The equations of type II_1 unprojection

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Abstract

The type II_1 unprojection is, by definition, the generic complete intersection type II unprojection, in the sense of [P] Section 3.1, for the parameter value $k = 1$, and depends on a parameter $n \geq 2$. Our main results are the explicit calculation of the linear relations of the type II_1 unprojection for any value $n \geq 2$ (Theorem 3.16) and the explicit calculation of the quadratic equation for the case $n = 3$ (Theorem 4.1). In addition, Section 5 contains applications to algebraic geometry.

1 Introduction

The first appearance of the type II unprojection was in the study of elliptic involutions between Fano 3-fold hypersurfaces in [CPR], while the theoretical foundations were developed in [P] using valuations. Valuations were useful to prove the existence of certain relations, but didn't provide any explicit formulas for them. The present paper is an effort towards the explicit calculations for the type II unprojection.

We define the type II_k unprojection to be the generic complete intersection type II unprojection, in the sense of [P] Section 3.1, for the parameter value $k \geq 1$. According to [P], it depends on a parameter $n \geq 2$, increases the codimension from $nk-1$ to $nk-1+(k+1)$ and preserves the Gorenstein property. To our knowledge, the only previously done explicit calculation for the type II unprojection was for the type II_1 case for $n = 2$ ([CPR], [R]), which is reproduced for completeness in Subsection 4.4 below.

Section 3 contains the calculation, using homological and multilinear algebra, of the linear equations of the type II_1 unprojection for any value $n \geq 2$. The main result is Theorem 3.16, which provides explicit formulas for the linear relations.

The main result of Section 4 is Theorem 4.1, which provides an explicit symmetric – in the sense of SL_3 invariance, see Subsection 4.2 – formula for the quadratic relation for the case $n = 3$. The proof of Theorem 4.1 is based on the explicit equality (4.6) which was verified using the computer algebra program Macaulay 2 [GS93-08]. In Subsection 4.3 we briefly sketch how we arrived to the formulas contained in Theorem 4.1 by computer assisted calculations (using the computer algebra program Maple).

As an application of the above results, we sketch in Section 5 the construction of two codimension 4 Fano 3-folds. The quasismoothness checking, using the explicit equations obtained in Sections 3 and 4, was done by the computer algebra program Singular [GPS01].

The calculation of the quadratic equation for the type II_1 unprojection for $n \geq 4$ remains open. We believe that the explicit formulas of the linear relations obtained in the present work together with representation and invariant theoretic techniques could, perhaps, lead to their calculation, and also to a better understanding of the complicated looking formula (4.2). To our knowledge, the problem of calculating the type II_k unprojection for $k \geq 2$ is also open.

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2 Notation

As already mentioned in Section 1 by type II_1 unprojection we mean the generic complete intersection type II unprojection in the sense of [P] Section 3.1 with fixed parameter value $k = 1$. According to [P], the type II_1 unprojection depends on an integral parameter $n \geq 2$, the initial data for the unprojection is specified by a triple

$$I_X \subset I_D \subset \mathcal{O}_{amb} \tag{2.1}$$

(where I_X and I_D are ideals of $\mathcal{O}_{amb}$), and the unprojection constructs an ideal

$$I_Y \subset \mathcal{O}_{amb}[s_0, s_1], \tag{2.2}$$

where s_0, s_1 are new variables.

Fix $n \geq 2$. The ambient ring $\mathcal{O}_{amb}$ will be the polynomial ring

$$\mathcal{O}_{amb} = \mathbb{Z}[x_1, \dots, x_n, y_1, \dots, y_n, z, A^p_{ij}, B^p_{lm}]$$

where $1 \leq p \leq n-1$, $1 \leq i < j \leq n$ and $1 \leq l \leq m \leq n$. (In Section 4 $\mathcal{O}_{amb}$ will change slightly to allow 2 to be invertible.)

We denote by M the $2 \times 2n$ matrix

$$M = \begin{pmatrix} y_1 & \dots & y_n & zx_1 & \dots & zx_n \\ x_1 & \dots & x_n & y_1 & \dots & y_n \end{pmatrix}. \tag{2.3}$$

The ideal $I_D \subset \mathcal{O}_{amb}$ in (2.1) is the ideal generated by the 2×2 minors of M.

For $1 \leq p \leq n-1$, we denote by A^p the $n \times n$ skew–symmetric matrix with (ij)-entry (for $i < j$) equal to A^p_{ij} (and zero diagonal entries), and by B^p the $n \times n$ symmetric matrix with (ij)-entry (for $i \leq j$) equal to B^p_{ij}.

We set

$$x = (x_1, \dots, x_n), \quad y = (y_1, \dots, y_n), \tag{2.4}$$

and, for $1 \leq p \leq n-1$, we set

$$f_p = \sum_{1 \leq i < j \leq n} A^p_{ij} F_{ij} + \sum_{i=1}^{n} B^p_{ii} G_{ii} + \sum_{1 \leq i < j \leq n} B^p_{ij} G_{ij},$$

where,

$$F_{ij} = x_i y_j - x_j y_i, \qquad G_{ij} = 2(y_i y_j - z x_i x_j), \qquad G_{mm} = y_m^2 - z x_m^2$$

for $1 \leq i < j \leq n$ and $1 \leq m \leq n$. In matrix notation

$$f_p = x A^p y^t + y B^p y^t - z x B^p x^t. \tag{2.5}$$

The ideal I_X in (2.1) is the ideal

$$I_X = (f_1, \dots, f_{n-1}).$$

Remark 2.1 According to [P] Proposition 2.16, the ideal I_Y of $\mathcal{O}_{amb}[s_0, s_1]$ contains I_X as a subset and is generated by I_X together with n polynomials $l_1, \dots, l_n \in \mathcal{O}_{amb}[s_0, s_1]$ affine linear in s_0, s_1 of the form

$$l_i = z x_i s_0 + y_i s_1 + \sigma_{1i},$$

where $1 \leq i \leq n$ and $\sigma_{1i} \in \mathcal{O}_{amb}$, together with n polynomials $l_{n+1}, \dots, l_{2n} \in \mathcal{O}_{amb}[s_0, s_1]$ affine linear in s_0, s_1 of the form

$$l_{i+n} = y_i s_0 + x_i s_1 + \sigma_{2i},$$

where $1 \leq i \leq n$ and $\sigma_{2i} \in \mathcal{O}_{amb}$, together with a single affine quadratic polynomial $q \in \mathcal{O}_{amb}[s_0, s_1]$ of the form

$$q = s_1^2 - zs_0^2 + \text{ lower terms},$$

where 'lower terms' means an affine linear polynomial in s_0, s_1, i.e., of the form $e_1 s_0 + e_2 s_1 + e_3$ with $e_1, e_2, e_3 \in \mathcal{O}_{amb}$.

In other words, we have the equality

$$I_Y = (f_1, \ldots, f_{n-1}) + (l_1, \ldots, l_{2n}) + (q) \tag{2.6}$$

of ideals of $\mathcal{O}_{amb}[s_0, s_1]$. Calculating I_Y means providing explicit formulas for $l_1, l_2, \ldots, l_{2n}$ and q.

For simplicity, we call the elements of I_X the original relations, we call $l_1, \ldots, l_{2n}$ the linear relations and, finally, we call q the quadratic relation.

For the parameter value $n = 2$ the ideal I_Y has been explicitly calculated by Reid in [R] Section 9.5. For completeness, we reproduce his results in Subsection 4.4.

In Theorem 3.16 we calculate, for any $n \geq 2$, the linear relations $l_1, \ldots, l_{2n}$. In addition, in Theorem 4.1 we calculate the quadratic polynomial q for the parameter value $n = 3$.

The calculation of the quadratic polynomial q for $n \geq 4$ remains open.

3 Linear relations of Type II_1 unprojection

In this section we use the notations of Section 2.

3.1 Generalities

Assume R is a ring, L is an R-module, and $h\colon L \to R$ a homomorphism of R-modules. For $p \geq 1$, [BH] p. 43 defines an R-homomorphism $d^p h : \wedge^p L \to \wedge^{p-1} L$ by

$$d^p h(v_1 \wedge \cdots \wedge v_p) = \sum_{i=1}^{p} (-1)^{i+1} h(v_i) v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_p, \tag{3.1}$$

for all $v_1, \ldots, v_p \in L$. (For $p = 1$ we set $\wedge^0 L = R$ and $d^1 h = h$.) The maps $d^p h$ define the Koszul complex

$$\cdots \to \wedge^p L \xrightarrow{d^p h} \wedge^{p-1} L \to \cdots \to \wedge^2 L \xrightarrow{d^2 h} L \xrightarrow{h} R \to 0$$

associated to the homomorphism h.

Proposition 3.1 *Assume $h_1, h_2 \colon L \to R$ are two homomorphisms of R-modules. For $p \geq 2$ we have the equality*

$$d^{p-1}h_1 \circ d^p h_2 + d^{p-1}h_2 \circ d^p h_1 = 0,$$

of maps $\wedge^p L \to \wedge^{p-2} L$.

Proof Indeed,

$$0 = d^{p-1}(h_1 + h_2) \circ d^p(h_1 + h_2) = d^{p-1}h_1 \circ d^p h_2 + d^{p-1}h_2 \circ d^p p_1,$$

since

$$d^{p-1}h_1 \circ d^p h_1 = d^{p-1}h_2 \circ d^p h_2 = 0.$$

QED

3.2 The second complex

Assume L is a free $\mathcal{O}_{amb}$-module of rank n, and let $e_1, \dots, e_n$ be a fixed basis of L. We define four $\mathcal{O}_{amb}$-homomorphisms $h_1, \dots, h_4 : L \to \mathcal{O}_{amb}$, with

$$h_1(e_i) = y_i, \quad h_2(e_i) = x_i, \quad h_3(e_i) = zx_i, \quad h_4(e_i) = y_i,$$

for $1 \leq i \leq n$. In addition, we define, for $1 \leq p \leq n$, homomorphisms

$$\phi_p \colon \wedge^p L \oplus \wedge^p L \to \wedge^{p-1} L \oplus \wedge^{p-1} L$$

as follows:

$$\phi_p(a, b) = \begin{pmatrix} d^p h_1 & d^p h_3 \\ d^p h_2 & d^p h_4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (d^p h_1(a) + d^p h_3(b), d^p h_2(a) + d^p h_4(b)),$$

for $a, b \in \wedge^p L$.

Proposition 3.2 *For $p \geq 2$ we have*

$$\phi_{p-1} \circ \phi_p = 0.$$

Proof Using the equalities $h_4 = h_1$ and $h_3 = zh_2$ the result follows from Proposition 3.1. QED

The proof of the following proposition will be given in the Subsection 3.3.

Proposition 3.3 *The complex*

$$\mathbf{L}^* : \quad \mathcal{O}_{amb} \oplus \mathcal{O}_{amb} \xleftarrow{\phi_1} L \oplus L \xleftarrow{\phi_2} \wedge^2 L \oplus \wedge^2 L \xleftarrow{\phi_3} \wedge^3 L \oplus \wedge^3 L \leftarrow \dots \quad (3.2)$$

is exact.

3.3 Proof of Proposition 3.3

The proof of Proposition 3.3 will be based on the Buchsbaum–Eisenbud acyclicity criterion as stated in [BH] Theorem 1.4.13. We first need the following combinatorial lemma.

Lemma 3.4 *Let p be an integer with $1 \leq p \leq n$. Then*

$$\sum_{j=p}^{n} (-1)^{j-p} \binom{n}{j} = \binom{n-1}{p-1}.$$

Proof We have

$$\begin{aligned}
\sum_{j=p}^{n} (-1)^{j-p} \binom{n}{j} &= (-1)^{n-p} + \sum_{j=p}^{n-1} (-1)^{j-p} \binom{n}{j} \\
&= (-1)^{n-p} + \sum_{j=p}^{n-1} (-1)^{j-p} (\binom{n-1}{j-1} + \binom{n-1}{j}) \\
&= (-1)^{n-p} + \binom{n-1}{p-1} + (-1)^{n-1-p} = \binom{n-1}{p-1},
\end{aligned}$$

where we used the Pascal's rule $\binom{n}{j} = \binom{n-1}{j-1} + \binom{n-1}{j}$ which is valid whenever $1 \leq j \leq n-1$. QED

For the following we fix an integer p with $1 \leq p \leq n$. We set

$$r_p = \sum_{j=p}^{n} (-1)^{j-p} \operatorname{rank} (\wedge^p L \oplus \wedge^p L),$$

where for a finitely generated free $\mathcal{O}_{amb}$-module N we denote by $\operatorname{rank} N$ the minimal number of generators of N as $\mathcal{O}_{amb}$-module. Using Lemma 3.4 we get

$$r_p = 2\binom{n-1}{p-1}. \tag{3.3}$$

Denote by $I_{r_p}(\phi_p)$ the r_p-th Fitting ideal of the map ϕ_p, that is the ideal of $\mathcal{O}_{amb}$ generated by the $r_p \times r_p$ minors of any matrix representation of ϕ_p. (For more information about Fitting ideals see, for example, [BH] Section 1.4.)

We will need the following general property of Koszul complexes.

Lemma 3.5 *Fix $t \in \{1,2\}$ and s, p with $1 \leq s, p \leq n$. Denote by M_t^p the matrix representation of the map*

$$d^p h_t \colon \wedge^p L \to \wedge^{p-1} L$$

with respect to the bases $e_{i_1} \wedge \cdots \wedge e_{i_p}$ of $\wedge^p L$ and $e_{i_1} \wedge \cdots \wedge e_{i_{p-1}}$ of $\wedge^{p-1} L$ induced by the basis $(e_1, \ldots, e_n)$ of L. For a suitable ordering of the two bases there exists an $(r_p/2) \times (r_p/2)$ submatrix of M_t^p which is diagonal with diagonal entries equal to a_t or $-a_t$, where $a_1 = y_s$ and $a_2 = x_s$.

Proof Consider the subset of the above mentioned basis of $\wedge^p L$ consisting of the elements where e_s appears on the wedge. This subset has $\binom{n-1}{p-1}$ elements, hence $(r_p/2)$ elements using (3.3). The result follows by restricting the homomorphism $d^p h_t$ to the $\mathcal{O}_{amb}$-submodule of $\wedge^p L$ generated by this subset and noticing that the corresponding matrix has a diagonal submatrix of the desired form. QED

Proposition 3.6 *Fix s, p with with $1 \leq s, p \leq n$. There exists a polynomial $g_s \in I_{r_p}(\phi_p)$ of the form*

$$g_s = y_s^{r_p} + g_s^1, \tag{3.4}$$

where g_s^1 involves only the variables y_s, z, x_s and has degree in y_s strictly less than r_p.

Proof It follows immediately from the definition of ϕ_p and Lemma 3.5. QED

The following codimension bound result will be be used in Proposition 3.8 to derive a similar bound for $I_{r_p}(\phi_p)$.

Proposition 3.7 *Let F be any field. The ideal $\widetilde{I}$ of $\mathcal{O}_{amb} \otimes_{\mathbb{Z}} F$ generated by the image of $I_{r_p}(\phi_p)$ under the natural map $\mathcal{O}_{amb} \to \mathcal{O}_{amb} \otimes_{\mathbb{Z}} F$ has codimension greater or equal than n.*

Proof For a lexicographical monomial ordering of the variables y_j, z, x_j (indices $1 \leq j \leq n$) with variables y_j big, Proposition 3.6 implies that the leading term of the polynomial g_s appearing in (3.4) is equal to $y_s^{r_p}$, for $1 \leq s \leq n$. Since over a field F an ideal $\widetilde{I}$ and the leading term ideal of $\widetilde{I}$ have the same codimension the result follows. QED

Recall (cf. [BH] Section 1.2) that the grade of a proper ideal $I \subset \mathcal{O}_{amb}$ is the length of a maximal $\mathcal{O}_{amb}$-regular sequence contained in I.

Proposition 3.8 *The ideal $I_{r_p}(\phi_p)$ has grade greater or equal than n.*

Proof Consider the restriction to $V(I_{r_p}(\phi_p)) \subset \operatorname{Spec} \mathcal{O}_{amb}$ of the morphism $\operatorname{Spec} \mathcal{O}_{amb} \to \operatorname{Spec} \mathbb{Z}$ induced by the ring inclusion $\mathbb{Z} \subset \mathcal{O}_{amb}$. Combining [L] Theorem 4.3.12 with Proposition 3.7 we get that $I_{r_p}(\phi_p)$ has codimension in $\mathcal{O}_{amb}$ greater or equal than n. Being a polynomial ring over the integers, the ring $\mathcal{O}_{amb}$ is Cohen–Macaulay. Using [BH] Corollary 2.1.4 the result follows. QED

We now finish the proof of Proposition 3.3. Combining Lemma 3.4 and Proposition 3.8, Proposition 3.3 follows from the Buchsbaum–Eisenbud acyclicity criterion as stated in [BH] Theorem 1.4.13.

3.4 The first complex

Fix $n \geq 2$. Let N be a free $\mathcal{O}_{amb}$-module of rank $n-1$, with basis $\widetilde{e}_1, \dots, \widetilde{e}_{n-1}$.

We define an $\mathcal{O}_{amb}$-homomorphism $\psi : N \to \mathcal{O}_{amb}$, with $\psi(\widetilde{e}_p) = f_p$, for $1 \leq p \leq n-1$, where f_p was defined in (2.5). As in Subsection 3.1, we have the Koszul complex

$$\mathbf{N}^* : \qquad \mathcal{O}_{amb} \xleftarrow{\psi} N \xleftarrow{d^2\psi} \wedge^2 N \xleftarrow{d^3\psi} \wedge^3 N \leftarrow \dots \tag{3.5}$$

Proposition 3.9 *The sequence $f_1, \dots, f_{n-1}$ is an $\mathcal{O}_{amb}$-regular sequence. As a consequence, the complex $\mathbf{N}^*$ is exact.*

Proof For an element g of $\mathcal{O}_{amb}$ we denote by $\eta(g)$ the result of substituting to g zero for z, zero for A^p_{ij} with $1 \leq p \leq n-1$ and $1 \leq i < j \leq n$, and zero for B^p_{ij} for $1 \leq p \leq n-1$ and $1 \leq i \leq j \leq n$ with $(i,j) \neq (p,p)$. That is, we set zero the variable z and all possible A^p_{ij} and all possible B^p_{ij} with the exception of B^p_{pp} (for $1 \leq p \leq n-1$).

It is clear that for $1 \leq p \leq n-1$ we have

$$\eta(f_p) = B^p_{pp}(y_p)^2.$$

Therefore, the ideal $(\eta(f_1), \dots, \eta(f_{n-1})) \subset \mathcal{O}_{amb}$ has codimension in $\mathcal{O}_{amb}$ equal to $n-1$. As a consequence, the ideal $(f_1, \dots, f_{n-1})$ of $\mathcal{O}_{amb}$ has also codimension $n-1$. Since $\mathcal{O}_{amb}$ is Cohen–Macaulay the sequence $f_1, \dots, f_{n-1}$ is an $\mathcal{O}_{amb}$-regular sequence. The exactness of the Koszul complex $\mathbf{N}^*$ follows from [BH] Corollary 1.6.14. QED

3.5 The first commutative square

We define two $\mathcal{O}_{amb}$-homomorphisms $u_1, u_2 \colon N \to L$, that will make the following diagram

$$\begin{array}{ccc} \mathcal{O}_{amb} & \xleftarrow{\ \psi\ } & N \\ \begin{pmatrix}1\\0\end{pmatrix}\Big\downarrow & & \begin{pmatrix}u_1\\u_2\end{pmatrix}\Big\downarrow \\ \mathcal{O}_{amb}\oplus\mathcal{O}_{amb} & \xleftarrow{\ \phi\ } & L\oplus L \end{array}$$

commutative. That is, they will satisfy the two conditions

$$\psi = h_1 \circ u_1 + z h_2 \circ u_2 \tag{3.6}$$
$$0 = h_1 \circ u_2 + h_2 \circ u_1 \tag{3.7}$$

We set, for $1 \leq j \leq m$,

$$u_1(\widetilde{e}_j) = \sum_{i=1}^{n} c_{ij} e_i \qquad \text{and} \qquad u_2(\widetilde{e}_j) = \sum_{i=1}^{n} c'_{ij} e_i,$$

with

$$c_{ij} = -\sum_{l=1}^{i-1} A^j_{li} x_l + \sum_{l=i+1}^{n} A^j_{il} x_l + \sum_{l=1}^{i} B^j_{li} y_l + \sum_{l=i+1}^{n} B^j_{il} y_l,$$

and

$$c'_{ij} = -\sum_{l=1}^{i} B^j_{li} x_l - \sum_{l=i+1}^{n} B^j_{il} x_l$$

for $1 \leq i \leq n$ and $1 \leq j \leq n-1$.

In matrix notation, u_1 and u_2 are given, with respect to the above defined bases $(\widetilde{e}_j)$ of N and (e_i) of L, by the following two $n \times (n-1)$ matrices:

$$\begin{aligned} u_1 &= (-A^1 x^t + B^1 y^t \mid -A^2 x^t + B^2 y^t \mid \ldots \mid -A^{n-1} x^t + B^{n-1} y^t) \\ u_2 &= (-B^1 x^t \mid -B^2 x^t \mid \ldots \mid -B^{n-1} x^t). \end{aligned} \tag{3.8}$$

That is, for $1 \leq j \leq n-1$, u_1 is the matrix with j-th column equal to $-A^j x^t + B^j y^t$, while u_2 is the matrix with j-th column equal to $-B^j x^t$.

Proposition 3.10 *The maps u_1 and u_2 satisfy the equations (3.6) and (3.7).*

Proof An easy calculation using (3.8). QED

3.6 The basic formula

The key result of the present subsection is Proposition 3.12, essentially a formal consequence of (3.7), which relates the differentials and the alternating products of the homomorphisms h_i and u_j. We need the following definitions.

Assume $a_1, a_2 \colon N \to L$ are two $\mathcal{O}_{amb}$-homomorphisms.

For $p \geq 0$ we have a natural induced homomorphism $\wedge^p a_1 \colon \wedge^p N \to \wedge^p L$. By definition, $\wedge^0 a_1$ is the identity map $\mathcal{O}_{amb} \to \mathcal{O}_{amb}$, For $p \geq 1$, the map $\wedge^p a_1$ is uniquely specified by the property

$$\wedge^p a_1(c_1 \wedge \cdots \wedge c_p) = a_1(c_1) \wedge \cdots \wedge a_1(c_p)$$

for all $c_1, \ldots, c_p \in N$. We denote the map $\wedge^p a_1$ also by $\mathrm{alt}(a_1^p)$.

We will now define for $p, q \geq 1$ an $\mathcal{O}_{amb}$-homomorphism

$$\mathrm{alt}(a_1^p, a_2^q) \colon \wedge^{p+q} N \to \wedge^{p+q} L.$$

We set

$$\mathrm{alt}(a_1^p, a_2^q) = \sum w_I,$$

where the sum is over all subsets $I \subseteq \{1, \ldots, p+q\}$, with $|I| = p$ and, by definition, $w_I = w_1 \wedge \cdots \wedge w_{p+q}$ with $w_i = a_1$ if $i \in I$ and $w_i = a_2$ if $i \notin I$. More precisely, even though the tensor product map

$$w_1 \otimes \cdots \otimes w_{p+q} \colon N^{\otimes(p+q)} \to L^{\otimes(p+q)}$$

does not induce a map $\wedge^{p+q} N \to \wedge^{p+q} L$, taking the sum of those maps over the family of subsets I as above induces a well–defined map $\wedge^{p+q} N \to \wedge^{p+q} L$, and this is the map denoted by $\mathrm{alt}(a_1^p, a_2^q)$.

Remark 3.11 For $p = q = 1$ we have

$$\mathrm{alt}(a_1, a_2)(c_1 \wedge c_2) = a_1(c_1) \wedge a_2(c_2) + a_2(c_1) \wedge a_1(c_2),$$

for all $c_1, c_2 \in N$. We will also denote $\mathrm{alt}(a_1, a_2)$ by $a_1 \wedge a_2 + a_2 \wedge a_1$.
For $p = 2, q = 1$ we have

$$\mathrm{alt}(a_1^2, a_2) = a_1 \wedge a_1 \wedge a_2 + a_1 \wedge a_2 \wedge a_1 + a_2 \wedge a_1 \wedge a_1,$$

in the sense that

$$\mathrm{alt}(a_1^2, a_2)(c_1 \wedge c_2 \wedge c_3) = a_1(c_1) \wedge a_1(c_2) \wedge a_2(c_3) + a_1(c_1) \wedge a_2(c_2) \wedge a_1(c_3) + a_2(c_1) \wedge a_1(c_2) \wedge a_1(c_3)$$

for all $c_1, c_2, c_3 \in N$.

For the following proposition, which will be proved in Subsection 3.7, recall that $u_1, u_2 \colon N \to L$ were defined in Subsection 3.5, and $h_1, h_2 \colon L \to \mathcal{O}_{amb}$ were defined in Subsection 3.2. We also use the notational conventions $\mathrm{alt}(u_1^p, u_2^0) = \mathrm{alt}(u_1^p)$, $\mathrm{alt}(u_1^0, u_2^q) = \mathrm{alt}(u_2^q)$, $\mathrm{alt}(u_1^{p-2}, u_2^q) = 0$ for $p = 1$ and $q \geq 1$, and $\mathrm{alt}(u_1^p, u_2^{q-2}) = 0$ for $q = 1$ and $p \geq 1$.

Proposition 3.12 *a) Let $u \colon M \to L$ and $h \colon L \to \mathcal{O}_{amb}$ be two (arbitrary) homomorphisms of $\mathcal{O}_{amb}$-modules. Assume $p \geq 2$. We have*

$$d^{p-1}h \circ \mathrm{alt}(u^{p-1}) = \mathrm{alt}(u^{p-2}) \circ d^{p-1}(h \circ u).$$

b) For $p, q \geq 1$ we have

$$(d^{p+q-1}h_2) \circ \mathrm{alt}(u_1^p, u_2^{q-1}) + (d^{p+q-1}h_1) \circ \mathrm{alt}(u_1^{p-1}, u_2^q) = \\ \mathrm{alt}(u_1^{p-2}, u_2^q) \circ d^{p+q-1}(h_1 \circ u_1) + \mathrm{alt}(u_1^p, u_2^{q-2}) \circ d^{p+q-1}(h_2 \circ u_2), \tag{3.9}$$

where the equality is, of course, as maps $\wedge^{p+q-1}N \to \wedge^{p+q-2}L$.

3.7 Proof of Proposition 3.12

We will use the following two lemmas.

Lemma 3.13 *Assume $p, q \geq 1$ and $u_1, u_2 \colon N \to L$ are two $\mathcal{O}_{amb}$-module homomorphisms. We have*

$$\mathrm{alt}(u_1^p, u_2^q) = u_1 \wedge \mathrm{alt}(u_1^{p-1}, u_2^q) + u_2 \wedge \mathrm{alt}(u_1^p, u_2^{q-1}),$$

in the sense that

$$\mathrm{alt}(u_1^p, u_2^q)(c \wedge c_1) = \\ u_1(c) \wedge \Big[\mathrm{alt}(u_1^{p-1}, u_2^q)(c_1)\Big] + u_2(c) \wedge \Big[\mathrm{alt}(u_1^p, u_2^{q-1})(c_1)\Big]$$

for all $c \in N$ and $c_1 \in \wedge^{p+q-1}N$.

Proof Immediate from the definitions. QED

Lemma 3.14 *Assume $p, q \geq 1$, $u_1, u_2 \colon N \to L$ are two $\mathcal{O}_{amb}$-module homomorphisms, and $h \colon N \to \mathcal{O}_{amb}$ is an $\mathcal{O}_{amb}$-homomorphism. We have*

$$\mathrm{alt}(u_1^p, u_2^q) \circ (d^{p+q+1}h) = h \wedge \mathrm{alt}(u_1^p, u_2^q) - u_1 \wedge \Big[\mathrm{alt}(u_1^{p-1}, u_2^q) \circ d^{p+q}h\Big] \\ -u_2 \wedge \Big[\mathrm{alt}(u_1^p, u_2^{q-1}) \circ d^{p+q}h\Big],$$

in the sense that

$$\Big[\mathrm{alt}(u_1^p, u_2^q) \circ (d^{p+q+1}h)\Big](c \wedge c_1) =$$
$$h(c)\Big[\mathrm{alt}(u_1^p, u_2^q)(c_1)\Big] - u_1(c) \wedge (\mathrm{alt}(u_1^{p-1}, u_2^q) \circ d^{p+q}h)(c_1)$$
$$-u_2(c) \wedge (\mathrm{alt}(u_1^p, u_2^{q-1}) \circ d^{p+q}h)(c_1)$$

for all $c \in N$ *and* $c_1 \in \wedge^{p+q} N$*.*

Proof Since

$$d^{p+q+1}h(c \wedge c_1) = h(c)c_1 - c \wedge d^{p+q}h(c_1), \tag{3.10}$$

we have, using Lemma 3.13, that

$$\mathrm{alt}(u_1^p, u_2^q) \circ (d^{p+q+1}h)(c \wedge c_1) = h(c)(\mathrm{alt}(u_1^p, u_2^q)(c_1))$$
$$-\Big[u_1 \wedge \mathrm{alt}(u_1^{p-1}, u_2^q) + u_2 \wedge \mathrm{alt}(u_1^p, u_2^{q-1})\Big](c \wedge d^{p+q}h(c_1))$$

and the result follows. QED

We will now prove part a) of Proposition 3.12. Assume $p \geq 2$. For $c_1, \ldots, c_{p-1} \in N$ we have

$$d^{p-1}h \circ \mathrm{alt}(u^{p-1})(c_1 \wedge \cdots \wedge c_{p-1}) = d^{p-1}h(u(c_1) \wedge \cdots \wedge u(c_{p-1}))$$
$$= \sum_{i=1}^{p-1} (-1)^{i-1} h(u(c_i))(u(c_1) \wedge \cdots \wedge \widehat{u(c_i)} \cdots \wedge u(c_{p-1}))$$
$$= \mathrm{alt}(u^{p-2}) \circ d^{p-1}(h \circ u)(c_1 \wedge \cdots \wedge c_{p-1}).$$

We will now prove part b) of Proposition 3.12. Assume that $p, q \geq 1$, that N, L are $\mathcal{O}_{amb}$-modules and that $u_1, u_2 \colon N \to L$ and $h_1, h_2 \colon L \to \mathcal{O}_{amb}$ are four $\mathcal{O}_{amb}$-module homomorphisms with the property

$$h_1 \circ u_2 + h_2 \circ u_1 = 0, \tag{3.11}$$

cf. (3.7). We will show by induction on $p + q$ that (3.9) holds. If $p = q = 1$ the result is clear, since it is exactly our hypothesis (3.11).

Assume now that $p + q \geq 3$ and that (3.9) holds for the $(p-1, q)$ and $(p, q-1)$ cases.

Using Lemma 3.13 we have

$$\begin{aligned}
d^{p+q-1}h_2 \circ \operatorname{alt}(u_1^p, u_2^{q-1}) + d^{p+q-1}h_1 \circ \operatorname{alt}(u_1^{p-1}, u_2^q) = \\
d^{p+q-1}h_2 \circ [u_1 \wedge \operatorname{alt}(u_1^{p-1}, u_2^{q-1}) + u_2 \wedge \operatorname{alt}(u_1^p, u_2^{q-2})] \\
+ d^{p+q-1}h_1 \circ [u_1 \wedge \operatorname{alt}(u_1^{p-2}, u_2^q) + u_2 \wedge \operatorname{alt}(u_1^{p-1}, u_2^{q-1})]
\end{aligned}$$

which, using (3.10), is equal to

$$\begin{aligned}
(h_2 \circ u_1) \wedge \operatorname{alt}(u_1^{p-1}, u_2^{q-1}) - u_1 \wedge [d^{p+q-2}h_2 \circ \operatorname{alt}(u_1^{p-1}, u_2^{q-1})] \\
+(h_2 \circ u_2) \wedge \operatorname{alt}(u_1^p, u_2^{q-2}) - u_2 \wedge [d^{p+q-2}h_2 \circ \operatorname{alt}(u_1^p, u_2^{q-2})] \\
+(h_1 \circ u_1) \wedge \operatorname{alt}(u_1^{p-2}, u_2^q) - u_1 \wedge [d^{p+q-2}h_1 \circ \operatorname{alt}(u_1^{p-2}, u_2^q)] \\
+(h_1 \circ u_2) \wedge \operatorname{alt}(u_1^{p-1}, u_2^{q-1}) - u_2 \wedge [d^{p+q-2}h_1 \circ \operatorname{alt}(u_1^{p-1}, u_2^{q-1})]
\end{aligned}$$

which, using (3.11), is equal to

$$\begin{aligned}
-u_1 \wedge [d^{p+q-2}h_2 \circ \operatorname{alt}(u_1^{p-1}, u_2^{q-1}) + d^{p+q-2}h_1 \circ \operatorname{alt}(u_1^{p-2}, u_2^q)] \\
-u_2 \wedge [d^{p+q-2}h_2 \circ \operatorname{alt}(u_1^p, u_2^{q-2}) + d^{p+q-2}h_1 \circ \operatorname{alt}(u_1^{p-1}, u_2^{q-1})] \\
+(h_2 \circ u_2) \wedge \operatorname{alt}(u_1^p, u_2^{q-2}) + (h_1 \circ u_1) \wedge \operatorname{alt}(u_1^{p-2}, u_2^q)
\end{aligned}$$

which, using the inductive hypothesis, is equal to

$$\begin{aligned}
(h_2 \circ u_2) \wedge \operatorname{alt}(u_1^p, u_2^{q-2}) - u_1 \wedge [\operatorname{alt}(u_1^{p-1}, u_2^{q-2}) \circ d^{p+q-2}(h_2 \circ u_2)] \\
-u_2 \wedge [\operatorname{alt}(u_1^p, u_2^{q-3}) \circ d^{p+q-2}(h_2 \circ u_2)] \\
+(h_1 \circ u_1) \wedge \operatorname{alt}(u_1^{p-2}, u_2^q) - u_1 \wedge [\operatorname{alt}(u_1^{p-3}, u_2^q) \circ d^{p+q-2}(h_1 \circ u_1)] \\
-u_2 \wedge [\operatorname{alt}(u_1^{p-2}, u_2^{q-1}) \circ d^{p+q-2}(h_1 \circ u_1)]
\end{aligned}$$

which, using Lemma 3.14, is equal to

$$\operatorname{alt}(u_1^{p-2}, u_2^q) \circ d^{p+q-1}(h_1 \circ u_1) + \operatorname{alt}(u_1^p, u_2^{q-2}) \circ d^{p+q-1}(h_2 \circ u_2)$$

which finishes the proof of Proposition 3.12.

3.8 The connecting homomorphisms

For an integer p, with $1 \leq p \leq n-1$, we set

$$T_p = \begin{pmatrix} \operatorname{alt}(u_1^p) + z \operatorname{alt}(u_1^{p-2}, u_2^2) + z^2 \operatorname{alt}(u_1^{p-4}, u_2^4) + \dots \\ \operatorname{alt}(u_1^{p-1}, u_2) + z \operatorname{alt}(u_1^{p-3}, u_2^3) + z^2 \operatorname{alt}(u_1^{p-5}, u_2^5) + \dots \end{pmatrix},$$

with the summation as long as all exponents are nonnegative. More precisely, using the notational convention $z^0 = 1$, and denoting, for $j = 1, 2$, by T_p^j the j-th row of T_p, we have

$$T_p^1 = \sum_{i=0}^{[\frac{p}{2}]} z^i \operatorname{alt}(u_1^{p-2i}, u_2^{2i}), \qquad T_p^2 = \sum_{i=0}^{[\frac{p-1}{2}]} z^i \operatorname{alt}(u_1^{p-2i-1}, u_2^{2i+1}).$$

We consider T_p as an $\mathcal{O}_{amb}$-homomorphism $\wedge^p N \to \wedge^p L \oplus \wedge^p L$. Taking the two rows of T_p we get two $\mathcal{O}_{amb}$-homomorphisms $T_p^1, T_p^2 \colon \wedge^p N \to \wedge^p L$. For example,

$$\begin{aligned} T_1 &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad T_2 = \begin{pmatrix} \wedge^2 u_1 + z \wedge^2 u_2 \\ u_1 \wedge u_2 + u_2 \wedge u_1 \end{pmatrix}, \quad T_3 = \begin{pmatrix} \wedge^3 u_1 + z \operatorname{alt}(u_1, u_2^2) \\ \operatorname{alt}(u_1^2, u_2) + z \wedge^3 u_2 \end{pmatrix}, \\ T_4 &= \begin{pmatrix} \wedge^4 u_1 + z \operatorname{alt}(u_1^2, u_2^2) + z^2 \wedge^4 u_2 \\ \operatorname{alt}(u_1^3, u_2) + z \operatorname{alt}(u_1, u_2^3) \end{pmatrix}, \\ T_5 &= \begin{pmatrix} \wedge^5 u_1 + z \operatorname{alt}(u_1^3, u_2^2) + z^2 \operatorname{alt}(u_1, u_2^4) \\ \operatorname{alt}(u_1^4, u_2) + z \operatorname{alt}(u_1^2, u_2^3) + z^2 \wedge^5 u_2 \end{pmatrix}. \end{aligned}$$

For the following proposition recall that the map ϕ_p was defined in Subsection 3.2 while the map ψ was defined in Subsection 3.4.

Proposition 3.15 *For any p, with $2 \le p \le n-1$, we have a commutative diagram*

$$\begin{array}{ccc} \wedge^{p-1} N & \xleftarrow{d^p\psi} & \wedge^p N \\ {\scriptstyle T_{p-1}}\downarrow & & {\scriptstyle T_p}\downarrow \\ \wedge^{p-1} L \oplus \wedge^{p-1} L & \xleftarrow{\phi_p} & \wedge^p L \oplus \wedge^p L \end{array}$$

That is,

$$\phi_p \circ T_p = T_{p-1} \circ d^p \psi$$

as maps $\wedge^p N \to \wedge^{p-1} L \oplus \wedge^{p-1} L$.

Proof Fix $p \ge 2$. For $i \in \mathbb{Z}$ we set

$$C_i = \operatorname{alt}(u_1^i, u_2^{p-i})$$

if $0 \le i \le p$ and $C_i = 0$ otherwise, and we set

$$D_i = \operatorname{alt}(u_1^i, u_2^{p-1-i})$$

if $0 \le i \le p-1$ and $D_i = 0$ otherwise.

For the rest of the proof all sums are for $i \in \mathbb{Z}$.

We have

$$T_p^1 = \sum z^i C_{p-2i}, \qquad T_p^2 = \sum z^i C_{p-2i-1},$$
$$T_{p-1}^1 = \sum z^i D_{(p-1)-2i}, \qquad T_{p-1}^2 = \sum z^i D_{(p-1)-2i-1}.$$

By Proposition 3.12, for any $i \in \mathbb{Z}$ we have

$$d^p h_1 \circ C_{i-1} + d^p h_2 \circ C_i = D_{i-2} \circ d^p(h_1 \circ u_1) + D_i \circ d^p(h_2 \circ u_2). \tag{3.12}$$

Using (3.6), we need to show that

$$d^p h_1 \circ T_p^1 + z d^p h_2 \circ T_p^2 = T_{p-1}^1 \circ d^p(h_1 \circ u_1 + z h_2 \circ u_2)$$

and that

$$d^p h_2 \circ T_p^1 + d^p h_1 \circ T_p^2 = T_{p-1}^2 \circ d^p(h_1 \circ u_1 + z h_2 \circ u_2).$$

Using (3.12) we get

$$d^p h_1 T_p^1 + z d^p h_2 T_p^2 = \sum [z^i (d^p h_1 C_{p-2i} + d^p h_2 C_{p-2i+1})]$$
$$= \sum [z^i (D_{p-2i-1} d^p(h_1 u_1) + D_{p-2i+1} d^p(h_2 u_2))]$$
$$= \sum [z^i (D_{(p-1)-2i}(d^p(h_1 u_1) + z d^p(h_2 u_2)))] = T_{p-1}^1 d^p(h_1 u_1 + z h_2 u_2).$$

Similarly,

$$d^p h_2 T_p^1 + d^p h_1 T_p^2 = \sum [z^i (d^p h_2 C_{p-2i} + d^p h_1 C_{p-2i-1})]$$
$$= \sum [z^i (D_{p-2i-2} d^p(h_1 u_1) + D_{p-2i} d^p(h_2 u_2))]$$
$$= \sum [z^i (D_{(p-1)-2i-1}(d^p(h_1 u_1) + z d^p(h_2 u_2)))] = T_{p-1}^2 d^p(h_1 u_1 + z h_2 u_2).$$

QED

3.9 Linear equations of type II_1 unprojection

Assume $n \geq 2$ is given. We use the notations of the previous subsections. In Subsection 3.8 we defined two $\mathcal{O}_{amb}$-module homomorphisms $T^1_{n-1}, T^2_{n-1} : \wedge^{n-1} N \to \wedge^{n-1} L$.

There exist, for $j = 1, 2$ and $1 \leq i \leq n$, unique element $\sigma_{ji} \in \mathcal{O}_{amb}$ such that

$$T^j_{n-1}(\widetilde{e}_1 \wedge \cdots \wedge \widetilde{e}_{n-1}) = \sum_{i=1}^{n} (-1)^{i-1} \sigma_{ji} e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_n.$$

For $1 \leq i \leq n$ we define the polynomials $l_i, l_{i+n} \in \mathcal{O}_{amb}[s_0, s_1]$ with

$$\begin{aligned} l_i &= z x_i s_0 + y_i s_1 + \sigma_{1i} \\ l_{i+n} &= y_i s_0 + x_i s_1 + \sigma_{2i} \end{aligned} \tag{3.13}$$

Using Proposition 3.15 and arguing as in [R] Section 9.5 we get the following Theorem.

Theorem 3.16 *Fix a parameter value $n \geq 2$. The polynomials $l_1, \ldots, l_{2n} \in \mathcal{O}_{amb}[s_0, s_1]$ specified in (3.13) are the linear relations of the type* II_1 *unprojection in the sense of Remark 2.1.*

4 Quadratic relation of Type II_1 for $n = 3$

Assume now that we are in the type II_1 unprojection with $n = 3$. We follow the notations of Section 2 with only one change: Since our symmetric formula (4.2) for the quadratic relation will need 1/2 as coefficient, our ambient ring $\mathcal{O}_{amb}$ will now be

$$\mathcal{O}_{amb} = \mathbb{Z}[\frac{1}{2}][x_1, \ldots, x_3, y_1, \ldots, y_3, z, A^p_{ij}, B^p_{lm}],$$

with indices $1 \leq p \leq 2$ and $1 \leq i < j \leq 3,\ 1 \leq l \leq m \leq 3$.

Recall that in Section 2 we defined two symmetric 3×3 matrices B^1 and B^2 by

$$B^j = \begin{pmatrix} B^j_{11} & B^j_{12} & B^j_{13} \\ & B^j_{22} & B^j_{23} \\ \text{sym} & & B^j_{33} \end{pmatrix},$$

for $j=1,2$, two skew–symmetric 3×3 matrices A^1 and A^2 by

$$A^j = \begin{pmatrix} 0 & A^j_{12} & A^j_{13} \\ & 0 & A^j_{23} \\ -\text{sym} & & 0 \end{pmatrix},$$

for $j=1,2$, and two 1×3 matrices x,y with

$$x=(x_1,x_2,x_3),\quad y=(y_1,y_2,y_3).$$

We now define three induced 3×3 symmetric matrices $\operatorname{ad} B^1,\ \operatorname{ad} B^2$ and $\operatorname{ad} B^{1,2}$.

For a square $n\times n$ matrix M with entries in a commutative ring R with unit, we denote by $\operatorname{ad} M$ the $n\times n$ matrix with kl-th entry, for $1\le k,l\le n$, equal to $(-1)^{k+l}$ times the determinant of the submatrix of M obtained by deleting the l-th row and the k-th column. It is well-known that we have the identities $M(\operatorname{ad} M)=(\operatorname{ad} M)M=(\det M)I_n$, where I_n is the identity $n\times n$ matrix over R.

In our situation, since B^1 and B^2 are symmetric 3×3 matrices we have that $\operatorname{ad} B^1$ and $\operatorname{ad} B^2$ are symmetric 3×3 matrices, such that for $j=1,2$

$$\operatorname{ad} B^j = \begin{pmatrix} B^j_{22}B^j_{33}-(B^j_{23})^2 & -(B^j_{12}B^j_{33}-B^j_{13}B^j_{23}) & B^j_{12}B^j_{23}-B^j_{13}B^j_{22} \\ & B^j_{11}B^j_{33}-(B^j_{13})^2 & -(B^j_{11}B^j_{23}-B^j_{13}B^j_{12}) \\ \text{sym} & & B^j_{11}B^j_{22}-(B^j_{12})^2 \end{pmatrix}.$$

Notice that for $1\le k\le 3,\ 1\le l\le 3$, we have

$$(\operatorname{ad} B^1)_{kl} = (-1)^{k+l}(B^1_{ij}B^1_{mn}-B^1_{in}B^1_{mj})$$

where $\{i,k,m\}=\{j,l,n\}=\{1,2,3\}$ and $i<m,\ j<n$, and similarly for $\operatorname{ad} B^2$.

We define $\operatorname{ad} B^{1,2}$ to be the 3×3 symmetric matrix, such that, for $1\le k\le 3,\ 1\le l\le 3$,

$$(\operatorname{ad} B^{1,2})_{kl} = (-1)^{k+l}(B^1_{ij}B^2_{mn}+B^2_{ij}B^1_{mn}-B^1_{in}B^2_{mj}-B^2_{in}B^1_{mj}),$$

where $\{i,k,m\}=\{j,l,n\}=\{1,2,3\}$ and $i<m,\ j<n$. By the above definition we have

$$\begin{aligned}
\operatorname{ad} B^{1,2}_{11} &= B^1_{22}B^2_{33}+B^2_{22}B^1_{33}-2B^1_{23}B^2_{23}\\
\operatorname{ad} B^{1,2}_{12} &= -B^1_{12}B^2_{33}-B^2_{12}B^1_{33}+B^1_{13}B^2_{23}+B^2_{13}B^1_{23}\\
\operatorname{ad} B^{1,2}_{13} &= B^1_{12}B^2_{23}+B^2_{12}B^1_{23}-B^1_{13}B^2_{22}-B^2_{13}B^1_{22}\\
\operatorname{ad} B^{1,2}_{22} &= B^1_{11}B^2_{33}+B^2_{11}B^1_{33}-2B^1_{13}B^2_{13}\\
\operatorname{ad} B^{1,2}_{23} &= -B^1_{11}B^2_{23}-B^2_{11}B^1_{23}+B^1_{13}B^2_{12}+B^2_{13}B^1_{12}\\
\operatorname{ad} B^{1,2}_{33} &= B^1_{11}B^2_{22}+B^2_{11}B^1_{22}-2B^1_{12}B^2_{12}
\end{aligned}$$

and the rest of the entries of $\mathrm{ad}\, B^{1,2}$ are determined by symmetry.

We now define two polynomials $a, b \in \mathcal{O}_{amb}$. We set

$$\begin{aligned} a &= \det \begin{pmatrix} x_1 & x_2 & x_3 \\ A^1_{23} & -A^1_{13} & A^1_{12} \\ A^2_{23} & -A^2_{13} & A^2_{12} \end{pmatrix} \\ &= x_1(A^1_{12}A^2_{13} - A^2_{12}A^1_{13}) + x_2(A^1_{12}A^2_{23} - A^2_{12}A^1_{23}) \\ &+ x_3(A^1_{13}A^2_{23} - A^2_{13}A^1_{23}), \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} b &= \frac{1}{2}\Big[yB^1(\mathrm{ad}\, B^2)B^1y^t + yB^2(\mathrm{ad}\, B^1)B^2y^t \\ &+ 2xA^1(\mathrm{ad}\, B^2)(A^1)^tx^t + 2xA^2(\mathrm{ad}\, B^1)(A^2)^tx^t \\ &- 2xA^1(\mathrm{ad}\, B^{1,2})(A^2)^tx^t \\ &- zxB^1(\mathrm{ad}\, B^2)B^1x^t - zxB^2(\mathrm{ad}\, B^1)B^2x^t \\ &+ 4yB^1(\mathrm{ad}\, B^2)(A^1)^tx^t + 4yB^2(\mathrm{ad}\, B^1)(A^2)^tx^t \\ &- \Big(\sum_{1\le i<j\le 3} A^1_{ij}(x_iy_j - x_jy_i)\Big)\Big(\sum_{1\le i,j\le 3}^{3} B^1_{ij}\,\mathrm{ad}\, B^2_{ij}\Big) \\ &- \Big(\sum_{1\le i<j\le 3} A^2_{ij}(x_iy_j - x_jy_i)\Big)\Big(\sum_{1\le i,j\le 3} B^2_{ij}\,\mathrm{ad}\, B^1_{ij}\Big)\Big]. \end{aligned} \tag{4.2}$$

The two polynomials f_1, f_2 corresponding to the ones in (2.5), are

$$f_j = xA^jy^t + yB^jy^t - zxB^jx^t, \tag{4.3}$$

for $j = 1, 2$.

We denote by u_1 the 3×2 matrix

$$u_1 = (-A^1x^t + B^1y^t \mid -A^2x^t + B^2y^t)$$

and by u_2 the 3×2 matrix

$$u_2 = (-B^1x^t \mid -B^2x^t).$$

These matrices correspond to the maps with the same name defined in (3.8).

For $j = 1, 2$, we set $\widetilde{\phi_1}(u^j)$ to be the 3×1 matrix

$$\widetilde{\phi_1}(u^j) = \begin{pmatrix} u^j_{21}u^j_{32} - u^j_{22}u^j_{31} \\ -u^j_{11}u^j_{32} + u^j_{12}u^j_{31} \\ u^j_{11}u^j_{22} - u^j_{12}u^j_{21} \end{pmatrix}$$

and we also define the 3×1 matrix

$$\widetilde{\phi_2}(u^1, u^2) = \begin{pmatrix} u^1_{21}u^2_{32} + u^2_{21}u^1_{32} - u^1_{22}u^2_{31} - u^2_{22}u^1_{31} \\ -u^1_{11}u^2_{32} - u^2_{11}u^1_{32} + u^1_{12}u^2_{31} + u^2_{12}u^1_{31} \\ u^1_{11}u^2_{22} + u^2_{11}u^1_{22} - u^1_{12}u^2_{21} - u^2_{12}u^1_{21} \end{pmatrix}.$$

The matrix $\widetilde{\phi_1}(u^j)$ corresponds to the map $\wedge^2 u_j$ and the matrix $\widetilde{\phi_2}(u^1, u^2)$ corresponds to the map $\mathrm{alt}(u_1, u_2)$ which were defined in Subsection 3.6.

We denote, for $1 \leq i \leq 3$,

$$\begin{aligned} l_i &= zx_is_0 + y_is_1 + [\widetilde{\phi_1}(u^1) + z\widetilde{\phi_1}(u^2)]_{i1} \\ l_{i+3} &= y_is_0 + x_is_1 + [\widetilde{\phi_2}(u^1, u^2)]_{i1}, \end{aligned} \tag{4.4}$$

where we use the usual notation M_{ij} for the ij-th entry of a matrix M. Clearly, for $1 \leq i \leq 6$, $l_i \in \mathcal{O}_{amb}[s_0, s_1]$ is affine linear with respect to s_0 and s_1. The polynomials $l_1, \ldots, l_6$ correspond to the polynomials with the same name in (3.13), hence by Theorem 3.16 they are the linear relations of the unprojection in the sense of Remark 2.1.

The main result of this section is the following theorem, which specifies the quadratic relation of the unprojection ring for the parameter value $n = 3$. It will be proved in Subsection 4.1.

Theorem 4.1 *The polynomial*

$$q = s_1^2 - zs_0^2 - as_0 + b \in \mathcal{O}_{amb}[s_0, s_1], \tag{4.5}$$

where a was defined in (4.1) and b was defined in (4.2), is the quadratic relation of the type II_1 *unprojection for the parameter value $n = 3$ in the sense of Remark 2.1.*

In other words, the ideal $I_Y \subset \mathcal{O}_{amb}[s_0, s_1]$ defining, in the sense of Section 2, the type II_1 *unprojection is equal to*

$$I_Y = (f_1, f_2) + (l_1, \ldots, l_6) + (q),$$

where the two polynomials f_1, f_2 generating I_X were defined in (4.3) and the six polynomials $l_1, \ldots, l_6$ were calculated in (4.4).

4.1 Proof of Theorem 4.1

We first prove the following reduction lemma.

Lemma 4.2 *To prove Theorem 4.1 it is enough to prove that that the element $(x_1 + x_2 + x_3)q$ of $\mathcal{O}_{amb}[s_0, s_1]$ is inside the ideal $(f_1, f_2, l_1, \ldots, l_6)$ of $\mathcal{O}_{amb}[s_0, s_1]$.*

Proof Indeed, by [P] Proposition 2.16 there exists $\widetilde{q} \in \mathcal{O}_{amb}[s_0, s_1]$ of the form

$$\widetilde{q} = s_1^2 - z s_0^2 + \widetilde{u}$$

where $\widetilde{u}$ is affine linear in s_0, s_1, such that the ideal of the unprojection ring is equal to J, with

$$J = (f_1, f_2, l_1, \dots, l_6, \widetilde{q}) \subset \mathcal{O}_{amb}[s_0, s_1].$$

Since by the assumptions of the lemma $(x_1 + x_2 + x_3)q \in J$, and by [P] Section 3.1 J is a prime ideal of $\mathcal{O}_{amb}[s_0, s_1]$ we get $q \in J$. Therefore, the affine linear with respect to s_0, s_1 element $q - \widetilde{q}$ is in J, as a consequence (compare proof of [P] Proposition 2.13) we get

$$q - \widetilde{q} \in (f_1, f_2, l_1, \dots, l_6),$$

hence

$$J = (f_1, f_2, l_1, \dots, l_6, q)$$

which finishes the proof of Lemma 4.2. QED

For the rest of the proof we define some more notation:
We define two polynomials $\widetilde{g_1}, \widetilde{g_2}$, such that, for $j \in \{1, 2\}$, we have

$$\begin{aligned} \widetilde{g_j} \;=\; & (x_1 B_{11}^j + x_2 B_{12}^j + x_3 B_{13}^j)(\operatorname{ad} B_{11}^p + \operatorname{ad} B_{12}^p + \operatorname{ad} B_{13}^p) \\ & + (x_1 B_{21}^j + x_2 B_{22}^j + x_3 B_{23}^j)(\operatorname{ad} B_{21}^p + \operatorname{ad} B_{22}^p + \operatorname{ad} B_{23}^p) \\ & + (x_1 B_{31}^j + x_2 B_{32}^j + x_3 B_{33}^j)(\operatorname{ad} B_{31}^p + \operatorname{ad} B_{32}^p + \operatorname{ad} B_{33}^p) \end{aligned}$$

where $p \in \{1, 2\}$ is uniquely specified by $\{j, p\} = \{1, 2\}$.

Moreover, we define two 1×3 matrices

$$L_1 = (l_1, l_2, l_3), \quad L_2 = (l_4, l_5, l_6)$$

and two 3×3 matrices M_1 and M_2 by

$$(M_1)_{ij} = \widetilde{\phi_3}(B_{i1}^1, B_{i2}^1, B_{i3}^1, B_{j1}^1, B_{j2}^1, B_{j3}^1, B_{i1}^2, B_{i2}^2, B_{i3}^2, B_{j1}^2, B_{j2}^2, B_{j3}^2)$$

and

$$\begin{aligned} (M_2)_{ij} \;=\; & \widetilde{\phi_3}(A_{i1}^1, A_{i2}^1, A_{i3}^1, A_{i1}^1, A_{i2}^1, A_{i3}^1, B_{j1}^2, B_{j2}^2, B_{j3}^2, B_{j1}^2, B_{j2}^2, B_{j3}^2) \\ & - \widetilde{\phi_3}(A_{i1}^2, A_{i2}^2, A_{i3}^2, A_{i1}^2, A_{i2}^2, A_{i3}^2, B_{j1}^1, B_{j2}^1, B_{j3}^1, B_{j1}^1, B_{j2}^1, B_{j3}^1) \end{aligned}$$

where

$$\widetilde{\phi_3}(a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3,d_1,d_2,d_3) = (a_1(d_3-d_2)+a_2(d_1-d_3) \\ +a_3(d_2-d_1)) - (c_1(b_3-b_2)+c_2(b_1-b_3)+c_3(b_2-b_1)).$$

Finally, we define the polynomial

$$\widetilde{w} = xM_1(L_1)^t + yM_1(L_2)^t + xM_2(L_2)^t.$$

Proposition 4.3 *We have the following equality*

$$2(x_1+x_2+x_3)q = -\widetilde{w}+\widetilde{g_1}f_1+\widetilde{g_2}f_2-2s_0(l_1+l_2+l_3)+2s_1(l_4+l_5+l_6) \quad (4.6)$$

of elements of $\mathcal{O}_{amb}[s_0,s_1]$.

Proof Both sides of (4.6) are polynomials with coefficients in $\mathbb{Q}$ and have been already explicitly written down. So, in principle, the validity of the Proposition can be checked by an easy but laborious hand calculation, and certainly by any computer algebra system. For example, the coefficient of s_1^2 on the left–hand side is equal to $2(x_1+x_2+x_3)$, while on the right–hand side there are contributions only from the term $2s_1(l_4+l_5+l_6)$ and it is easy to see that they are also equal to $2(x_1+x_2+x_3)$.

We did the verification of (4.6) using the computer algebra program Macaulay 2 [GS93-08][1]. QED

The proof of Theorem 4.1 follows now by combining Proposition 4.3 with Lemma 4.2.

4.2 Invariance under SL_3

Let $P \in \mathrm{SL}_3(\mathbb{Z})$ be a 3×3 matrix with integer coefficients and determinant one. Define two 1×3 matrices $\widetilde{x} = (\widetilde{x_1},\widetilde{x_2},\widetilde{x_3})$ and $\widetilde{y} = (\widetilde{y_1},\widetilde{y_2},\widetilde{y_3})$ by

$$\widetilde{x} = xP^{-t}, \qquad \widetilde{y} = yP^{-t},$$

(where P^{-t} is, of course, the transpose matrix of the inverse of P), two 3×3 skew–symmetric matrices $\widetilde{A^j}$ and two 3×3 symmetric matrices $\widetilde{B^j}$, for $j=1,2$, by

$$\widetilde{A^j} = P^tA^jP, \qquad \widetilde{B^j} = P^tB^jP$$

[1]The interested reader can find the relevant Macaulay 2 code on the author's webpage http://www.math.ist.utl.pt/∼papadak/

and finally set $\widetilde{z} = z$. Moreover, we denote by $\widetilde{A^j_{lm}}$ the lm-th entry of the matrix $\widetilde{A^j}$ and by $\widetilde{B^j_{lm}}$ the lm-th entry of the matrix $\widetilde{B^j}$.

It can be checked that if we denote by $\widetilde{a}$ and $\widetilde{b}$ the polynomials of Expressions (4.1) and (4.2) respectively, with the variables without tilde replaced by the corresponding expressions with tilde, we have $\widetilde{a} = a$ and $\widetilde{b} = b$.

Remark 4.4 The SL_3 symmetry was useful for our derivation of the concise formula of Equation (4.2), cf. Subsection 4.3 below. In addition, as already mentioned in the Introduction, we believe that a more substantial use of the SL_n invariance could, perhaps, lead to the calculation of the quadratic equations of the type II_1 unprojection for all values $n \geq 4$.

4.3 How we got to Theorem 4.1

In this subsection we briefly sketch the method that allowed us to calculate, using the computer program Maple, the expressions for a and b appearing in Theorem 4.1.

Step 1. Recall the explicit expressions for l_i in (4.4). We started from the expression

$$2\left[-s_0(l_1 + l_2 + l_3) + s_1(l_4 + l_5 + l_6)\right]$$

which is equal to

$$2(x_1 + x_2 + x_3)(s_1^2 - zs_0^2) + 2H,$$

where

$$H = -s_0 \sum_{i=1}^{3} [\phi_1(u^1) + z\phi_1(u^2)]_{i1} + s_1 \sum_{i=1}^{3} [\phi_2(u^1, u^2)]_{i1}.$$

It is easy to see that $2H$ is a homogeneous quadratic polynomial with respect to the variables A^t_{ij}, B^s_{kl}. More precisely, $2H$ has a (unique) natural representation as a sum of terms of the form $A^1_{ij}A^2_{kl}c^A_{ijkl}$ plus sum of terms of the form $A^1_{ij}B^2_{kl}c^{AB}_{ijkl}$ plus sum of terms of the form $A^2_{ij}B^1_{kl}c^{BA}_{ijkl}$ plus sum of terms of the form $B^1_{ij}B^2_{kl}c^B_{ijkl}$.

Step 2. We first handle the 'subpart' of $2H$ which is sum of terms of the form $A^1_{ij}A^2_{kl}c^A_{ijkl}$. These term appear only as contributions from $s_0(l_1+l_2+l_3)$ and, moreover, this subpart is equal to $2as_0$, where a was defined in (4.1).

Step 3. We now handle the 'subpart' of $2H$ which is sum of terms of the form $A^1_{ij}B^2_{kl}c^{AB}_{ijkl}$. We notice that there is a kind of complementarity of terms for the pairs $(l_1, l_4), (l_2, l_5), (l_3, l_6)$ in the sense of the following example:

A partial sum of the coefficient of $A^1_{13}B^2_{12}$ coming from l_1 and l_4 is $2y_1x_1s_0 + 2x_1^2s_1$ which can be written as $2x_1(y_1s_0 + x_1s_1)$. Using l_4 we get an expression not involving s_0 and s_1. Similarly, another partial sum coming from l_2 and l_5 is $2x_1(y_2s_0 + x_2s_1)$ and we can use l_5 to to get an expression not involving s_0 and s_1. A similar procedure can be done for the the partial sum coming from l_3 and l_6.

By symmetry, the handling of the 'subpart' of $2H$ which is sum of terms of the form $A^2_{ij}B^1_{kl}c^{BA}_{ijkl}$ is similar.

The handling of the 'subpart' of $2H$ which is sum of terms of the form $B^1_{ij}B^2_{kl}c^B_{ijkl}$ is similar, but we need to take care of preserving the symmetries. This is the only part where we actually need the coefficient 2 in $2H$.

To give an example, a partial sum of the coefficient of $B^1_{13}B^2_{23}$ coming from l_1 and l_4 is

$$2\left[-s_0(-x_1x_3z - y_1y_3) + s_1(y_3x_1 + y_1x_3)\right]$$

which we first write as

$$y_3(y_1s_0 + x_1s_1) + x_1(zx_3s_0 + y_3s_1) + y_1(y_3s_0 + x_3s_1) + x_3(zx_1s_0 + y_1s_1)$$

and then we use the appropriate linear equations l_i to get an expression not involving s_0 and s_1.

Step 4. At the end of step 3 we arrived to an expression not involving s_0, s_1. By suitable subtraction (in a symmetric way) of multiples of f_1 and f_2 we get a polynomial divisible by $x_1 + x_2 + x_3$. The quotient is b, which after some further term by term effort can be written in the form (4.2).

4.4 Type II_1 unprojection for $n = 2$

This subsection contains the explicit equations obtained by Miles Reid for the type II_1 unprojection with parameter value $n = 2$. It is taken from [R] Section 9.5.

For $n = 2$ the ambient ring is

$$\mathcal{O}_{amb} = \mathbb{Z}[x_1, x_2, y_1, y_2, z, A_{11}, B_{11}, B_{12}, B_{22}],$$

the matrix M corresponding to the one in (2.3) is equal to

$$M = \begin{pmatrix} y_1 & y_2 & zx_1 & zx_2 \\ x_1 & x_2 & y_1 & y_2 \end{pmatrix},$$

there is a unique polynomial

$$f = A_{12}(x_1y_2 - x_2y_1) + B_{11}(y_1^2 - zx_1^2) + 2B_{12}(y_1y_2 - zx_1x_2) + B_{22}(y_2^2 - zx_2^2)$$

corresponding to (2.5), the linear equations $l_1, \dots, l_4$ are given by

$$\begin{aligned} l_1 &= zx_1s_0 + y_1s_1 + (x_1A_{12} + y_1B_{12} + y_2B_{22}) \\ l_2 &= zx_2s_0 + y_2s_1 + (x_2A_{12} - y_1B_{11} - y_2B_{12}) \\ l_3 &= y_1s_0 + x_1s_1 + (-x_1B_{12} - x_2B_{22}) \\ l_4 &= y_2s_0 + x_2s_1 + (x_1B_{11} + x_2B_{12}) \end{aligned}$$

and the quadratic equation q is given by

$$q = s_1^2 - zs_0^2 - A_{12}s_0 - (B_{12})^2 + B_{11}B_{22}.$$

In other words, using the notations of Section 2 we have the equality of ideals of $\mathcal{O}_{amb}[s_0, s_1]$

$$I_Y = (f) + (l_1, \dots, l_4) + (q).$$

Remark 4.5 Since $f = y_2l_1 - y_1l_2 + zx_2l_3 - zx_1l_4,$ we even get

$$I_Y = (l_1, \dots, l_4) + (q).$$

Remark 4.6 By [P] the ideal I_Y is Gorenstein codimension 3. It is easy to see that it is equal to the ideal generated by the 5 submaximal Pfaffians of the 5×5 skew–symmetric matrix (with entries in $\mathcal{O}_{amb}[s_0, s_1]$)

$$\begin{pmatrix} 0 & x_1 & x_2 & y_1 & y_2 \\ & 0 & -s_0 & -B_{22} & s_1 + B_{12} \\ & & 0 & -s_1 + B_{12} & -B_{11} \\ & & & 0 & -zs_0 - A_{12} \\ & \text{–sym} & & & 0 \end{pmatrix}.$$

(For a discussion about the Pfaffians of a skew–symmetric matrix see, for example, [BH] Section 3.4.)

5 Applications to algebraic geometry

We believe that the explicit formulas of Theorem 4.1 can be used together with Gavin Brown's online database of graded rings [Br] for the proof of the existence of a number of (singular) Fano 3-folds with anticanonical ring of codimension 4. We discuss below two such examples, the first of which is due to Selma Altınok, and has the interesting property $|-K_X| = \emptyset$. We also expect that the explicit formulas of Theorem 4.1 together with the orbifold Riemann–Roch theorem obtained in [BS] can lead to the construction of new codimension 4 Calabi–Yau 3-folds.

5.1 The example of Altınok

This example is taken from [R] Example 9.14 and is due to Altınok. Consider the weighted projective space $\mathbb{P}^2(1,3,5)$ with coordinates u, v, w and the weighted projective space $\mathbb{P}^5(2,3,4,5,6,7)$ with coordinates x, v, y, w, z, t. We define the map $\mathbb{P}(1,3,5) \to \mathbb{P}(2,3,4,5,6,7)$ given by

$$x = u^2, \quad v = v, \quad y = uv, \quad w = w, \quad z = uw, \quad t = u^7.$$

The equations of the image D of the map are

$$\operatorname{rank} \begin{pmatrix} y & z & t & xv & xw & x^4 \\ v & w & x^3 & y & z & t \end{pmatrix} \le 1,$$

that is,

$$\begin{array}{l} yw = vz, \quad yx^3 = vt, \quad zx^3 = wt, \quad y^2 = v^2x, \quad yz = vwx \\ yt = vx^4, \quad z^2 = w^2x, \quad zt = wx^4, \quad t^2 = x^7. \end{array}$$

They correspond to the specialisation

$$(x_1, x_2, x_3) = (v, w, x^3), \qquad (y_1, y_2, y_3) = (y, z, t), \qquad z = x$$

of the variables of the generic format defined in Section 2.

A general complete intersection $X_{12,14}$ containing D is obtained by choosing two general combinations, one of degree 12 and the other of degree 14, of the above equations of D. This corresponds to a specialisation of the variables A^p_{ij}, B^p_{ij} defined in Section 2.

We perform a type II_1 unprojection of the pair $D \subset X_{12,14}$ to get a codimension 4 3-fold

$$Y \subset \mathbb{P}^7(2,3,4,5,6,7,8,9).$$

Notice that the new variable s_0 has degree 8 and the new variable s_1 has degree 9.

After substituting to the formulas (3.13) and (4.5) we checked the quasismoothness of Y using the computer algebra program Singular [GPS01].

5.2 The second Fano example

In this subsection we sketch a construction, suggested by Brown's online database of graded rings [Br], of a codimension 4 Fano 3-fold

$$Y \subset \mathbb{P}^7(1,1,2,2,2,2,3,3)$$

starting from a (nongeneric) codimension 2 complete intersection Fano 3-fold $X_{4,6} \subset \mathbb{P}^5(1,1,2,2,2,3)$.

Write $x_1, x_2, y_1, y_2, y_3, w$ for the coordinates of $\mathbb{P}^5(1,1,2,2,2,3)$, and consider the subscheme $D \subset \mathbb{P}^5(1,1,2,2,2,3)$ with equations

$$\operatorname{rank} \begin{pmatrix} y_1 & y_2 & w & y_3 x_1 & y_3 x_2 & y_3^2 \\ x_1 & x_2 & y_3 & y_1 & y_2 & w \end{pmatrix} \leq 1.$$

By definition, the ideal I_D of D is generated by the 2×2 minors of the above matrix. It corresponds to the specialisation

$$(x_1, x_2, x_3) = (x_1, x_2, y_3), \qquad (y_1, y_2, y_3) = (y_1, y_2, w), \qquad z = y_3$$

of the variables of the generic format defined in Section 2.

Denote by $X_{4,6} \subset \mathbb{P}^5(1,1,2,2,2,3)$ a general codimension 2 complete intersection (with equations of degrees 4 and 6) containing D. The equations of $X_{4,6}$ are obtained by choosing two general combinations, one of degree 4 and the other of degree 6, of the above equations of D. They correspond to a specialisation of the variables A^p_{ij}, B^p_{ij} defined in Section 2.

We perform a type II_1 unprojection of the pair $D \subset X_{4,6}$ to get a codimension 4 3-fold $Y \subset \mathbb{P}^5(1,1,2,2,2,2,3,3)$. Notice that here the new variable s_0 has degree 2 and the new variable s_1 has degree 3.

After substituting to the explicit formulas (3.13) and (4.5) we checked the quasismoothness of Y using the computer algebra program Singular [GPS01].

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