III European Conference on Computational Mechanics
Solids, Structures and Coupled Problems in Engineering
C.A. Mota Soares et.al. (eds.)
Lisbon, Portugal, 5–8 June 2006

THE METHOD OF FUNDAMENTAL SOLUTIONS APPLIED TO THE CALCULATION OF EIGENSOLUTIONS FOR SIMPLY CONNECTED PLATES

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Keywords: eigensolutions, plate problem, method of fundamental solutions.

Abstract. *In this work we study the application of the Method of Fundamental Solutions (MFS) to the numerical calculation of the eigenvalues and eigenfunctions for the 2D Bilaplacian in simply connected plates. This problem was considered in [20] using wave-type functions and in [14] using radial basis function for a circular and rectangular domain. The MFS is a mesh-free method that was already applied to the calculation of the eigenvalues and eigenfunctions associated to the Laplace operator (cf. [21], [10] and [2]). The application of this method to the Bilaplacian was already considered in [13], but only for simple shapes. Here we apply an algorithm for the choice of point-sources, as in [2], that leads to very good results for a fairly general class of domains.*

1 Introduction

The determination of the eigenvalues and eigenfunctions associated to elliptic operators in a bounded domain Ω is a well known problem with a lot of applications in engineering and acoustics. In some particular domains we have an explicit formula for the eigenvalues and eigenfunctions. However, when the shape is non trivial the use of a numerical method for PDEs is required. A standard finite differences method can produce good results when dealing with a particular type of shapes defined on rectangular grids, while for other type of shapes the finite element method or the boundary element method are more appropriate (e.g. [24]). These classical methods require extra computational effort; in one case, the construction of the mesh and the associated rigid matrix, and in the other, the integration of weakly singular kernels. Here we propose a meshfree method for solving the eigenvalue problem of a simply connected clamped plate using the method of fundamental solutions (MFS). The MFS has been mainly applied to boundary problems in PDEs, starting in the 1960s (e.g. [23]). This method can be applied to Helmholtz equation (eg. [21], [2]), Stokes (cf. [3]), elasticity (cf. [23]) and biharmonic problems (eg. [22]). The application of the MFS to the calculation of the eigenfrequencies of a membrane problem was introduced by Karageorghis in [21], and applied for simple shapes. In [2] it was proposed a particular choice for the point-sources which can lead to very good results for a general class of simply connected domains. J T Chen et al. studied the application of the MFS for the eigencalculation of multiply connected domains (cf. [10]). They find the appearance of spurious solutions and to filter them out, they applied the singular value decomposition (SVD) and the Burton & Miller method. In [21] it is presented a comparison with the boundary element method used by De Mey in [24], and the results obtained for simple shapes (circles, squares), show a better performance for the MFS. The application of other meshless methods to the determination of eigenfunctions and eigenmodes has also been subject to recent research, mainly using radial basis functions (e.g. [7], [8]) or the method of particular solutions (cf. [4]).

The plate problem was already considered using meshfree methods. Kang and Lee proposed a method based on non-dimensional influence functions (cf. [20]). That method results in spurious solutions that need a special treatment. Later, J T Chen et al. proposed a method based on radial basis functions and made an analytical study for the circular plate (cf. [14]). The MFS was already applied for the calculation of the eigenvalues of an annular domain (cf. [13]). It was shown that spurious eigenvalues appear when dealing with multiply connected domains. To extract the true eigenvalues it was applied the Burton & Miller method.

In this work we consider the application of the MFS to general simply connected shapes. In this case the choice of the source points in the MFS becames more important to retrieve with accuracy the eigenvalues. We adopt the choice proposed in [2] which revealed to lead to very accurate results. Having determined an approximation of the eigenvalue, we apply an algorithm based on the MFS to obtain the associated eigenmodes.

2 The plate problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain with regular boundary $\partial\Omega$. We will consider the 2D eigenvalue problem for the Bilaplacian. This is equivalent to obtain the frequencies λ that satisfies the problem

$$\begin{cases} \Delta^2 u - \lambda^4 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_n u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

for a non null function u, where $\partial_n u$ is the normal derivative of the function u. If a pair (λ, u) satisfies the problem 2.1, then we say that λ is an eigenfrequency and u is an eigenmode. As an application, this corresponds to recover the resonance frequencies $\lambda > 0$ associated with a particular shape of a plate Ω.

A fundamental solution Φ_ω of the biharmonic eigenvalue equation verifies $(\Delta^2 - \omega^4)\Phi_\omega = -\delta$, where δ is the Dirac delta distribution. In the 2D case, we consider

$$\Phi_\omega(x) = \frac{i}{8\omega^2}(H_0^{(1)}(i\omega|x|) + H_0^{(2)}(\omega|x|)) \tag{2.2}$$

where $H_0^{(1)}$ and $H_0^{(2)}$ are Hänkel functions of the first and second type. We will consider $\hat{\Gamma}$ an admissible source set as defined in [1], for instance, the boundary of a bounded open set $\hat{\Omega} \supset \overline{\Omega}$, considering $\hat{\Gamma}$ surrounding $\partial\Omega$ and using the approximation ($x \in \Omega$),

$$u(x) = \int_{\hat{\Gamma}} \varphi(y)\Phi_\omega(x-y)\, ds_y + \int_{\hat{\Gamma}} \varpi(y)\partial_{n_y}\Phi_\omega(x-y)\, ds_y \approx$$

$$u_m(x) = \sum_{j=1}^{m} \alpha_{m,j}\Phi_\omega(x - y_{m,j}) + \sum_{j=1}^{m} \beta_{m,j}\partial_{n_{y_{m,j}}}\Phi_\omega(x - y_{m,j}). \tag{2.3}$$

where ∂_{n_y} is the normal derivative at the point $y \in \hat{\Gamma}$.

3 Numerical Method using the MFS

3.1 Determination of the eigenfrequencies

Now we present a procedure to find the eigenfrequencies. We define m collocation points $x_i \in \partial\Omega$ and m source points $y_{m,j} \in \hat{\Gamma}$ and obtain the system

$$\begin{cases} u_m(x_i) = 0 \\ \partial_n u(x_i) = 0 \end{cases} \tag{3.1}$$

As described in [2], an arbitrary choice of source points may lead to worst results than the expected with the MFS applied to simple shapes as in [21]. We will choose uniformly on the boundary $\partial\Omega$ the points $x_1, \ldots, x_m$ and for each of these points we calculate the unitary vector n_{x_i} that is normal to the boundary at the point x_i. Then, we calculate the point sources

$$y_i = x_i + \beta n_{x_i}$$

The parameter β is a constant choosen such that $y_j \notin \Omega,\ j = 1, \ldots, m$. We will assume that $n_{y_{m,j}} = n_{x_j}$ and denote this vector simply by n_j and define $d_{i,j} = x_i - y_{m,j}$. Using this notation the system 3.1 becames

$$\begin{cases} \sum_{j=1}^{m} \alpha_{m,j}\Phi_\omega(d_{i,j}) + \sum_{j=1}^{m} \beta_{m,j}\left(n_j \bullet \nabla\Phi_\omega(d_{i,j})\right) = 0 \\ \sum_{j=1}^{m} \alpha_{m,j}\left(n_i \bullet \nabla\Phi_\omega(d_{i,j})\right) + \sum_{j=1}^{m} \beta_{m,j}\left(n_i \bullet \nabla\left(n_j \bullet \nabla\Phi_\omega(d_{i,j})\right)\right) = 0 \end{cases} \tag{3.2}$$

Therefore a straighforward procedure is to find the values ω for which the $(2m) \times (2m)$ matrix $M(\omega)$ has a null determinant.

$$M(\omega) = \begin{bmatrix} A(\omega) & B(\omega) \\ C(\omega) & D(\omega) \end{bmatrix}$$

This matrix is composed by the following four $m \times m$ blocks

$$A(\omega) = \left[\Phi_\omega(d_{i,j})\right]_{m\times m} \qquad B(\omega) = \left[n_j \bullet \nabla\Phi_\omega(d_{i,j})\right]_{m\times m}$$

$$C(\omega) = \left[n_i \bullet \nabla\Phi_\omega(d_{i,j})\right]_{m\times m} \qquad D(\omega) = \left[n_i \bullet \nabla\left(n_j \bullet \nabla\Phi_\omega(d_{i,j})\right)\right]_{m\times m}$$

Remark 1. *As described in [20], one possibility to reduce the system matrix is to calculate the $N(\omega) = C - D.B^{-1}.A$. This reduces the dimensions of the system matrix from $(2m)\times(2m)$ to $m \times m$, however we prefered to avoid the calculation of the inverse matrix.*

The components of the matrix $A(\omega)$ are complex numbers, so the determinant is also a complex number. We consider the real function $g(\omega) = |Det[M(\omega)]|$. It is clear that the function g will be very small in any case, since the MFS is highly ill conditioned and the determinant is quite small.

First we plot the graph of $\log(g(\omega))$ using a fewer number of points to choose an interval $]a,b[$ where there is only one eigenfrequency.

Golden Ratio Search. To search the point where the local minimum is attained we use an algorithm based on the golden ratio search method. See [2] for details.

3.2 Determination of the eigenmodes

To obtain an eigenmode associated with a certain eigenfrequency λ we use a collocation method on $n+1$ points, with $x_1,\cdots,x_n$ on $\partial\Omega$, a point $x_{n+1}\in\Omega$ and an extra point source $y_{n+1}\in\bar{\Omega}^C$. The approximation of the eigenmode is given by

$$\tilde{u}(x) = \sum_{j=1}^{m+1}\alpha_{m+1,j}\Phi_\omega(x-y_{m+1,j}) + \sum_{j=1}^{m}\beta_{m,j}\partial_{n_{y_{m,j}}}\Phi_\omega(x-y_{m,j}). \tag{3.3}$$

To exclude the solution $\tilde{u}(x)\equiv 0$, the coefficients α_j are determinated by the solution of the system

$$\begin{cases} \tilde{u}(x_i) = 0, & i = 1,\ldots,n \\ \tilde{u}(x_{n+1}) = 1 & \\ \partial_n u(x_i) = 0, & i = 1,\ldots,n \end{cases} \tag{3.4}$$

This is equivalent to solve the $(2m+1)\times(2m+1)$ system

$$\left[\begin{array}{c:c} A(\omega) & B(\omega) \\ \hdashline \Phi_\omega\left(d_{m+1,j}\right) & 0 \\ \hdashline C(\omega) & D(\omega) \end{array}\right]\left[\begin{array}{c} \alpha_{m+1,j} \\ \hdashline \alpha_{m+1,m+1} \\ \hdashline \beta_{m,j} \end{array}\right]=\left[\begin{array}{c} 0 \\ \hdashline 1 \\ \hdashline 0 \end{array}\right]$$

Remark 2. *This procedure may fail if the selected point x_{n+1} is on the nodal line (cf. [11], [12]). Depending on the multiplicity of the eigenvalue, we will add one or more collocation points to make the linear system well determined.*

3.3 Error bounds

An error estimate can be derived using the following result which is attributed by Collatz (cf. [17]) to Kryloff and Bogoliubov and to D. H. Weinstein (See [19]).

Theorem. *Let A be a self-adjoint operator on a Hilbert space with inner product $(.,.)$. Let λ_n and u_n be the eigenvalues and orthonormal eigenfunctions of A. Assume $\{\lambda_n\}$ as no finite accumulation point. Let v be any element in the space spanned by $\{u_n\}$ and let*

$$\rho = \frac{(v, Av)}{\|v\|^2} \quad (\textit{the Rayleigh quocient}) \tag{3.5}$$

and

$$\sigma = \frac{\|Av\|}{\|v\|}. \tag{3.6}$$

Then $\sigma \geq \rho$ and there exists at least one eigenvalue λ_k satisfying

$$\rho - \sqrt{\sigma^2 - \rho^2} \leq \lambda_k \leq \rho + \sqrt{\sigma^2 - \rho^2} \tag{3.7}$$

As the bilaplacian is a self-adjoint operator in H_0^2, the previous theorem can be used to obtain a bound for the error of the approximations of the eigenfrequencies.

Corollary. *Let Ω be a Lipschitz domain and $\tilde{\kappa}$ and $\tilde{u} \in C^4(\Omega) \cap C^1(\bar{\Omega})$ be an approximate eigenfrequency and eigenfunction which satisfy the following problem:*

$$\begin{cases} \Delta^2 \tilde{u} - \tilde{\kappa}^4 \tilde{u} = 0 & \textit{in } \Omega, \\ \tilde{u} = \varepsilon(x) & \textit{on } \partial\Omega, \\ \partial_n \tilde{u} = \delta(x) & \textit{on } \partial\Omega, \end{cases} \tag{3.8}$$

with $\|\tilde{u}\|_{L^2(\partial\Omega)} = 1$. Then there exists an eigenfrequency κ_p such that

$$\frac{|\kappa_p - \tilde{\kappa}|}{|\tilde{\kappa}|} \leq \frac{\sqrt{2}\gamma + \gamma^2}{1 - 2\gamma + \gamma^2} \tag{3.9}$$

where

$$\gamma = c\left(\|\varepsilon\|_{L^1(\partial\Omega)} + \|\delta\|_{L^2(\partial\Omega)}\right) \tag{3.10}$$

where $|\Omega|$ *is the area of the domain* Ω.

4 Numerical Results

In Figure 1 we plot the graph of $log(g(\omega))$, $\omega \in [2.9, 6.2]$ for the unit ball obtained with $m = 15$ and $\beta = 1.5$.

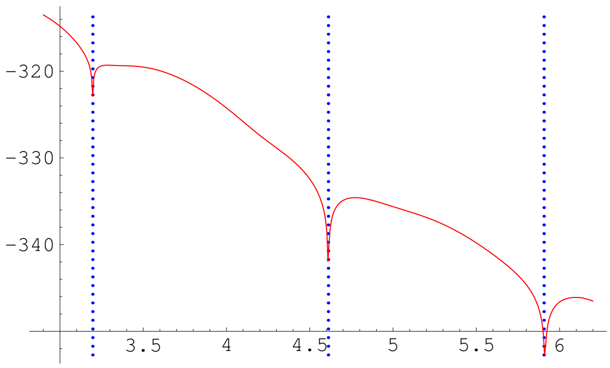

Figure 1: Plot of the graph of of $log(g(\omega))$ for the unit ball.

In Table 4 we present the results of the absolute errors for the unit ball (with $\beta = 1.5$).

m	abs. error (λ_1)		m	abs. error (λ_2)		m	abs. error (λ_3)
20	4.23176×10^{-6}		20	7.88573×10^{-5}		20	5.54069×10^{-3}
25	4.17119×10^{-8}		25	8.80722×10^{-7}		25	7.58151×10^{-5}
30	3.66573×10^{-10}		30	3.85887×10^{-8}		30	3.57842×10^{-6}

Table 1: Absolute errors for the unit ball.

Now we test the method for the domain with a boundary given implicity by

$$\sin^2(y) = \frac{3 + \frac{3-4\sin^2(x)}{3}}{4}$$

We will by Ω_1 this domain, which is plotted in Figure 2. We will now consider three cases of different choices for the point-sources. In the first case we consider as artificial boundary the "expansion" of the boundary of the domain; in the second case we consider the boundary of a circular domain and in the last case we consider our choice with $\beta = 0.8$ (Figure 3). In Figure 4 we present the plot of $\log(g(\omega))$ with the points plotted in Figure 3. We note that in Figure 4 the first two plots present rounding errors generated by the ill conditioned matrix. With the proposed choice of points the ill conditioning decreases and the rounding errors are much smaller (third plot). This phenomenon also occurs in the membrane problem (cf. [2]).

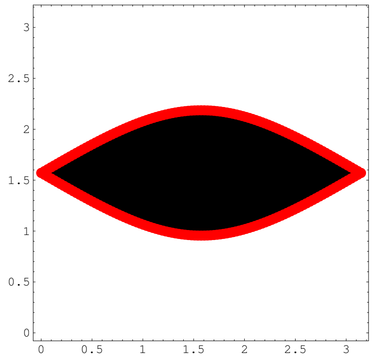

Figure 2: Plot of the domain Ω_1.

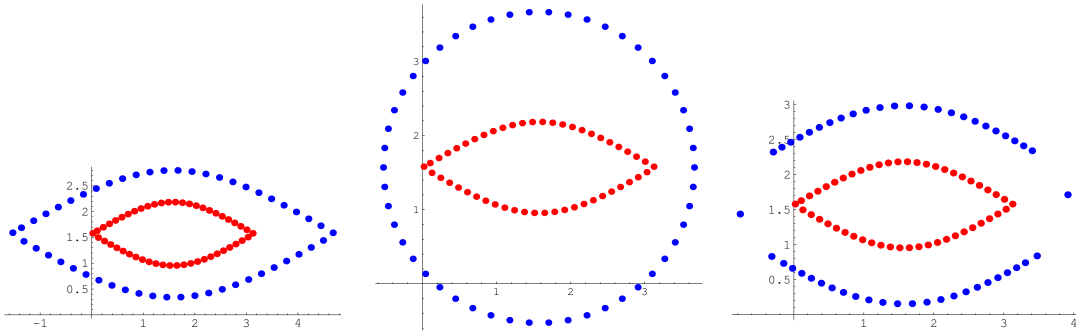

Figure 3: Collocation points and three different choices for the point-sources with $m = 70$.

Will call by Ω_2 and Ω_3 the domains which can be parametrized (resp.) by

$$t \mapsto \left(\cos(t), \sin(t) + \frac{5\sin(t)\cos(2t)}{9}\right)$$

and

$$t \mapsto \left(\cos(t), \sin(t) + \frac{\sin(2t)}{3}\right).$$

In Figure 5 we plot the eigenmodes and the respective nodal domains associated to the first and second eigenfrequencies of the domain Ω_2. In Figure 6 we plot the eigenmodes and the respective nodal domains associated to the 14-th and 16-th eigenfrequencies of the domain Ω_3.

In Figure 7 we plot $\tilde{u}|_{\partial\Omega}$ and $\partial_n\tilde{u}|_{\partial\Omega}$ for the eigenmode associated to the first eigenfrequency of the domain Ω_2.

It's well known that the eigenmode associated to the first eigenfrequency of the laplacian in a domain Ω does not change the sign in Ω. For the bilaplacian it was proven that this is not true (cf. [5] and [16]). For some poligonal domains with a sufficiently small internal angle θ any eigenmode changes sign an infinite number of times in the neighbourhood of this corner. One of these domains is the equilateral triangle. In Figure 8 we plot the eigenmode associated to the first eigenfrequency of the equilateral triangle, the respective nodal domains and the behaviour of the eigenmode near one of the corners.

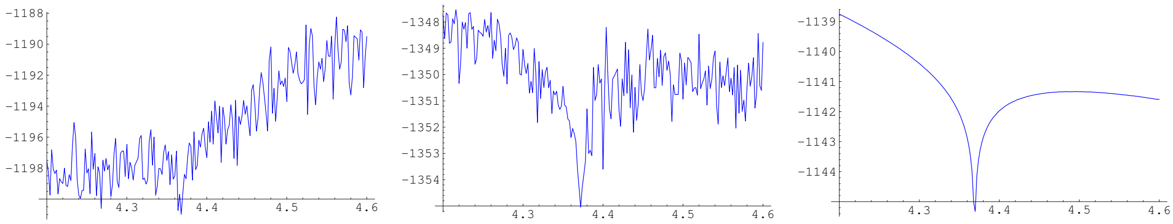

Figure 4: Plot of the function $\log(g(\omega))$ with $m = 70$ for three choices of points.

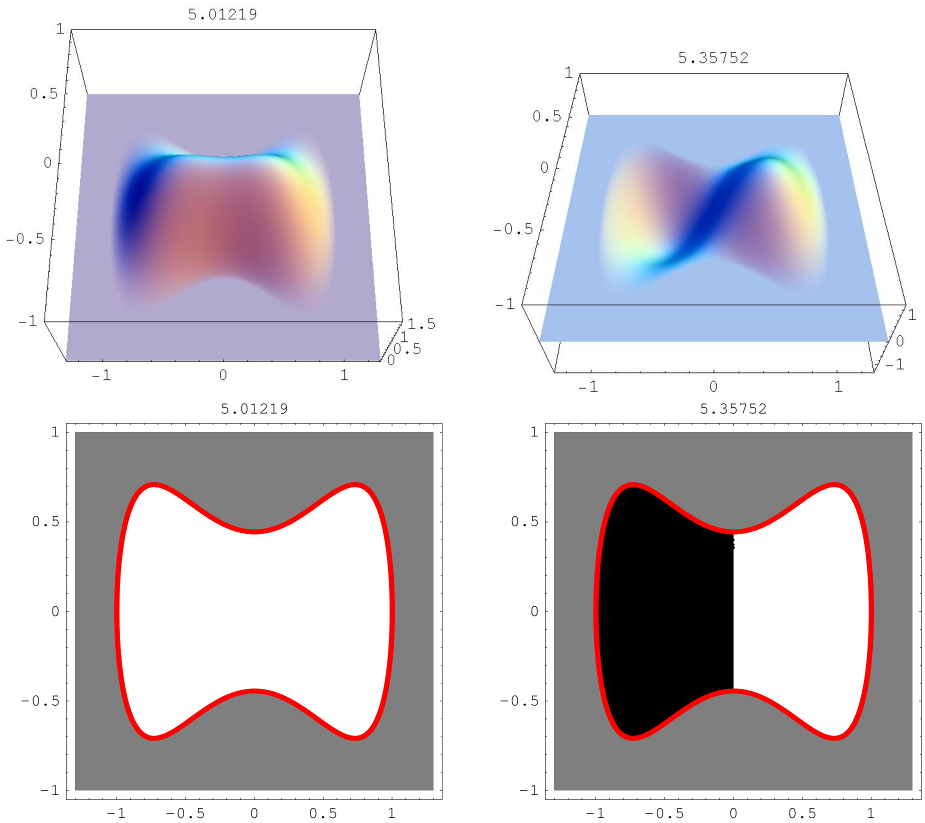

Figure 5: *Plots of eigenmodes and the respective nodal domains associated to the first and second eigenfrequencies of the domain* Ω_2.

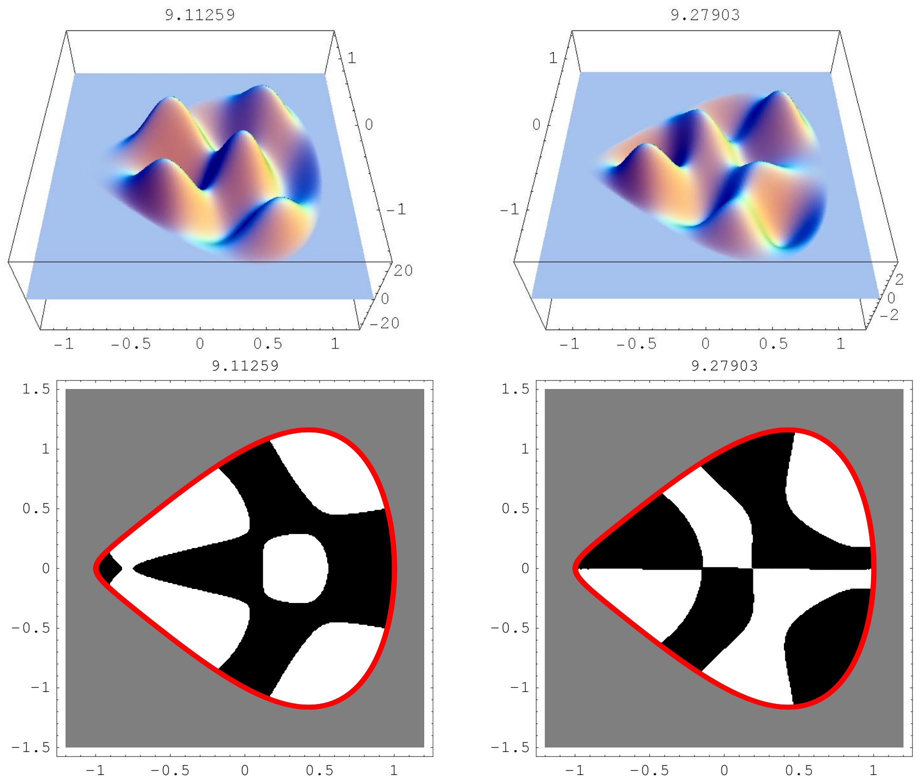

Figure 6: *Plots of eigenmodes and the respective nodal domains associated to the 14-th and 16-th eigenfrequencies of the domain* Ω_3.

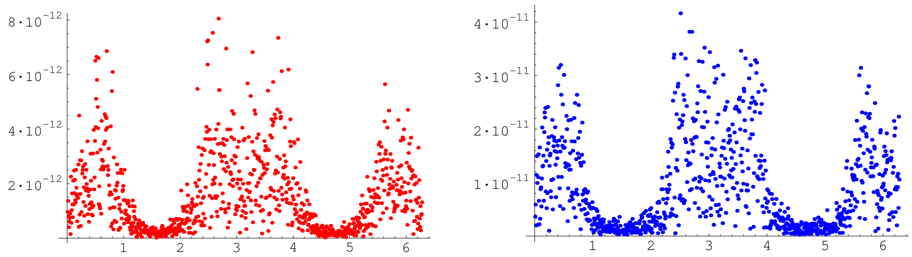

Figure 7: *Plot of* $\tilde{u}|_{\partial\Omega}$ *and* $\partial_n\tilde{u}|_{\partial\Omega}$ *for the eigenmode associated to the first eigenfrequency of the domain* Ω_2.

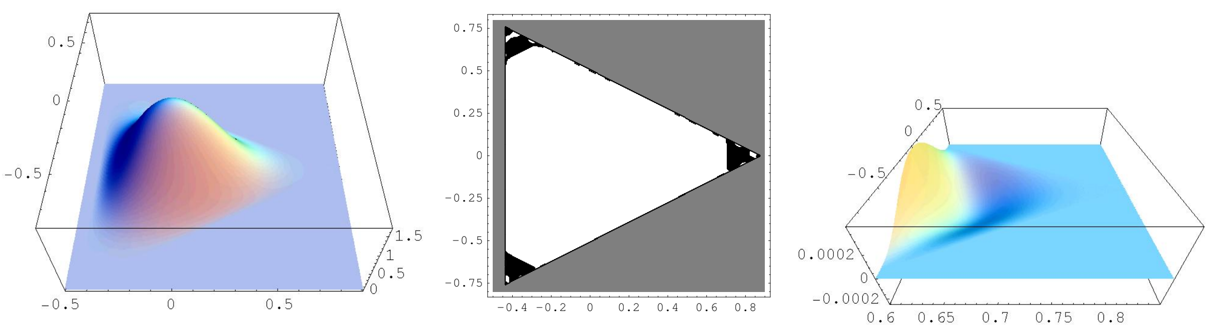

Figure 8: *Plot of the eigenmode associated to the first eigenfrequency, the respective nodal domains and the behaviour of the eigenmode near one of the corners.*

REFERENCES

[1] Alves C J S, *Density results for the Helmholtz equation and the method of fundamental solutions.*, in Advances in Computational Engineering & Sciences, Volume I, (Editors: S. N. Atluri and F. W. Brust). Tech Sc. Press, 45-50 (2000).

[2] Alves C J S and Antunes P R S, *The method of fundamental solutions applied to the calculation of eigenfrequencies and eigenmodes of 2D simply connected shapes.*, Computers, Materials & Continua Vol. 2, no. 4, 251-266 (2005).

[3] Alves C J S and Silvestre A L, *Density results using Stokeslets and a method of fundamental solutions for the Stokes equations.*, Engineering Analysis with Boundary Elements 28, 1245-1252 (2004).

[4] Betcke T and Trefethen, L N, *Reviving the method of particular solutions.* Preprint Oxford University (2003), to appear in SIAM Review.

[5] Blum H and Rannacher R, *On the boundary value problem of the biharmonic operator on domains with angular corners.*, Math. Meth. in Appl. Sci. 2, 556-581 (1980).

[6] Bogomolny A,*Fundamental solutions method for elliptic boundary value problems.*, SIAM J. Numer. Anal. 22, 4, 644-669 (1985).

[7] Chen J T, Chang M H, Chen K H and Lin S R, *Boundary collocation method with meshless concept for acoustic eigenanalysis of two-dimensional cavities using radial basis function.*, Journal of Sound and Vibration 257, 4, 667-711 (2002).

[8] Chen J T, Chang M H, Chen K H and Chen I L, *Boundary collocation method for acoustic eigenalysis of three-dimensional cavities using radial basis function.*, Computational Mechanics 29, 392-408 (2002).

[9] Chen J T, Chen I L, Chen K H, Lee Y T and Yeh Y T, *A meshless method for free vibration analysis of circular and rectangular clamped plates using radial basis function.*, Engineering Analysis with Boundary Elements, Vol. 28, no. 5, 535-545 (2004).

[10] Chen J T, Chen I L and Lee Y T, *Eigensolutions of multiply connected membranes using the method of fundamental solutions.*, Engineering Analysis with Boundary Elements Vol. 29, no. 2, 166-174 (2005).

[11] Chen J T, Chen K H, Chyuan S W, *Numerical experiments for acoustic modes of a square cavity using the dual BEM.*, Applied Acoustics, Vol. 57, No 4, 293-325 (1999).

[12] Chen J T, Huang C X, Chen K H, *Determination of spurious eigenvalues and multiplicities of the true eigenvalues using the real-part BEM.*, Computational Mechanics, Vol 24, No 1 41-51 (1999).

[13] Chen J T and Lee Y T, *True and spurious eigensolutions for membrane and plate problems by using the method of fundamental solutions.*, ECCOMAS Thematic Conference on Meshless Methods 2005.

[14] Chen J T, Chen I L, Chen K H, Lee Y T and Yeh Y T, *A meshless method for free vibration analysis of circular and rectangular clamped plates using radial basis function.*, Eng. Anal. Bound. Elem 28, 535-545 (2004).

[15] Chen J T, Kuo S R, Chen K H and Cheng Y C, *Comment on free vibration analysis of arbitrarily shaped plates with clamped edges using wave-type function.*, J. Sound Vibr. 262, 370-378 (2003).

[16] Coffman C V, *On the structure of solutions* $\Delta^2 u = \lambda u$ *which satisfy the clamped plate conditions on a rigth angle.*, SIAM J. Math. Anal. 13, 746-757 (1982).

[17] Collatz L, *Eigenwertprobleme und ihre Numerische Behandlung.*, Akademische Verlagsgesellschaft, Leipzig p.208 (1945).

[18] Dahlberg B E J, Kenig C E and Verchota G C, *The Dirichlet Problem for the Biharmonic Equation in a Lipschitz Domain.*, Ann. Inst. Fourier, Grenoble 36, 109-135 (1986).

[19] Fox L, Henrici P and Moler C B, *Approximations and bounds for eigenvalues of elliptic operators.*, SIAM J. Numer. Anal. 4, 89-102 (1967).

[20] Kang S W and Lee J M, *Free vibration analysis of arbitrary shaped plates with clamped edges using wave-type functions.*, J. Sound Vibr. 242(1), 9-16 (2001).

[21] Karageorghis A, *The method of fundamental solutions for the calculation of the Eigenvalues of the Helmholtz Equation.*, Appl Math. Letters 14 (7), 837-842 (2001).

[22] Karageorghis A and Fairweather G, *The method of fundamental solutions for the numerical solution of the biharmonic equation.*, J. Comput. Phys. 69, 434-459 (1987).

[23] Kupradze V D, Aleksidze M A, *The method of functional equations for the approximate solution of certain boundary value problems.*, U.S.S.R. Computational Mathematics and Mathematical Physics, 4, 82-126 (1964).

[24] Mey G De, *Calculation of the eigenvalues of the Helmholtz equation by an Integral Equation.*, Int. J. Num. Meth. Eng., 10, 59-66 (1976).

[25] Moler C B, Payne L E, *Bounds for Eigenvalues and Eigenfunctions of Symmetric Operators.*, SIAM J. Numer. Anal., 5, 64-70 (1968).