The Method of Fundamental Solutions applied to the numerical calculation of eigenfrequencies and eigenmodes for 3D simply connected domains

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Abstract: *In this work we consider the Method of Fundamental Solutions (MFS) applied to the calculation of eigenfrequencies and eigenmodes of 3D shapes. This meshless method was considered for 2D shapes (e.g. [1], [8]) and the application to 3D simply connected domains is analysed. We propose a choice of collocation and source points in 3D adapted from the algorithm presented in [1]. An example with Dirichlet boundary conditions is considered to illustrate the convergence and the good approximations obtained with the proposed algorithm.*

Keywords: method of fundamental solutions, Helmholtz equation, eigenfrequencies, eigenmodes.

1 Introduction

The calculation of the eigenvalues and eigenfunctions associated to the Laplace operator in a bounded domain is a problem with applications in acoustics. There are some domains with simple geometries for which we have an explicit formula, such as balls and rectangular domains. However, for more general domains, the calculation of the eigenfrequencies and eigenmodes requires a numerical method. There are some classical methods that have already been applied to this calculation. A standard finite differences method can produce good results when dealing with a particular type of shapes defined on rectangular grids, while for other type of shapes the finite element method is more appropriated. However these methods require extra computational effort because of the generation of the mesh and the calcul of integrals over the domain Ω. Other possibility is to apply the boundary element method (e.g. [10]). It avoids the calcul of integrals over all the domain but we need to deal with the integration of weakly singular kernels on the boundary. Recently, researchers have paid attention to the meshless methods. These methods avoids both the built of the mesh and the integral calculation and it's very easy to apply to domains with complicated geometry. As disadvantage the system matrices are ill conditioned. Here we propose a meshless method, the method of fundamental solutions (MFS) for this calculation. The MFS has been applied since the 1960s (e.g. [9] or [4]). This method is applicable to any elliptic boundary value problem, provided the fundamental solution of the PDE is known. The basic idea is to approximate the solution in terms of a set of fundamental solutions of the PDE centered on some "point-sources" placed on the exterior of the domain Ω. The coefficients of the linear combination are obtained such that the approximation satisfies the boundary condition at some points on the boundary of the domain. The application

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of the MFS to the calculation of the eigenfrequencies was introduced by Karageorghis in [8], and applied for simple shapes in the two-dimensional problem. It is well known that the location of the point-sources is very important to obtain an accurate approximation. In [1] it was proposed a particular choice of the "point-sources" which leads to very accurate results in the eigencalculation. In [8] it is presented a comparison with the boundary element method used by De Mey in [10], and the results obtained for simple shapes (circles, squares), show a better performance for the MFS. The application of other meshless methods to the determination of eigenfrequencies and eigenmodes has also been subject to recent research, mainly using radial basis functions (e.g. [6]). The three-dimensional problem was already considered by J T Chen et al. using radial basis functions (cf. [7]). That method is very similar to the MFS, but only the imaginary part of the fundamental solution is used. However, the method results in spurious eigenvalues. To deal with this problem JT Chen et al. applied the singular value decomposition (SVD) and the Fredholm alternative theorem. Here we consider the application of the MFS for the eigenproblem with a general three dimensional domain. A choice for the collocation and source points is proposed which is an extension of the choice considered in the two-dimensional problem (cf. [1]). Having determined an approximation of the eigenfrequency, we apply an algorithm based on the MFS to obtain the associated eigenmodes. Several examples with Dirichlet and Neumann boundary conditions are considered to show the validity of the proposed method.

2 Helmholtz equation

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with regular boundary $\partial\Omega$. We will consider the 3D - Dirichlet/Neumann eigenvalue problem for the Laplace operator. This is equivalent to obtain the resonance frequencies κ that satisfies the Helmholtz equation

$$\begin{cases} \Delta u + \kappa^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \partial_n u = 0 & \text{on } \Gamma_2, \end{cases} \tag{2.1}$$

with $\partial\Omega = \Gamma_1 \cup \Gamma_2$ for a non null function u. As an application, this corresponds to recover the resonance frequencies $\kappa > 0$ associated with a cavity or a room.
A fundamental solution Φ_ω of the Helmholtz equation verifies $(\Delta + \omega^2)\Phi = -\delta$, where δ is the Dirac delta distribution. In the 3D case, we take

$$\Phi_\omega(x) = \frac{e^{i\kappa\|x\|}}{\|x\|} \tag{2.2}$$

where $\|.\|$ is the euclidean norm in $\mathbb{R}^3$. A density result in [2] states that if ω is not an eigenfrequency then

$$L^2(\partial\Omega) = \overline{\text{span}\left\{\Phi_\omega(x-y)|_{x\in\partial\Omega} : y \in \hat{\Gamma}\right\}}, \tag{2.3}$$

where $\hat{\Gamma}$ is an admissible source set, for instance, the boundary of a bounded open set $\hat{\Omega} \supset \bar{\Omega}$. This allows to justify the approximation of a L^2 function, with complex values, defined on $\partial\Omega$, using a sequence of functions

$$u_m(x) = \sum_{j=1}^{m} \alpha_{m,j}\Phi_\omega(x - y_{m,j}) \tag{2.4}$$

that converges to $u|_\Gamma$ in $L^2(\partial\Omega)$. This is a partial justification to the convergence of the Method of Fundamental Solution (MFS) based on density results. It is similar to Bogomolny's approach in [5],

but it avoids the use of boundary layer potentials. As pointed out in [2] or [5], the convergence of the MFS, in a general case, is not completely related to the discretization of a single layer potential, although there is a straightforward relation. A single layer potential defined on $\hat{\Gamma}$ is an analytic function in Ω, and therefore such an approach would only be appropriate for analytic functions. Since $u|_\Gamma \equiv 0$ is an analytic function, it makes sense to consider the approach of the MFS as being related to the discretization of the single layer potential, for $x \in \Gamma$,

$$\mathcal{S}_\omega \varphi(x) = \int_{\hat{\Gamma}} \Phi_\omega(x-y)\varphi(y)\, ds_y \approx u_m(x) = \sum_{j=1}^{m} \alpha_{m,j} \Phi_\omega(x - y_{m,j}). \tag{2.5}$$

If ω is not an eigenfrequency of the interior Dirichlet problem then

$$\dim(Ker(\mathcal{S}_\omega)) = 0,$$

and therefore (see [1]) we search for ω such that $\dim(Ker(\mathcal{S}_\omega)) \neq 0$.

3 Numerical Algorithm

3.1 Determination of the eigenfrequencies

From the previous considerations we may sketch a procedure of finding the eigenfrequencies by checking the frequencies ω for which $\dim(Ker(\mathcal{S}_\omega)) \neq 0$. Defininning m collocation points $x_i \in \partial\Omega$ and m source points $y_{m,j} \in \hat{\Gamma}$, we obtain the system

$$\sum_{j=1}^{m} \alpha_{m,j} \Phi_\omega(x_i - y_{m,j}) = 0, \quad (x_i \in \partial\Omega). \tag{3.1}$$

Therefore a straighforward procedure is to find the values ω for which the $m \times m$ matrix

$$A(\omega) = \left[\Phi_\omega(x_i - y_j)\right]_{m\times m} \tag{3.2}$$

has a null determinant. However, an arbitrary choice of source points may lead to worst results than the expected with the MFS applied to simple shapes. We will place m collocation points on the boundary of the domain. The "point-sources" are calculated in the following way, for each collocation point x_i we calculate $y_i = x_i + \alpha\tilde{n}$, where $\tilde{n}$ is an approximation of the unitary vector which is normal to the boundary $\partial\Omega$ at the point x_i. When the boundary of the domain is given by a parametrization $p(t,s)$ this can be calculated by

$$\tilde{n} = \pm\partial_t p(t,s) \times \partial_s p(t,s)$$

chosing the sign such that the vector points to the exterior of the domain on each collocation point x_i. The components of the matrix $A(\omega)$ are complex numbers, so the determinant is also a complex number. We consider the real function $g(\omega) = |Det[A(\omega)]|$. If κ is an eigenfrequency, κ is a point of minimum where $g(\kappa) = 0$. To calculate this point we use the procedure described in [1], and to search the point where the minimum is attained we use an algorithm based on the *golden ratio search* method.

3.2 Determination of the eigenmodes

To obtain an eigenfunction associated with a certain resonance frequency κ we use a collocation method on $n+1$ points, with $x_1,\cdots,x_n$ on $\partial\Omega$ and a point $x_{n+1}\in\Omega$. Then, the approximation of the eigenfunction is given by

$$\tilde{u}(x)=\sum_{j=1}^{n+1}\alpha_j\Phi_\kappa(x-y_j). \tag{3.3}$$

To exclude the solution $\tilde{u}(x)\equiv 0$, the coefficients α_j are determinated by the solution of the system

$$\tilde{u}(x_i)=\delta_{i,n+1},\quad i=1,\ldots,n+1 \tag{3.4}$$

where $\delta_{i,j}$ is the Kronecker delta. When we take $n=m$ this resumes to add one line and one column to the matrix $A(\omega)$ defined in (3.2). Depending on the multiplicity of the eigenvalue, we will add one or more collocation points to make the linear system well determined.

4 Numerical Results

We will call Ω_1 the 3D bounded simply connected domain with parametrized boundary given by

$$\left\{(2\cos(t)\cos(u),\sin(t)cos(u),2\sin(u)-\sin(u)\cos(2u)):\,(t,u)\in[0,2\pi\,[\,\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right\}$$

Since the values of the Dirichlet eigenfrequencies for the unit ball are well known, given by Bessel functions, we will first test the results of this method for the three first resonance frequencies considering $\alpha=5$

m	abs. error (κ_1)		m	abs. error (κ_2)		m	abs. error (κ_3)
112	1.25003×10^{-8}		112	9.20966×10^{-7}		112	8.57032×10^{-6}
158	8.60556×10^{-12}		158	1.97913×10^{-9}		158	6.53597×10^{-8}
212	2.18626×10^{-14}		212	1.61488×10^{-13}		212	9.46334×10^{-11}

and for the unit cube we obtain the following results for the three first resonance frequencies considering $\alpha=2$

m	abs. error (κ_1)		m	abs. error (κ_2)		m	abs. error (κ_3)
152	2.11668×10^{-7}		152	3.75642×10^{-5}		152	4.86049×10^{-5}
218	6.13927×10^{-10}		218	9.27686×10^{-7}		218	1.55639×10^{-6}
386	9.15445×10^{-12}		386	5.25961×10^{-9}		386	1.95312×10^{-10}

In Figure 1 we present the results for the eigenmode associated with the fourth eigenfrequency $\kappa_4\approx 3.263078$ of domain Ω_1.

Conclusions: We presented the MFS applied to the calculation of eigenfrequencies and eigenmodes for 3D shapes. The algorithm for the choice of source and collocation points was adapted from [1] and lead to accurate results for a quite general class of tested domains.

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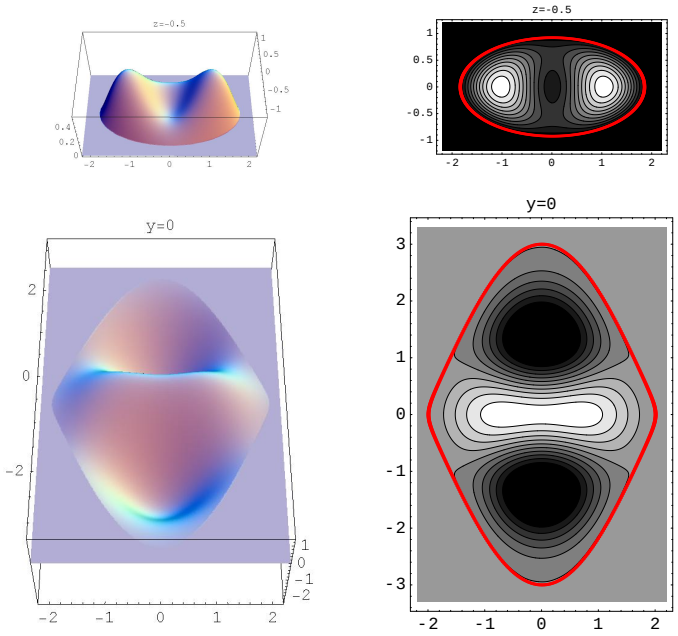

Figure 1: Plot of the eigenfunction and the respective contourplot for $z = -0.5$ and $y = 0$.

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