REZK'S NERVE IS A MODEL FOR ∞-LOCALIZATION

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ABSTRACT. We give a direct proof that Rezk's nerve is a model for the ∞-localization of a category with respect to a set of weak equivalences.

Let $\mathbf{C}$ be a category and $\mathbf{W} \subset \mathbf{C}$ be a subcategory containing all objects of $\mathbf{C}$. Let $N(\mathbf{C}, \mathbf{W})$ the Rezk's classification diagram construction [4]. The purpose of this note is to give a direct proof that $N(\mathbf{C}, \mathbf{W})$ is a model for the ∞-localization of $\mathbf{C}$ with respect to $\mathbf{W}$. This result is is implicit in the work of Barwick-Kan, and it has been established precisely by Hinich [3], following an argument of Chris Schommer-Pries. Our argument is more direct and, we believe, well adapted to be generalized to ∞-operads and (∞, n)-categories.

Theorem 0.1. *There is an equivalence of ∞-categories between the category of functors $N(\mathbf{C}, \mathbf{W}) \to \mathbf{S}$ and the category of functors $\mathbf{C} \to \mathbf{S}$ that send all morphisms in $\mathbf{W}$ to weak equivalences.*

We take the point of view that a functor $\mathbf{C} \to \mathbf{S}$ is the same data as a left fibration $X \to \mathbf{C}$. So, the statement paraphrases to: the ∞-category of left fibrations over $\mathbf{C}$ is equivalent to the ∞-category of left fibrations over $\mathbf{C}$ which are also right fibrations when restricted to $\mathbf{W}$ (i.e. are Kan fibrations over $\mathbf{W}$).

Step 1: From a left fibration over $N(\mathbf{C}, \mathbf{W})$ to a left fibration over $\mathbf{C}$

Lemma 0.2. *Let $p : X \to N(\mathbf{C}, \mathbf{W})$ be a left fibration. Then the pullback of p along $i : W \hookrightarrow C \hookrightarrow N(\mathbf{C}, \mathbf{W})$ is a Kan fibration.*

Proof. We assume without loss of generality that p is a Reedy fibration (so in particular a degreewise Kan fibration). Since left fibrations are stable under homotopy base change, $i^*X \to W$ is a left fibration. The claim is that it is also a right fibration. This means showing that for each $f : c \to d$ in $\mathbf{W}$, the map of sets

$$(i^*X)_f \to (i^*W)_d$$

is surjective. That is, given $x \in (i^*W)_d \subset X_{0,0}$ we must find $\alpha \in (i^*W)_f \subset X_{1,0}$ such that $d_1(\alpha) = x$. Since $p : X_0 \to N(\mathbf{C}, \mathbf{W})_0$ is a Kan fibration, there exists a 1-simplex $\sigma \in X_{0,1}$ over $f \in N(\mathbf{C}, \mathbf{W})_{0,1}$ whose image under d_1 is x. Now consider the following element F in $N(\mathbf{C}, \mathbf{W})_{1,1}$:

$$\begin{array}{ccc} d & \xleftarrow{f} & c \\ \| & & \downarrow{\scriptstyle f} \\ d & = & d \end{array}$$

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Then the square of simplicial spaces

$$\begin{array}{ccc} \Delta^{0,1} & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle d_0} & & \downarrow \\ \Delta^{1,1} & \xrightarrow{F} & N(\mathbf{C}, \mathbf{W}) \end{array}$$

commutes, i.e. $d_0(F) = p(\sigma) = f$. Since p is a left fibration, a lift $\tilde{\sigma} \in X_{1,1}$ exists. The equation is $p(\tilde{\sigma}) = F$ in $N(\mathbf{C}, \mathbf{W})_{1,1}$. Then we set α to be the image of $\tilde{X}$ by the map $d_0 : X_{1,1} \to X_{1,0}$. This lives over $f : c \to d$ in $N(\mathbf{C}, \mathbf{W})_{1,0}$, viewed as the right-hand column of F in the diagrammatic representation above. □

Step 2: From a left fibration over $\mathbf{C}$ *to a left fibration over* $N(\mathbf{C}, \mathbf{W})$.

We describe the left adjoint $i_!$ to i^*. To do this, we view the category $N(\mathbf{C}, \mathbf{W})_n$ as the pullback of $\mathbf{C}^{\Delta[n]} \to \coprod_n \mathbf{C} \leftarrow \coprod_n \mathbf{W}$, where the left-hand map is induced by $\coprod_n \Delta[0] \hookrightarrow \Delta[n]$.

Let $Y \to \mathbf{C}$ be a left fibration. Then $Y^{\Delta[n]} \to \mathbf{C}^{\Delta[n]}$ is a left fibration, and by pullback we obtain a left fibration, which we denote by

$$Y_n^{\mathbf{W}} \to N(\mathbf{C}, \mathbf{W})_n \ .$$

Then we define

$$(i_!Y)_n := \operatorname*{hocolim}_{N(\mathbf{C},\mathbf{W})_n} Y_n^{\mathbf{W}} = |Y_n^{\mathbf{W}}|$$

where, for a simplicial space Z, the vertical bars $|Z|$ denote the diagonal simplicial set $[n] \mapsto Z_{n,n}$ (equivalently, the homotopy colimit over Δ^{op} or the geometric realization). Moreover, there is a canonical map $i_!Y \to N(\mathbf{C}, \mathbf{W})$ given by in degree n by $|Y_n^{\mathbf{W}}| \to |N(\mathbf{C}, \mathbf{W})_n| = N(\mathbf{C}, \mathbf{W})_n$.

Proposition 0.3. *The map* $i_!Y \to N(\mathbf{C}, \mathbf{W})$ *is a left fibration.*

Proof. Since each map $Y_n^{\mathbf{W}} \to N(\mathbf{C}, \mathbf{W})_n$ is a left fibration, the first map in the composite

$$|Y_n^{\mathbf{W}}| \to |Y_0^{\mathbf{W}} \times^h_{N(\mathbf{C},\mathbf{W})_0} N(\mathbf{C}, \mathbf{W})_n| \to |Y_0^{\mathbf{W}}| \times^h_{N(\mathbf{C},\mathbf{W})_0} N(\mathbf{C}, \mathbf{W})_n$$

is a weak equivalence. The second map is a weak equivalence by proposition 0.4, since $Y \to \mathbf{C}$ restricts to a Kan fibration on $\mathbf{W}$. □

Step 3: completing the proof.

The theorem follows from two statements:

(1) Let $p : Y \to \mathbf{C}$ be a left fibration whose restriction to $\mathbf{W}$ is a Kan fibration. Then the *unit map*

$$Y \to i^* i_! Y$$

is a weak equivalence of left fibrations over $\mathbf{C}$.

(2) Let $p : X \to N(\mathbf{C}, \mathbf{W})$ be a left fibration. Then the *counit map*

$$i_! i^* X \to X$$

is a weak equivalence of left fibrations over $N(\mathbf{C}, \mathbf{W})$.

A standard argument is that (2) is implied by (1) if i^* is homotopy conservative. To see that i^* is homotopy conservative, i.e. if $X \to Y$ is a map between left fibrations on $N(\mathbf{C}, \mathbf{W})$ such that $i^*X \to i^*Y$ is a weak equivalence of left fibrations on $\mathbf{C}$ (i.e. a degreewise weak equivalence), then

Let us begin with (1). Let $c \in \mathbf{C}$. We must show that

$$Y_0 \to (i^* i_! Y)_0 \tag{0.1}$$

is a weak equivalence in $\mathbf{S}$. In other words, we must show that the homotopy cartesian square of simplicial spaces

$$\begin{array}{ccc} Y_0 \times \Delta[0] & \longrightarrow & Y_0^{\mathbf{W}} \\ \downarrow & & \downarrow \\ C_0 \times \Delta[0] & \xrightarrow{\ i\ } & \mathbf{W} \end{array}$$

remains homotopy cartesian after realization, i.e. applying $|-|$. This is a special case of the following (perhaps not very well) known fact:

Proposition 0.4. *Suppose $X \to D$ is a Kan fibration of simplicial spaces. Then for every map $Y \to D$ the canonical map*

$$|Y \times^h_D X| \to |Y| \times^h_{|D|} |X|$$

is a weak equivalence.

Proof. We assume that $X \to D$ is a Reedy fibration between Reedy fibrant objects. We are allowed to do so since the conclusion of the lemma is invariant under degreewise weak equivalences of simplicial spaces. Then the map in the statement is weakly equivalent to the non-derived map

$$|Y \times_D X| \to |Y| \times_{|D|} |X| \, .$$

For the source, this follows from the Reedy assumption. For the target, this follows from lemma 0.5 below. But the non-derived map is an isomorphism since the geometric realization (diagonal) functor commutes with strict pullbacks. □

Lemma 0.5. *Let $X \to C$ be a Reedy fibration between Reedy fibrant simplicial spaces. If $X \to C$ is a left/right fibration in $\mathbf{S}$, then the map $|X| \to |C|$ of simplicial sets is a left/right fibration in* **Sets**.

Proof. The Reedy assumptions guarantee that the derived mapping spaces can be substituted by honest mapping spaces:

$$\operatorname{map}(\Delta[n], X) \to \operatorname{map}(\Lambda^k[n], X) \times_{\operatorname{map}(\Lambda^k[n], C)} \operatorname{map}(\Delta[n], C)$$

and that this map is a Kan fibration between Kan simplicial sets. A Kan fibration between Kan simplicial sets $p : E \to B$ that is surjective on π_0 is surjective on each degree. Therefore, under the left/right fibration hypothesis, the map in the display is surjective on all degrees m provided $0 \le k < n$, respectively $0 < k \le n$. Viewing X and Y as bisimplicial sets, this condition can be rephrased as saying that the map

$$X_{\bullet,m} \to Y_{\bullet,m}$$

is a left/right fibration for every m. But, by [1, Lemma B.9] or [2, Lemma IV.4.8], for a map $X \to Y$ which is a Reedy fibration and such that the "horizontal" maps

$X_{\bullet,m} \to Y_{\bullet,m}$ are left/right fibrations (for all $m \geq 0$), its diagonal is a left/right fibration. $\square$

References

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