4^{a} Ficha de Exercícios de AMIII

Resolução Sumária

1. Escreva o tensor-2 em $\mathbb{R}^3$ dado por $T(\mathbf{v},\mathbf{w}) = 3(v^1+v^2)(w^2+w^3) - \frac{1}{2}v^3w^2 + \sqrt{5}v^3w^3$ como combinação linear dos elementos da base $\{dx^i \otimes dx^j\}_{i,j=1}^3$ para $T^2(\mathbb{R}^3)$.

   Resolução:

   $$T = 3dx^1 \otimes dx^2 + 3dx^1 \otimes dx^3 + 3dx^2 \otimes dx^2 + 3dx^2 \otimes dx^3 - \frac{1}{2}dx^3 \otimes dx^2 + \sqrt{5}dx^3 \otimes dx^3$$

2. Mostre que $\mathrm{Alt}(T) \in \Lambda^k(\mathbb{R}^n)$, $\forall T \in T^k(\mathbb{R}^n)$.

   Resolução: Seja $\tau \in \Sigma_k$. Temos de mostrar

   $$\mathrm{Alt}(T)(\mathbf{v}_{\tau(1)},\ldots,\mathbf{v}_{\tau(k)}) = \mathrm{sgn}(\tau)\,\mathrm{Alt}(T)(\mathbf{v}_1,\ldots,\mathbf{v}_k).$$

   Temos,

   $$\begin{aligned}\mathrm{Alt}(T)(\mathbf{v}_{\tau(1)},\ldots,\tau_{\tau(k)}) &= \sum_{\sigma\in\Sigma_k}\mathrm{sgn}(\sigma)\,T(\mathbf{v}_{\sigma(\tau(1))},\ldots,\mathbf{v}_{\sigma(\tau(k))})\\ &= \mathrm{sgn}(\tau)\sum_{\sigma\in\Sigma_k}\mathrm{sgn}(\sigma\circ\tau)\,T(\mathbf{v}_{\sigma\circ\tau(1)},\ldots,\mathbf{v}_{\sigma\circ\tau(k)})\\ &= \mathrm{sgn}(\tau)\sum_{\sigma\in\Sigma_k}\mathrm{sgn}(\sigma)\,T(\mathbf{v}_{\sigma(1)},\ldots,\mathbf{v}_{\sigma(k)})\\ &= \mathrm{sgn}(\tau)\,\mathrm{Alt}(T)(\mathbf{v}_1,\ldots,\mathbf{v}_k),\end{aligned}$$

   onde usámos o facto de que a aplicação $\sigma \mapsto \sigma\circ\tau$ é uma bijecção $\Sigma_k \to \Sigma_k$.

3. Calcule os seguintes produtos exteriores

   (a) $(2dx - dy) \wedge (dx + 3dy)$

   (b) $(ydx + dy + 3dz) \wedge (2dx + \pi dy + \frac{1}{3}dz)$

   (c) $(ydx + xdy + z^2dz) \wedge (2dx \wedge dy + xdy \wedge dz + z^4 dx \wedge dz)$

   (d) $(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \wedge (dx^1 \wedge dx^2 + dx^3 \wedge dx^4 - dx^1 \wedge dx^3)$

   Resolução:

   (a) $(2dx - dy) \wedge (dx + 3dy) = 7dx \wedge dy$

   (b) $(ydx+dy+3dz)\wedge(2dx+\pi dy+\frac{1}{3}dz) = (\pi y-2)dx\wedge dy+(\frac{y}{3}-6)dx\wedge dz+(\frac{1}{3}-3\pi)dy\wedge dz$

   (c) $(xy - xz^4 + 2z^2)dx \wedge dy \wedge dz$

(d) $2dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$

4. Sejam $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ e $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. Mostre que

(a) $dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}) = 1$;

(b) $dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ é o determinante da matriz $k \times k$ que se obtém seleccionando as colunas $i_1, \ldots, i_k$ da matriz $\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_k \end{pmatrix}$.

Resolução:

(a) Recorde-se que

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k} = k! \operatorname{Alt}(dx^{i_1} \otimes \cdots \otimes dx^{i_k}).$$

Logo,

$$\begin{aligned} dx^{i_1} \wedge \cdots \wedge dx^{i_k}(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}) &= k! \operatorname{Alt}(dx^{i_1} \otimes \cdots \otimes dx^{i_k})(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}) \\ &= k! \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \operatorname{sgn} \sigma \; dx^{i_1}(\mathbf{e}_{i_{\sigma(1)}}) \cdots dx^{i_k}(\mathbf{e}_{i_{\sigma(k)}}) \\ &= \sum_{\sigma \in \Sigma_k} \operatorname{sgn} \sigma \; \delta^{i_1}_{i_{\sigma(1)}} \cdots \delta^{i_k}_{i_{\sigma(k)}} \\ &= 1, \end{aligned}$$

onde $\delta^p_q = \begin{cases} 1, & p = q \\ 0, & p \neq q \end{cases}$.

(b) Seja $A(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ a matriz que se obtém seleccionando as colunas $i_1, \ldots, i_k$ da matriz cujas linhas são $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Pela definição de produto exterior, $dx^{i_1} \wedge \cdots \wedge dx^{i_k}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ é uma função multilinear alternante das linhas de $A(\mathbf{v}_1, \ldots, \mathbf{v}_k)$. Pela alínea anterior, o valor desta função quando $A = I_k$ é 1. O determinante é a única função que satisfaz estas três propriedades, portanto

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \det A(\mathbf{v}_1, \ldots, \mathbf{v}_k).$$

5. Dado um vector $\mathbf{v} = \sum_{i=1}^3 v^i \mathbf{e}_i \in \mathbb{R}^3$ é possível associar-lhe um covector-1, $\omega_{\mathbf{v}}$, dado por

$$\omega_{\mathbf{v}} = v^1 dx^1 + v^2 dx^2 + v^3 dx^3,$$

e um covector-2, $\Omega_{\mathbf{v}}$, dado por

$$\Omega_{\mathbf{v}} = v^1 dx^2 \wedge dx^3 + v^2 dx^3 \wedge dx^1 + v^3 dx^1 \wedge dx^2.$$

Mostra-se que

(i) $\omega_{\mathbf{v}} \wedge \omega_{\mathbf{w}} = \Omega_{\mathbf{v} \times \mathbf{w}}, \quad \forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ (onde $\times$ designa o produto externo em $\mathbb{R}^3$)

(ii) $\omega_{\mathbf{v}} \wedge \Omega_{\mathbf{w}} = (\mathbf{v} \cdot \mathbf{w}) dx^1 \wedge dx^2 \wedge dx^3, \quad \forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$

Verfique estas fórmulas no caso em que $\mathbf{v} = \sqrt{2}\mathbf{e}_1 + \sqrt{3}\mathbf{e}_2 + \sqrt{5}\mathbf{e}_3$ e $\mathbf{w} = \pi \mathbf{e}_1 + \pi^2 \mathbf{e}_2 + \pi^3 \mathbf{e}_3$.

Resolução:

(i) Temos

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \sqrt{2} & \sqrt{3} & \sqrt{5} \\ \pi & \pi^2 & \pi^3 \end{vmatrix} = (\sqrt{3}\pi^3 - \sqrt{5}\pi^2)\mathbf{e}_1 + (\sqrt{5}\pi - \sqrt{2}\pi^3)\mathbf{e}_2 + (\sqrt{2}\pi^2 - \sqrt{3}\pi)\mathbf{e}_3,$$

logo

$$\Omega_{\mathbf{v}\times\mathbf{w}} = (\sqrt{3}\pi^3 - \sqrt{5}\pi^2)dy\wedge dz - (\sqrt{5}\pi - \sqrt{2}\pi^2)dx\wedge dz + (\sqrt{2}\pi^2 - \sqrt{3}\pi)dx\wedge dy.$$

Por outro lado,

$$\begin{aligned} \omega_{\mathbf{v}}\wedge\omega_{\mathbf{w}} &= (\sqrt{2}dx + \sqrt{3}dy + \sqrt{5}dz)\wedge(\pi dx + \pi^2 dy + \pi^3 dz) \\ &= (\sqrt{2}\pi^2 - \sqrt{3}\pi)dx\wedge dy + (\sqrt{2}\pi^3 - \sqrt{5}\pi)dx\wedge dz + (\sqrt{3}\pi^3 - \sqrt{5}\pi^2)dy\wedge dz \\ &= \Omega_{\mathbf{v}\times\mathbf{w}}. \end{aligned}$$

(ii)

$$\begin{aligned} \omega_{\mathbf{v}}\wedge\Omega_{\mathbf{w}} &= (\sqrt{2}dx + \sqrt{3}dy + \sqrt{5}dz)\wedge(\pi dy\wedge dz - \pi^2 dx\wedge dz + \pi^3 dx\wedge dy) \\ &= \sqrt{2}\pi dx\wedge dy\wedge dz - \sqrt{3}\pi^2 dy\wedge dx\wedge dz + \sqrt{5}\pi^3 dz\wedge dx\wedge dy \\ &= (\sqrt{2}\pi + \sqrt{3}\pi^2 + \sqrt{5}\pi^3)dx\wedge dy\wedge dz \\ &= (\mathbf{v}\cdot\mathbf{w})dx\wedge dy\wedge dz \end{aligned}$$

6. Calcule os *pull-backs* das formas seguintes pelas aplicações indicadas

(a) $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ por $(x,y) = \mathbf{g}(\theta) = (\cos\theta, \operatorname{sen}\theta)$
(b) $\omega = x^2dx + y^2dy$ por $(x,y) = \mathbf{g}(r,\theta) = (r\cos\theta, r\operatorname{sen}\theta)$
(c) $\omega = xdx\wedge dy + ydz\wedge dx + zdx\wedge dy$ por $(x,y,z) = \mathbf{g}(\theta,\varphi) = (\operatorname{sen}\varphi\cos\theta, \operatorname{sen}\varphi\operatorname{sen}\theta, cos\varphi)$
(d) $\omega = (x^2+y^2)dx\wedge dy\wedge dz$ por $(x,y,z) = \mathbf{g}(\rho,\theta,z) = (\rho\cos\theta, \rho\operatorname{sen}\theta, z)$

Resolução:

(a) $\mathbf{g}^*(\omega) = d\theta$;
(b) $\mathbf{g}^*(\omega) = r^2(\cos^3\theta + \operatorname{sen}^3\theta)dr + r^3\cos\theta\operatorname{sen}\theta(\operatorname{sen}\theta - \cos\theta)d\theta$;
(c) $\mathbf{g}^*(\omega) = -(\operatorname{sen}^2\varphi\cos\varphi\cos\theta + \operatorname{sen}^3\varphi\operatorname{sen}^2\theta + \operatorname{sen}\varphi\cos^2\varphi)d\theta\wedge d\varphi$;
(d) $\mathbf{g}^*(\omega) = \rho^3 d\rho\wedge d\theta\wedge dz$.

7. Calcule a derivada exterior das seguintes formas

(a) $\omega = xy + e^{x^2+y^2}$
(b) $\omega = (x + \log(x+z))dx + xzdy + \operatorname{arctg}(y+z)dz$
(c) $\omega = x^3dy\wedge dz + ye^x dx\wedge dz$
(d) $\omega = \sum_{i=1}^n x^i dx^i$

Resolução:

(a) $d\omega = \left(2xe^{x^2+y^2} + y\right) dx + \left(2ye^{x^2+y^2} + x\right) dy;$

(b) $d\omega = \left(\frac{1}{1+(y+z)^2)} - x\right) dy \wedge dz - \frac{1}{x+z} dx \wedge dz + z dx \wedge dy;$

(c) $d\omega = \left(3x^2 - e^x\right) dx \wedge dy \wedge dz;$

(d) $d\omega = 0.$