mathbb{H}}}$ is compatible the canonical equivariant homotopy equivalence $\mathcal{Z}^j_{\mathbb{H}} \cong \mathcal{Z}^{j+2}_{\mathbb{H}}$, given by inclusion of hyperplanes.

Suppose now that $j$ is even. We need to show that $\widetilde{c_{\mathbb{H}}}$ induces an injective map

$$(\mathrm{id} \times \widetilde{c_{\mathbb{H}}})^* : H^{*,*}(\mathcal{Z}^j_{\mathbb{H}} \times \mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}}) \to H^{*,*}(\mathcal{Z}_0\left(\mathrm{BU}^j_{\mathbb{H}}\right)_1) \times \mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}}).$$

The same argument as before shows that we may replace $\mathrm{BU}^j_{\mathbb{H}}$ by $\mathrm{BU}^{ev}_{\mathbb{H}}$ . Recall from Remark 5.2 that there are equivariant homotopy equivalences $\Psi : \mathrm{BU}^{ev}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H}) \to \mathrm{BU}^{odd}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H})$ and $\Psi : \mathcal{Z}^{\mathrm{ev}}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H}) \to \mathcal{Z}^{\mathrm{odd}}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H})$. By construction $\Psi$ and $\Phi$ are compatible with $c_{\mathbb{H}}$ so that the following diagram commutes

$$\begin{tikzcd}
H^{*,*}(\mathcal{Z}^{\mathrm{ev}}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}}) \arrow[r, "(\mathrm{id} \times c_{\mathbb{H}})^*"] & H^{*,*}(\mathcal{Z}_0\left(\mathrm{BU}^{ev}_{\mathbb{H}}\right)_1 \times \mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}}) \\
H^{*,*}(\mathcal{Z}^{\mathrm{odd}}_{\mathbb{H}} \times \mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}}) \arrow[r, "(\mathrm{id} \times c_{\mathbb{H}})^*"] \arrow[u, "\Phi^*"] & H^{*,*}(\mathcal{Z}_0\left(\mathrm{BU}^{odd}_{\mathbb{H}}\right)_1 \times \mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}}). \arrow[u, "\Psi^*"']
\end{tikzcd}$$

It follows that the map on top is injective, as desired.

**Proposition 5.10.** *Let $X$ be a $\mathbb{Z}/2$-space, and let $a \cdot b$ denote the product of elements $a, b$ in $\mathfrak{Z}^0_{\mathbb{H}}(X)$. Consider $\mathfrak{Z}^0_{\mathbb{H}}(X)$ included in*

$$\bigoplus_{j \in \mathbb{Z}} \prod_{r,s \geq 1} H^{r,s}(X \times \mathbb{P}(\mathbb{H}), \underline{\mathbb{Z}}),$$

*as in* (60)*. Then, under this inclusion we have,*

$$a \cdot b = a \cup b + pr^*(a/z) \cup pr^*(b/z),$$

*where $\mathbf{z} \in H_{2,1}(\mathbb{P}(\mathbb{H}); \underline{\mathbb{Z}})$ is the fundamental class $\mathbb{P}(\mathbb{H})$, $-/z$ denotes slant product with $z$ and $pr$ is the projection onto the first factor in the product $X \times \mathbb{P}(\mathbb{H})$.*

*Proof.* By the preceeding remarks it suffices to show the following formula for the quaternionic Chern class of a Whitney sum holds

$$c_{\mathbb{H}}(E \oplus F) = c_{\mathbb{H}}(E) \cup c_{\mathbb{H}}(F) + pr^*(c_{\mathbb{H}}(E)/z) \cup pr^*(c_{\mathbb{H}}(F)/z).$$

Recall that $c_{\mathbb{H}}(E)$ is defined as a combination of Chern classes of the real bundle $E \otimes \mathcal{O}(1)$, where $\mathcal{O}(1)$ is the hyperplane bundle over $\mathbb{P}(\mathbb{H})$. To simplify notation we will use $\bar{E}$ and $\bar{F}$ to denote $E \otimes \mathcal{O}(1)$ and $F \otimes \mathcal{O}(1)$, respectively. Let $e$ , $f$ be the dimensions of $E$ and $F$, respectively. We have,

$$
\begin{aligned}
(61)\quad c_{\mathbb{H}}(E)\cup c_{\mathbb{H}}(F) &= \sum_{i=0}^{\lfloor\frac{e}{2}\rfloor}\Big[\widetilde{c}_{e-2i}(\bar{E}) - i\widetilde{c}_{e-2i-1}(\bar{E})\mathbf{x}\Big] \cup \sum_{j=0}^{\lfloor\frac{f}{2}\rfloor}\Big[\widetilde{c}_{f-2j}(\bar{F}) - j\widetilde{c}_{f-2j-1}(\bar{F})\mathbf{x}\Big] \\
&= \sum_{r=0}^{\lfloor\frac{e+f}{2}\rfloor}\sum_{s=0}^{r}\widetilde{c}_{e-2s}(\bar{E})\cup\widetilde{c}_{f-2(r-s)}(\bar{F}) \\
&\quad - \Big[s\widetilde{c}_{e-2s-1}(\bar{E})\cup\widetilde{c}_{f-2(r-s)}(\bar{F}) + (r-s)\widetilde{c}_{e-2s}(\bar{E})\cup\widetilde{c}_{f-2(r-s)-1}(\bar{F})\Big]\mathbf{x},
\end{aligned}
$$

where we've set $\widetilde{c}_k(-)$ equal to 1 if $k$ is zero and $\widetilde{c}_k(-)$ equal to zero if $k<0$. By Lemma 5.6 we have

$$
\begin{aligned}
(62)\quad pr^*(c_{\mathbb{H}}(E)/z)\cup pr^*(c_{\mathbb{H}}(F)/z) &= \sum_{i=0}^{\lfloor\frac{e}{2}\rfloor} d_{e-2i-1}(E)\cup\sum_{j=0}^{\lfloor\frac{f}{2}\rfloor} d_{f-2j-1}(F) \\
&= \sum_{r=0}^{\lfloor\frac{e+f}{2}\rfloor}\sum_{s=0}^{r}\widetilde{c}_{e-2s-1}(\bar{E})\cup\widetilde{c}_{f-2(r-s)-1}(\bar{F}) \\
&\quad - \Big[s\widetilde{c}_{e-2s-2}(\bar{E})\cup\widetilde{c}_{f-2(r-s)-1}(\bar{F}) + (r-s)\widetilde{c}_{e-2s-1}(\bar{E})\cup\widetilde{c}_{f-2(r-s)-2}(\bar{F})\Big]\mathbf{x}.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
(63)\quad & c_{\mathbb{H}}(E)\cup c_{\mathbb{H}}(F) + pr^*(c_{\mathbb{H}}(E)/z)\cup pr^*(c_{\mathbb{H}}(F)/z) \\
&= \sum_{r=0}^{\lfloor\frac{e+f}{2}\rfloor}\sum_{s=0}^{2r}\widetilde{c}_{e-s}(\bar{E})\cup\widetilde{c}_{f-2r+s}(\bar{F}) - \Big[r\widetilde{c}_{e-2s-1}(\bar{E})\cup\widetilde{c}_{f-2r+s}(\bar{F}) + r\widetilde{c}_{e-2s}(\bar{E})\cup\widetilde{c}_{f-2r+s-1}(\bar{F})\Big]\mathbf{x} \\
&= \sum_{r=0}^{\lfloor\frac{e+f}{2}\rfloor}\widetilde{c}_{e+f-2r}(\bar{E}\oplus\bar{F}) - r\widetilde{c}_{e+f-2r-1}(\bar{E}\oplus\bar{F})\mathbf{x} = c_{\mathbb{H}}(E\oplus F).
\end{aligned}
$$

The Proposition follows. □

5.5. **Remarks on the space $(\mathbb{Z}\times BU)_{\mathbb{H}}$.** Here are two facts about $(\mathbb{Z}\times BU)_{\mathbb{H}}$ which seem to be quite interesting. Both of them are particular cases of results of Karoubi [Kar00].

**Remark 5.11.** In [Kar00] , Karoubi observes that there is an involution on BU such that $\mathrm{BU}^{h\mathbb{Z}/2} = \mathrm{BSp}$. We claim that the involution on $\mathrm{BU}^{ev}_{\mathbb{H}}$ satisfies this. It is clear that $\mathrm{BU}^{ev}_{\mathbb{H}}$ is homotopy equivalent to BU and that $\{\mathrm{BU}^{ev}_{\mathbb{H}}\}^{\mathbb{Z}/2} = \mathrm{BSp}$. We now proceed to show that $\{\mathrm{BU}^{ev}_{\mathbb{H}}\}^{\mathbb{Z}/2} = \{\mathrm{BU}^{ev}_{\mathbb{H}}\}^{h\mathbb{Z}/2}$.

The proof mimics one of the proofs of the well known fact $\mathrm{BO} = \mathrm{BU}^{h\mathbb{Z}/2}$, using Dupont's quaternionic K-theory instead of $KR$-theory. Let $X$ be a $\mathbb{Z}/2$-space. From [Dup69], we know that there is a natural splitting

$$KH(X\times\mathbb{P}(\mathbb{H})) \cong KH(X)\oplus KR(X).$$

This implies that there is a natural equivariant homotopy equivalence

$$F(X \times \mathbb{P}(\mathbb{H})_+, (\mathbb{Z} \times BU)_{\mathbb{H}}) \cong F(X_+, \mathbb{Z} \times \mathrm{BU}) \vee F(X_+, (\mathbb{Z} \times BU)_{\mathbb{H}}).$$

Applying this equivalence to $X \times E\mathbb{Z}/2$ instead of $X$ and using the equivariant homotopy equivalence

$$X \times \mathbb{P}(\mathbb{H}) \times E\mathbb{Z}/2 \cong X \times \mathbb{P}(\mathbb{H}),$$

we obtain an equivariant homotopy equivalence

$$F(X \times E\mathbb{Z}/2_+, (\mathbb{Z} \times BU)_{\mathbb{H}}) \cong F(X_+, (\mathbb{Z} \times BU)_{\mathbb{H}}).$$

**Remark 5.12.** Another interesting fact about $(\mathbb{Z} \times BU)_{\mathbb{H}}$ is that

$$(\mathbb{Z} \times BU)_{\mathbb{H}} \cong \Omega^{4,0}(\mathbb{Z} \times \mathrm{BU}).$$

To see this, consider a map $f : S^4 \to \mathrm{BSp}$ which represents a generator of $\pi_4(\mathrm{BSp})$. The tensor product induces an equivariant map

$$S^{4,0} \wedge (\mathbb{Z} \times BU)_{\mathbb{H}} \to \mathbb{Z} \times \mathrm{BU}.$$

Its adjoint $(\mathbb{Z} \times BU)_{\mathbb{H}} \to \Omega^{4,0}(\mathbb{Z} \times \mathrm{BU})$ is an equivariant map which is a non-equivariant homotopy equivalence, and whose restriction to the fixed points induces the homotopy equivalence

$$\mathbb{Z} \times \mathrm{BSp} \cong \Omega^4\, \mathrm{BO};$$

see [Bot59].

## References


[Ati66] M. F. Atiyah, *K-theory and reality*, Quart. J. Math. Oxford **17** (1966), 367–386.

[BLLF$^+$93] C. Boyer, H. B. Lawson, Jr., P.C. Lima-Filho, B. Mann, and M.-L. Michelsohn, *Algebraic cycles and infinite loop spaces*, Inventiones Math. **113** (1993), 373–388.

[Bot59] R. Bott, *The stable homotopy of the classical groups*, Ann. of Math. **70** (1959), 313–337.

[CW91] S.R. Costenoble and S. Waner, *Fixed set systems of equivariant infinite loop spaces*, Trans. Amer. Math. Soc. **326** (1991), no. 2, 485–505.

[CW92] ______, *Equivariant Poincaré dulity*, Michigan Math. J. **39** (1992), no. 2, 325–351.

[dS99a] Pedro F. dos Santos, *Algebraic cycles on real varieties and $\mathbb{Z}/2$-equivariant homotopy theory*, Ph.D. thesis, State University of New York at Stony Brook, August 1999.

[dS99b] ______, *A note on the equivariant Dold-Thom theorem*, Preprint, October 1999.

[dS00] ______, *Algebraic cycles on real varieties and $\mathbb{Z}/2$-homotopy theory*, Preprint, July 2000.

[dSLF01] Pedro F. dos Santos and Paulo Lima-Filho, *Quaternionic $K$-theory for real varieties*, In preparation, 2001.

[Dug01] D. Dugger, *An Atiyah-Hirzebruch spectral sequence for $KR$-theory*, Preprint, 2001.

[Dup69] Johan L. Dupont, *Symplectic bundles and KR-theory*, Math.Scand. **24** (1969), 27–30.

[Dup99] ______, *A note on characteristic classes for Real vector bundles*, Preprint, 1999.

[FL92] Eric Friedlander and H. B. Lawson, Jr., *A theory of algebraic cocycles*, Ann. of Math **136** (1992), no. 2, 361–428.

[FW01] Eric Friedlander and Mark Walker, *Semi-topological k-theory of real varieties*, Preprint, 2001.

[Gro57] Alexander Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. **9** (1957), no. 2, 119–221.

[Kah87] B. Kahn, *Construction de classes de chern équivariantes pour un fibré vectoriel réel*, Comm. in Algebra **15** (1987), no. 4, 695–711.

[Kar00] M. Karoubi, *A descent theorem in topological $K$-theory*, Preprint, 2000.

[Lam90] T. K. Lam, *Spaces of real algebraic cycles and homotopy theory*, Ph.D. thesis, SUNY at Stony Brook, 1990.

[Law89] H. B. Lawson, Jr., *Algebraic cycles and homotopy theory*, Ann. of Math. **129** (1989), no. 2, 253–291.

[LF94] Paulo Lima-Filho, *The topological group structure of algebraic cycles*, Duke Math. J. **75** (1994), no. 2, 467–491.

[LF97] ———, *On the equivariant homotopy of free abelian groups on g-spaces and g-spectra*, Math. Z. **224** (1997), 567–601.

[LLFM96] H. B. Lawson, Jr., P.C. Lima-Filho, and M.-L. Michelsohn, *Algebraic cycles and equivariant cohomology theories*, Proc. London Math. Soc. **73** (1996), no. 3, 679–720.

[LLFM98a] ———, *On equivariant algebraic suspension*, J. Alg. Geometry **7** (1998), 627–650.

[LLFM98b] ———, *Spaces of algebraic cycles and classical groups, Part I: Real cycles*, Preprint, 1998.

[LLFM98c] ———, *Spaces of algebraic cycles and classical groups, Part II: Quaternionic cycles*, Preprint, 1998.

[LM88] H. B. Lawson, Jr. and M. L. Michelsohn, *Algebraic cycles, Bott periodicity, and the Chern characteristic map*, Proc. of Symp. in Pure Math. **48** (1988), 241–264.

[LM91] ———, *Algebraic cycles and group actions*, Differential Geometry, Longman Press, 1991, pp. 261–278.

[May77] J. P. May, *$e_\infty$-ring spaces and $e_\infty$-ring spectra*, Lecture Notes in Mathematics, vol. 577, Springer-Verlag, 1977.

[May86] J. P. et al. May, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, vol. 1213, Springer-Verlag, 1986.

[May96] ———, *Equivariant homotopy and cohomology theory*, CBMS, vol. 91, American Mathematical Society, 1996.

[Mos98] J. Mostovoy, *Algebraic cycles and anti-holomorphic involutions on projective spaces*, UNAM Preprint, 1998.

[Qui73] D. Quillen, *Higher algebraci k-theory*, Algebraic $K$-theory I (New York), Lecture Notes in Mathematics, vol. 341, Springer-Verlag, 1973, pp. 85–147.

[Seg68] G. Segal, *Equivariant k-theory*, Inst. Hautes Études Sci. Publ. Math. **34** (1968), 129–151.

[Sey73] R. M. Seymour, *The real k-theory of Lie groups and homogeneous spaces*, Quart. J. Math. Oxford **24** (1973), 7–30.

[Wan86] Stefan Waner, *Periodicity in the cohomology of universal g-spaces*, Illinois J. Math. **30** (1986), 468–478.



Department of Mathematics, Texas A&M University, USA

Departamento de Matemática, Instituto Superior Técnico, Portugal

Department of Mathematics, Texas A&M University, USA