# A NOTE ON THE EQUIVARIANT DOLD-THOM THEOREM

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ABSTRACT. In this note we prove a version of the classical Dold-Thom theorem for the $RO(G)$-graded equivariant homology functors $H_*^G(-;\underline{M})$, where $G$ is a finite group, $M$ is a discrete $\mathbb{Z}[G]$-module, and $\underline{M}$ is the Mackey functor associated to $M$. In the case where $M = \mathbb{Z}$ with the trivial $G$-action, our result says that, for a $G$-CW-complex $X$, and for a finite dimensional $G$-representation $V$, there is a natural isomorphism

$$[S^V, \mathcal{Z}_0(X)]_G \cong H_V^G(X;\underline{\mathbb{Z}});$$

where $\mathcal{Z}_0(X)$ denotes the free abelian group on $X$.

## 1. INTRODUCTION

It is a classical result of Dold and Thom that for a CW-complex $X$ there is a natural isomorphism

$$H_n(X;\mathbb{Z}) \cong \pi_n(\mathcal{Z}_0(X)),$$

where $\mathcal{Z}_0(X)$ denotes the free abelian group on $X$. More generally, for a discrete abelian group $M$ one can consider the topological abelian group $M \otimes X$ — see Definition 2.1 below. In [11] it is proved that

$$\pi_n(M \otimes X) \cong \widetilde{H}_n(X;M).$$

In [8, Thm.4.5] Lima-Filho proved an equivariant version of the Dold-Thom Theorem. It says that if $X$ is a G-CW-complex and $G$ is a finite group then

$$H_n^G(X;\underline{\mathbb{Z}}) \cong \pi_n(\mathcal{Z}_0(X)^G), \tag{1}$$

where $H_n^G(X;\underline{\mathbb{Z}})$ denotes the Bredon homology of $X$ with coefficients in the Mackey functor $\underline{\mathbb{Z}}$. For the definition of Mackey functor and Bredon homology see [10].

In this paper $G$ will always denote a finite group.

Our goal in this note is to generalize Lima-Filho's result [8, Thm.4.5] in the following two directions.

(1) We replace $\underline{\mathbb{Z}}$ by the Mackey functor $\underline{M}$ associated to a discrete $\mathbb{Z}[G]$-module $M$ as follows: the value of $\underline{M}$ on $G/H$ is $M^H$ and the value on the projection $G/K \to G/H$, for $K \le H \le G$, is the inclusion of $M^H$ in $M^K$. The functor $\underline{\mathbb{Z}}$ corresponds to the case $M = \mathbb{Z}$ with the trivial action.

(2) We replace Bredon homology with the corresponding $RO(G)$-graded homology theory [10]. For a finite dimensional $G$-representation $V$, the value of this homology theory in dimension $V$ is denoted by $\widetilde{H}_V^G(-;\underline{M})$. The space $M \otimes X$ has a natural $G$-action. We replace the homotopy group

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in (1) by the set of equivariant homotopy classes $[S^V, M \otimes X]_G$, where $S^V = V \cup \{\infty\}$. The set $[S^V, M \otimes X]_G$ is also denoted by $\pi_V^G(M \otimes X)$.

Our main result is the following.

**Theorem 1.1.** *Let $X$ be a based $G$-CW-complex and let $V$ be a finite dimensional $G$-representation, then $M \otimes X$ is an equivariant infinite loop space and there is a natural equivalence*

$$\pi_V^G(M \otimes X) \cong \widetilde{H}_V^G(X; \underline{M}). \tag{2}$$

As a corollary to this theorem we see that $M \otimes S^V$ is a $K(\underline{M}, V)$ space. Thus we have identified a simple model for the equivariant Eilenberg-Mac Lane spectrum $H\underline{M}$.

Observe that, if $S^n$ is equipped with the trivial $G$-action, the right hand side of (1) can also be described as the set of equivariant homotopy classes $[S^n, \mathcal{Z}_0(X)]_G$. Thus (1) is a particular case of (2) with $M = \mathbb{Z}$ and $V$ a trivial $G$-representation of dimension $n$.

The paper is organized as follows. In section 2 we define the functor $M \otimes -$ and prove that Theorem 1.1 holds in the case where $V = \mathbb{R}^n$. In section 3 we show that if $R$ is a discrete commutative $G$-ring then $R \otimes -$ is an FSP and that if $M$ is a discrete $R[G]$-module then $M \otimes -$ is a module over $R \otimes -$. Using these facts we define a $G$-spectrum $M \otimes \mathbb{S}$ and identify it as the Eilenberg-Mac Lane spectrum $H\underline{M}$. This model for $H\underline{M}$ is used in section 4 to prove Theorem 1.1. In section 5 we describe briefly an application of Theorem 1.1 to the study of algebraic cycles on real projective varieties.

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## 2. Equivariant Dold-Thom for Bredon homology with $\underline{M}$ coefficients

In this paper $G\mathcal{T}$ denotes the category of based compactly generated $G$-spaces and based $G$-maps. Following [9] we consider the category $\mathcal{T}_G$ which has the same objects as $G\mathcal{T}$ and whose morphisms are non-equivariant based maps. Note that $\mathcal{T}_G$ is enriched over $G\mathcal{T}$: its morphisms are $G$-spaces ($G$ acts by conjugation) and composition is given by $G$-maps. A functor $F : \mathcal{C} \to \mathcal{D}$, between $G\mathcal{T}$-enriched categories, is $G\mathcal{T}$-enriched if $F : \mathcal{C}(X,Y) \to \mathcal{D}(F(X), F(Y))$ is a map of $G$-spaces for all $X, Y$.

We start by defining functors $M \otimes -$ in the categories of $G$-sets, simplicial $G$-sets and $G$-spaces.

**Definition 2.1.** Let $M$ be a discrete $\mathbb{Z}[G]$-module.

(a) Given a based $G$-set $X$ (with base point $*$) we let $M \otimes X$ denote the $\mathbb{Z}[G]$-module $\bigoplus_{x \in X - \{*\}} M$ (the action of $g \in G$ is given by $(g \cdot m)_x = g \cdot m_{g^{-1} \cdot x}$, where $m_x$ denotes the $x$th coordinate of $m \in \bigoplus_{x \in X - \{*\}} M$). The correspondence $X \mapsto M \otimes X$ defines a covariant functor from based $G$-sets to $\mathbb{Z}[G]$-modules.
(b) Given a simplicial set $X_\bullet$, the correspondence $n \mapsto M \otimes X_n$ defines a $G$-simplicial abelian group which we denote by $M \otimes X_\bullet$.
(c) Given $X$ a based $G$-space, we endow the $\mathbb{Z}[G]$-module $M \otimes X$ with a topology as follows. The group $M \otimes X$ can be equivalently defined as the quotient $\coprod_{n \geq 0} M^n \times X^n / \sim$, where $\sim$ is the equivalence relation generated by:

*(i)* $(r, \phi^* x) \sim (\phi_* r, x)$, for each based map[1] $\phi : \{0, \dots, n\} \to \{0, \dots, m\}$, $n, m \in \mathbb{N}$, where $\phi^* x = x \circ \phi$ and $(\phi_* r)_i = \sum_{k \in \phi^{-1}(i)} r_k$ .

*(ii)* $((r, r'), (x, *)) \sim (r, x)$, for each $r \in M^n, r' \in M, x \in X^n$ with $*$ denoting the base point of $X$.

We give the discrete topology to $M$ and endow $M \otimes X$ with the quotient topology corresponding to the relation $\sim$. The correspondence $X \mapsto M \otimes X$ defines a $G\mathfrak{T}$-enriched functor from $\mathfrak{T}_G$ to the category of topological $\mathbb{Z}[G]$-modules.

**Notation 2.2.** (a) Given $m \in M^n$ and $x \in X^n$, the image of $(m, x)$ in $M \otimes X$ will be denoted $\sum_i m_i x_i$.

(b) If $f : X \to Y$ is a map of based $G$-spaces, the induced homomorphism $M \otimes X \to M \otimes Y$ will be denoted $M \otimes f$.

(c) If $m \in M$ and $f \colon X \to Y$, the map $X \to M \otimes Y$ defined by $x \mapsto m f(x)$, $x \in X$, will be denoted $m \otimes f$.

(d) In [8], $\mathbb{Z} \otimes X$ is denoted $AG(X)$, for a based space $X$ and, for any space $X$, $\mathbb{Z} \otimes X_+$, is denoted $\mathcal{Z}_0(X)$.

(e) In [11] $M \otimes X$ is denoted $B(M, X)$.

**Remark 2.3.** (a) The space $M \wedge X$ – where $M$ is considered as a based $G$-space with the discrete topology having 0 as the base point – includes in $M \otimes X$ as the image of the natural map $M \times X \to M \otimes X$. This inclusion is denoted $\iota_X$ and will be used throughout.

(b) If $X$ has the trivial action then $(M \otimes X)^H = M^H \otimes X$, for all $H \leq G$. In particular, it follows from [11] that $M \otimes S^p$ is an equivariant Eilenberg-Mac Lane space $K(\underline{M}, p)$.

(c) For a pointed simplicial set $X_\bullet$ the $G$-simplicial group $M \otimes X_\bullet$ can be alternatively defined as the quotient $\coprod_n M^n \times X_\bullet^n / \sim$, where $\sim$ is the equivalence relation generated by $(i)$ and $(ii)$, in Definition 2.1(c). Thus, since the realization functor $| - |$ commutes with colimits and finite products, if $X$ is the realization of $X_\bullet$ then $M \otimes X$ is $G$-homeomorphic to $|M \otimes X_\bullet|$.

Our goal in this section is to prove that, for a discrete $\mathbb{Z}[G]$-module $M$,

$$\pi_n^G(M \otimes (X/A)) \cong H_n^G(X, A; \underline{M}). \tag{3}$$

In other words, we want to prove the Dold-Thom Theorem for Bredon homology with coefficients in the Mackey functor $\underline{M}$.

The following simple observation, due to Lima-Filho will be used throughout.

**Lemma 2.4.** *Let $S$ be a based finite $G$-set.*

*(a) There is a $G$-homeomorphism $F(S, M \otimes X) \xrightarrow{\varphi} M \otimes (X \wedge S)$, where $F(S, M \otimes X)$ denotes the space of based maps with the $G$-action given by conjugation.*

*(b) The homeomorphism $\varphi$ of (a) is natural with respect to the variable $S$: let $h \colon S \to T$ be a $G$-map between finite $G$-sets and let $h^* \colon F(T, M \otimes X) \to F(S, M \otimes X)$ be the map induced by $h$. Then the map $\hat{h} \colon M \otimes (X \wedge T) \to M \otimes (X \wedge S)$ that corresponds to $h^*$ under $\varphi$ is given by*

$$\hat{h}(m(x \wedge t)) = \sum_{h(s)=t} m(x \wedge s);$$

[1] 0 is the base point

*and the map* $\check{h}\colon F(S, M\otimes X)\to F(T, M\otimes X)$ *that corresponds to* $\mathbb{Z}\otimes(\mathrm{id}\wedge h)$ *under* $\varphi$ *is given by*

$$\check{h}(f)(t)=\sum_{h(s)=t} f(s).$$

*Proof.* (a) There is a map $R_{S,X}\colon (M\otimes X)\wedge S\to M\otimes(X\wedge S)$ defined by

$$R_{S,X}\left(\left(\sum_i m_i x_i\right)\wedge s\right)=\sum_i m_i(x_i\wedge s).$$

The map $\varphi$ is defined by

$$\varphi(f)=\sum_{s\in S} R_{S,X}(f(s)\wedge s).$$

It is easy to check that $\varphi$ is a $G$-homeomorphism.

(b) Straightforward from the definition of $\varphi$.

□

Our approach to proving (3) is to show that the functor $(X,A)\mapsto\pi_*^G(M\otimes(X/A))$ satisfies the axioms for an equivariant homology theory with $\underline{M}$ coefficients. Just as in the case of $\underline{\mathbb{Z}}$ coefficients, the main step in this proof is to show that the functor $X\mapsto M\otimes X$ transforms $G$-cofiber sequences into $G$-fiber sequences.

**Proposition 2.5.** *For a $G$-CW-pair $(X,A)$ the projection $M\otimes X\to M\otimes(X/A)$ is naturally $G$-homotopy equivalent to an equivariant Serre fibration.*

*Proof.* Let $(X_\bullet, A_\bullet)$ be the singular complex of $(X,A)$. Then $(X,A)$ is naturally $G$-equivalent to $|(X_\bullet,A_\bullet)|$ and, by continuity of $M\otimes -$, $(M\otimes X, M\otimes A)$ is naturally $G$-equivalent to $(M\otimes|X_\bullet|, M\otimes|A_\bullet|)$. By Remark 2.3, the pair $(M\otimes|X_\bullet|, M\otimes|A_\bullet|)$ is naturally $G$-homeomorphic to $|(M\otimes X_\bullet, M\otimes A_\bullet)|$. It is easy to see that, for any $H\le G$, the natural map

$$\{M\otimes X_\bullet\}^H/\{M\otimes A_\bullet\}^H\to\{M\otimes(X_\bullet/A_\bullet)\}^H,$$

is an isomorphism (let $[x]$ denote the class of $x\in X_\bullet$ in $X_\bullet/A_\bullet$ and use the fact that every nonzero element of $M\otimes(X_\bullet/A_\bullet)$ is of the form $\sum_i m_i\otimes[x_i]$ with $x_i\in X_\bullet - A_\bullet$). Therefore the projection $\{M\otimes X_\bullet\}^H\to\{M\otimes(X_\bullet/A_\bullet)\}^H$ is a fibration for it is a surjection of simplicial abelian groups (see [4, III.2.10]). Since the functors $|-|$ and $-^H$ commute, it follows that the projection $|M\otimes X_\bullet|^H\to|M\otimes(X_\bullet/A_\bullet)|^H$ is a Serre fibration, for any $H\le G$. □

**Remark 2.6.** It is possible to show that the projection $M\otimes X\to M\otimes(X/A)$ is a locally trivial $(G,\alpha,M\otimes A)$-bundle, where $\alpha: G\to\mathrm{Aut}(M\otimes A)$ is induced by the action of $G$ on $M\otimes A$ (cf. [8, Thm. 2.7]). We will not prove this result here as it will not be needed.

**Corollary 2.7.** *For a $G$-$CW$ pair $(X,A)$ there is a natural equivalence*

$$\pi_n^G(M\otimes(X/A))\cong H_n^G(X,A;\underline{M}).$$

*Proof.* Consider the functors $\mathfrak{h}_k(X,A)\overset{\mathrm{def}}{=}\pi_k^G(M\otimes(X/A))$, $k\ge 0$, defined on the category of $G$-CW-pairs[2]. We will show that $\mathfrak{h}_*$ satisfies the axioms of an ordinary $G$-homology theory with coefficients $\underline{M}$:

[2]For $A=\emptyset$ we set $X/A=X_+$.

*$G$-homotopy axiom.* A pair of $G$-homotopic maps $f_0, f_1 : (X, A) \to (Y, B)$ induces equivariant abelian group homomorphisms $M \otimes f_0, M \otimes f_1 : M \otimes (X/A) \to M \otimes (Y/B)$ which are $G$-homotopic: if $f : (X, A) \times I \to (Y, B)$ is a homotopy, then $t \mapsto M \otimes f_t$ is a $G$-homotopy from $M \otimes f_0$ to $M \otimes f_1$, hence $M \otimes f_0$, $M \otimes f_1$ induce the same map on $\mathfrak{h}_*(X, A)$.
*Excision axiom.* If $f : (X, A) \to (Y, B)$ is a relative homeomorphism of $G$-pairs then $M \otimes f : M \otimes (X/A) \to M \otimes (Y/B)$ is a $G$-homeomorphism which induces an isomorphism $\mathfrak{h}_*(X, A) \to \mathfrak{h}_*(Y, B)$
*Exact sequences.* By Proposition 2.5 there is a natural transformation $\delta : \mathfrak{h}_k(X, A) \to \mathfrak{h}_{k-1}(A, \emptyset)$ which fits in an exact sequence

$$\cdots \to \mathfrak{h}_k(A) \to \mathfrak{h}_k(X) \to \mathfrak{h}_k(X, A) \to \mathfrak{h}_{k-1}(A) \to \cdots$$

*Dimension axiom.* For $X = G/H$ we have

$$\mathfrak{h}_k(X) = \pi_k^G(M \otimes (G/H_+)) \cong [S^k, F(G/H_+, M)]_G \cong [S^k \wedge G/H_+, M]_G \cong [S^k, M]_H,$$

where we used Lemma 2.4. It follows that $\mathfrak{h}_k(G/H) = 0$ for $k > 0$ and $\mathfrak{h}_0(G/H) \cong M^H$. □

## 3. The $G$-spectrum $M \otimes \mathbb{S}$

In this section we use the functor $M \otimes -$ to produce a spectrum $M \otimes \mathbb{S}$ which we identify as the Eilenberg-Mac Lane spectrum $H\underline{M}$ in Proposition 3.7. In order to define $M \otimes \mathbb{S}$ we will use the fact that, for a discrete commutative $G$-ring $R$, the functor $R \otimes -$ is a functor with smash products (FSP).

**Definition 3.1** ([9])**.** A commutative $\mathcal{T}_G$-FSP is a $G\mathcal{T}$-enriched functor $F : \mathcal{T}_G \to \mathcal{T}_G$ with natural transformations $\mu_{X,Y} : F(X) \wedge F(Y) \to F(X \wedge Y)$ and $\eta_X : X \to F(X)$ such that the composite

$$F(X) \simeq S^0 \wedge F(X) \xrightarrow{\eta_{S^0} \wedge \mathrm{id}_{F(X)}} F(S^0) \wedge F(X) \xrightarrow{\mu_{S^0,X}} F(S^0 \wedge X) \simeq F(X),$$

is the identity and

(i) Unit Property: $\mu_{X \wedge Y} \circ (\eta_X \wedge \eta_Y) = \eta_{X \wedge Y}$.
(ii) Associativity: $\mu_{X,Y \wedge Z} \circ (\mathrm{id}_{F(X)} \wedge \mu_{Y,Z}) = \mu_{X \wedge Y,Z} \circ (\mu_{X,Y} \wedge \mathrm{id}_{F(Z)})$.
(iii) Commutativity: $\mu_{Y,X} \circ \tau_{F(X),F(Y)} = F(\tau_{X,Y}) \circ \mu_{X,Y}$, where $\tau$ is the permutation isomorphism.

A (left) module over $F$ is a $G\mathcal{T}$-enriched functor $E : \mathcal{T}_G \to \mathcal{T}_G$ with continuous natural maps $\sigma_{X,Y} : F(X) \wedge E(Y) \to E(X \wedge Y)$, such that the composite

$$E(X) \simeq S^0 \wedge E(X) \xrightarrow{\eta_{S^0} \wedge \mathrm{id}_{E(X)}} F(S^0) \wedge E(X) \xrightarrow{\sigma_{S^0,X}} E(S^0 \wedge X) \simeq E(X),$$

is the identity and $\sigma_{X,Y \wedge Z} \circ (\mathrm{id}_{F(X)} \wedge \sigma_{Y,Z}) = \sigma_{X \wedge Y,Z} \circ (\mu_{X,Y} \wedge \mathrm{id}_{E(Z)})$.

**Proposition 3.2.** *Let $R$ be a discrete commutative $G$-ring and let $M$ be a discrete $R[G]$-module. Then the functor $R \otimes -$ is a commutative $\mathcal{T}_G$-FSP and $M \otimes -$ is a module over $R \otimes -$.*

*Proof.* Let $\mathbf{1}$ denote the identity element of $R$. We define $\eta_X : X \to R \otimes X$ by $\eta_X(x) = \mathbf{1}x$. Considering $X \wedge Y$ included in $R \otimes X \wedge R \otimes Y$ via $\eta_X \wedge \eta_Y$ we define $\mu_{X,Y} : (R \otimes X) \wedge (R \otimes Y) \to R \otimes (X \wedge Y)$ as the $R$-bilinear extension of $\eta_{X \wedge Y}$; *i.e.* $\mu_{X,Y}\left((\sum r_i' x_i) \wedge (\sum r_j'' y_j)\right) = \sum_{i,j} r_i' r_j'' (x_i \wedge y_j)$.

By construction, the composite

$$R\otimes X \simeq S^0\wedge(R\otimes X) \xrightarrow{\eta_{S^0}\wedge \mathrm{id}_{R\otimes X}} R\otimes S^0\wedge R\otimes X \xrightarrow{\mu_{S^0,X}} R\otimes(S^0\wedge X)\simeq R\otimes X$$

is the identity map. Moreover, the transformations $\eta$ and $\mu$ satisfy:

(i) Unit Property: follows by construction of $\mu_{X,Y}$.
(ii) Associativity: follows because both maps are $R$-trilinear extensions of $\eta_{X\wedge Y\wedge Z}$.
(iii) Commutativity: follows easily from the definition of $\mu_{Y,X}$ as an $R$-bilinear extension of $\eta_{Y\wedge X}$ (where the commutativity of $R$ was used implicitly).

Hence we conclude that $R\otimes -$ is a commutative $\mathcal{T}_G$-FSP.

Given a discrete $R[G]$-module $M$ we define $\sigma_{X,Y} : (R\otimes X)\wedge(M\otimes Y)\to M\otimes(X\wedge Y)$ by the formula $\sigma_{X,Y}\left((\sum r_i x_i)\wedge(\sum m_j y_j)\right)=\sum_{i,j} r_i m_j (x_i\wedge y_j)$. As in the case where $M=R$ it follows that the composite

$$M\otimes X\simeq S^0\wedge(M\otimes X)\xrightarrow{\eta_{S^0}\wedge\mathrm{id}_{M\otimes X}} R\otimes S^0\wedge M\otimes X\xrightarrow{\sigma_{S^0,X}} M\otimes(S^0\wedge X)\simeq M\otimes X$$

is the identity map. The equality $\sigma_{X,Y\wedge Z}\circ(\mathrm{id}_{R\otimes X}\wedge\sigma_{Y,Z})=\sigma_{X\wedge Y,Z}\circ(\mu_{X,Y}\wedge\mathrm{id}_{M\otimes Z})$ follows as in (ii) above. □

We can now define the $G$-spectrum $M\otimes\mathbb{S}$ and proceed to identify its equivariant homotopy type. The problem of identifying the equivariant homotopy type of the $M\otimes\mathbb{S}$ in the case where $M=\mathbb{Z}$ was first addressed by Lima-Filho in [8]. Unfortunately, a small computational error led him to incorrectly identify it as the Eilenberg-Mac Lane spectrum of the Burnside ring Mackey functor.

**Definition 3.3.** Let $X$ be a $G$-CW-complex. The correspondence $V\mapsto M\otimes\,\Sigma^V X$ where $V$ runs over an indexing set $\mathcal{A}$ of a complete $G$-universe $\mathcal{U}$, defines a $G$-prespectrum, whose structural maps are the following composites

$$S^W\wedge(M\otimes\Sigma^V X)\xrightarrow{\eta_{S^W}\wedge\mathrm{id}_{M\otimes\Sigma^V X}}(R\otimes S^W)\wedge(M\otimes\Sigma^V X)\xrightarrow{\sigma_{S^W,\Sigma^V X}} M\otimes\Sigma^{W+V}X.$$

This $G$-prespectrum will be denoted by $M\otimes S^\infty X$. The associated $G$-spectrum will be denoted $M\otimes(\Sigma^\infty X)$, except if $X=S^0$ in which case it will be denoted $M\otimes\mathbb{S}$.

**Remark 3.4.** Note that, for a discrete commutative $G$-ring $R$, $R\otimes\mathbb{S}$ is an equivariant $E^\infty$-ring spectrum as it is obtained from a commutative FSP (see [9]).

Before we can proceed we need to recall a few more facts from equivariant homotopy theory.

**Definition 3.5** ([5])**.** Let $X,Y$ be $G$-spaces, let $f:X\to Y$ be a $G$-map and let $V$ be a finite dimensional $G$-representation. For each subgroup $H$ of $G$, let $V(H)$ denote an $H$-invariant complement of $V^H$.

(i) $f$ is a $|V^*|$-equivalence if, for every subgroup $H$ of $G$, $f^H : X^H\to Y^H$ is a $|V^H|$-equivalence.
(ii) $f$ is a $V$-equivalence if it is a $|\mathbf{0}^*|$-equivalence and for every subgroup $H$ of $G$, $f_*:\pi^H_{V(H)+q}X\to\pi^H_{V(H)+q}Y$ is an isomorphism for $0\le q<|V^H|$ and is an epimorphism for $q=|V^H|$.
(iii) $X$ is $|V^*|$-connected if $X^H$ is $|V^H|$-connected, for every subgroup $H$ of $G$.
(iv) $X$ is $V$-connected if, $\pi^H_{V(H)+q}X=0$ for $0\le q\le |V^H|$, for every subgroup $H$ of $G$.

**Lemma 3.6** ([5])**.** *Let $X, Y$ be $G$-spaces, $V$ be a finite dimensional $G$-representation, and $f : X \to Y$ be a $G$-map. Then*

*(i) $X$ is $V$-connected if and only if it is $|V^*|$-connected.*
*(ii) $f$ is a $V$-equivalence if and only if it is a $|V^*|$-equivalence.*

**Proposition 3.7.** *For a finite group $G$, the $G$-spectrum $M \otimes \mathbb{S}$ is an Eilenberg-Mac Lane spectrum $H\underline{M}$.*

*Proof.* In Lemma 3.8 we prove that, for any representation $V$ of $G$ and for $n > 0$,

$$\pi^G_{V+n}(M \otimes S^V) = 0.$$

The proof of Lemma 3.9(a) implies that $M \otimes S^V$ is $(|V^*| - 1)$-connected hence, for $n < 0$, we have

$$\pi^G_n(M \otimes \mathbb{S}) = \operatorname*{colim}_V [S^{V+n}, M \otimes S^V]_G = 0.$$

Hence $\pi^G_n(M \otimes \mathbb{S}) = 0$, for $n \neq 0$. Since this holds for every finite group $G$ it follows that $\underline{\pi_n}(M \otimes \mathbb{S}) = 0$ for $n \neq 0$. It remains to show that $\underline{\pi_0}(M \otimes \mathbb{S}) = \underline{M}$.

We start by observing that the inclusion $\Psi : M \hookrightarrow (M \otimes \mathbb{S})(0)$ determines a map of coefficient systems $\Psi_* : \underline{M} \to \underline{\pi_0}(M \otimes \mathbb{S})$. Thus, for each $H \leq G$, there is a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\Psi^e_*"] & \pi_0(M \otimes \mathbb{S}) \\ M^H \arrow[u, hook] \arrow[r, "\Psi^H_*"] & \pi^H_0(M \otimes \mathbb{S}). \arrow[u] \end{tikzcd}$$

Since $\Psi^e_*$ is an isomorphism, the maps $\Psi^H_*$ are injective. In Lemma 3.9 we show that $\Psi^H_*$ is also surjective. It follows that $\Psi_*$ is an isomorphism of coefficient systems. Since $\underline{M}$ is completely determined by the groups $\underline{M}(G/H)$ and the restriction maps (see the proof of [6, Prop.V.9.10] for $\underline{M} = \underline{\mathbb{Z}}$), it follows that $\underline{\pi_0}(M \otimes \mathbb{S}) = \underline{M}$ as Mackey functors. □

**Lemma 3.8.** *Let $V$ be a finite dimensional $G$-representation. Then for all $k > 0$,*

$$\pi^G_{V+k}(M \otimes S^V) = 0.$$

*Proof.* In the case of $M = \mathbb{Z}$ this is proved in [8]. The proof for a general $M$ is exactly the same. We include the details for completeness.

Let $(X_p)_{p \leq d}$ denote the skeleta of a $G$-CW decomposition of $S^V$. Consider the tower

$$\begin{tikzcd} F(S^V, M \otimes X_0)^G \arrow[d] & \\ F(S^V, M \otimes X_1)^G \arrow[d] \arrow[r] & F(S^V, M \otimes (X_1/X_0))^G \\ \vdots \arrow[d] & \\ F(S^V, M \otimes X_d)^G \arrow[r] & F(S^V, M \otimes (X_d/X_{d-1}))^G. \end{tikzcd}$$

The $E^1$ term of the spectral sequence associated to the homotopy groups of this tower is given by

$$E^1_{p,q} = \pi^G_{V+p+q}\left(M \otimes (X_p/X_{p-1})\right),$$

and the sequence converges to $\pi^G_{V+p+q}(M \otimes S^V)$ in the range $(p,q)$ where $p+q > 0$.

Denoting the set of $p$-cells of $X$ by $\Lambda_p$, we have

$$\begin{aligned} E^1_{p,q} &= \pi^G_{V+p+q}\left(M \otimes \left(\bigvee_{\alpha\in\Lambda_p} S^p \wedge G/H_{\alpha+}\right)\right) \\ &\cong \pi^G_{V+p+q}\left(\bigoplus_{\alpha\in\Lambda_p} M \otimes (S^p \wedge G/H_{\alpha+})\right) \\ &\cong \bigoplus_{\alpha\in\Lambda_p} \pi^G_{V+p+q} M \otimes (S^p \wedge G/H_{\alpha+}). \end{aligned}$$

Now let $k > 0$ and let $\alpha \in \Lambda_p$ be a cell of type $H_\alpha$. We have,

$$\pi^G_{V+k}(M \otimes (S^p \wedge G/H_{\alpha+})) \cong \pi^G_{V+k}(F(G/H_{\alpha+}, M \otimes S^p)) \cong \pi^{H_\alpha}_{V+k}(M \otimes S^p).$$

Write $V$ as $V^{H_\alpha} \oplus V(H_\alpha)$. Then, since $M \otimes S^p$ is an Eilenberg-Mac Lane space $K(\underline{M}, p)$,

$$[S^{V+k}, M \otimes S^p]_{H_\alpha} \cong \widetilde{H}^{p-|V^{H_\alpha}|-k}_{H_\alpha}(S^{V(H_\alpha)}; \underline{M}). \tag{4}$$

Note that $k > 0$ and $p \leq |V^{H_\alpha}|$ because the cell $\alpha$ is contained in $\bigcup_{g\in G}\{gS^{V^{H_\alpha}}g^{-1}\}$. Hence the right-hand group in (4) is zero. We conclude that $E^1_{p,q} = 0$, if $p+q = k$, which implies that $\pi^G_{V+k}(M \otimes S^V) = 0$. □

**Lemma 3.9.** *The inclusion*

$$M \xrightarrow{\Psi} \operatorname*{colim}_V \Omega^V(M \otimes S^V)$$

*induces a surjective map* $M^G \to \operatorname{colim}_V \pi^G_V(M \otimes S^V)$.

*Proof.* For each $m \in M$, $\Psi(m)$ is represented in the colimit above by maps of the form $m \otimes \mathrm{id}$ (see 2.2(c)), where $\mathrm{id} : S^V \to S^V$ is the identity. The lemma will follow if we show that

(a) The inclusion $i_V : M \wedge S^V \to M \otimes S^V$ is surjective on $\pi^G_V$
(b) Any $G$-equivariant map $S^V \to M \wedge S^V$ is $G$-homotopic in $\operatorname{colim}_V \Omega^V(M \otimes S^V)$ to $\Psi(m)$, for some $m \in M$

*Proof of (a).* It suffices to prove that $i_V$ is a $|V^*|$-equivalence — see Lemma 3.6.

Fix $H \leq G$, set $n(H) = |V^H|$ and write $V$ as $V^H \oplus V(H)$. The inclusion $(M \wedge S^V)^H \subset (M \otimes S^V)^H$ factors as follows

$$(M \wedge S^V)^H = M^H \wedge S^{V^H} \subset (M \otimes S^{V^H})^H \subset (M \otimes S^V)^H.$$

Since the first inclusion is clearly an $n(H)$-equivalence, we need only show that the same is true of the second inclusion. By Corollary 2.7, this translates into a statement about the map induced in Bredon homology with $\underline{M}$ coefficients by the inclusion $S^{V^H} \subset S^V$. For $k < n(H)$, we have

$$\pi^H_k(M \otimes S^V) \cong \widetilde{H}^H_k(S^V; \underline{M}) \cong \widetilde{H}_{k-n(H)}(S^{V(H)}; \underline{M}) = 0.$$

For $k = n(H)$, we have to show that the map

$$H_k^H(S^{V^H}; \underline{M}) \to H_k^H(S^V; \underline{M})$$

is onto. Now observe that $S^V \cong S^{V^H} * S(V(H))$ — where $*$ denotes the unreduced join and $S(V(H))$ is the unit sphere. Thus

$$S^V/S^{V^H} \cong S^{V^H+1} \wedge S(V(H))_+,$$

and this gives $\widetilde{H}^H_{n(H)}(S^V/S^{V^H}; \underline{M}) = 0$. The Bredon homology exact sequence of the pair $(V, V^H)$ shows that $\widetilde{H}^H_{n(H)}(S^{V^H}; \underline{M}) \to \widetilde{H}^H_{n(H)}(S^V; \underline{M})$ is onto, as required.

*Proof of (b).* We start by considering the case where $M = \mathbb{Z}$. It suffices to show that given an element $[f] \in \{S^0, S^0\}_G$, there is a $G$-representation $V$ and an integer $k$ such that the map $1 \otimes f$ is $G$-homotopic in $\mathbb{Z} \otimes S^V$ to the map $k \otimes \mathrm{id}$ (the proof of (a) shows that $S^V \subset \mathbb{Z} \otimes S^V$ is surjective on $\pi_V^G$). Recall that $\{S^0, S^0\}_G$ is isomorphic to the Burnside ring $A(G)$. We will use the ring isomorphism $I : A(G) \to \{S^0, S^0\}_G$ constructed in [12, §II.8]. In particular we will use the additive basis $\{I(G/H) : H \leq G\}$ for $\{S^0, S^0\}_G$. We will show that $1 \otimes I(G/H)$ is $G$-homotopic to $|G/H| \otimes \mathrm{id}$. Since $I(G/G) = \mathrm{id}$ we can assume $H < G$ and $|G| > 1$. The proof proceeds by induction on $|G|$. Now, the element $I(G/H)$ can be represented by a composite:

$$S^V \xrightarrow{\tau(G/H)} S^V \wedge G/H_+ \xrightarrow{pr} S^V, \tag{5}$$

where $V$ is a large enough $G$-representation and $pr$ is the projection — see [12, §II.8]. It follows that $1 \otimes I(G/H)$ factors as

$$S^V \xrightarrow{1 \otimes \tau(G/H)} \mathbb{Z} \otimes (S^V \wedge G/H_+) \xrightarrow{\mathbb{Z} \otimes pr} \mathbb{Z} \otimes S^V.$$

By Lemma 2.4(a), there is an isomorphism $\varphi_* : [S^V, \mathbb{Z} \otimes S^V]_H \cong [S^V, \mathbb{Z} \otimes (S^V \wedge G/H_+)]_G$ and so, assuming $V$ is large enough, it follows by the induction hypothesis that $1 \otimes \tau(G/H) = \varphi_*(k \otimes \mathrm{id})$, for some integer $k$. From Lemma 2.4(b), it follows that $(\mathbb{Z} \otimes pr) \circ (1 \otimes \tau(G/H)) = k|G/H| \otimes \mathrm{id}$. This completes the proof of the case $M = \mathbb{Z}$. Since non-equivariantly $I(G/H) = |G/H|\,\mathrm{id}$, we have $k = 1$.

We now consider the case of a general $M$. Decomposing $M$ into $G$-orbits the problem can be reduced to proving the following assertion: given an inclusion $\zeta : G/H_+ \to M$ and $[f] \in \{S^0, G/H_+\}_G$ there is a large enough representation $V$ such that the composite

$$S^V \xrightarrow{f} G/H_+ \wedge S^V \xrightarrow{\zeta \wedge \mathrm{id}} M \wedge S^V \xrightarrow{\iota_{S^V}} M \otimes S^V \tag{6}$$

is $G$-homotopic to $m \otimes \mathrm{id}$, for some $m \in M^G$. We note that the composite (6) factors as follows

$$S^V \xrightarrow{f} G/H_+ \wedge S^V \xrightarrow{\eta_{G/H_+ \wedge S^V}} \mathbb{Z} \otimes (G/H_+ \wedge S^V) \xrightarrow{\zeta_\sharp} M \otimes S^V,$$

where $\zeta_\sharp$ is the group homomorphism induced by $\iota_{G/H_+ \wedge S^V} \circ (\zeta \wedge \mathrm{id})$. From the equivalence $\varphi_* : [S^V, \mathbb{Z} \otimes S^V]_H \to [S^V, \mathbb{Z} \otimes (S^V \wedge G/H_+)]_G$ of Lemma 2.4 and from the case $M = \mathbb{Z}$, we see that $\eta_{G/H_+ \wedge S^V} \circ f$ is homotopic to $\varphi_*(k \otimes \mathrm{id})$, for some integer $k$. A simple computation shows that setting $m = \sum_{r \in \zeta(G/H)} r$, we have $\zeta_\sharp \circ (\varphi_*(k \otimes \mathrm{id})) = km \otimes \mathrm{id}$. This completes the proof. $\square$

## 4. The $RO(G)$-graded version of the Dold-Thom Theorem

Recall that, since $\underline{M}$ is a Mackey functor, the homology theory $H_*^G(-;\underline{M})$ extends to an $RO(G)$-graded theory which is represented by $H\underline{M}$ — see [10]. Having identified $M \otimes \mathbb{S}$ as an Eilenberg-Mac Lane $G$-spectrum $H\underline{M}$ it follows that for a finite dimensional $G$-representation $V$ and a based $G$-CW-complex $X$, we have

$$\widetilde{H}_V^G(X;\underline{M}) \cong \pi_V((M \otimes \mathbb{S}) \wedge X).$$

We will show that actually,

$$\widetilde{H}_V^G(X;\underline{M}) \cong \pi_V^G(M \otimes X).$$

**Definition 4.1.** For a $G$-CW-complex $X$ and $V \in \mathcal{U}$ there is a map

$$\begin{aligned} R_{X,S^V} : (M \otimes S^V) \wedge X &\to M \otimes (S^V \wedge X) \\ \left(\sum_i m_i x_i\right) \wedge y &\mapsto \sum_i m_i (x_i \wedge y). \end{aligned}$$

It is clear that the maps $R_{X,S^V}$ assemble into a map of $G$-spectra

$$(M \otimes \mathbb{S}) \wedge X \to M \otimes \Sigma^\infty X$$

(see Definition 3.3) which we denote by $R_{\Sigma^\infty X}$. The zero component of this map of spectra will be denoted by $R^0_{\Sigma^\infty X}$.

**Proposition 4.2.** *Let $X$ be a based $G$-CW-complex. The inclusion*

$$\Psi_X : M \otimes X \to \operatorname*{colim}_V \Omega^V \{M \otimes (S^V \wedge X)\} = (M \otimes \Sigma^\infty X)(0)$$

*and the map*

$$R^0_{\Sigma^\infty X} : ((M \otimes \mathbb{S}) \wedge X)(0) \to (M \otimes \Sigma^\infty X)(0)$$

*are $G$-homotopy equivalences.*

*Proof.* From what was proved so far we know that

$$\pi_n^G(M \otimes X) \cong \widetilde{H}_n^G(X;\underline{M}) \cong \pi_n^G(\operatorname*{colim}_V \Omega^V \{(M \otimes S^V) \wedge X\}).$$

Consider the functors $\mathcal{H}_*^G$ from pointed G-CW-complexes to abelian groups, defined by

$$\mathcal{H}_n^G(X) \stackrel{\text{def}}{=} \operatorname*{colim}_V [S^{V+n}, M \otimes (S^V \wedge X)]_G = \pi_n^G((M \otimes \Sigma^\infty X)(0)).$$

Using Proposition 2.5 and the fact that colimits preserve exact sequences, one can easily show that $\mathcal{H}_*^G(-)$ defines an equivariant homology theory. Also $\Psi_-$ induces a transformation of equivariant cohomology theories which, for each $X$, is defined by

$$\Psi_{X*} : \pi_*^G(M \otimes X) \to \pi_*^G(M \otimes \Sigma^\infty X)(0) = \mathcal{H}_n^G(X).$$

Hence it suffices to show that $\Psi_{X*}$ is an isomorphism for $X = G/H_+$, $H \leq G$. By Propositions 3.7 and 2.4, we have

$$\mathcal{H}_n^G(G/H_+) \cong \operatorname*{colim}_V [S^{V+n}, M \otimes S^V]_H \cong \begin{cases} M^H & n = 0 \\ 0 & n \neq 0, \end{cases}$$

and it is easy to check that this isomorphism is $\Psi_{G/H_{+*}}$. This completes the proof of the first assertion.

The proof of the statement concerning $R^0_{\Sigma^\infty X}$ is similar. We observe that $R^0_{\Sigma^\infty -}$ induces a map on Bredon homotopy groups which is is a self-transformation of the equivariant homology theory $\widetilde{H}^G_*(-;\underline{M})$. Hence it suffices to show that $R^0_{\Sigma^\infty G/H_+}$ is an equivariant homotopy equivalence, for each $H \leq G$. Consider the composite

$$\{M \otimes \mathbb{S}\} \wedge G/H_+ \xrightarrow{R_{\Sigma^\infty G/H_+}} M \otimes \{\mathbb{S} \wedge G/H_+\} \xrightarrow{\varphi^{-1}} F(G/H_+, M \otimes \mathbb{S}),$$

where $\varphi$ is the equivalence induced by the space level equivalence of Lemma 2.4. We claim that $\varphi^{-1} \circ R_{\Sigma^\infty G/H_+}$ is the $G$-map $\widetilde{f}$ determined by the $H$-map $f \colon \{M \otimes \mathbb{S}\} \wedge G/H_+ \to M \otimes \mathbb{S}$ which collapses the complement of $\{M \otimes \mathbb{S}\} \wedge eH$ to the base point. Indeed, on the space level, we have

$$\begin{aligned} \varphi^{-1}\Big(R_{G/H_+, S^V}\big((mx) \wedge g_1H\big)\Big)(g_2H) &= \begin{cases} mx & \text{if } g_1H = g_2H \\ 0 & \text{otherwise} \end{cases} \\ &= \widetilde{f}\Big((mx) \wedge g_1H\Big)(g_2H). \end{aligned}$$

Thus, $\varphi^{-1} \circ R_{\Sigma^\infty G/H_+}$ is the inverse of the Wirthmüller isomorphism (see [6, Lemma II.6.10 and Thm II.6.2] or [12, Prop. II.6.12]). We conclude that, for any based $G$-CW-complex $X$, $R^0_{\Sigma^\infty X}$ induces an isomorphism on Bredon homotopy groups, and hence it is an equivariant homotopy equivalence. $\square$

**Theorem 1.1** *Let $X$ be a based $G$-CW-complex and let $V$ be a finite dimensional $G$-representation, then $M \otimes X$ is an equivariant infinite loop space and there is a natural equivalence*

$$\pi^G_V(M \otimes X) \cong \widetilde{H}^G_V(X; \underline{M}).$$

*Proof.* This is an immediate consequence of the Proposition above. $\square$

**Definition 4.3.** Let $V \in RO(G)$. A $K(\underline{M}, V)$ space is a classifying space for the functor $\widetilde{H}^V_G(-;\underline{M})$.

**Corollary 4.4.** *Let $V$ be a finite dimensional representation of $G$. The space $M \otimes S^V$ is a $K(\underline{M}, V)$-space.*

## 5. Application: Lawson homology for real varieties

An algebraic $p$-cycle on a complex projective variety $X$ is a finite formal sum $\sigma = \sum_i n_i V_i$ where the $n_i$'s are integers and the $V_i$'s are (irreducible) subvarieties of dimension $p$ in $X$. The group of $p$-cycles $\mathcal{Z}_p(X)$ can be equipped with a Hausdorff topology making it a topological group. The homotopy groups of $\mathcal{Z}_p(X)$ form a set of invariants called the Lawson homology of $X$ [2], [7]. In [3] it is shown that there is a natural map $s^p : \mathcal{Z}_p(X) \to \Omega^{2p}\mathcal{Z}_0(X)$. Passing to homotopy groups and applying the classical Dold-Thom theorem we get a map

$$\pi_k \mathcal{Z}_p(X) \to \pi_{k+2p}\mathcal{Z}_0(X) \cong H_{k+2p}(X; \mathbb{Z}).$$

Thus Lawson homology maps to singular homology. This map is very useful in computations.

If $X$ is a real projective variety (*i.e.* defined by real equations) then the Galois group $\mathbb{Z}/2 = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathcal{Z}_p(X)$. In [1] we define a version of Lawson homology for real projective varieties as $\mathbb{Z}/2$-equivariant homotopy groups of $\mathcal{Z}_p(X)$. One can show the map $s^p$ is actually an equivariant map $s^p : \mathcal{Z}_p(X) \to \Omega^{\mathbb{C}^p}\mathcal{Z}_0(X)$, where

$\mathbb{C}^p$ is considered as a $\mathbb{Z}/2$-representation under the action of complex conjugation. Passing to homotopy groups and applying Theorem 1.1 we get a map

$$\pi_V^{\mathbb{Z}/2}\mathcal{Z}_p(X) \to \pi_{V+\mathbb{C}^p}^{\mathbb{Z}/2}\mathcal{Z}_0(X) \cong H_{V+\mathbb{C}^p}^{\mathbb{Z}/2}(X;\underline{\mathbb{Z}}),$$

so that real Lawson homology maps to equivariant homology with $\underline{\mathbb{Z}}$ coefficients. This map is very useful in the computation of real Lawson homology for real varieties such as products of projective spaces, grassmanianns and certain quadrics.

In the case of projective space $\mathbb{P}^n$ one can use Theorem 1.1 to show [1] that there is a $\mathbb{Z}/2$-homotopy equivalence

$$\mathcal{Z}_p(\mathbb{P}^n) \cong K(\underline{\mathbb{Z}},0) \times K(\underline{\mathbb{Z}},\mathbb{C}) \times \cdots \times K(\underline{\mathbb{Z}},\mathbb{C}^{n-p}).$$

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