Oscillatory Integrals with Polynomial Phase

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To my parents,
Romylos and Diatsenta

and to Popi.

Introductory

This PhD thesis belongs to the area of real harmonic analysis. In particular, the objects we are interested in are the so called oscillatory integrals, that is integrals of functions that oscillate between their positive and negative values. The result of these oscillations is the cancellation of positive and negative values. A suitable implementation of this heuristic principle leads to "good" estimates for the integrals in question.

The primordial example of an oscillatory integral is the Fourier transform:

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi} f(x)dx.$$

A classical result that takes advantage of the oscillation of the factor $e^{-2\pi i x\cdot\xi}$ is Riemmann-Lebesgue Lemma which for $f \in L^1(\mathbb{R})$ says that $\lim_{|\xi|\to+\infty} \mathcal{F}(f)(\xi) = 0$.

‘ Already, the Fourier transform plays a central role when someone wants to study the boundedness properties of singular integral operators like the Hilbert transform

$$H(f)(x) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} f(x-y)\frac{dy}{y} = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{|y|>\epsilon} f(x-y)\frac{dy}{y},$$

as well as the corresponding maximal operator

$$H^*(f)(x) = \sup_{\epsilon>0}\frac{1}{\pi}\int_{|y|>\epsilon} f(x-y)\frac{dy}{y}.$$

The central role of the interplay between these three objects (Fourier transform, singular integrals and corresponding maximal operators) is already well known from the beginning of the development of the real methods in harmonic analysis. The successive generalizations and refinements of singular integral operators and corresponding maximal operators created the need to study more general oscillatory integrals. These integrals come in multiple forms (Fourier integral operators, convolution operators as well as variants of the Fourier transform) which makes their classification quite difficult and generates a whole new study area in harmonic analysis.

In this thesis we study three problems coming from this study area:

Problem A Let P be a polynomial of degree at most d. Consider the integral

$$I(P) = p.v. \int_{\mathbb{R}} e^{iP(t)} \frac{dt}{t}.$$

Stein and Wainger showed in [16] that $|I(P)| \leq c_d$, where the constant c_d depends only on the degree d of the polynomial and is independent of its coefficients. We want to determine the optimal dependence of $I(P)$ on the parameter d.

This problem finds the answer in theorem 3.3 and in particular we have

$$\sup_{P \in \mathcal{P}_d} |I(P)| \sim \log d.$$

Theorem 3.3 answers positively to the corresponding conjecture stated by Carbery, Wainger and Wright in [5].

Problem B This is the $n-$dimensional analogue of problem A. Let P be a real polynomial on $\mathbb{R}^n$, of degree at most d and K a homogeneous function on $\mathbb{R}^n$ of degree $-n$, having zero mean value on the unit sphere S^{n-1}. The function K can be written in the form $K(x) = \frac{\Omega(x/|x|)}{|x|^n}$ where Ω is a real function defined on the unit sphere S^{n-1}. We now consider the integral

$$I_n(P) = p.v. \int_{\mathbb{R}^n} e^{iP(x)} K(x) dx.$$

Here we wish to obtain estimates of the form

$$|I_n(P)| \leq c_d \|\Omega\|_{S^{n-1}},$$

where $\|\Omega\|_{S^{n-1}}$ is a suitable norm of the function Ω on the unit sphere S^{n-1}. Again, the constant c_d depends only on the parameter d.

Stein proved in [14] that if Ω is bounded and has zero mean value on the unit sphere then $\sup_{P \in \mathcal{P}_{d,n}} |I_n(P)| \leq c_d \|\Omega\|_{L^\infty(S^{n-1})}$.

We improve Stein's result in theorem 3.9 where we prove that if the function Ω has zero mean value and belongs to the class $L \log L$ on the unit sphere, then

$$|I_n(P)| \leq c \log d \|\Omega\|_{L \log L(S^{n-1})},$$

where c is an absolute positive constant.

Problem C Let P be a real polynomial of degree at most d on $\mathbb{R}^n$ and $Q = [0,1]^n$ be the unit cube on $\mathbb{R}^n$. The writers in [4] prove that, if the polynomial P has zero mean value on the unit cube Q, then

$$\left|\int_Q e^{iP(x)}dx\right| \leq c_{d,n}\|P\|_{L^1(Q)}^{-\frac{1}{d}}.$$

On the other hand Carbery Wright have conjectured in [6] that the constant $c_{d,n}$ could be replaced by $c\min(n,d)$ for some absolute constant c. In theorem 2.12 we show that

$$\left|\int_Q e^{iP(x)}dx\right| \leq c\min(d,n)n^{\frac{1}{2d}}\|P\|_{L^1(Q)}^{-\frac{1}{d}}$$

for some absolute positive constant c. This result answer positively to the conjecture in the case $n \leq c^d$ for some absolute positive constant c while it misses the conjectured constant by the factor $n^{\frac{1}{2d}}$ in the general case.

There are three chapters in this thesis. The first chapter is a general introduction to oscillatory integrals and contains some general results and techniques. Several known results are presented here that will either be used in what follows, or are there in order to give a flavor of the type of results we are interested in. This chapter does not contain any new result.

In the second chapter, we study oscillatory integrals with polynomial phase. Here, the analytic properties of polynomials allow us to obtain precise estimates for the corresponding oscillatory integrals. This is being done by using well known estimates contained in the first chapter as well as other results that cant be found in the bibliography. The analysis of chapter 2 leads to theorem 2.12 which is a new result and partially answers the question raised in problem C. At the same time it provides us with a series of tools that come in handy when one wants to study oscillatory integrals with polynomial phase.

In chapter 3 we study singular oscillatory integrals. These usually come up as multipliers of singular integral operators like the generalized Hilbert transform $f \mapsto p.v. \int_{\mathbb{R}} f(x-P(y))\frac{dy}{y}$. Theorems 3.3 and 3.9 answer problems A and B respectively and are both new results.

Contents

CHAPTER 1

Oscillatory integrals in harmonic analysis

Oscillatory integrals have always played a central role in harmonic analysis. We won't try here to give an extensive review of the relative results. We refer the interested reader to [14] and [15] for a more thorough description of the theory. Following the terminology introduced in [14], we divide oscillatory integrals into those of the first and those of the second kind.

For oscillatory integrals of the first kind, we study the behavior of only function which typically can be written in the form

$$I_\psi(\lambda) = \int e^{i\lambda\phi(x)}\psi(x)dx.$$

Here, we want to study the behavior of $I_\psi(\lambda)$ as the real parameter λ tends to infinity.

We also consider oscillatory integrals of the first kind, related to some singular integrals. More precisely, if we consider the Hilbert transform along a polynomial curve

$$H_P(f)(x) = p.v. \int_{\mathbb{R}} \frac{f(x-P(y))}{y} dy,$$

where P is a real polynomial of degree at most d, then the corresponding multiplier can be written in the form

$$m_P(\xi) = p.v. \int_{\mathbb{R}} e^{i\xi P(t)} \frac{dt}{t}.$$

We want to estimate the L^∞-norm of the multiplier by a constant depending only on the degree d of the polynomial and independent of its coefficients.

In oscillatory integrals of the second kind, we are dealing with the boundedness of an operator which carries an oscillatory factor in its kernel. Oscillatory integrals of the second kind can be usually written in the form

$$T_\lambda(f)(x) = \int e^{i\lambda\Phi(x,y)} K(x,y)dy.$$

The purpose here is the description of the norm of the operator T_λ as $\lambda \to \pm\infty$.

1.1 Oscillatory integrals of the first kind, one variable

We are interested in the behavior of the integral

$$I_\psi(\lambda) = \int_a^b e^{i\lambda\phi(t)}\psi(t)dt \tag{1.1}$$

for large positive λ. Here, ϕ is a real-valued smooth (phase) and ψ is complex-valued and smooth on (a,b). Often, but not always, one assumes that ψ is compactly supported in (a,b).

In one variable the theory is complete. There are three basic principles that govern the behavior of $I_\psi(\lambda)$, as $\lambda \to +\infty$. These are: the main contributions to $I_\psi(\lambda)$ come from the critical points of the phase ϕ; in addition, supposing there is only one critical point, there is a complete description of the asymptotic behavior of $I_\psi(\lambda)$ which is determined by the order of vanishing of ϕ' at this critical point; finally, there is a universal estimate for the decay of $I_\psi(\lambda)$, consistent with the asymptotic description, in terms of some uniform lower bound for some derivative of the phase function ϕ.

The principle that is studied and exploited here is the last one. Suppose we only know that

$$\left|\frac{d^k\phi(t)}{dt^k}\right| \geq 1, \ t \in [a,b],$$

for some fixed k, and we wish to obtain an estimate for the integral

$$I(\lambda) = \int_a^b e^{i\lambda\phi(t)}dt,$$

independent of a, b as well as any other quantitative attribute of the function ϕ. The change of variable $t \mapsto \lambda^{-\frac{1}{k}}t'$ shows that the only possible estimate for $I(\lambda)$ is

$$I(\lambda) \leq O(\lambda^{-\frac{1}{k}}).$$

The fact that this is indeed the case goes back to van der Corput, and is the content of the following proposition:

Proposition 1.1. *Let $\phi : [a,b] \to \mathbb{R}$ be a C^k-function and suppose that $|\phi^{(k)}(t)| \geq 1$ for some $k \geq 1$ and all $t \in [a,b]$. If $k = 1$ suppose in addition that ϕ' is monotonic. Then, for every $\lambda > 0$,*

$$\left| \int_a^b e^{i\lambda\phi(t)} dt \right| \leq \frac{c\,k}{\lambda^{\frac{1}{k}}} \tag{1.2}$$

where c is an absolute positive constant, independent of a,b,k and ϕ.

Before we proceed to the proof of Proposition 1.1, several remarks are in hand.

One way to obtain estimates for the integral in (1.2) is to successively integrate by parts. This was the original idea of van der Corput, which proves Proposition 1.1 with the factor ck being replaced by some constant c_k. The proof of Proposition 1.1 with this method is contained foe example in [15].

Let us suppose that for functions ϕ satisfying the condition

$$\left| \frac{d^k \phi(t)}{dt^k} \right| \geq 1$$

we know ((good)) estimates for the Lebesgue measure of the sublevel set

$$E_\alpha = \{t \in [a,b] : |\phi(t)| < \alpha\}, \tag{1.3}$$

in terms of $\alpha > 0$. Then, the estimate (1.2) is a direct consequence of the sublevel set estimate. This principle first appears in [1] while it's fully exploited in [4]. Using this principle we will next prove Proposition 1.1, following the proof that can be found in [1] and which proves the linear dependence on k on the right hand of (1.2). It is also worth noting that this argument will be frequently used in the present work, in the chapters that follow.

The ((good)) estimate for the sublevel set (1.3) is the content of the following Proposition:

Proposition 1.2. *Let $\phi : [a,b] \to \mathbb{R}$ be a C^k-function and suppose that $|\phi^{(k)}(t)| \geq 1$ for some $k \geq 1$ and all $t \in [a,b]$. Then*

$$|\{t \in [a,b] : |\phi(t)| \leq \alpha\}| \leq 2k\alpha^{\frac{1}{k}} \tag{1.4}$$

Proof. Let $E_\alpha = \{t \in [a,b] : |\phi(t)| \leq \alpha\}$. E_α is a finite union of closed intervals. We slide them in order to create a single interval I of length $|E_\alpha|$ and then pick $k+1$ equally spaced points in I. If we bring back the interval in their original position, we end up with $k+1$ points $x_0, x_1, x_2, \ldots, x_k$ in E_α which satisfy

$$|x_j - x_l| \geq |E_\alpha| \frac{|j-l|}{k}. \tag{1.5}$$

Consider the Lagrange interpolation polynomial $h(x)$, which interpolates $\phi(x_0), \phi(x_1), \ldots, \phi(x_k)$:

$$h(x) = \sum_{n=0}^{k} \phi(x_n) \frac{(x - x_0)\cdots(x - x_{n-1})(x - x_{n+1})\cdots(x - x_k)}{(x_n - x_0)\cdots(x_n - x_{n-1})(x_n - x_{n+1})\cdots(x_n - x_k)}.$$

The function $F(x) = h(x) - \phi(x)$ is k times differentiable and vanishes at each of the points $x_0, x_1, \ldots, x_k$. Thus, there are k points $\xi_1, \xi_2, \ldots, \xi_k$, where $x_0 < \xi_1 < x_1 < \xi_2 < \cdots < \xi_k < x_k$, such that $F'(\xi_1) = F'(\xi_2) = \cdots = F'(\xi_k) = 0$. Using the same argument k times we conclude that there is a point $\xi \in (a, b)$ such that $F^{(k)}(\xi) = h^{(k)}(\xi) - \phi^{(k)}(\xi) = 0$. We therefore have that

$$\frac{\phi^{(k)}(\xi)}{k!} = \sum_{n=0}^{k} \frac{\phi(x_n)}{(x_n - x_0)\cdots(x_n - x_{n-1})(x_n - x_{n+1})\cdots(x_n - x_k)}.$$

Now, from the hypothesis $|\phi^{(k)}(t)| \geq 1$ and the property (1.5) of the points $x_0, x_1, x_2, \ldots, x_k$, we get that

$$\begin{aligned}\frac{1}{k!} \leq \frac{\left|\phi^{(k)}(\xi)\right|}{k!} &\leq \alpha \sum_{n=0}^{k} \frac{1}{|x_n - x_0|\cdots|x_n - x_{n-1}||x_n - x_{n+1}|\cdots|x_n - x_k|} \\ &\leq \alpha \sum_{n=0}^{k} \frac{k^k}{n!(k-n)!|E_\alpha|^k} \leq \frac{\alpha k^k}{|E_\alpha|^k k!} \sum_{n=0}^{k} \binom{k}{n} = \frac{\alpha k^k}{|E_\alpha|^k k!} 2^k,\end{aligned}$$

and thus

$$|E_\alpha|^k \leq \alpha k^k 2^k.$$

Taking $k-$th roots in both sides of the above estimate, completes the proof. □

The proof of Proposition 1.1 is now immediate.

Proof of Proposition 1.1. We will first prove the Proposition in the case $k = 1$. We have

$$\int_a^b e^{i\lambda\phi(t)} dt = \frac{e^{i\lambda\phi(b)}}{i\lambda\phi'(b)} - \frac{e^{i\lambda\phi(a)}}{i\lambda\phi'(a)} - \frac{1}{i\lambda} \int_a^b e^{i\lambda\phi(t)} \frac{d}{dt}\left(\frac{1}{\phi'(t)}\right) dt.$$

Therefore

$$\left|\int_a^b e^{i\lambda\phi(t)} dt\right| \leq \frac{1}{\lambda\,|\phi'(b)|} + \frac{1}{\lambda\,|\phi'(a)|} + \frac{1}{\lambda} \int_a^b \left|\frac{d}{dt}\left(\frac{1}{\phi'(t)}\right)\right| dt.$$

By the monotonicity of ϕ' we now get

$$\frac{1}{\lambda} \int_a^b \left|\frac{d}{dt}\left(\frac{1}{\phi'(t)}\right)\right| dt = \frac{1}{\lambda} \left|\int_a^b \frac{d}{dt}\left(\frac{1}{\phi'(t)}\right) dt\right| = \frac{1}{\lambda} \left|\frac{1}{\phi'(b)} - \frac{1}{\phi'(a)}\right| \leq \frac{1}{\lambda}.$$

From the last estimates now give:

$$\left|\int_a^b e^{i\lambda\phi(t)}dt\right| \leq \frac{2}{\lambda}\max\left(\frac{1}{|\phi'(a)|}, \frac{1}{|\phi'(b)|}\right) \leq \frac{2}{\lambda}.$$

This proves the proposition in the case $k = 1$ with $c = 2$.

Suppose now that $k \geq 2$. For some $\alpha > 0$ to be defined later, we write

$$\left|\int_a^b e^{i\lambda\phi(t)}dt\right| \leq \left|\int_{\{t\in[a,b]:|\phi'(t)|\geq\alpha\}} e^{i\lambda\phi(t)}dt\right| + \int_{\{t\in[a,b]:|\phi'(t)|<\alpha\}} dt = I_1 + I_2.$$

In order to estimate $I_2 = |\{t \in [a,b] : |\phi'(t)| < \alpha\}|$ we use Proposition 1.2 for the function ϕ' to get $I_2 \leq 2(k-1)\alpha^{\frac{1}{k-1}}$. For I_1, observe that the hypothesis $\left|\phi^{(k)}(t)\right| \geq 1$ means that the set $\{t \in [a,b] : |\phi'(t)| \geq \alpha\}$ is a union of at most $2k$ intervals on each of which ϕ' is monotone. Using the result for $k = 1$ in each of these intervals we get that $I_1 \leq 4\frac{k}{\lambda\alpha}$. Thus

$$\left|\int_a^b e^{i\lambda\phi(t)}dt\right| \leq 4\frac{k}{\lambda\alpha} + 2(k-1)\alpha^{\frac{1}{k-1}}.$$

Optimizing in α, yields that

$$\left|\int_a^b e^{i\lambda\phi(t)}dt\right| \leq \frac{4k}{\lambda^{\frac{1}{k}}}.$$

□

Remark 1.3. *The linear dependence of the constants on k in Propositions 1.1 and 1.2 is optimal as may be seen by testing them against the function $\phi(t) = \frac{t^k}{k!}$.*

The estimate 1.1 leads to the corresponding estimate for integrals of the form (1.1).

Corollary 1.4. *If ϕ satisfies the hypotheses of Proposition 1.1 and ψ is a C^1-function, then*

$$\left|\int_a^b e^{\lambda i\phi(t)}\psi(t)dt\right| \leq \frac{c\,k}{\lambda^{\frac{1}{k}}}\left\{|\psi(b)| + \int_a^b \left|\psi'(t)\right| dt\right\} \tag{1.6}$$

Proof. We just write $F(t) = \int_a^t e^{i\lambda\phi(x)}dx$. Then, from Proposition 1.1 we have that

$$\begin{aligned}\left|\int_a^b e^{\lambda i\phi(t)}\psi(t)dt\right| &= \left|\int_a^b F'(t)\psi(t)dt\right| = \left|F(b)\psi(b) - \int_a^b F(t)\psi'(t)dt\right| \\ &\leq \frac{c\,k}{\lambda^{\frac{1}{k}}}\left\{|\psi(b)| + \int_a^b \left|\psi'(t)\right| dt\right\}.\end{aligned}$$

□

1.2 Oscillatory integrals of the first kind, many variables

In many variables we are interested in the behavior of the integral

$$I_\psi(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x)}\psi(x)dx \tag{1.7}$$

as $\lambda \to +\infty$, where the phase function ϕ is real and smooth and ψ is a complex-valued smooth function with compact support. The situation in many variables is far from being as clear as in one variable and the corresponding results are quite far from being optimal.

The first step here is to consider the integral

$$\int_{\mathbb{R}^n} e^{i\lambda\phi(x)}dx \tag{1.8}$$

and then try to estimate integrals of the form (1.7). However, in dimensions greater than one, satisfactory results for (1.8) are generally valid only for functions ϕ defined on suitable bounded subsets of $\mathbb{R}^n$. For simplicity we work on the unit cube of $\mathbb{R}^n$, $Q = [0,1]^n$, and we study integrals of the form

$$I(\lambda) = \int_Q e^{i\lambda\phi(x)}dx. \tag{1.9}$$

Ideally here we would like to suppose that for some multi-index $\beta \in \mathbb{N}_o^n$ with $|\beta| > 0$ we have

$$|D^\beta \phi(x)| \geq 1 \tag{1.10}$$

on Q and conclude that

$$\left| \int_Q e^{i\lambda\phi(x)}dx \right| \leq c_{n,\beta}\lambda^{-\epsilon}, \tag{1.11}$$

uniformly for all ϕ satisfying (1.10). This means that the constant $c_{n,\beta}$ and the exponent ϵ should depend only on the dimension n and the multi-index β. The seemingly ((natural)) exponent in (1.11) is $\epsilon = \frac{1}{|\beta|}$.

The main idea is the one already developed in the first section of this chapter in the case of one variable. If we know ((good)) estimates for the sublevel sets of the function ϕ,

$$\{x \in Q : |\phi(x)| \leq \alpha\},$$

then we can get respectively good estimates for the integrals of the form (1.9). For example, we have the following Theorems.

Theorem 1.5. *For each multi-index $\beta \in \mathbb{N}_o^n$ with $|\beta| > 0$ there is an $\epsilon = \epsilon_{\beta,n} > 0$ and some positive constant $c_{\beta,n}$, depending only on β and n, such that for every real function $\phi \in C^{|\beta|}(Q)$ satisfying $\left|D^\beta \phi(x)\right| \geq 1$ on Q and every $\alpha > 0$:*

$$|\{x \in Q : |\phi(x)| < \alpha\}| \leq c_{\beta,n}\, \alpha^{\epsilon}. \tag{1.12}$$

Theorem 1.6. *Let $\beta = (\beta_1, \ldots, \beta_n) \neq 0$ be a multi-index and suppose that $\beta_j \geq 2$ for at least one $j \in \{1, 2, \ldots, n\}$. Then, there exists an $\epsilon = \epsilon_{\beta,n} > 0$ and some positive constant $c_{\beta,n}$, depending only on β and n, such that for every real function $\phi \in C^{|\beta|}(Q)$ satisfying $\left|D^\beta \phi(x)\right| \geq 1$ on Q and every $\lambda > 0$:*

$$\left|\int_Q e^{i\lambda\phi(x)} dx\right| \leq c_{\beta,n} \lambda^{-\epsilon}. \tag{1.13}$$

The results above were stated and proved in [4]. The writers in [4] also prove the above theorems with the stronger exponent $\epsilon = \frac{1}{|\beta|}$ under some additional ((convexity)) hypotheses for the phase function ϕ. This is the content of the following Theorems.

Theorem 1.7. *Let $\phi \in C^{|\beta|}(Q)$, $\left|D^\beta \phi(x)\right| \geq 1$ on Q, and moreover that for some indices $N_2 > \beta_2, N_3 > \beta_3, \ldots, N_n > \beta_n$ the partial derivatives*

$$D^{(0,0,\ldots,N_n)}\phi,\ D^{(0,0,\ldots,0,N_{n-1},\beta_n)}\phi,\ \ldots,\ D^{(0,N_2,\beta_3,\ldots,\beta_n)}\phi,$$

are all single-signed. Then, there exists a constant $c_{\beta,N_2,\ldots,N_n}$ which depends only on $\beta, N_2, \ldots, N_n$ such that for every $\alpha > 0$,

$$|\{x \in Q : |\phi(x)| < \alpha\}| \leq c_{\beta,N_2,\ldots,N_n} \alpha^{\frac{1}{|\beta|}}. \tag{1.14}$$

Proof. We prove the theorem by induction on n. The case $n = 1$ is Proposition 1.2 and therefore we suppose that the statement of the theorem holds true for dimensions up to $n-1$. Let $E = \{x \in Q : |\phi(x)| < \alpha\}$, for some $\gamma > 0$ to be defined later we write

$$\begin{aligned}
|E| &= \int_Q \chi_E(x', x_n) dx = \int_{Q^1} \int_{Q^{n-1}} \chi_E(x', x_n) dx' dx_n \\
&= \int_{Q^1} \int_{\{x' \in Q^{n-1} : |\partial^{\beta_n}\phi(x)/\partial x_n^{\beta_n}| \geq \gamma\}} \chi_E(x', x_n) dx' dx_n \\
&+ \int_{Q^1} \int_{\{x' \in Q^{n-1} : |\partial^{\beta_n}\phi(x)/\partial x_n^{\beta_n}| < \gamma\}} \chi_E(x', x_n) dx' dx_n = I + II.
\end{aligned}$$

We also write $\beta = (\beta', \beta_n)$. For II we use the inductive hypothesis for $D^{\beta'}$ and $(\partial/\partial x_n)^{\beta_n}\phi$ replacing ϕ and we get $II \leq c_{\beta',N_2,\ldots,N_{n-1}} \gamma^{\frac{1}{|\beta'|}}$. For I, the fact that the partial derivative $\partial^{N_n}\phi/\partial x_n^{N_n}$ has constant sign assures that $[0,1]$ can be written as a union of boundedly many intervals on each of which $|\partial^{\beta_n}\phi(x)/\partial x_n^{\beta_n}| \geq \gamma$. Using the one dimensional estimate, that Proposition 1.2, we get $I \leq c_{N_n}(\alpha/\gamma)^{\frac{1}{\beta_n}}$. We thus have $|E| \leq c_{\beta,N_2,\ldots,N_n}(\gamma^{\frac{1}{|\beta'|}} + (\alpha/\gamma)^{\frac{1}{\beta_n}})$. Optimizing in γ, $\gamma = \alpha^{\frac{|\beta'|}{|\beta|}}$, completes the proof. □

As before, Theorem 1.7 implies implies the corresponding estimate for the integral (1.9). This is the content of the following Theorem.

Theorem 1.8. *Under the hypotheses of Theorem 1.7, there exists a constant $c_{\beta,N_2,\ldots,N_n}$ such that*

$$|I(\lambda)| = \left|\int_Q e^{i\lambda\phi(x)}dx\right| \leq c_{\beta,N_2,\ldots,N_n}\lambda^{-\frac{1}{|\beta|}} \tag{1.15}$$

Proof. We have that

$$|I(\lambda)| \leq \left|\int_{\{x\in Q:|(\partial\phi(x)/\partial x_n)^{\beta_n}|\geq\gamma\}} e^{i\lambda\phi(x)}dx\right| + |\{x \in Q : |(\partial\phi(x)/\partial x_n)^{\beta_n}| < \gamma\}| = I + II.$$

To estimate II we use Theorem 1.7 to get $II \leq c_{\beta',N_2,\ldots,N_n}\gamma^{\frac{1}{|\beta'|}}$. For I we use Proposition 1.1 since for every $x' \in Q^{n-1}$ the set $\{x_n \in [0,1] : \left|\frac{\partial^{\beta_n}\phi}{\partial x_n^{\beta_n}}(x', x_n)\right| \geq \gamma\}$ consists of boundedly many intervals. We thus get $I \leq c_{N_n}(\lambda\gamma)^{-\frac{1}{\beta_n}}$. Choosing $\gamma = \lambda^{-\frac{|\beta'|}{|\beta|}}$ we get (1.15). □

Finally, it is worth mentioning a result of Stein for estimating integrals of the form (1.7) which can be found in [15]. Stein gives the optimal estimate as far as the exponent $\epsilon = \frac{1}{|\beta|}$ is concerned , but his result is not uniform over all ϕ satisfying (1.10). More precisely, we have the following theorem.

Theorem 1.9. *Let ψ be a smooth function supported inside the unit ball of $\mathbb{R}^n$. We suppose that ϕ is a real-valued function, such that for some multi-index β with $|\beta| > 0$ it satisfies*

$$|D^\beta\phi| \geq 1$$

inside the support of ψ. Then

$$\left|\int_{\mathbb{R}^n} e^{i\lambda\phi(x)}\psi(x)dx\right| \leq c_k(\phi)\lambda^{-\frac{1}{k}}\{\|\psi\|_{L^\infty} + \|\nabla\psi\|_{L^1}\} \tag{1.16}$$

where $k = |\beta|$; the constant $c_k(\phi)$ is independent of λ and ψ and remains bounded as long as the C^{k+1} norm of ϕ remains bounded.

1.3 Notes and references

1.3.1 References

The basic references for this chapter are [14],[15], [1] and [4]. The proof of propositions 1.1 and 1.2 which we presented here can be found in [1]. Essentially the same proofs are contained in [4]. In [4], Proposition 1.2 is inductively exploited in order to give results like Theorems 1.5, 1.6, 1.7 and 1.8. Theorem 1.9 is proved in [14] and [15].

1.3.2 Sublevel sets in many dimensions

The problem of estimating the $n-$dimensional Lebesgue measure of the set

$$E_\alpha = \{x \in Q : |\phi(x)| \leq \alpha\},$$

for all ϕ with $|D^\beta \phi| \geq 1$ on Q and no additional hypotheses, finds an answer in Theorem 1.5:

$$|\{x \in Q : |\phi(x)| \leq \alpha\}| \leq c_{\beta,n}\alpha^\epsilon.$$

We don't know if the exponent ϵ can take the ((optimal)) value $\epsilon = \frac{1}{|\beta|}$. In dimension $n = 2$, we have the following theorem which is proved in [4].

Theorem 1.10. *Let $\phi \in C^2(Q^2)$ with $|\frac{\partial^2\phi(x)}{\partial x \partial y}| \geq 1$ on Q^2. Then*

$$|\{x \in Q^2 : |\phi(x)| \leq \alpha\}| \leq c\alpha^{\frac{1}{2}}|\log\alpha|^{\frac{1}{2}}.$$

Note that the above estimate is a logarithmic factor away from the seemingly optimal value $\alpha^{\frac{1}{2}}$.

1.3.3 Connection with combinatorial problems

The problem of estimating the Lebesgue measure of a sublevel set has a natural connection with some combinatorial problems.

Question 1.11. *Let $E \subseteq Q^2 = [0,1]^2$ with $|E| > 0$. Is it true for some absolute constant c, we can always find points $A, B, C, D \in E$ such that the quadrilateral $ABCD$ has sides parallel to the coordinate axes, and area at least $c|E|^2$?*

Question 1.11 has an equivalent discrete formulation

Question 1.12. *Let M, N be positive integers, A a $N \times N$ matrix with at least M entries equal to 1 and the rest equal to 0. Is there an absolute constant $c > 0$ and some 2×2 sub-matrix of A, with all its entries equal to 1 and ((area)) at least cM^2/N^2?*

A positive answer to the above question would have as an immediate consequence the estimate

$$|\{x \in Q : |\phi(x)| \leq \alpha\}| \leq c\alpha^{\frac{1}{2}},$$

for all ϕ with $|\frac{\partial^2 \phi}{\partial x \partial y}| \geq 1$ on Q. Indeed, if there are four points (x_i, y_i), $i, j \in \{0, 1\}$ in the set $E_\alpha = \{x \in Q : |\phi(x)| \leq \alpha\}$, such that the area of the parallelogram T they form is at least $c|E_\alpha|^2$, then

$$c|E_\alpha|^2 \leq |T| \leq \int_T \left| \frac{\partial^2 \phi(x)}{\partial x \partial y} \right| dx dy \leq 4\alpha,$$

and thus $|E_\alpha| \leq c\alpha^{\frac{1}{2}}$. For more details, we refer the interested reader to [4].

CHAPTER 2

Oscillatory integrals with polynomial phase on $\mathbb{R}^n$

In this chapter, we focus on oscillatory integrals of the first kind, where the phase function is a real polynomial on $\mathbb{R}^n$. We denote by $\mathcal{P}_{d,n}$ the vector space of all real polynomial on $\mathbb{R}^n$, of degree at most d. The typical element $P \in \mathcal{P}_{d,n}$ can be written in the form

$$P(x) = \sum_{0 \le |\alpha| \le d} c_\alpha x^\alpha = c_0 + \sum_{0 < |\alpha| \le d} c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}. \tag{2.1}$$

Let $P \in \mathcal{P}_{d,n}$ and we write $Q = [0,1]^n$ for the unit cube on $\mathbb{R}^n$. We want to study the behavior of the integral

$$I(\lambda) = \int_Q e^{i\lambda P(x)} dx \tag{2.2}$$

as $\lambda \to +\infty$.

2.1 Van der Corput estimates and polynomials

The analytic properties of polynomials will let us obtain estimates for $I(\lambda)$, essentially with no additional hypothesis. For example, the additional convexity hypotheses imposed on the phase functions of theorems 1.7 and 1.8 are automatically satisfied. Thus, we have the following.

Theorem 2.1. *For every d, n there exists an absolute constant $c_{d,n}$ such that for every multi-index β and every $P \in \mathcal{P}_{d,n}$ which satisfies $|D^\beta P(x)| \ge 1$ for every $x \in Q$, we have that*

$$|\{x \in Q : |P(x)| < \alpha\}| \le c_{d,n} \alpha^{\frac{1}{|\beta|}}. \tag{2.3}$$

Now theorem 1.8 gives us a van der Corput type estimate for polynomials on $\mathbb{R}^n$.

Theorem 2.2. *For every d, n there exists a positive constant $c_{d,n}$ such that for every multi-index β and every polynomial $P \in \mathcal{P}_{d,n}$ satisfying $|D^\beta P(x)| \geq 1$ for every $x \in Q$, we have that*

$$\left|\int_Q e^{i\lambda P(x)} dx\right| \leq c_{d,n} \lambda^{-\frac{1}{|\beta|}}. \tag{2.4}$$

If $P \in \mathcal{P}_{d,n}$ we can always find some $\beta \in \mathbb{N}_o^n$ with $|\beta| = d$ such that $|D^\beta P(x)| \geq |c_\beta|$ everywhere on $\mathbb{R}^n$. As a result, theorem 2.2 immediately gives that for every polynomial $P \in \mathcal{P}_{d,n}$ there exists a multi-index β with $|\beta| = d$ such that

$$\left|\int_Q e^{iP(x)} dx\right| \leq c_{d,n} |c_\beta|^{-\frac{1}{d}}, \tag{2.5}$$

where the constant $c_{d,n}$ depends only on d and n. Estimate (2.5) can be strengthened.

Corollary 2.3. *Let $P \in \mathcal{P}_{d,n}$, $P(x) = \sum_{0 \leq |\alpha| \leq d} c_\alpha x^\alpha$. Then*

$$\left|\int_Q e^{iP(x)} dx\right| \leq c_{d,n} \left(\sum_{0<|\alpha|\leq d} |c_\alpha|\right)^{-\frac{1}{d}}, \tag{2.6}$$

where the constant $c_{d,n}$ depends only on d and n.

Proof. We can assume, without loss of generality, that the polynomial P has no constant term, that is $P(0) = 0$. Consider the space $\mathcal{P} = \{P \in \mathcal{P}_{d,n} : P(0) = 0\}$ endowed with the norm $\|P\| = \sum_{0<|\alpha|\leq d} |c_\alpha|$. The functional

$$\theta(P) = \max_{0<|\alpha|<d} \inf_{x\in Q} |D^\alpha P(x)|$$

is continuous on $\mathcal{P}$ and homogeneous of degree 1. It is clear that for every non zero polynomial $P \in \mathcal{P}$ we have that $\theta(P) \neq 0$. As a consequence, there exists a positive constant $c_{d,n}$ which depends only on n and d such that $\theta(P) \geq c_{d,n} \|P\|$ for all $\mathcal{P}$. Theorem 2.2 now gives the desired result. □

Remark 2.4. *The writers in [4] also prove that in dimension 1 we have the estimate $c_{d,1} \leq cd$ for some absolute positive constant c. We did not include that proof since we will prove a stronger result in what follows.*

We consider the vector space $\mathcal{P}^o_{d,n} = \{P \in \mathcal{P}_{d,n} : \int_Q P(x)dx = 0\}$, that is the space of all real polynomials on $\mathbb{R}^n$of degree at most d, having zero mean value on the unit cube Q. For $P \in \mathcal{P}^o_{d,n}$ we have that:

$$\sum_{0\leq|\alpha|\leq d}|c_\alpha| \leq 2\sum_{0<|\alpha|\leq d}|c_\alpha|,$$

and thus, for $P \in \mathcal{P}^o_{d,n}$, corollary 2.3 gives that

$$\left|\int_Q e^{iP(x)}dx\right| \leq c_{d,n}\big(\sum_{0\leq|\alpha|\leq d}|c_\alpha|\big)^{-\frac{1}{d}}.$$

Since all norms in the space $\mathcal{P}^o_{d,n}$ are equivalent, with constants depending only on d and n, corollary 2.3 can be equivalently written in the form:

Corollary 2.5. *Let $P \in \mathcal{P}^o_{d,n}$ and $\|\cdot\|$ a norm in the space $\mathcal{P}^o_{d,n}$. Then*

$$\left|\int_Q e^{iP(x)}dx\right| \leq c_{d,n}\|P\|^{-\frac{1}{d}}, \tag{2.7}$$

where the constant $c_{d,n}$ depends only on d,n and the choice of the norm.

The purpose of this chapter from now on is to study the constants in estimates of the form (2.7). For that reason we will deviate a bit in order to give some basic estimates on the constants involved in the equivalence of polynomial norms on $\mathbb{R}^n$.

2.2 Equivalent polynomial norms and sublevel sets

We consider the vector space of polynomials $\mathcal{P}_{d,n}$ and some convex body $K$of volume one 1 on $\mathbb{R}^n$. Since the space $\mathcal{P}_{d,n}$ is finite dimensional , the norms $(\int_K|P(x)|^p dx)^{\frac{1}{p}}$, $1 \leq p \leq \infty$, are all equivalent. In this section we are going to study the constants involved in these equivalences. We refer the interested reader to [6] and the references therein.

Let $1 \leq p \leq q \leq \infty$.Hölder's inequality then gives

$$\|P\|_{L^p(K)} \leq \|P\|_{L^q(K)}, \tag{2.8}$$

and the constant 1 in the above inequality is best possible. For the opposite inequality we have the following theorem which is proved in [6].

Theorem 2.6. *Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree at most d and K a convex body on $\mathbb{R}^n$ of volume 1. Let $1 \leq p \leq q \leq \infty$. Then there exist an absolute positive constant c, such that*

(i) *If $\frac{n}{d} \leq p \leq q$ then*

$$\|P\|^{\frac{1}{d}}_{L^q(K)} \leq c\,\|P\|^{\frac{1}{d}}_{L^p(K)}.$$

(ii) *If* $p \leq \frac{n}{d} \leq q$ *then*

$$\|P\|_{L^q(K)}^{\frac{1}{d}} \leq c\,\frac{n}{pd}\,\|P\|_{L^p(K)}^{\frac{1}{d}}.$$

(iii) *If* $p \leq q \leq \frac{n}{d}$, *then*

$$\|P\|_{L^q(K)}^{\frac{1}{d}} \leq c\,\frac{q}{p}\,\|P\|_{L^p(K)}^{\frac{1}{d}}.$$

The constants arising in the right hand sides of the above theorem are best possible modulo the arithmetic constant c when one looks for inequalities over **arbitrary** convex bodies K of volume 1 on $\mathbb{R}^n$. We do not know for example if these constants can be sharpened if we fix as our convex body the unit cube on $\mathbb{R}^n$, that is when $K = Q$.

For $0 \leq p \leq 1 \leq q \leq \infty$, theorem 2.6 has a stronger formulation in terms of distribution inequalities. This is the content of theorem 2.7. In this theorem we estimate the Lebesgue measure of a polynomial sublevel set on a convex body in $\mathbb{R}^n$ in terms of the $L^q(K)$ norm of the polynomial. From the previous discussion, we know that sublevel set estimates usually imply estimates for corresponding oscillatory integrals. The estimates contained in the following theorem of [6] will prove particularly useful in what follows.

Theorem 2.7. *Let* $P : \mathbb{R}^n \to \mathbb{R}$ *be a polynomial of degree at most* d *and* K *be a convex body on* $\mathbb{R}^n$ *of volume* 1. *Let* $1 \leq q \leq \infty$. *Then there exists an absolute positive constant* c, *independent of* P, d, K, q *and* n, *such that, for every* $\alpha > 0$,

$$|\{x \in K : |P(x)| \leq \alpha\}| \leq c \min(n, qd)\alpha^{\frac{1}{d}}\,\|P\|_{L^q(K)}^{-\frac{1}{d}}. \tag{2.9}$$

Again, the constant $c\min(n, qd)$ in the right-hand side of (2.9) is best possible in the context of arbitrary convex bodies of volume 1.

Looking back at the discussion in chapter 1, it is immediately clear that sublevel set estimates are used for some derivative of the phase function and not for the function itself. If we look at equation (2.9) we will see that its use for some derivative of the polynomial P on $\mathbb{R}^n$ will produce the corresponding derivative norm on the right-hand side. For that reason we state now a simple lemma which will allow us to compare the $L^2(Q)$ norm of a polynomial P with the $L^2(Q)$ norm of ∇P. This is a Poincaré inequality for polynomials on the unit cube of $\mathbb{R}^n$.

Lemma 2.8. *Let* $P : \mathbb{R}^n \to \mathbb{R}$ *be a polynomial with zero mean value on the unit cube* $Q = [0, 1]^n$. *Then*

$$\|P\|_{L^2(Q)} \leq c\,\|\nabla P(x)\|_{L^2(Q)},$$

for some absolute positive constant c, *independent of* P *and* n.

The proof of lemma 2.8 is a simple application of Fourier series on the unit cube Q of $\mathbb{R}^n$.

We conclude this section by giving another application of theorem 2.7. Let $1 \leq q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. A weight function ω (that is a non negative measurable function) for which there exists some constant $K < +\infty$ such that

$$\frac{1}{|B|} \int_B \omega(x)dx \left(\frac{1}{|B|} \int_B \omega(x)^{-\frac{p}{q}} dx \right)^{\frac{q}{p}} \leq K < +\infty$$

for every ball B is said to belong to the weight class A_q. When $q = 1$ the quantity $\left(\frac{1}{|B|} \int_B \omega(x)^{-\frac{p}{q}} dx\right)^{\frac{q}{p}}$ should be replaced by $esssup_B\left(\frac{1}{\omega}\right)$. The smallest constant K for which the above estimate holds is called the A_q constant of the weight ω and it is denoted by $A_q(\omega)$. For polynomials on $\mathbb{R}^n$ we have the following result toward that direction.

Corollary 2.9. *Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree at most d and K a convex body on $\mathbb{R}^n$ of volume 1. Let $1 \leq p \leq \infty$ and $0 < \epsilon < \frac{1}{d}$. Then there exists an absolute positive constant c, independent of P, d, K, p, ϵ and n such that*

$$\left(\int_K |P(x)|^{-\epsilon} dx \right)^{\frac{1}{\epsilon}} \left(\int_K |P(x)|^p dx \right)^{\frac{1}{p}} \leq \left(c \min(n, pd) \frac{d\epsilon}{1 - d\epsilon} \right)^d.$$

Proof. Let$P$ be a polynomial of degree at most d and K be a convex body on $\mathbb{R}^n$ of volume 1. For $1 \leq p \leq \infty$ and some $\lambda > 0$ to be defined later, we write

$$\begin{aligned} \int_K |P(x)|^{-\epsilon} dx &= \int_0^\infty |\{x \in K : |P(x)|^{-\epsilon} > \alpha\}| d\alpha \\ &\leq \lambda + \int_\lambda^\infty |\{x \in K : |P(x)| < \alpha^{-\frac{1}{\epsilon}}\}| d\alpha. \end{aligned}$$

Now, using theorem 2.7 for the sublevel set of the polynomial P we get

$$\begin{aligned} \int_K |P(x)|^{-\epsilon} dx &\leq \lambda + c \min(n, pd) \|P\|_{L^p(K)}^{-\frac{1}{d}} \int_\lambda^\infty \alpha^{-\frac{1}{d\epsilon}} d\alpha \\ &= \lambda + c \min(n, pd) \|P\|_{L^p(K)}^{-\frac{1}{d}} \frac{\lambda^{-\frac{1}{d\epsilon}+1}}{\frac{1}{d\epsilon} - 1}, \end{aligned}$$

since $\epsilon < \frac{1}{d}$. Optimizing in λ the above inequality becomes,

$$\int_K |P(x)|^{-\epsilon} dx \leq \left(c \min(n, pd) \frac{d\epsilon}{1 - d\epsilon} \right)^{d\epsilon} \|P\|_{L^p(K)}^{-\epsilon},$$

which is the desired estimate. ∎

The corollary above says that for a polynomial $P \in \mathcal{P}_{d,n}$, the function $\omega = |P|$ is a weight on $\mathbb{R}^n$ with

$$A_q(|P|) \leq \left(c \frac{d \min(n,d)}{q-(d+1)} \right)^d$$

for every $q > d+1$. For another application of theorem 2.9 see section 3.3.

2.2.1 Estimates for polynomials in dimension one

As is usually the case, the one dimensional estimates are more precise and complete. For example, theorem 2.6 says that, for $1 \leq p \leq q \leq \infty$, we have that

$$\|P\|_{L^p[0,1]}^{\frac{1}{d}} \leq \|P\|_{L^q[0,1]}^{\frac{1}{d}} \leq c \|P\|_{L^p[0,1]}^{\frac{1}{d}}.$$

If $P(t) = b_0 + b_1 t + \cdots + b_d t^d$, proposition 1.2 gives for the sublevel set of the polynomial P that

$$|\{x \in [0,1] : |P(x)| \leq \alpha\}| \leq c \alpha^{\frac{1}{d}} |b_d|^{-\frac{1}{d}}. \tag{2.10}$$

On the other hand, theorem 2.7 in the case $n = 1$ gives the estimate

$$|\{x \in [0,1] : |P(x)| \leq \alpha\}| \leq c \alpha^{\frac{1}{d}} \, \|P\|_{L^p[0,1]}^{-\frac{1}{d}}. \tag{2.11}$$

The ((worse)) norm, which gives the best estimate for the measure of the polynomial sublevel set is the L^∞ norm. We will now show that this estimate can be strengthened in dimension one , and the L^∞ norm can be replaced by a bigger norm. More precisely, we have the following result due to Vinogradov.

Lemma 2.10. *Let $h(t) = b_0 + b_1 t + \cdots + b_d t^d$ be a real polynomial of degree at most d. Then,*

$$|\{t \in [a,b] : |h(t)| \leq \alpha\}| \leq c \max(|a|,|b|) \left(\frac{\alpha}{\max_{0 \leq k \leq d} |b_k|} \right)^{\frac{1}{d}}.$$

Proof. The set $E_\alpha = \{t \in [a,b] : |h(t)| \leq \alpha\}$ is a finite union of closed intervals. We slide them together to form a single interval I of length $|E_\alpha|$ and pick $d+1$ equally spaced points in I. If we slide the intervals back to their original position, we end up with $d+1$ points $x_0, x_1, x_2, \ldots, x_d \in E_\alpha$ which satisfy

$$|x_j - x_k| \geq |E_\alpha| \frac{|j-k|}{d}. \tag{2.12}$$

The Lagrange interpolation polynomial which interpolates the values $h(x_0)$, $h(x_1)$,…, $h(x_d)$ coincides with $h(x)$:

$$h(x) = \sum_{j=0}^{d} h(x_j) \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_d)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_d)}.$$

We thus get for the coefficients of h that

$$b_k = \sum_{j=0}^{d} h(x_j) \frac{(-1)^{d-k}\sigma_{d-k}(x_0,\ldots,\hat{x_j},\ldots,x_d)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_d)}$$

for $k = 0, 1, \ldots, d$. In the above equality, $\sigma_{d-k}(x_0,\ldots,\hat{x_j},\ldots,x_d)$ is the $(d-k)$-th elementary symmetric function of $x_0,\ldots,\hat{x_j},\ldots,x_d$ where x_j is omitted. Using the estimate $\sigma_{d-k}(x_0,\ldots,\hat{x_j},\ldots,x_d) \le \binom{d}{d-k}\max(|a|,|b|)^{d-k}$ together with (2.12), we get that, for every $k = 0, 1, \ldots, d$,

$$\begin{aligned} |b_k| &\le \binom{d}{d-k}\max(|a|,|b|)^{d-k} d^d \frac{\alpha}{|E_\alpha|^d} \sum_{j=0}^{d} \frac{1}{j!(d-j)!} \\ &= \binom{d}{d-k}\max(|a|,|b|)^{d-k} 2^d \frac{d^d}{d!}\frac{\alpha}{|E_\alpha|^d} \ \le\ c\frac{(4\max(|a|,|b|))^d}{\sqrt{d}}\frac{d^d}{d!}\frac{\alpha}{|E_\alpha|^d}, \end{aligned}$$

where we have used the estimate $\binom{d}{d-k} \le \binom{d}{[\frac{d}{2}]} \le c\frac{2^d}{\sqrt{d}}$. Consequently

$$\max_{0\le k\le d} |b_k| \le c\frac{(4\max(|a|,|b|))^d}{\sqrt{d}}\frac{d^d}{d!}\frac{\alpha}{|E_\alpha|^d}$$

and solving with respect to $|E_\alpha|$ we get

$$|E_\alpha| \le c\max(|a|,|b|)\left(\frac{\alpha}{\max_{0\le k\le d}|b_k|}\right)^{\frac{1}{d}}.$$

□

Lemma 2.10 has the following consequence

$$|\{x \in [0,1] : |P(x)| \le \alpha\}| \le c\alpha^{\frac{1}{d}}\{\max_{0\le j\le d}|b_j|\}^{-\frac{1}{d}}. \tag{2.13}$$

A moments' reflection will now show that (2.13) implies the estimates (2.10) and (2.11).

2.3 A conjecture by Carbery and Wright

In this section we fix as our convex body the unit cube Q of $\mathbb{R}^n$. If $P \in \mathcal{P}^o_{d,n}$, corollary 2.5 says that there exists some constant $c_{d,n}$, which depends only on d and n, such that

$$\left|\int_Q e^{iP(x)}dx\right| \leq c_{d,n}\|P\|_{L^1(Q)}^{-\frac{1}{d}}. \tag{2.14}$$

On the other hand, theorem 2.7 gives for the sublevel set of the polynomial P that

$$|\{x \in Q : |P(x)| \leq \alpha\}| \leq c\min(d,n)\alpha^{\frac{1}{d}}\|P\|_{L^1(Q)}^{-\frac{1}{d}}. \tag{2.15}$$

Comparing the estimates (2.14) and (2.15) and baring in mind the general principle that sublevel set estimates imply oscillatory integral estimates motivated Carbery and Wright to state the following question in [6].

Question 2.11. *Can we replace the constant $c_{d,n}$ in (2.14) with $c\min(d,n)$ for some absolute positive constant c?*

We partially answer question 2.11 by means of the following theorem.

Theorem 2.12. *Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree at most d, with zero mean value on the unit cube Q. Then, there is an absolute constant c, independent of P, n and d, such that*

$$\left|\int_Q e^{iP(x)}dx\right| \leq c\min(n,2d)n^{\frac{1}{2d}}\|P\|_{L^2(Q)}^{-\frac{1}{d}}.$$

Proof. Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree at most d with zero mean value on the unit cube Q. There exists $j \in \{1, 2, \ldots n\}$ such that $\left\|\frac{\partial P}{\partial x_j}\right\|_{L^2(Q)} \geq \frac{1}{\sqrt{n}}\|\nabla P\|_{L^2(Q)}$. For some $\alpha > 0$ to be defined later, we write

$$I = \int_Q e^{i\lambda P(x)}dx = \int_{\{x\in Q:|\frac{\partial P}{\partial x_j}(x)|\leq\alpha\}} e^{i\lambda P(x)}dx + \int_{\{x\in Q:|\frac{\partial P}{\partial x_j}(x)|>\alpha\}} e^{i\lambda P(x)}dx.$$

We thus have that

$$|I| \leq |\{x \in Q : |\partial P(x)/\partial x_j| \leq \alpha\}| + \left|\int_{\{x\in Q:|\frac{\partial P}{\partial x_j}(x)|>\alpha\}} e^{i\lambda P(x)}dx\right|.$$

Using theorem 2.7 we now get

$$\begin{aligned}
|\{x \in Q : |\partial P(x)/\partial x_j| \leq \alpha\}| &\leq c\min(2(d-1),n)\alpha^{\frac{1}{d-1}}\|\partial P/\partial x_j\|_{L^2(Q)}^{-\frac{1}{d-1}} \\
&\leq cn^{\frac{1}{2(d-1)}}\min(2d,n)\alpha^{\frac{1}{d-1}}\|\nabla P\|_{L^2(Q)}^{-\frac{1}{d-1}} \\
&\leq cn^{\frac{1}{2(d-1)}}\min(2d,n)\alpha^{\frac{1}{d-1}}\|P\|_{L^2(Q)}^{-\frac{1}{d-1}},
\end{aligned}$$

where in the last inequality we used lemma 2.8.

For $\int_{\{x\in Q:|\frac{\partial P}{\partial x_j}(x)|>\alpha\}} e^{i\lambda P(x)}dx$ we observe that for every $(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)$ the set $\{x_j \in [0,1] : |\frac{\partial P}{\partial x_j}(x)| > \alpha\}$ consists of at most $O(d)$ interval where the function $\frac{\partial P}{\partial x_j}(x)$ is monotonic. Using proposition 1.1 we get

$$\left|\int_{\{x\in Q:|\frac{\partial P}{\partial x_j}(x)|>\alpha\}} e^{i\lambda P(x)}dx\right| \leq c\frac{d}{\alpha}.$$

Summing up the estimates we get

$$|I| \leq c\Bigg(\min(2d,n)n^{\frac{1}{2(d-1)}}\alpha^{\frac{1}{d-1}}\|P\|_{L^2(Q)}^{-\frac{1}{d-1}} + \frac{d}{\alpha}\Bigg).$$

Optimizing in α completes the proof. □

Although we stated theorem 2.12 for the L^2 norm, it is easy to see that it is equivalent with the same statement for the L^1 norm. Indeed, it follows from the discussion in the section 2.2 that there is an absolute positive constant c such that

$$\|P\|_{L^1(Q)}^{\frac{1}{d}} \leq \|P\|_{L^2(Q)}^{\frac{1}{d}} \leq c\|P\|_{L^1(Q)}^{\frac{1}{d}}.$$

On the other hand it is obvious that $\min(n,d) \sim \min(n,2d)$. Thus, theorem 2.12 answers positively in question 2.11 in the case $n \leq c^d$ for some absolute positive constant c. In the case $n > c^d$, theorem 2.12 misses the constant of question 2.11 by the factor $n^{\frac{1}{2d}}$. More generally, one could ask the following question:

Question 2.13. *Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree at most d, with zero mean value on the unit cube Q. Let $1 \leq p \leq \infty$. Corollary 2.5 sat that there exists an absolute positive constant $c_{d,n,p}$, independent of P, such that*

$$\left|\int_Q e^{iP(x)}dx\right| \leq c_{d,n,p}\|P\|_{L^p(Q)}^{-\frac{1}{d}}. \tag{2.16}$$

Can we replace the constant $c_{d,n,p}$ in (2.16) by $c\min(n,pd)$ for some absolute positive constant $c > 0$?

For example, using lemma 2.10, we can easily show that in the case of dimension $n = 1$, the answer in question 2.13 is positive.

2.4 Notes and References

2.4.1 References

The basic references for the development of this chapter are [4] and [6]. Theorems 2.1 and 2.2 are contained in [4]. Theorem 2.2 can also be proved by the methods in [1]. See also [12], [18] for a van der Corput type result for oscillatory integrals with polynomial phase. Theorems 2.6 and 2.7 are contained in [6]. The basic tool used in [6] is a strong extremal result of Kannan, Lovász and Simonovits [8]. Analogous estimates by using the same tool but different methods can be found in [9]. Corollary 2.9 exists in [6] since it's an immediate consequence of the results therein. Analogous estimates are contained in [9], which improve older results from [14] and [13]. Lemma 2.10 is due to Vinogradov and the proof we presented here is based on [19]. Question 2.11 is stated in [6] while theorem 2.12 is a new result.

2.4.2 The KLS lemma

A central role in the proof of theorems 2.6 and 2.7 plays the following theorem which is contained in [8].

Theorem 2.14. *For $a, b \in \mathbb{R}^n$ and $\lambda \geq 1$ we define the measures $\mu_{a,b,\lambda}$ by means of the formula*

$$< \phi, \mu_{a,b,\lambda} >= \int_0^1 \phi(a(1-t)+bt)(\lambda - t)^{n-1} dt.$$

Let f_1, f_2, f_3, f_4 be continuous, non negative functions on $\mathbb{R}^n$ and $\alpha, \beta > 0$. Suppose that for $a, b \in \mathbb{R}^n$ and $\lambda \geq 1$,

$$\left(\int f_1 d\mu_{a,b,\lambda}\right)^{\alpha} \left(\int f_2 d\mu_{a,b,\lambda}\right)^{\beta} \leq \left(\int f_3 d\mu_{a,b,\lambda}\right)^{\alpha} \left(\int f_4 d\mu_{a,b,\lambda}\right)^{\beta}.$$

Then, for every convex body K on $\mathbb{R}^n$,

$$\left(\int_K f_1\right)^{\alpha} \left(\int_K f_1\right)^{\beta} \leq \left(\int_K f_3\right)^{\alpha} \left(\int_K f_4\right)^{\beta}.$$

Using theorem 2.14, theorems 2.6 and 2.7 reduce to one dimensional weighted inequalities. Theorem 2.14 is also used, in a different way, in [9].

2.4.3 Poincaré inequalities on convex bodies

Results like lemma 2.8 are classic when we look at functions with zero mean value on a convex body. For example, we have that for every convex body K and a ((good)) function u with zero mean value on K,

$$\|u\|_{L^2(K)} \leq c\, d(K) \|\nabla u\|_{L^2(K)},$$

where $d(K)$ is the diameter of K. This is a classical result which can be found for example in [11],[3]. The dependence on the diameter is sharp in this case. If $K = Q$, we have that $d(Q) = \sqrt{n}$. However, if we restrict ourself in functions on the unit cube, the corresponding estimate holds with an absolute constant replacing $d(Q)$.

2.4.4 Polynomials and BMO

We consider the following definition of the norm of the space BMO:

$$\|u\|_{BMO} = \sup_{K \subset \mathbb{R}^n, K \text{ convex}} \inf_{c \in \mathbb{R}} \frac{1}{|K|} \int_K |u(x) - c| dx.$$

It is a classical result that, if a function ω is an A_q weight, then $\log|\omega| \in BMO$. More precisely, if we know estimates like the one in 2.9, we can estimate the BMO norm of $\log|\omega|$. For polynomials, it is well known that $\log|P| \in BMO$. See for example [14] and [13]. Using estimates analogous to the ones in [6], the writers in [9] show that if $P \in \mathcal{P}_{d,n}$ then $\|\log|P|\|_{BMO} \sim d$.

CHAPTER 3

Singular oscillatory integrals

If $\gamma(t)$ is a ((nice)) curve, we can define the Hilbert transform along γ,

$$H_\gamma f(x) = p.v. \int_{\mathbb{R}} f(x - \gamma(t)) \frac{dt}{t}.$$

A classical question in harmonic analysis is the following.

Question 3.1. *Can we have an estimate of the form*

$$\|H_\gamma f\|_{L^p} \le c_p \|f\|_p,$$

for some p?

Following the standard argument, one is first going to show the L^2 bound. On the level of the Fourier transform, this amounts to showing that the multiplier of H_γ,

$$m_\gamma(\xi) = p.v. \int_{\mathbb{R}} e^{i\xi\gamma(t)} \frac{dt}{t},$$

belongs to L^∞. In what follows, we will estimate the L^∞ norm of the multiplier $m_P(\xi)$, in the case where $\gamma = P$, is a real polynomial of degree at most d.

Let $\mathcal{P}_d$ be the vector space of all real polynomials of degree at most d in $\mathbb{R}$. For $P \in \mathcal{P}_d$ we consider the principal value integral

$$I(P) = \left| p.v. \int_{\mathbb{R}} e^{iP(t)} \frac{dt}{t} \right|.$$

We wish to estimate the quantity $I(P)$ by a constant $C(d)$ depending only on the degree of the polynomial d. This amounts to estimating the integral

$$I_{(\epsilon,R)}(P) = \left| \int_{\epsilon\leq|t|\leq R} e^{iP(t)}\frac{dt}{t} \right|$$

by some constant $C(d)$ independent of ϵ, R and P.

This problem is quite old and in fact has been answered some thirty years ago by Stein and Wainger in [16] and [20]. They showed that the quantity $I(P)$ is bounded by a constant C_d depending only on d. Their proof is very simple and uses a combination of induction and Van der Corput's lemma.

On the other hand, Carbery, Wainger and Wright stated in [5] the following conjecture.

Conjecture 3.2. *The order of magnitude of the principal value integral $I(P)$ is $\log d$.*

The main result of this chapter is the proof of this conjecture. This is the content of

Theorem 3.3. *There exist two absolute positive constants c_1 and c_2 such that*

$$c_1 \log d \leq \sup_{P\in\mathcal{P}_d} \left| p.v. \int_{\mathbb{R}} e^{iP(x)}\frac{dx}{x} \right| \leq c_2 \log d.$$

3.1 The lower bound in theorem 3.3

In this section we will construct a real polynomial P of degree at most d such that the inequality

$$I(P) = \left| p.v. \int_{\mathbb{R}} e^{iP(t)}\frac{dt}{t} \right| \geq c \log d. \tag{3.1}$$

holds. The general plan of the construction is as follows. We will first construct a *function* f (which will not be a polynomial) such that $I(f) \geq c\log n$. We will then construct a polynomial P of degree $d = 2n^2 - 1$ that approximates the function f in a way that $|I(f) - I(P)|$ is small (small means $o(\log n)$ here). Since $\log n \sim \log d$ this will yield our result.

Lemma 3.4. *For n a large positive integer, let $f(t)$ be the continuous function which is equal to 1 for $\frac{1}{n} \leq t \leq 1 - \frac{1}{n}$, equal to -1 for $-1 + \frac{1}{n} \leq t \leq -\frac{1}{n}$, equal to 0 for $|t| \geq 1$ and linear in each interval $[-1, -1 + \frac{1}{n}]$, $[-\frac{1}{n}, \frac{1}{n}]$ and $[1 - \frac{1}{n}, 1]$. Then,*

$$I(f) = \left| p.v. \int_{\mathbb{R}} e^{if(t)}\frac{dt}{t} \right| \geq c \log n. \tag{3.2}$$

Proof. The proof is more or less straightforward.

$$\begin{aligned} I(f) &= 2\Big|\int_0^1 \frac{\sin f(t)}{t}dt\Big| \\ &\geq 2\Big|\int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{\sin f(t)}{t}dt\Big| - 2\Big|\int_0^{\frac{1}{n}} \frac{\sin f(t)}{t}dt\Big| - 2\Big|\int_{1-\frac{1}{n}}^{1} \frac{\sin f(t)}{t}dt\Big| \\ &\geq 2\sin 1\log(n-1) - 2\int_0^{\frac{1}{n}} \frac{f(t)}{t}dt - 2\int_{1-\frac{1}{n}}^{1} \frac{f(t)}{t}dt \\ &= 2\sin 1\log(n-1) - 2 - 2n\log\frac{n}{n-1} + 2 \\ &\geq 2\sin 1\log(n-1) - 4 \quad\geq\quad c\log n. \end{aligned}$$

□

We now want to construct a polynomial which approximates the function f. We will do so by convolving the function f with a "polynomial approximation to the identity". To be more specific, for $k \in \mathbb{N}$ and $x \in \mathbb{R}$ define the function

$$\phi_k(x) = c_k\left(1 - \frac{x^2}{4}\right)^{k^2} \tag{3.3}$$

where the constant c_k is defined by means of the normalization

$$\int_{-2}^{2} \phi_k(x)dx = 1. \tag{3.4}$$

Observe that

$$1 = c_k\int_{-2}^{2}\left(1 - \frac{x^2}{4}\right)^{k^2}dx = 4c_k\int_0^1 (1-x^2)^{k^2}dx = 2c_kB\left(\frac{1}{2}, k^2+1\right),$$

where $B(\cdot,\cdot)$ is the beta function. Using standard estimates for the beta function we see that $c_k \sim k$.

Define, next, the functions P_k in $\mathbb{R}$ as

$$P_k(t) = \int_{-1}^{1} f(x)\phi_k(t-x)dx, \tag{3.5}$$

where f is the function of Lemma **??**. It is clear that the functions P_k are *polynomials* of degree at most $2k^2$. The following lemma deals with some technical issues concerning the polynomials P_k.

Lemma 3.5. *Let P_k be defined as in (**??**) above.*

(i) P_k is an odd polynomial of degree $2k^2-1$ with leading coefficient

$$a_k = (-1)^{k^2+1}\frac{2c_k k^2}{4^{k^2}}\left(1-\frac{1}{n}\right).$$

That is

$$P_k(t) = a_k t^{2k^2-1} + \cdots .$$

(ii) As a consequence of (i) we have for all t

$$|P_k^{(2k^2-1)}(t)| \geq c(2k^2-1)!\frac{k^3}{4^{k^2}}.$$

(iii) For $t \in [-1,1]$ we have

$$P_k(t) = \int_0^2 \big(f(t+x)+f(t-x)\big)\phi_k(x)dx.$$

Proof. (i) Using (3.5) we have

$$\begin{aligned} P_k(-t) &= \int_{-1}^1 f(x)\phi_k(-t-x)dx = \int_{-1}^1 f(x)\phi_k(t+x)dx \\ &= \int_{-1}^1 f(-x)\phi_k(t-x)dx = -P_k(t). \end{aligned}$$

Next, from (3.5) we have that

$$\begin{aligned} P_k(t) &= c_k\int_{-1}^1 f(x)\sum_{m=0}^{k^2}\binom{k^2}{m}\left(-\frac{(t-x)^2}{4}\right)^m dx \\ &= c_k\sum_{m=0}^{k^2}\binom{k^2}{m}\frac{(-1)^m}{4^m}\int_{-1}^1 f(x)(t-x)^{2m}dx \\ &= c_k\frac{(-1)^{k^2}}{4^{k^2}}\int_{-1}^1 f(x)(x-t)^{2k^2}dx \\ &+ c_k\sum_{m=0}^{k^2-1}\binom{k^2}{m}\frac{(-1)^m}{4^m}\int_{-1}^1 f(x)(t-x)^{2m}dx. \end{aligned}$$

It is now easy to see that the two highest order terms come from the first summand in the above formula. Therefore,

$$\begin{aligned} P_k(t) &= c_k\frac{(-1)^{k^2}}{4^{k^2}}\int_{-1}^1 f(x)dx\ t^{2k^2} - c_k\frac{(-1)^{k^2}2k^2}{4^{k^2}}\int_{-1}^1 f(x)xdx\ t^{2k^2-1} + \cdots \\ &= (-1)^{k^2+1}\frac{2c_k k^2}{4^{k^2}}\left(1-\frac{1}{n}\right)t^{2k^2-1} + \cdots . \end{aligned}$$

(ii) We just use the result of (i) and that $c_k \sim k$.
(iii) Fix a $t \in [-1, 1]$. Then,

$$\begin{aligned}\int_{-2}^{2} f(t-x)\phi_k(x)dx &= \int_{\mathbb{R}} f(t-x)\phi_k(x)\chi_{[-2,2]}(x)dx \\ &= \int_{-1}^{1} f(x)\phi_k(t-x)\chi_{[-2,2]}(t-x)dx \\ &= \int_{-1}^{1} f(x)\phi_k(t-x)dx \\ &= P_k(t).\end{aligned}$$

However, since ϕ_k is even,

$$P_k(t) = \int_{-2}^{2} f(t-x)\phi_k(x)dx = \int_{0}^{2} \big(f(t+x) + f(t-x)\big)\phi_k(x)dx.$$

□

We are now ready to prove the lower bound for $I(P)$.

Proposition 3.6. *Let P_n be the polynomial defined in (3.5) where n is the large positive integer used to define the function f in Lemma* **??**. *Then P_n is a polynomial of degree $d = 2n^2 - 1$ and*

$$I(P_n) = \left| p.v. \int_{\mathbb{R}} e^{iP_n(t)} \frac{dt}{t} \right| \geq c \log d.$$

Proof. Since P_n is odd,

$$I(P_n) = 2\left| \int_{0}^{+\infty} \frac{\sin P_n(t)}{t} dt \right|,$$

and it suffices to show that for all $R \geq 1$

$$\left| \int_{0}^{R} \frac{\sin P_n(t)}{t} dt \right| \geq c \log d \sim c \log n. \tag{3.6}$$

By part (ii) of Lemma (ii) of lemma 3.5 and corollary 1.4, we see that

$$\left| \int_{1}^{R} \frac{\sin P_n(t)}{t} dt \right| \leq c$$

for every $R \geq 1$. As a result, the proof will be complete if we show that

$$I_1(P_n) = \left| \int_{0}^{1} \frac{\sin P_n(t)}{t} dt \right| \geq c \log n. \tag{3.7}$$

Using lemma 3.4 and the triangle inequality we get

$$I_1(P_n) \geq c \log n - |I_1(P_n) - I(f)| \tag{3.8}$$

and, in order to show (3.7), it suffices to show that

$$|I_1(P_n) - I(f)| = o(\log n). \tag{3.9}$$

We have that

$$\begin{aligned} |I_1(P_n) - I(f)| &= \left| \int_0^1 \frac{\sin P_n(t) - \sin f(t)}{t} dt \right| \\ &\leq \int_0^1 \frac{|P_n(t) - f(t)|}{t} dt. \end{aligned}$$

Using part (iii) of lemma 3.5 and (3.4), we get

$$|P_n(t) - f(t)| \leq \int_0^2 |f(t+x) + f(t-x) - 2f(t)| \phi_n(x) dx$$

for $0 \leq t \leq 1$. Hence,

$$|I_1(P_n) - I(f)| \leq \int_0^2 \int_0^1 \frac{|f(t+x) + f(t-x) - 2f(t)|}{t} \ dt \ \phi_n(x) dx.$$

Now, the desired result, condition(3.9), is the content of the following lemma. □

Lemma 3.7. *Let* $A(x,t) = |f(t+x) + f(t-x) - 2f(t)|$. *Then,*

$$\int_0^2 \int_0^1 \frac{A(x,t)}{t} \ dt \ \phi_n(x) dx = o(\log n).$$

Proof. Firstly, it is not difficult to establish that

$$\begin{aligned} A(x,t) &\leq 4 \min(nx, nt, 1) && (3.10) \\ A(x,t) &= 0, \qquad \text{when } \frac{1}{n} \leq t - x \leq t + x \leq 1 - \frac{1}{n} \ . && (3.11) \end{aligned}$$

Indeed,

$$\begin{aligned} A(x,t) &\leq |f(t+x) - f(t)| + |f(t-x) - f(t)| \\ &\leq nx + nx \ \leq \ 2nx. \end{aligned}$$

On the other hand,

$$\begin{aligned} A(x,t) &= |f(t+x)-f(x)+f(t-x)-f(-x)-2f(t)| \\ &\leq |f(t+x)-f(x)|+|f(t-x)-f(-x)|+2|f(t)| \\ &\leq nt+nt+2nt \;=\; 4nt. \end{aligned}$$

Inequality (3.10) now follows by the fact that $|f|$ is bounded by 1 and (3.11) is trivial to prove.

We split the integral $\int_0^2\int_0^1 \cdots dtdx$ into seven integrals:

$$\int_0^2\int_{\frac{1}{2}}^1 \cdots dtdx + \int_0^{\frac{1}{n}}\int_0^x \cdots dtdx + \int_{\frac{1}{n}}^2\int_0^{\frac{1}{n}} \cdots dtdx + \int_0^{\frac{1}{n}}\int_x^{x+\frac{1}{n}} \cdots dtdx$$
$$+\int_0^{\frac{1}{2}-\frac{1}{n}}\int_{x+\frac{1}{n}}^{\frac{1}{2}} \cdots dtdx + \int_{\frac{1}{n}}^{\frac{1}{2}-\frac{1}{n}}\int_{\frac{1}{n}}^{x+\frac{1}{n}} \cdots dtdx + \int_{\frac{1}{2}-\frac{1}{n}}^2\int_{\frac{1}{n}}^{\frac{1}{2}} \cdots dtdx.$$

We estimate each of the seven integrals separately.

$$\int_0^2\int_{\frac{1}{2}}^1 \frac{A(x,t)}{t}dt\phi_n(x)dx \;\leq\; 4\log 2\int_0^2 \phi_n(x)dx \;=\; 2\log 2.$$

$$\begin{aligned} \int_0^{\frac{1}{n}}\int_0^x \frac{A(x,t)}{t}dt\phi_n(x)dx &\leq \int_0^{\frac{1}{n}}\int_0^x \frac{4nt}{t}dt\phi_n(x)dx \\ &= \int_0^{\frac{1}{n}} 4nx\phi_n(x)dx \;\leq\; 2. \end{aligned}$$

$$\begin{aligned} \int_{\frac{1}{n}}^2\int_0^{\frac{1}{n}} \frac{A(x,t)}{t}dt\phi_n(x)dx &\leq \int_{\frac{1}{n}}^2\int_0^{\frac{1}{n}} \frac{4nt}{t}dt\phi_n(x)dx \\ &= \int_0^{\frac{1}{n}} 4\phi_n(x)dx \;\leq\; 2. \end{aligned}$$

$$\begin{aligned} \int_0^{\frac{1}{n}}\int_x^{x+\frac{1}{n}} \frac{A(x,t)}{t}dt\phi_n(x)dx &\leq \int_0^{\frac{1}{n}}\int_x^{x+\frac{1}{n}} \frac{4nx}{t}dt\phi_n(x)dx \\ &= \int_0^{\frac{1}{n}} 4nx\log\left(1+\frac{1}{nx}\right)\phi_n(x)dx \;\leq\; 2. \end{aligned}$$

For $\int_0^{\frac{1}{2}-\frac{1}{n}}\int_{x+\frac{1}{n}}^{\frac{1}{2}}$ we have $\frac{1}{n}\le t-x\le t+x\le 1-\frac{1}{n}$ and, from (3.11), $A(x,t)=0$. Hence

$$\int_0^{\frac{1}{2}-\frac{1}{n}}\int_{x+\frac{1}{n}}^{\frac{1}{2}}\frac{A(x,t)}{t}dt\phi_n(x)dx=0.$$

Next,

$$\begin{aligned}\int_{\frac{1}{n}}^{\frac{1}{2}-\frac{1}{n}}\int_{\frac{1}{n}}^{x+\frac{1}{n}}\frac{A(x,t)}{t}dt\phi_n(x)dx &\le \int_{\frac{1}{n}}^{\frac{1}{2}-\frac{1}{n}}\int_{\frac{1}{n}}^{x+\frac{1}{n}}\frac{4}{t}dt\phi_n(x)dx\\ &\le 4\int_{\frac{1}{n}}^{1}\log(nx+1)\phi_n(x)dx.\end{aligned}$$

Now, fix some $\alpha\in(0,1)$. Write

$$\begin{aligned}\int_{\frac{1}{n}}^{1}\log(nx+1)\phi_n(x)dx &= \int_{\frac{1}{n}}^{\frac{1}{n^\alpha}}\cdots dx+\int_{\frac{1}{n^\alpha}}^{1}\cdots dx\\ &\le \frac{\log(n^{1-\alpha}+1)}{2}+c_n\log(n+1)\int_{\frac{1}{n^\alpha}}^{1}\left(1-\frac{x^2}{4}\right)^{n^2}dx\\ &\le \frac{\log(n^{1-\alpha}+1)}{2}+cn\log(n+1)\ e^{-\frac{1}{4}n^{2(1-\alpha)}}.\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty}\frac{\int_{\frac{1}{n}}^{1}\log(nx+1)\phi_n(x)dx}{\log n}\le\frac{1-\alpha}{2}$$

and, since α is arbitrary in $(0,1)$,

$$\int_{\frac{1}{n}}^{\frac{1}{2}-\frac{1}{n}}\int_{\frac{1}{n}}^{x+\frac{1}{n}}\frac{A(x,t)}{t}dt\phi_n(x)dx=o(\log n).$$

Finally,

$$\begin{aligned}\int_{\frac{1}{2}-\frac{1}{n}}^{2}\int_{\frac{1}{n}}^{\frac{1}{2}}\frac{A(x,t)}{t}dt\phi_n(x)dx &\le \int_{\frac{1}{2}-\frac{1}{n}}^{2}\int_{\frac{1}{n}}^{\frac{1}{2}}\frac{4}{t}dt\phi_n(x)dx\\ &\le 4\log\frac{n}{2}\ c_n\int_{\frac{1}{2}-\frac{1}{n}}^{2}\left(1-\frac{x^2}{4}\right)^{n^2}dx\\ &\le cn\log ne^{-\frac{1}{16}n^2}\ =\ o(1).\end{aligned}$$

□

3.2 The upper bound in theorem 3.3

We set

$$K_d = \sup_{P\in\mathcal{P}_d,\epsilon,R} \left| \int_{\epsilon\le|t|\le R} e^{iP(t)}\frac{dt}{t}\right|. \tag{3.12}$$

We take any real polynomial P of degree at most d which we can assume has no constant term, that is, $P(0)=0$. We set $k=[\frac{d}{2}]$ and we write

$$\begin{aligned} P(t) &= a_1t + a_2t^2 + \cdots + a_kt^k + a_{k+1}t^{k+1} + \cdots + a_dt^d \\ &= Q(t) + R(t), \end{aligned}$$

where $Q(t) = a_1t+a_2t^2+\cdots+a_kt^k$ and $R(t) = a_{k+1}t^{k+1}+\cdots+a_dt^d$. Let $|a_l| = max_{k+1\le j\le d}|a_j|$ for some $k+1\le l\le d$. By a change of variables in the integral in (3.12) we can assume that $|a_l|=1$ and thus that $|a_j|\le 1$ for every $k+1\le j\le d$. Now split the integral in (3.12) in two parts as follows

$$\left|\int_{\epsilon\le|t|\le R} e^{iP(t)}\frac{dt}{t}\right| \le \left|\int_{\epsilon\le|t|\le 1} e^{iP(t)}\frac{dt}{t}\right| + \left|\int_{1\le|t|\le R} e^{iP(t)}\frac{dt}{t}\right| = I_1 + I_2. \tag{3.13}$$

For I_1 we have that

$$\begin{aligned} I_1 &\le \left|\int_{\epsilon\le|t|\le 1}\left[e^{iP(t)} - e^{iQ(t)}\right]\frac{dt}{t}\right| + \left|\int_{\epsilon\le|t|\le 1} e^{iQ(t)}\frac{dt}{t}\right| \\ &\le \int_{\epsilon\le|t|\le 1}\left|e^{iP(t)} - e^{iQ(t)}\right|\frac{dt}{t} + K_{[\frac{d}{2}]} \\ &\le \int_{0\le|t|\le 1}\frac{|R(t)|}{t}dt + K_{[\frac{d}{2}]} \\ &\le 2\sum_{j=k+1}^{d}\frac{|a_j|}{j} + K_{[\frac{d}{2}]} \;\le\; 2\sum_{j=k+1}^{d}\frac{1}{j} + K_{[\frac{d}{2}]} \;\le\; c + K_{[\frac{d}{2}]}. \end{aligned}$$

For the second integral in (3.13) we have that

$$I_2 \le \left|\int_{1\le t\le R} e^{iP(t)}\frac{dt}{t}\right| + \left|\int_{-R\le t\le -1} e^{iP(t)}\frac{dt}{t}\right| = I_2^+ + I_2^-.$$

For some $\alpha>0$ to be defined later, split I_2^+ into two parts as follows:

$$I_2^+ \le \int_{\{t\in[1,+\infty):|P'(t)|\le\alpha\}}\frac{dt}{t} + \left|\int_{\{t\in[1,R]:|P'(t)|>\alpha\}} e^{iP(t)}\frac{dt}{t}\right|.$$

Since $\{t \in [1,R] : |P'(t)| > \alpha\}$ consists of at most $O(d)$ intervals where P' is monotonic, using Corollary 1.4 we get the bound

$$\Big| \int_{\{t\in[1,R]:|P'(t)|>\alpha\}} e^{iP(t)} \frac{dt}{t} \Big| \leq c\frac{d}{\alpha}.$$

For the logarithmic measure of the set $\{t \in [1,+\infty) : |P'(t)| \leq \alpha\}$, observe that

$$\begin{aligned}
\int_{\{t\in[1,+\infty):|P'(t)|\leq\alpha\}} \frac{dt}{t} &\leq \sum_{m=0}^{\infty} \int_{\{t\in[2^m,2^{m+1}]:|P'(t)|\leq\alpha\}} \frac{dt}{t} \\
&= \sum_{m=0}^{\infty} \int_{\{2^m t\in[2^m,2^{m+1}]:|P'(2^m t)|\leq\alpha\}} \frac{dt}{t} \\
&= \sum_{m=0}^{\infty} \int_{2^m\{t\in[1,2]:|P'(2^m t)|\leq\alpha\}} \frac{dt}{t} \\
&= \sum_{m=0}^{\infty} \int_{\{t\in[1,2]:|P'(2^m t)|\leq\alpha\}} \frac{dt}{t}.
\end{aligned}$$

We have thus showed that

$$\int_{\{t\in[1,+\infty):|P'(t)|\leq\alpha\}} \frac{dt}{t} \leq \sum_{m=0}^{\infty} |\{t \in [1,2] : |P'(2^m t)| \leq \alpha\}|. \tag{3.14}$$

In order to finish the proof we need a suitable estimate for the sublevel set of a polynomial. This is the content of lemma 2.10.

Consider the polynomial $P'(2^m t)$ with coefficients $ja_j 2^{m(j-1)}$, $1 \leq j \leq d$. Clearly,

$$\max_{1\leq j\leq d} |ja_j 2^{m(j-1)}| \geq |la_l 2^{m(l-1)}| \geq ([\frac{d}{2}]+1)2^{m[\frac{d}{2}]}.$$

Using lemma 2.10 and (3.14), we get

$$\int_{\{t\in[1,+\infty):|P'(t)|\leq\alpha\}} \frac{dt}{t} \leq c\alpha^{\frac{1}{d-1}} \sum_{m=0}^{\infty} \left(\frac{1}{([\frac{d}{2}]+1)2^{m[\frac{d}{2}]}} \right)^{\frac{1}{d-1}} \leq c\alpha^{\frac{1}{d-1}}.$$

Obviously, a similar estimate holds for I_2^-. Summing up the estimates we get

$$\Big| \int_{\epsilon\leq|t|\leq R} e^{iP(t)} \frac{dt}{t} \Big| \leq c + c\frac{d}{\alpha} + c\alpha^{\frac{1}{d-1}} + K_{[\frac{d}{2}]}.$$

Optimizing in α we get that

$$\left| \int_{\epsilon \le |t| \le R} e^{iP(t)} \frac{dt}{t} \right| \le c + K_{[\frac{d}{2}]} \tag{3.15}$$

and hence

$$K_d \le c + K_{[\frac{d}{2}]}.$$

In particular we have

$$K_{2^n} \le c + K_{2^{n-1}}.$$

Using induction on n we get that $K_{2^n} \le cn$. It is now trivial to show the inequality for general d. Indeed, if $2^{n-1} < d \le 2^n$ then $K_d \le K_{2^n} \le cn \le c \log d$.

3.3 Singular oscillatory integrals on $\mathbb{R}^n$

In this section we want to generalize theorem 3.3 in many variables. The problem is stated as follows. Let $K : \mathbb{R}^n \to \mathbb{R}$ be a $-n$ homogeneous function on $\mathbb{R}^n$. The function K can be written in the form

$$K(x) = \frac{\Omega(x')}{|x|^n}, \tag{3.16}$$

where $x' = x/|x| \in S^{n-1}$ and the function Ω is S^{n-1}. Let $P \in \mathcal{P}_{d,n}$. We are interested in estimates of the form

$$I_n(P) = \left| p.v. \int_{\mathbb{R}^n} e^{iP(x)} K(x) dx \right| \le c_d \|\Omega\|_{S^{n-1}}, \tag{3.17}$$

where the constant c_d depends only on the degree d of the polynomial and $\|\Omega\|_{S^{n-1}}$ is a suitable norm of the function Ω on the unit sphere. This problem was also studied by Stein who showed in [14] that (3.17) holds for some constant c_d, depending only on d and $\|\Omega\|_{S^{n-1}} = \|\Omega\|_{L^\infty(S^{n-1})}$.

In the case where Ω is an odd function, (3.17) is an immediate consequence of the one dimensional estimate. We introduce the notation

$$\mathcal{W}_{\mathcal{L}} = \left\{ \Omega : S^{n-1} \to \mathbb{R},\ \Omega \text{ is odd}, \|\Omega\|_{L^1(S^{n-1})} \le 1 \right\}.$$

Corollary 3.8. *There exists an absolute positive constant c such that*

$$\sup_{P \in \mathcal{P}_{d,n}, \Omega \in \mathcal{W}_{\mathcal{L}}} \left| p.v. \int_{\mathbb{R}^n} e^{iP(x)} \frac{\Omega(x/|x|)}{|x|^n} dx \right| \le c \log d.$$

Proof. We simply write

$$\begin{aligned}\int_{\epsilon\leq|x|\leq R} e^{iP(x)}\frac{\Omega(x/|x|)}{|x|^n}dx &= \int_{S^{n-1}}\int_{\epsilon}^{R} e^{P(rx')}\frac{dr}{r}\Omega(x')d\sigma_{n-1}(x') \\ &= \frac{1}{2}\int_{S^{n-1}}\int_{\epsilon}^{R} e^{P(rx')}\frac{dr}{r}(\Omega(x')-\Omega(-x'))d\sigma_{n-1}(x') \\ &= \frac{1}{2}\int_{S^{n-1}}\int_{\epsilon\leq|r|\leq R} e^{P(rx')}\frac{dr}{r}\Omega(x')d\sigma_{n-1}(x'),\end{aligned}$$

and use theorem 3.3. □

For the general case we will follow Stein's proof from [14], incorporating the ideas that gave the proof in dimension one. Here, the suitable class of functions Ω on the unit sphere is

$$\mathcal{W}_{\mathcal{L}\log\mathcal{L}} = \left\{\Omega : S^{n-1}\to\mathbb{R},\ \int_{S^{n-1}}\Omega(x')d\sigma_{n-1}(x') = 0, \|\Omega\|_{L\log L}\leq 1\right\},$$

where

$$\|\Omega\|_{L\log L} = \int_{S^{n-1}}|\Omega(x')|(1+\log^{+}|\Omega(x')|)d\sigma_{n-1}(x').$$

For functions $\Omega\in\mathcal{W}_{\mathcal{L}\log\mathcal{L}}$ we have the analogue of corollary 3.8.

Theorem 3.9. *There exists an absolute positive constant c such that*

$$\sup_{P\in\mathcal{P}_{d,n},\Omega\in\mathcal{W}_{\mathcal{L}\log\mathcal{L}}}\left|p.v.\int_{\mathbb{R}^n} e^{iP(x)}\frac{\Omega(x/|x|)}{|x|^n}dx\right| \leq c\log d.$$

The proof of theorem 3.9 is based on corollary 2.9 and on the following lemmas.

Lemma 3.10. *Let $P(t) = a_0 + a_1t + \cdots + a_dt^d$ and suppose that for some $\frac{d}{2} < j \leq d$, $|a_j|\geq 1$. Then, there exists an absolute positive constant c such that for every $R > 0$,*

$$\left|\int_1^R e^{iP(t)}\frac{dt}{t}\right| \leq c. \tag{3.18}$$

Proof. The proof is contained in the proof of theorem 3.3 in section 3.2. □

Lemma 3.11. *Let $P_a(t) = a_1t + a_2t^2 + \cdots + a_dt^d$, $P_b(t) = b_1t + b_2t^2 + \cdots + b_dt^d$ be real polynomials of degree at most $d = 2^n$, with $a_l, b_l \neq 0$ for every $l\in\{1,2,\ldots,d\}$. Then there exists an absolute positive constant c such that*

$$\left|\int_{\epsilon}^{R}\left\{e^{iP_a(t)} - e^{iP_b(t)}\right\}\frac{dt}{t}\right| \leq c_n + \sum_{j=0}^{n}\left|\log\frac{\max_{2^{j-1}<l\leq 2^j}|a_l|^{\frac{1}{l}}}{\max_{2^{j-1}<l\leq 2^j}|b_l|^{\frac{1}{l}}}\right|, \tag{3.19}$$

for every $0 < \epsilon < R$ and $c_n \leq cn$.

Proof. We prove the lemma by induction on n. For $n=0$ the estimate

$$\left|\int_\epsilon^R \{e^{iat}-e^{ibt}\}\frac{dt}{t}\right| \le c+\left|\log\frac{|a|}{|b|}\right|$$

is classical. Suppose now that (3.19) holds for $n-1$. We set $\delta=\max_{2^{n-1}<l\le 2^n}(|a_l|^{\frac{1}{l}},|b_l|^{\frac{1}{l}})$. By the change of variables $t\mapsto t/\delta$ in the integral in (3.19) we get

$$\begin{aligned}
\left|\int_\epsilon^R \left\{e^{iP_a(t)}-e^{iP_b(t)}\right\}\frac{dt}{t}\right| &= \left|\int_{\epsilon\delta}^{R\delta} \left\{e^{iP_a(t/\delta)}-e^{iP_b(t/\delta)}\right\}\frac{dt}{t}\right| \\
&\le \left|\int_{\epsilon\delta}^{1} \left\{e^{iP_a(t/\delta)}-e^{iP_b(t/\delta)}\right\}\frac{dt}{t}\right| + \left|\int_{1}^{R\delta} \left\{e^{iP_a(t/\delta)}-e^{iP_b(t/\delta)}\right\}\frac{dt}{t}\right| \\
&= I_1+I_2.
\end{aligned}$$

We write $Q_a(t)=\sum_{1\le l\le 2^{n-1}} a_l t^l$, $Q_b(t)=\sum_{1\le l\le 2^{n-1}} b_l t^l$. For I_1 we have that

$$\begin{aligned}
I_1 &\le \left|\int_{\epsilon\delta}^{1} \left\{e^{iQ_a(t/\delta)}-e^{iQ_b(t/\delta)}\right\}\frac{dt}{t}\right| \\
&+ \left|\int_{\epsilon\delta}^{1} \left\{e^{iP_a(t/\delta)}-e^{iQ_a(t/\delta)}\right\}\frac{dt}{t}\right| + \left|\int_{\epsilon\delta}^{1} \left\{e^{iP_b(t/\delta)}-e^{iQ_b(t/\delta)}\right\}\frac{dt}{t}\right|
\end{aligned}$$

From the inductive hypothesis we now have

$$\left|\int_{\epsilon\delta}^{1} \left\{e^{iQ_a(t/\delta)}-e^{iQ_b(t/\delta)}\right\}\frac{dt}{t}\right| \le c_{n-1}+\sum_{j=0}^{n-1}\left|\log\frac{\max_{2^{j-1}<l\le 2^j}|a_l|^{\frac{1}{l}}}{\max_{2^{j-1}<l\le 2^j}|b_l|^{\frac{1}{l}}}\right|. \tag{3.20}$$

On the other hand,

$$\begin{aligned}
\left|\int_{\epsilon\delta}^{1} \left\{e^{iP_a(t/\delta)}-e^{iQ_a(t/\delta)}\right\}\frac{dt}{t}\right| &\le \int_0^1 |P_a(t/\delta)-Q_a(t/\delta)|\,\frac{dt}{t} \le \sum_{2^{n-1}<l\le 2^n}\frac{1}{l}\frac{|a_l|}{\delta^l} \\
&\le \sum_{2^{n-1}<l\le 2^n}\frac{1}{l} \le c.
\end{aligned}$$

Similarly, we show that $\left|\int_{\epsilon\delta}^1 \left\{e^{iP_b(t/\delta)}-e^{iQ_b(t/\delta)}\right\}\frac{dt}{t}\right|\le c$, and thus that

$$I_1 \le c_{n-1}+c+\sum_{j=0}^{n-1}\left|\log\frac{\max_{2^{j-1}<l\le 2^j}|a_l|^{\frac{1}{l}}}{\max_{2^{j-1}<l\le 2^j}|b_l|^{\frac{1}{l}}}\right|.$$

Observer that I_2 is symmetrical with respect to P_a, P_b. Hence we can assume that $\delta = \max_{2^{n-1}<l\leq 2^n} |b_l|^{\frac{1}{l}}$ and thus that one of the 2^{n-1} last coefficients of the polynomial $P_b(t/\delta)$ is equal to 1. Lemma 3.10 now gives

$$\left|\int_1^{R\delta} e^{iP_b(t/\delta)}\frac{dt}{t}\right| \leq c.$$

'Estw $\tilde{P}_a(t) = P_a(t/\delta) = \tilde{a_1}t + \cdots + \tilde{a_d}t^d$. We set $\delta_a = \max_{2^{n-1}<l\leq 2^n} |\tilde{a_l}|^{\frac{1}{l}}$. Then

$$\left|\int_1^{R\delta} e^{iP_a(t/\delta)}\frac{dt}{t}\right| = \left|\int_1^{R\delta} e^{i\tilde{P}_a(t)}\frac{dt}{t}\right| \leq \left|\int_{\delta_\alpha}^{1} e^{i\tilde{P}_a(t/\delta_a)}\frac{dt}{t}\right| + \left|\int_1^{R\delta\delta_a} e^{i\tilde{P}_a(t/\delta_a)}\frac{dt}{t}\right|.$$

Now, one of the last 2^{n-1} coefficients of the polynomial $\tilde{P}_a(t/\delta_\alpha)$ is equal to 1 and thus, using lemma 3.10 again

$$\left|\int_1^{R\delta} e^{iP_a(t/\delta)}\frac{dt}{t}\right| \leq \log\frac{1}{\delta_\alpha} + c = \log\frac{1}{\max_{2^{n-1}<l\leq 2^n}|\tilde{a_l}|^{\frac{1}{l}}} + c = \log\frac{\max_{2^{n-1}<l\leq 2^n}|b_l|^{\frac{1}{l}}}{\max_{2^{n-1}<l\leq 2^n}|a_l|^{\frac{1}{l}}} + c.$$

We have supposed that

$$\max_{2^{n-1}<l\leq 2^n} |b_l|^{\frac{1}{l}} \geq \max_{2^{n-1}<l\leq 2^n} |a_l|^{\frac{1}{l}}.$$

A moment's reflection will let us write

$$I_2 \leq c + \left|\log\frac{\max_{2^{n-1}<l\leq 2^n}|b_l|^{\frac{1}{l}}}{\max_{2^{n-1}<l\leq 2^n}|a_l|^{\frac{1}{l}}}\right|.$$

Summing up the estimates for I_1 and I_2 we get

$$\left|\int_\epsilon^R \left\{e^{iP_a(t)} - e^{iP_b(t)}\right\}\frac{dt}{t}\right| \leq c_{n-1} + c + \sum_{j=0}^{n}\left|\log\frac{\max_{2^{j-1}<l\leq 2^j}|a_l|}{\max_{2^{j-1}<l\leq 2^j}|b_l|}\right|,$$

and thus that $c_n \leq c_{n-1} + c$. This also proves that $c_n \leq cn$. □

Proof of theorem 3.9. Let $P \in \mathcal{P}_{d,n}$ and $\Omega \in \mathcal{W}_{\mathcal{L}\log\mathcal{L}}$. It is enough to prove the theorem for polynomials of degree $d = 2^n$, by using the same argument we used in the proof of theorem 3.2. For $0 < \epsilon < R$ we have that

$$\int_{\epsilon\leq|x|\leq R} e^{iP(x)}\frac{\Omega(x/|x|)}{|x|^n}dx = \int_{S^{n-1}}\int_\epsilon^R e^{P(rx')}\frac{dr}{r}\Omega(x')d\sigma_{n-1}(x').$$

We write

$$P(rx') = \sum_{j=1}^{d} P_j(x')r^j,$$

where the polynomials P_j are homogeneous polynomials of degree j. We set $m_j = \|P_j\|_{L^\infty(S^{n-1})}$. Using the zero mean value condition on Ω we can write

$$\begin{aligned}
&\left|\int_{\epsilon\le|x|\le R} e^{iP(x)}\frac{\Omega(x/|x|)}{|x|^n}dx\right| = \left|\int_{S^{n-1}}\int_\epsilon^R \left\{e^{i\sum_{j=1}^d P_j(x')r^j} - e^{i\sum_{j=1}^d m_j r^j}\right\}\frac{dr}{r}\Omega(x')d\sigma_{n-1}(x')\right| \\
\le\ & \int_{S^{n-1}}\left|\int_\epsilon^R \left\{e^{i\sum_{j=1}^d P_j(x')r^j} - e^{i\sum_{j=1}^d m_j r^j}\right\}\frac{dr}{r}\right||\Omega(x')|d\sigma_{n-1}(x') \\
\le\ & cn\|\Omega\|_{L^1(S^{n-1})} + \sum_{j=0}^n \int_{S^{n-1}}\left|\log\frac{\max_{2^{j-1}<l\le 2^j}|m_l|^{\frac{1}{l}}}{\max_{2^{j-1}<l\le 2^j}|P_l(x')|^{\frac{1}{l}}}\right||\Omega(x')|d\sigma_{n-1}(x'),
\end{aligned}$$

where in the last inequality we used lemma 3.11. Now take any $j\in\{0,1,\ldots,n\}$. Observe that $\max_{2^{j-1}<l\le 2^j}|m_l|^{\frac{1}{l}} \ge \max_{2^{j-1}<l\le 2^j}|P_l(x')|^{\frac{1}{l}}$. We have that $\max_{2^{j-1}<l\le 2^j}|m_l|^{\frac{1}{l}} = |m_{l_o}|^{\frac{1}{l_o}}$ for some $2^{j-1}<l_o\le 2^j$. Then

$$\begin{aligned}
&\int_{S^{n-1}}\left|\log\frac{\max_{2^{j-1}<l\le 2^j}|m_l|^{\frac{1}{l}}}{\max_{2^{j-1}<l\le 2^j}|P_l(x')|^{\frac{1}{l}}}\right||\Omega(x')|d\sigma_{n-1}(x') \le \int_{S^{n-1}}\log\frac{|m_{l_o}|^{\frac{1}{l_o}}}{|P_{l_o}(x')|^{\frac{1}{l_o}}}|\Omega(x')|d\sigma_{n-1}(x') \\
=\ & 2\int_{S^{n-1}}\log\frac{|m_{l_o}|^{\frac{1}{2l_o}}}{|P_{l_o}(x')|^{\frac{1}{2l_o}}}|\Omega(x')|d\sigma_{n-1}(x') \\
\le\ & 2\int_{S^{n-1}}\frac{|m_{l_o}|^{\frac{1}{2l_o}}}{|P_{l_o}(x')|^{\frac{1}{2l_o}}}d\sigma_{n-1}(x') + 2\int_{S^{n-1}}|\Omega(x')|\log(|\Omega(x')|+1)d\sigma_{n-1}.
\end{aligned}$$

To see the last line observe that the functions$\phi(t)=e^t-1$, $\psi(t)=\log(t+1)$ both vanish at 0, they are non negative and monotonic, and the one is the inverse of the other for $t\ge 0$. From Young's inequality we get that for every $a,b\ge 0$,

$$ab \le \int_0^a \phi(t)dt + \int_0^b \psi(t)dt \le e^a + b\log(b+1).$$

The last inequality now follows if we set $a=\log\frac{|m_{l_o}|^{\frac{1}{2l_o}}}{|P_{l_o}(x')|^{\frac{1}{2l_o}}}$ and $b=|\Omega(x')|$. Furthermore, observer that

$$\int_{S^{n-1}}|\Omega(x')|\log(|\Omega(x')|+1)d\sigma_{n-1} \le c\|\Omega\|_{L\log L}.$$

For the first term in the last inequality we have the following lemma:

Lemma 3.12. *Let $P : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous polynomial of degree k. Then, there exists an absolute positive constant c such that*

$$\int_{S^{n-1}} \frac{\|P\|_{L^\infty(S^{n-1})}^{\frac{1}{2k}}}{|P(x')|^{\frac{1}{2k}}} d\sigma_{n-1}(x') \leq c. \tag{3.21}$$

We postpone the proof of lemma 3.12 until the end of the proof of theorem 3.9. Summing up the estimates we get that

$$\begin{aligned} \left| \int_{\epsilon \leq |x| \leq R} e^{iP(x)} \frac{\Omega(x/|x|)}{|x|^n} dx \right| &\leq cn\|\Omega\|_{L^1(S^{n-1})} + c\sum_{j=0}^{n}(1 + \|\Omega\|_{L \log L}) \\ &\leq cn\|\Omega\|_{L^1(S^{n-1})} + cn\|\Omega\|_{L \log L} + cn \leq cn, \end{aligned}$$

since $\|\Omega\|_{L \log L} \leq 1$. □

Proof of lemma 3.12. Let $B = B(0, \rho)$ be the ball of volume 1 in $\mathbb{R}^n$. The radius of B satisfies the equation $\rho^n = \frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}}}$. Now, from corollary 2.9 we have that

$$\int_B |P(x)|^{-\frac{1}{2k}} dx \|P\|_{L^\infty(B)}^{\frac{1}{2k}} \leq (cn)^{\frac{1}{2}}.$$

Using polar coordinates we get

$$\begin{aligned} \|P\|_{L^\infty(S^{n-1})}^{\frac{1}{2k}} \int_{S^{n-1}} |P(x')|^{-\frac{1}{2k}} d\sigma_{n-1}(x') &\leq c\frac{n^{\frac{3}{2}}}{\rho^n} = c\frac{n^{\frac{3}{2}}\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} \\ &\leq c\frac{n^{\frac{3}{2}}(e\pi)^{\frac{n}{2}}}{(\frac{n}{2}+1)^{\frac{n+1}{2}}} \leq c, \end{aligned}$$

from Stirling's formula. □

3.4 Notes and references

3.4.1 References

The problem of estimating the principal value integral $I(P)$ is posed for the first time in [16] and then in [14] and [20]. The corresponding problem on $\mathbb{R}^n$ is also contained in [14]. In these earlier results, the known estimates didn't describe the dependence of the constants on the degree d of the polynomials. Theorem 3.3 is contained in [10] and it is a new result as well as its $n-$dimensional analogues. In the proof of theorem 3.9, we follow the argument in the proof of Stein from [14], improving all the lemmas therin. The argument used in the proof of theorem 3.9 in order to pass to the $L \log L$ norm of the function Ω is classical. Young's inequality can be found for example in [21].

3.4.2 Singular oscillatory integrals with rational phase

Let $P, Q \in \mathcal{P}_d$. We consider the principal value integral

$$I(P,Q) = p.v. \int_{\mathbb{R}} e^{i\frac{P(t)}{Q(t)}} \frac{dt}{t}.$$

The following theorem is contained in [7].

Theorem 3.13. *Let $P, Q \in \mathcal{P}_d$. Then there exists an absolute positive constant c_d depending only on d such that*

$$\left| p.v. \int_{\mathbb{R}} e^{i\frac{P(t)}{Q(t)}} \frac{dt}{t} \right| \leq c_d.$$

3.4.3 A descreet analogue

For $P \in \mathcal{P}_d$ let us consider the symmetrical sums

$$S(P) = \sum_{n \neq 0} \frac{e^{iP(n)}}{n}.$$

The correspondind symmetrical partial sums are then

$$S_N(P) = \sum_{0<|n|<N} \frac{e^{iP(n)}}{n}.$$

We have the following theorem.

Theorem 3.14. *Let $d \geq 2$. Then*

$$\sup_{N=1,2,\ldots} \sup_{P \in \mathcal{P}_d} |S_N(P)| \leq c_d.$$

Whatsmore, for every polynomial $P \in \mathcal{P}_d$ the sequence $\{S_N(P)\}$ converges as $N \to \infty$ and thus the series $S(P)$, considered as the limit of the symmetrical partial sums $S_N(P)$ is everywhere defined and bounded on the space $\mathcal{P}_d$.

This theorem is proved in [2] and independently in [17]. The dependence of the constant c_d apo to d den e'inai gnwst'h.

Bibliography

[1] G. I. Arkhipov, A. A. Karacuba, and V. N. Čubarikov, *Trigonometric integrals*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 5, 971–1003, 1197. MR MR552548 (81f:10050)

[2] G. I. Arkhipov and K. I. Oskolkov, *A special trigonometric series and its applications*, Mat. Sb. (N.S.) **134(176)** (1987), no. 2, 147–157, 287. MR MR922412 (89a:42010)

[3] M. Bebendorf, *A note on the Poincaré inequality for convex domains*, Z. Anal. Anwendungen **22** (2003), no. 4, 751–756. MR MR2036927 (2004k:26025)

[4] A. Carbery, M. Christ, and J. Wright, *Multidimensional van der Corput and sublevel set estimates*, J. Amer. Math. Soc. **12** (1999), no. 4, 981–1015. MR MR1683156 (2000h:42010)

[5] A. Carbery, S. Wainger, and J. Wright, *Personal communication*, 2005.

[6] A. Carbery and J. Wright, *Distributional and L^q norm inequalities for polynomials over convex bodies in $\mathbb{R}^n$*, Math. Res. Lett. **8** (2001), no. 3, 233–248. MR MR1839474 (2002h:26033)

[7] M. Folch-Gabayet and J. Wright, *An oscillatory integral estimate associated to rational phases*, J. Geom. Anal. **13** (2003), no. 2, 291–299. MR MR1967028 (2004b:42025)

[8] R. Kannan, L. Lovász, and M. Simonovits, *Isoperimetric problems for convex bodies and a localization lemma*, Discrete Comput. Geom. **13** (1995), no. 3-4, 541–559. MR MR1318794 (96e:52018)

[9] F. Nazarov, M. Sodin, and A. Vol′berg, *The geometric Kannan-Lovász-Simonovits lemma, dimension-free estimates for the distribution of the values of polynomials, and the distribution*

of the zeros of random analytic functions, Algebra i Analiz **14** (2002), no. 2, 214–234. MR MR1925887 (2004e:60086)

[10] I. R. Parissis, *A sharp bound for the Stein-Wainger oscillatory integral*, Proc. Amer. Math. Soc. **to appear** (2006).

[11] L. E. Payne and H. F. Weinberger, *An optimal Poincaré inequality for convex domains*, Arch. Rational Mech. Anal. **5** (1960), 286–292 (1960). MR MR0117419 (22 #8198)

[12] D. H. Phong and E. M. Stein, *Oscillatory integrals with polynomial phases*, Invent. Math. **110** (1992), no. 1, 39–62. MR MR1181815 (93k:58215)

[13] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals. I. Oscillatory integrals*, J. Funct. Anal. **73** (1987), no. 1, 179–194. MR MR890662 (88g:42023)

[14] E. M. Stein, *Oscillatory integrals in Fourier analysis*, Beijing lectures in harmonic analysis (Beijing, 1984), Ann. of Math. Stud., vol. 112, Princeton Univ. Press, Princeton, NJ, 1986, pp. 307–355. MR MR864375 (88g:42022)

[15] ———, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. MR MR1232192 (95c:42002)

[16] E. M. Stein and S. Wainger, *The estimation of an integral arising in multiplier transformations.*, Studia Math. **35** (1970), 101–104. MR MR0265995 (42 #904)

[17] ———, *Discrete analogues of singular Radon transforms*, Bull. Amer. Math. Soc. (N.S.) **23** (1990), no. 2, 537–544. MR MR1056560 (92e:42010)

[18] ———, *Oscillatory integrals related to Carleson's theorem*, Math. Res. Lett. **8** (2001), no. 5-6, 789–800. MR MR1879821 (2002k:42038)

[19] I. M. Vinogradov, *Selected works*, Springer-Verlag, Berlin, 1985, With a biography by K. K. Mardzhanishvili, Translated from the Russian by Naidu Psv [P. S. V. Naidu], Translation edited by Yu. A. Bakhturin. MR MR807530 (87a:01042)

[20] S. Wainger, *Averages and singular integrals over lower-dimensional sets*, Beijing lectures in harmonic analysis (Beijing, 1984), Ann. of Math. Stud., vol. 112, Princeton Univ. Press, Princeton, NJ, 1986, pp. 357–421. MR MR864376 (89a:42026)

[21] A. Zygmund, *Trigonometric series. Vol. I, II*, third ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2002, With a foreword by Robert A. Fefferman. MR MR1963498 (2004h:01041)