Oscillatory integrals and extremal problems in harmonic analysis

by

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Abstract

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Oscillatory integrals appear naturally in a variety of problems related to harmonic analysis, and have been a part of the subject since its creation. One is generally interested in quantifying how certain degeneracies of the phase affect the behavior of the integral. Numerous generalizations of the so-called stationary phase method have been the focus of past and current research, and the study of multilinear oscillatory forms with rough amplitudes falls into this category.

In the first part of this thesis, in a joint work with Michael Christ, we examine a certain class of trilinear integral operators which incorporate oscillatory factors e^{iP}, where P is a real-valued polynomial, and prove smallness of such integrals in the presence of rapid oscillations.

Oscillatory integrals provide a link between geometric properties of manifolds and harmonic analysis related to them, as illustrated by a multitude of restriction theorems for the Fourier transform which have been the object of careful investigation since the late 1960's.

In the second part of this thesis, we establish the existence of extremizers for a Fourier restriction inequality on planar convex arcs whose curvature satisfies a natural assumption. More generally, we prove that any extremizing sequence of nonnegative functions has a subsequence which converges to an extremizer. By studying the three-fold convolution of arclength measure on the curve with itself we additionally show that, if the geometric assumption on the curvature fails in a strong sense, then extremizing sequences concentrate at a point on the curve and extremizers do not exist.

To my parents, for everything they taught me.

Contents

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Chapter 1

Decay of multilinear oscillatory integrals

Oscillatory integrals have played a central role in harmonic analysis since its very beginning. Besides the obvious fact that the Fourier transform is itself an example of an oscillatory integral, the early works on asymptotic analysis by Bessel, Airy, and Riemann, among others, tried to make use of a version of the principle of stationary phase. Oscillatory integrals arising in combination with kernels of singular integral operators appear in widely different contexts and occupy a prominent place in current research.

Notation. In this and the next chapter, if x, y are real numbers, we will write $x = O(y)$ or $x \lesssim y$ if there exists a finite constant C such that $|x| \leq C|y|$, and $x \asymp y$ if $C^{-1}|y| \leq |x| \leq C|y|$ for some finite constant $C \neq 0$. If we want to make explicit the dependence of the constant C on some parameter α, we will write $x = O_\alpha(y)$ or $x \lesssim_\alpha y$. As is customary the constant C is allowed to change from line to line.

Oscillatory integrals of the first kind

The well-known van der Corput's lemma dictates the behavior of the so-called oscillatory integrals of the first kind, in Stein's terminology [1]:

Lemma 1.1 (van der Corput)**.** *Let $k \in \mathbb{N}$. Let $I \subseteq \mathbb{R}$ be an interval and suppose that $\phi : I \to \mathbb{R}$ satisfies $|\phi^{(k)}(x)| \geq 1$ for $x \in I$. Then, for $\lambda \in \mathbb{R}$,*

$$\Big| \int_I e^{i\lambda\phi(x)} dx \Big| \leq C_k |\lambda|^{-1/k},$$

provided, in addition when $k = 1$, that ϕ' is monotonic on I.

A few remarks may help to further orient the reader. (i) The decay rate in λ is sharp, as is seen by taking $I = [0, 1]$ and $\phi(x) = x^k/k!$. (ii) Because of this, the inequality scales in the sense that knowing it for $\lambda = 1$ and arbitrary I automatically gives the inequality for arbitrary $\lambda \in \mathbb{R}$. (iii) The constants C_k are absolute, that is, independent of I, ϕ and λ. (iv) Higher dimensional versions of this lemma, though not completely satisfactory, were established in [2]. See also [3].

One proof uses basic sublevel set estimates which will turn out to be important in our discussion later on:

Proof of van der Corput's lemma. Fix a parameter $t \in (0, \infty)$ (to be determined) and write

$$\int_I e^{i\lambda\phi(x)}dx = \int_{I\cap\{|\phi'|\le t\}} e^{i\lambda\phi(x)}dx + \int_{I\cap\{|\phi'|>t\}} e^{i\lambda\phi(x)}dx = \mathbf{I} + \mathbf{II}.$$

To estimate the first term, observe that the hypothesis $\left|(\frac{d}{dt})^{k-1}\phi'\right| \ge 1$ implies

$$|\mathbf{I}| \le |\{x \in I : |\phi'(x)| \le t\}| \lesssim_k t^{\frac{1}{k-1}},$$

where the implicit constant depends only on k. This can easily be seen by induction on k. For the second term, integrate by parts on each of the at most k intervals where ϕ' is monotonic. This yields $|\mathbf{II}| \lesssim_k (|\lambda|t)^{-1}$, and the result follows by optimizing in t. □

Let ϕ satisfy the hypothesis of van der Corput's lemma. It is interesting to note that the sublevel set estimate

$$|\{x \in I : |\phi(x)| \le t\}| \le C_k t^{\frac{1}{k}} \tag{1.1}$$

is a corollary of van der Corput's lemma. To see why this is the case, let $h \in C_0^\infty(\mathbb{R})$ be such that $h \equiv 1$ on $[-1, 1]$, and choose a smooth cutoff $\psi \in C_0^\infty(\mathbb{R})$ which is $\equiv 1$

on the interval I. Then

$$\begin{aligned} |\{x \in I : |\phi(x)| \leq t\}| &\leq \int_{\mathbb{R}} h(t^{-1}\phi(x))\psi(x)dx \\ &= \frac{1}{2\pi}\int \Big(\int \widehat{h}(\xi)e^{i\xi t^{-1}\phi(x)}d\xi\Big)\psi(x)dx \\ &= \frac{1}{2\pi}\int \widehat{h}(\xi)\Big(\int e^{i\xi t^{-1}\phi(x)}\psi(x)dx\Big)d\xi \\ &\lesssim_k \int |\widehat{h}(\xi)|\langle \xi t^{-1}\rangle^{-\frac{1}{k}}d\xi \simeq_k t^{\frac{1}{k}}. \end{aligned}$$

On the other hand, the proof of van der Corput's lemma presented above used a special case of the sublevel set estimate (1.1) as a key ingredient. Thus sublevel set estimates and oscillatory integrals of the first kind should be seen as going hand-in-hand together.

Oscillatory integrals of the second kind

The classical theory of oscillatory integrals of the second kind establishes the boundedness of operators of the form

$$(T_\lambda f)(\xi) = \int_{\mathbb{R}^m} e^{i\lambda\Phi(x,\xi)}f(x)\psi(x,\xi)dx, \tag{1.2}$$

and predicts polynomial decay in the parameter λ as $|\lambda| \to \infty$. Here, ψ is a fixed smooth function of compact support in x and ξ, and the phase Φ is real-valued and smooth. If the Hessian of Φ is nonvanishing in the support of the cutoff ψ i.e.

$$\det\Big(\frac{\partial^2\Phi(x,\xi)}{\partial x_i \partial \xi_j}\Big) \neq 0 \quad \text{whenever } (x,\xi) \in \operatorname{supp}(\psi), \tag{1.3}$$

then one has that

$$|\langle T_\lambda f, g\rangle| \lesssim \lambda^{-m/2}\|f\|_{L^2(\mathbb{R}^m)}\|g\|_{L^2(\mathbb{R}^m)}. \tag{1.4}$$

To prove (1.4), it is enough to show that the operator norm of $T_\lambda T_\lambda^*$ is $O(\lambda^{-m})$. Write

$$(T_\lambda T_\lambda^* f)(\xi) = \int K_\lambda(\xi,\eta)f(\eta)d\eta$$

where

$$K_\lambda(\xi,\eta) = \int_{\mathbb{R}^m} e^{i\lambda(\Phi(x,\xi)-\Phi(x,\eta))}\psi(x,\xi)\overline{\psi}(x,\eta)dx.$$

An estimate for K_λ will give the required bound, and it turns out that

$$|K_\lambda(\xi,\eta)| \lesssim_N (1+\lambda|\xi-\eta|)^{-N}, \quad N \geq 0, \tag{1.5}$$

suffices. We omit the proof of (1.5) and refer the interested reader to [1, pp. 377−379].

The multilinear setting

Aiming at similar results in a somewhat different context, let us consider n-linear oscillatory expressions of the form

$$I(\lambda P; f_1,\ldots,f_n) = \int_{\mathbb{R}^m} e^{i\lambda P(x)} \prod_{j=1}^{n} f_j \circ \pi_j(x)\eta(x)dx, \tag{1.6}$$

where $\lambda \in \mathbb{R}$ is a typically large parameter, $P : \mathbb{R}^m \to \mathbb{R}$ is a real-valued polynomial, $\pi_j : \mathbb{R}^m \to V_j$ are orthogonal projections onto some subspaces V_j of $\mathbb{R}^m$, $f_j : V_j \to \mathbb{C}$ are locally integrable functions, and $\eta \in C_0^1(\mathbb{R}^m)$ is compactly supported. All the subspaces V_j are assumed to have the same dimension, which we denote by κ.

Christ, Li, Tao and Thiele initiated the study of expressions like (1.6) in [4], and established that certain algebraic conditions on the polynomial P and geometric conditions on the projections $\{\pi_j\}$ ensure power decay estimates of the form

$$|I(\lambda P; f_1,\ldots,f_n)| \lesssim \langle\lambda\rangle^{-\epsilon} \prod_{j=1}^{n} \|f_j\|_{L^\infty(V_j)}. \tag{1.7}$$

Their results were restricted to the comparatively extreme cases $\kappa = 1$ and $\kappa = m-1$, and the small codimension case $n \leq \frac{m}{m-\kappa}$, leaving most cases open.

Before stating the main results, let us introduce a key definition:

Definition 1.2. A real-valued polynomial $P : \mathbb{R}^m \to \mathbb{R}$ is *degenerate* with respect to the projections $\{\pi_j\}$ if there exist polynomials $p_j : V_j \to \mathbb{R}$ such that $P = \sum_j p_j \circ \pi_j$. Otherwise P is said to be *nondegenerate*.

This nondegeneracy condition is to replace hypothesis (1.3) of a nonvanishing derivative. It is a somewhat generic condition, as illustrated by the following facts. (i) For given dimensions m, κ, degree of multilinearity n, and subspaces $\{V_j\}$, there exist nondegenerate homogeneous polynomials P of all sufficiently high degrees. To see this, observe that the dimension of the vector space of all $\sum_j q_j \circ \pi_j$, where q_j is an arbitrary homogeneous polynomial of degree d on $\mathbb{R}^\kappa$, is $O(nd^\kappa)$. On the other hand, the space of all homogeneous polynomials $P : \mathbb{R}^m \to \mathbb{R}$ has dimension $\sim cd^m \gg d^\kappa$

as $d \to \infty$. (ii) If d and $\{\pi_j\}$ are held fixed, then the collection of all nondegenerate polynomials P of degrees $\leq d$ is an open subset of the vector space $\mathcal{P}(d)$ of all polynomials of degree $\leq d$. See [4, Lemma 3.2].

Example 1.3. Let $n \in \mathbb{N}$ be arbitrary. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $P(x) = x_3^2$. For $1 \leq j \leq n$, take light-cone unit vectors $v^j = (v_1^j, v_2^j, v_3^j) \in \mathbb{R}^3$ such that $(v_3^j)^2 = (v_1^j)^2 + (v_2^j)^2$. Let π_j be the orthogonal projection onto $V_j = \text{span}(v_j)$ i.e. $\pi_j(x) = x \cdot v^j = x_1 v_1^j + x_2 v_2^j + x_3 v_3^j$. One readily checks that P is nondegenerate with respect to $\{V_j\}_{j=1}^n$ *for every* $n \in \mathbb{N}$.

Example 1.3 is somewhat surprising in view of the following fact:

Example 1.4. In $\mathbb{R}^2$, any polynomial $Q : \mathbb{R}^2 \to \mathbb{R}$ of degree two is degenerate with respect to any family of three or more mappings of the form $\pi_j(x) = x \cdot w_j$, provided that none of the w_j is a multiple of any of the others.

One can expect decay like (1.7) only if P is nondegenerate with respect to $\{\pi_j\}$. To quantify this, consider the vector space of all degenerate polynomials of degree $\leq d$ as a subspace $\mathcal{P}_{degen}$ of the vector space $\mathcal{P}(d)$. Fix some norm on the quotient space $\mathcal{P}(d)/\mathcal{P}_{degen}$, and denote it by $\|\cdot\|_{nd}$. Finally, let $\|\cdot\|_{nc}$ denote some fixed choice of norm on the quotient space of polynomials of degree $\leq d$ modulo constants.

It is a consequence of the lemma of van der Corput that the norm $\|\cdot\|_{nc}$ controls oscillatory integrals of the first kind, as is well quantified in [2, Corollary 7.3]. The following result shows that the norm $\|\cdot\|_{nd}$ controls a certain class of multilinear oscillatory integrals:

Theorem 1.5 ([4]). *Suppose that $n < 2m$ and $d < \infty$. Then, for any family $\{V_j\}_{j=1}^n$ of one-dimensional subspaces of $\mathbb{R}^m$ which lie in general position*[1]*, there exist constants $C < \infty$ and $\epsilon > 0$ such that*

$$|I(P; f_1, \ldots, f_n)| \leq C \langle \|P\|_{nd} \rangle^{-\epsilon} \prod_{j=1}^{n} \|f_j\|_{L^2(V_j)} \tag{1.8}$$

for all polynomials $P : \mathbb{R}^m \to \mathbb{R}$ of degree $\leq d$ and for all functions $f_j \in L^2(\mathbb{R})$.

Since Theorem 1.5 was the starting point for our work on trilinear oscillatory integrals, we describe its proof in some detail in §1.2. Before doing so we dedicate the next section to describing some more elementary but related results from [4].

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[1] In this context, a collection $\{V_j\}$ of subspaces of $\mathbb{R}^m$ of dimension κ is said to lie in *general position* if any subcollection of cardinality $k \geq 1$ spans a subspace of dimension $\min\{k\kappa, m\}$.

1.1 The simply nondegenerate case

A concept related to the nondegeneracy of the polynomial phase described above turns out to be somewhat easier to analyze:

Definition 1.6. A polynomial P is *simply nondegenerate* with respect to a family of subspaces $\{V_j\}_{j=1}^n$ if there exists a differential operator L of the form

$$L = \prod_{j=1}^{n} (w_j \cdot \nabla), \text{ with each } w_j \in V_j^{\perp}, \tag{1.9}$$

such that $L(P)$ does not vanish identically.

Simple nondegeneracy implies nondegeneracy. Indeed, for any distribution $f \in \mathcal{D}'(V_j)$, the j-th factor of L annihilates $f \circ \pi_j$, and hence so does L since all factors commute. Thus $L(P - \sum_j f_j \circ \pi_j) \equiv L(P)$. The converse does not hold in general, as is shown by Example 1.3. However, it is proved in [4, Proposition 3.1] and [4, Lemma 3.5], respectively, that nondegeneracy does imply simple nondegeneracy in the following special cases:

- the codimension 1 case, $\kappa = m - 1$;
- the "small" codimension case, $n(m-\kappa) \le m$, provided that the subspaces $\{V_j\}$ lie in general position.

The next result illustrates the importance of the concept of simple nondegeneracy:

Theorem 1.7 ([4])**.** *Let m, κ be arbitrary, and let $U \subset \mathbb{R}^m$ be an open set. Then any simply nondegenerate polynomial P has the power decay property in U in the sense that for any $\eta \in C_0^1(U)$ there exist $\epsilon > 0$ and $C < \infty$ such that*

$$|I(\lambda P; f_1, \ldots, f_n)| \le C \langle \lambda \rangle^{-\epsilon} \prod_{j=1}^{n} \|f_j\|_{L^\infty(V_j)} \quad \text{for all } f_j \in L^\infty(V_j) \text{ and all } \lambda \in \mathbb{R}. \tag{1.10}$$

The power decay property just described is said to be *uniform* if (1.10) holds with uniform constants C, ϵ, for any family of real-valued polynomials of bounded degree with are uniformly nondegenerate relative to the subspaces $\{V_j\}$. An immediate consequence is the following:

Corollary 1.8 ([4])**.** *Any collection of codimension one subspaces $\{V_j\}$ has the uniform power decay property.*

1.2 The one-dimensional case

In this section we present the main ideas behind the proof of Theorem 1.5. We start with a few elementary remarks. (i) Without loss of generality we can assume that all the functions f_j have compact support, and that $\|f_j\|_2 \leq 1$ for every $j \leq n$. (ii) We will restrict our attention to polynomial phases of the form λP, where $\lambda \in (0, \infty)$ and $\|P\|_{nd} = 1$, for otherwise the conclusion is trivial. (iii) The cases $n = 0$ and $n = 1$ are straightforward consequences of the lemma of van der Corput. In the case $n = 2$ one is dealing with a bilinear form $\langle T_\lambda f_1, f_2 \rangle$, where the associated operator T_λ is of the form (1.2). However, if $n \geq 3$ and $m < n$, there arises a class of singular oscillatory integrals which have no direct analogues in the bilinear case. (iv) The condition $n < 2m$ is necessary for L^2 decay in (1.8), as is easily seen by taking $f_j = \chi_{(-\delta,\delta)}$ for every $j \leq n$ and letting $\delta \to 0^+$.

Preliminary remarks on uniformity

We now turn to a notion related to the concept of Gowers' uniformity that appeared in his celebrated work [5] on the Szemerédi's theorem. Fix a ball $B \subset \mathbb{R}^m$ of unit volume, an integer $d \in \mathbb{N}$ and a positive constant $\tau > 0$.

Definition 1.9. A function $f \in L^2(B)$ is λ-*nonuniform* if there exist a polynomial q of degree $\leq d$ and a scalar c such that

$$\|f - ce^{iq}\|_{L^2(B)} \leq (1 - |\lambda|^{-\tau})\|f\|_{L^2(B)}.$$

Otherwise f is said to be λ-*uniform*.

This definition depends on the parameters d and τ, but as long as they remain fixed, generalized Fourier coefficients of λ-uniform functions satisfy favorable bounds. More precisely, the following result holds:

Lemma 1.10. *If $f \in L^2(B)$ is λ-uniform, then*

$$\Big| \int_B f(x) e^{-iq(x)} dx \Big| \leq |\lambda|^{-\tau} \|f\|_{L^2(B)}$$

uniformly for all real-valued polynomials q of degree $\leq d$.

Proof. Aiming at a contradiction, let $f \in L^2(B)$ be λ-uniform, and suppose that $|\langle f, e^{iq} \rangle| \|f\|_2^{-1} > |\lambda|^{-\tau}$ for some polynomial $q \in \mathcal{P}(d)$. Consider the orthogonal decomposition in $L^2(B)$

$$f = \langle f, e^{iq} \rangle e^{iq} + g,$$

where $g \perp e^{iq}$. Then:

$$\begin{aligned} \|f - \langle f, e^{iq}\rangle e^{iq}\|_2^2 = \|g\|_2^2 &= \|f\|_2^2 - \|\langle f, e^{iq}\rangle e^{iq}\|_2^2 \\ &= \Big(1 - \frac{|\langle f, e^{iq}\rangle|^2}{\|f\|_2^2}\Big)\|f\|_2^2 \\ &< (1 - |\lambda|^{-2\tau})\|f\|_2^2, \end{aligned}$$

a contradiction. □

The proof of Theorem 1.5 proceeds by induction on the degree of multilinearity n. We lose no generality in assuming that f_1 is λ-uniform. Indeed, if that were not the case, we would have that

$$\begin{aligned} |I(\lambda P; f_1, \dots, f_n)| =& |I(\lambda P; f_1 - ce^{iq}, f_2, \dots, f_n) + cI(\lambda P; e^{iq}, f_2, \dots, f_n)| \\ \le& \mathbf{A}(\lambda)\|f_1 - ce^{iq}\|_2 \prod_{j>1} \|f_j\|_2 + |c||I(\lambda P; e^{iq}, f_2, \dots, f_n)| \\ \le& \mathbf{A}(\lambda)(1 - |\lambda|^{-\tau}) + C|\lambda|^{-\sigma}, \end{aligned} \tag{1.11}$$

where by $\mathbf{A}(\lambda)$ we denote the best constant in the inequality (1.8). The existence of $\sigma > 0$ for which (1.11) holds follows from the induction hypothesis. A typical "swallow-the-constant" argument would then finish the proof; see the discussion following (1.17) below.

Step 1: reduction to the trilinear case on "slices"

Phong, Stein and Sturm [6] proved an estimate which improves (1.8) in the case $n = m$. Since the case $n < m$ then follows immediately, we lose no generality in assuming that $n > m$.

Let $e_1 \in \mathbb{R}^m$ be a unit vector orthogonal to the span of $\{V_j\}_{j=2}^m$, and let $e_2 \in \mathbb{R}^m$ be a unit vector orthogonal to the span of $\{V_j\}_{j=m+1}^n$ and not orthogonal to V_1. Clearly e_1 cannot be orthogonal to V_1. Moreover, the general position hypothesis implies that e_1 and e_2 are linearly independent. Therefore, given $x \in \mathbb{R}^m$, we may write $x = t_1e_1 + t_2e_2 + y$ for some unique $(t_1, t_2) = t \in \mathbb{R}^2$ and $y \in \text{span}(e_1, e_2)^\perp \simeq \mathbb{R}^{m-2}$. Consequently, for some $a, b \neq 0$, we have that

$$\begin{aligned}
I(\lambda P; f_1, \ldots, f_n) &= \iint_{\mathbb{R}^m} e^{i\lambda P(t,y)} f_1(\pi_1(t,y)) \prod_{j=2}^{m} f_j(\pi_j(t,y)) \prod_{j=m+1}^{n} f_j(\pi_j(t,y))\eta(t,y)dtdy \\
&=: \int_{\mathbb{R}^{m-2}} \Big(\int_{\mathbb{R}^2} e^{i\lambda P^y(t)} \cdot G^y(at_1+bt_2) \cdot F_1^y(t_2) \cdot F_2^y(t_1) \cdot \eta(t,y)dt \Big) dy \\
&= \int_{\mathbb{R}^{m-2}} I(\lambda P^y; G^y, F_1^y, F_2^y) dy.
\end{aligned}$$

Cauchy-Schwarz's inequality, the hypotheses and Fubini's theorem together imply that

$$\begin{aligned}
\int_{\mathbb{R}^{m-2}} \|G^y\|_2 \|F_1^y\|_2 \|F_2^y\|_2 dy &\leq \Big(\int_{\mathbb{R}^{m-2}} \|G^y\|_2^2 \|F_1^y\|_2^2 dy \Big)^{1/2} \Big(\int_{\mathbb{R}^{m-2}} \|F_2^y\|_2^2 dy \Big)^{1/2} \\
&\lesssim \prod_{j=1}^{n} \|f_j\|_2 < \infty.
\end{aligned} \tag{1.12}$$

Step 2: analysis of the set of *good* parameters

Let $\mathcal{B} \subset \mathbb{R}^{m-2}$ denote the set of *bad* parameters, consisting of all $y \in \mathbb{R}^{m-2}$ for which the polynomial P^y has small norm in the quotient space of polynomials of degree $\leq d$ modulo degenerate polynomials with respect to the three projections $t \mapsto t_1$, $t \mapsto t_2$ and $t \mapsto \pi(t) := at_1 + bt_2$. More precisely, $y \in \mathcal{B}$ if P^y can be decomposed as

$$P^y(t) = Q_1(t_1) + Q_2(t_2) + Q_3(\pi(t)) + R(t), \tag{1.13}$$

for some real-valued polynomials Q_j and R of degree $\leq d$, with the additional requirement that $\|R\| \leq |\lambda|^{-\rho}$, where $\|\cdot\|$ denotes a given norm on the space $\mathcal{P}(d)$ and $\rho > 0$ is a small parameter to be chosen later on. If $y \in \mathbb{R}^{m-2} \setminus \mathcal{B}$, then we may apply Theorem 1.7 with $n = 3$, $m = 2$, $\kappa = m - 1 = 1$ and the phase $|\lambda|^{\rho} P^y$ to obtain

$$|I(\lambda P^y; G^y, F_1^y, F_2^y)| \lesssim (|\lambda|^{1-\rho})^{-\tilde{\rho}} \|G^y\|_2 \|F_1^y\|_2 \|F_2^y\|_2 \text{ for some } \tilde{\rho} > 0.$$

Together with (1.12), this implies

$$\int_{y \notin \mathcal{B}} |I(\lambda P^y; G^y, F_1^y, F_2^y)| dy \lesssim |\lambda|^{-(1-\rho)\tilde{\rho}}. \tag{1.14}$$

Step 3: analysis of the set of *bad* parameters

The set of bad parameters might have full measure as Example 1.3 shows. However, we will show that if ρ is small enough, then there exists $\tilde{\epsilon} > 0$ such that

$$|I(\lambda P^y; G^y, F_1^y, F_2^y)| \lesssim |\lambda|^{-\tilde{\epsilon}} \|G^y\|_2 \|F_1^y\|_2 \|F_2^y\|_2 \text{ uniformly for all } y \in \mathcal{B},$$

and this will suffice for our purposes.

Using decomposition (1.13) and Fourier inversion, we have that

$$\begin{aligned} I(\lambda P^y; G^y, F_1^y, F_2^y) &= \int_{\mathbb{R}^2} e^{i\lambda P^y(t)} G^y(\pi(t)) F_1^y(t_2) F_2^y(t_1) \eta(t,y) dt \\ &= \int_{\mathbb{R}^2} e^{i\lambda Q_3(\pi(t))} G^y(\pi(t)) \cdot e^{i\lambda Q_2(t_2)} F_1^y(t_2) \cdot e^{i\lambda Q_1(t_1)} F_2^y(t_1) \cdot e^{i\lambda R(t)} \eta(t,y) dt \\ &=: \int_{\mathbb{R}^2} \widetilde{G}(\pi(t)) \cdot \widetilde{F}_1(t_2) \cdot \widetilde{F}_2(t_1) \cdot \zeta(t) dt \\ &\simeq \iiint_{\mathbb{R}^3} \widehat{\widetilde{G}}(\xi_0) \widehat{\widetilde{F}}_1(-a\xi_0 - \xi_2) \widehat{\widetilde{F}}_2(-b\xi_0 - \xi_1) \widehat{\zeta}(\xi_1, \xi_2) d\xi_0 d\xi_1 d\xi_2. \end{aligned}$$

By the principle of non-stationary phase and the fact that λR is a polynomial of bounded degree which is $O(|\lambda|^{1-\rho})$ on $\mathrm{supp}(\zeta)$ (and the same holds for all its derivatives), we have that

$$|\widehat{\zeta}(\xi)| \lesssim_N |\lambda|^{(1-\rho)N} (1 + |\xi|)^{-N}, \quad \forall \xi \in \mathbb{R}^2, \forall N \in \mathbb{N},$$

provided[2] $\eta \in C^N$. Using this estimate with $N = 4$, Hölder's inequality and Plancherel's theorem, we obtain

$$|I(\lambda P^y; G^y, F_1^y, F_2^y)| \lesssim |\lambda|^{4(1-\rho)} \|\widehat{\widetilde{G}}\|_{L^\infty} \|F_1^y\|_{L^2} \|F_2^y\|_{L^2}.$$

We are now ready for the *coup de grâce*: recall that f_1 is λ-uniform. This implies favorable bounds for every generalized Fourier coefficient of f_1. In particular,

$$\|\widehat{\widetilde{G}}\|_{L^\infty} \lesssim |\lambda|^{-\tau} \|G^y\|_2,$$

[2]We lose no generality in assuming this extra smoothness on η: by the usual decomposition of a compactly supported Hölder continuous function $\zeta = f + g$ into a smooth part f such that $\|f\|_{C^N} = O(\lambda^{C_N \delta})$ and a bounded remainder g such that $\|g\|_\infty = O(\lambda^{-\delta})$, it is easy to see that if the result holds for some $\eta \in C_0^N$ ($N \in \mathbb{N}$) with a constant $C = O(\|\eta\|_{C^N})$, then it will continue to hold for all η which are compactly supported and Hölder continuous of order α.

and so

$$|I(\lambda P^y; G^y, F_1^y, F_2^y)| \lesssim |\lambda|^{4(1-\rho)-\tau} \|G^y\|_2 \|F_1^y\|_2 \|F_2^y\|_2. \tag{1.15}$$

In light of (1.12) this implies

$$\int_{y\in\mathcal{B}} |I(\lambda P^y; G^y, F_1^y, F_2^y)| dy \lesssim |\lambda|^{4(1-\rho)-\tau}. \tag{1.16}$$

We conclude this step by choosing $\rho < 1$ sufficiently close to 1 in order to ensure that $4(1-\rho) - \tau = -\tilde{\epsilon}$ is strictly negative.

The end of the proof

Estimates (1.11), (1.14) and (1.16) together imply that

$$\mathbf{A}(\lambda) \lesssim \max\{\mathbf{A}(\lambda)(1 - |\lambda|^{-\tau}) + C|\lambda|^{-\sigma}, |\lambda|^{-(1-\rho)\tilde{\rho}}, |\lambda|^{-\tilde{\epsilon}}\}. \tag{1.17}$$

Since $\mathbf{A}(\lambda)$ is certainly finite, this implies that it is majorized by a constant times some negative power of $|\lambda|$, as was to be proved.

1.3 Our contribution: the trilinear case

As we have seen, a crucial step in the proof of Theorem 1.5 amounts to a reduction to the trilinear situation, which is then analyzed using an idea taken from Gowers' seminal work [5]. This was the main motivation to further understand the trilinear case. Here is our result, which is the subject of the next chapter:

Theorem 1.11 ([7]). *Let $\kappa \geq 1$ and $d < \infty$. Let $\{\pi_j : 1 \leq j \leq 3\}$ be a collection of three surjective linear mappings from $\mathbb{R}^{2\kappa}$ to $\mathbb{R}^\kappa$, which lie in general position. Then*

$$|I(P; f_1, f_2, f_3)| \leq C\langle \|P\|_{nd}\rangle^{-\epsilon} \prod_{j=1}^{3} \|f_j\|_{L^2(\mathbb{R}^\kappa)},$$

for all polynomials $P : \mathbb{R}^{2\kappa} \to \mathbb{R}$ of degree $\leq d$ and for all functions $f_j \in L^2(\mathbb{R}^\kappa)$. Moreover, the constants $C, \epsilon \in \mathbb{R}^+$ can be taken to depend only on κ, d and η.

Note that the projections $\{\pi_j\}$ are onto half-dimensional subspaces of the ambient space, and so the situation contemplated by Theorem 1.11 is intermediate between the extreme cases $\kappa = 1$ and $\kappa = m - 1$ considered in [4].

Our result is certainly not definitive. To understand the trilinear case completely one needs, among other things, to determine the optimal decay exponents, to study the case where the three target spaces have different dimensions, and to remove the general position hypothesis which is not completely necessary. We plan to pursue this direction of research, keeping in mind the methods developed in [2, 8, 9, 10, 4, 6].

The following general problem remains open:

Question 1.12. Is the power decay property (1.7) equivalent to nondegeneracy?

Question 1.12 is not accessible by the methods of the previously mentioned papers alone, even if the negative power of λ in (1.7) is replaced by some slowly decaying function. There are indications that, as was emphasized for certain related bilinear problems in [2], the power decay property (1.7) is linked to combinatorial issues.

Chapter 2

On trilinear oscillatory integrals

This is joint work with Michael Christ.

2.1 Introduction

We continue the study of multilinear oscillatory integral expressions of the form

$$I(\lambda P; f_1, \dots, f_n) = \int_{\mathbb{R}^m} e^{i\lambda P(x)} \prod_{j=1}^{n} f_j \circ \pi_j(x) \eta(x) dx,$$

where $\lambda \in \mathbb{R}$ is a parameter, $P : \mathbb{R}^m \to \mathbb{R}$ is a real-valued polynomial, $\pi_j : \mathbb{R}^m \to V_j$ are orthogonal projections onto some subspaces V_j of $\mathbb{R}^m$, $f_j : V_j \to \mathbb{C}$ are locally integrable functions with respect to Lebesgue measure on V_j, and $\eta \in C_0^1(\mathbb{R}^m)$ is compactly supported. All the subspaces V_j are assumed to have the same dimension, which is denoted by κ.

Christ, Li, Tao, and Thiele [4] initiated this study, exploring conditions on the polynomial phase P and on the projections $\{\pi_j\}$ which ensure decay estimates of the form

$$|I(\lambda P; f_1, \dots, f_n)| \leq C \langle \lambda \rangle^{-\epsilon} \prod_{j=1}^{n} \|f_j\|_{L^\infty(V_j)}. \tag{2.1}$$

Their results were restricted to the comparatively extreme cases $\kappa = 1$ and $\kappa = m-1$, and the small codimension case $n \leq \frac{m}{m-\kappa}$, leaving most cases open.

In this chapter we consider the trilinear situation in $\mathbb{R}^m = \mathbb{R}^{2\kappa}$ for arbitrary $\kappa \geq 2$. A typical expression of this type is then

$$I(P; f_1, f_2, f_3) = \iint_{\mathbb{R}^{2\kappa}} e^{iP(x,y)} f_1(x) f_2(y) f_3(x+y) \eta(x,y) dx dy,$$

with coordinates $(x, y) \in \mathbb{R}^{\kappa+\kappa}$. Before stating our main theorem, we introduce some notation and recall relevant results from the literature.

Review

Let $d \geq 1$ be a positive integer, and let $\{\pi_j\}_{j=1}^3$ be surjective linear mappings from $\mathbb{R}^{2\kappa}$ to $\mathbb{R}^\kappa$. A polynomial $P : \mathbb{R}^{2\kappa} \to \mathbb{R}$ is said to be degenerate (with respect to the projections $\{\pi_j\}$) if there exist polynomials $p_j : \mathbb{R}^\kappa \to \mathbb{R}$ such that $P = \sum_{j=1}^3 p_j \circ \pi_j$. The vector space of all degenerate polynomials $P : \mathbb{R}^{2\kappa} \to \mathbb{R}$ of degree $\leq d$ is a subspace $\mathcal{P}_{degen}$ of the vector space $\mathcal{P}(d)$ of all polynomials $P : \mathbb{R}^{2\kappa} \to \mathbb{R}$ of degree $\leq d$. Denote the quotient space by $\mathcal{P}(d)/\mathcal{P}_{degen}$, by $[P]$ the equivalence class of P in $\mathcal{P}(d)/\mathcal{P}_{degen}$, and by $\|\cdot\|_{nd}$ some fixed choice of norm for this quotient space. In a similar way, let $\|\cdot\|_{nc}$ denote some fixed choice of norm for the quotient space of polynomials $P : \mathbb{R}^{2\kappa} \to \mathbb{R}$ of degree $\leq d$ modulo constants.

It will be convenient to work with norms defined by inner products. If $P(x, y) = \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha y^\beta$, then we set

$$\|P\|_{\mathcal{P}(d)} = \Big(\sum_{\alpha,\beta} |c_{\alpha\beta}|^2 \Big)^{1/2}, \quad \|P\|_{nc} = \Big(\sum_{(\alpha,\beta)\neq(0,0)} |c_{\alpha\beta}|^2 \Big)^{1/2}.$$

$\|\cdot\|_{nd}$ is defined by choosing some Hilbert space structure for $\mathcal{P}(d)/\mathcal{P}_{degen}$.

The norm $\|\cdot\|_{nc}$ controls oscillatory integrals of the first kind, in light of the following version of stationary phase which is a straightforward consequence of van der Corput's lemma:

Theorem 2.1 ([2]). *Let $p(t) = \sum_{|\alpha|\leq d} c_\alpha t^\alpha, c_\alpha \in \mathbb{R}$, be a polynomial in m variables of degree $d \geq 1$. Then*

$$\Big| \int_{[0,1]^m} e^{ip(t)} dt \Big| \leq C_{d,m} \Big(\sum_{0<|\alpha|\leq d} |c_\alpha| \Big)^{-1/d}.$$

simi

On the other hand, the norm $\|\cdot\|_{nd}$ controls multilinear oscillatory integrals (in particular, oscillatory integrals of the second kind), as is shown in [4]. The following theorem is most relevant to our discussion:

Theorem 2.2 ([4]). *Suppose that $n < 2m$ and $d < \infty$. Then, for any family $\{V_j\}_{j=1}^n$ of one-dimensional subspaces of $\mathbb{R}^m$ which lie in general position, there exist constants*

$C < \infty$ and $\epsilon > 0$ such that

$$|I(P; f_1, \ldots, f_n)| \leq C\langle\|P\|_{nd}\rangle^{-\epsilon} \prod_{j=1}^{n} \|f_j\|_{L^2(V_j)}$$

for all polynomials $P : \mathbb{R}^m \to \mathbb{R}$ of degree $\leq d$ and for all functions $f_j \in L^2(\mathbb{R})$. Moreover, ϵ can be taken to depend only on n, m and d.

Result

Let $\{\pi_j : 1 \leq j \leq 3\}$ be a collection of three surjective linear mappings from $\mathbb{R}^{2\kappa}$ to $\mathbb{R}^{\kappa}$. We say that these lie in general position if for any two indices $i \neq j \in \{1, 2, 3\}$, the nullspace of π_i is transverse to the nullspace of π_j.

Here is our main result:

Theorem 2.3. *Let $\kappa \geq 1$ and $d < \infty$. Let $\{\pi_j : 1 \leq j \leq 3\}$ be a collection of three surjective linear mappings from $\mathbb{R}^{2\kappa}$ to $\mathbb{R}^{\kappa}$, which lie in general position. Then*

$$|I(P; f_1, f_2, f_3)| \leq C\langle\|P\|_{nd}\rangle^{-\epsilon} \prod_{j=1}^{3} \|f_j\|_{L^2(\mathbb{R}^{\kappa})},$$

for all polynomials $P : \mathbb{R}^{2\kappa} \to \mathbb{R}$ of degree $\leq d$ and for all functions $f_j \in L^2(\mathbb{R}^{\kappa})$, with constants $C, \epsilon \in \mathbb{R}^+$ which depend only on κ, d and η.

A more general result is established in [10], but we hope that the quite different method presented here will be of some value.

2.2 First reduction

It is no loss of generality to restrict attention to the case where $\mathbb{R}^{2\kappa}$ is identified with $\mathbb{R}^{\kappa}_x \times \mathbb{R}^{\kappa}_y$, and $\pi_1(x, y) = x$, $\pi_2(x, y) = y$, and $\pi_3(x, y) = x + y$. Indeed, since the nullspaces of π_1, π_2 are transverse, we may adopt coordinates $(x, y) \in \mathbb{R}^{\kappa+\kappa}$ such that the nullspace of π_1 is $\{(0, y)\}$, while the nullspace of π_2 is $\{(x, 0)\}$. Writing $\pi_3(x, y) = Ax + By$ where $A, B : \mathbb{R}^{\kappa} \to \mathbb{R}^{\kappa}$ are linear, the transversality hypothesis implies that both A, B are injective. Therefore it is possible to make invertible changes of coordinates in $\mathbb{R}^{\kappa}_x, \mathbb{R}^{\kappa}_y$ so that A, B become the identity operator; $\pi_3(x, y) = x + y$. Next, $\pi_1(x, y) = Dx$ for some invertible $D : \mathbb{R}^{\kappa} \to \mathbb{R}^{\kappa}$. By making a change of variables in the range of D, we may achieve $\pi_1(x, y) \equiv x$. Finally, a corresponding change of coordinates in the range of π_2 makes $\pi_2(x, y) \equiv y$.

In order to keep the notation simple, we will discuss in detail the case $\kappa = 2$, then will indicate in § 2.7 how the analysis extends without additional difficulty to arbitrary dimensions.

2.3 Second reduction

We will restrict our attention to polynomial phases of the form λP, where $\lambda \in (0, \infty)$ and $\|P\|_{nd} = 1$; for if $\|P\|_{nd} = 0$, then the conclusion of Theorem 2.3 is trivial. In particular, P will henceforth be assumed nondegenerate with respect to the projections $\{\pi_j\}_{j=1}^3$.

In the following lemma, $P_{(x_2,y_2)}(x_1, y_1) := P(x_1, y_1, x_2, y_2)$, and $\| \cdot \|_{nd}$ denotes a norm on the space of polynomials of degree $\leq d$ in x_1 and y_1 modulo degenerate polynomials with respect to the projections $(x_1, y_1) \mapsto x_1, y_1, x_1 + y_1$. We will sometimes write $I(P)$ as shorthand for $I(P; f_1, f_2, f_3)$.

Let $K \subset \mathbb{R}^2_{x_2,y_2}$ be the projection of the support of η onto the (x_2, y_2) plane.

Lemma 2.4. *Let $P : \mathbb{R}^4 \to \mathbb{R}$ be a real-valued polynomial of degree $\leq d$. If the polynomial $(x_1, y_1) \mapsto P(x, y)$ is nondegenerate with respect to the one-dimensional projections $(x_1, y_1) \mapsto x_1, y_1, x_1 + y_1$ for some $(x_2, y_2) \in \mathbb{R}^2$, then there exists a constant $C < \infty$ such that:*

$$|I(\lambda P; f_1, f_2, f_3)| \leq C(\lambda \cdot \sup_{(x_2,y_2)\in K} \|P_{(x_2,y_2)}\|_{nd})^{-\sigma} \prod_{j=1}^{3} \|f_j\|_2,$$

for all functions $f_j \in L^2(\mathbb{R}^2)$, where $\sigma > 0$ is a constant which depends only on d.

Proof. We can apply Theorem 2.2 with $m = 2$ and $n = 3$ to conclude that

$$J_{x_2,y_2}(\lambda P) := \iint_{\mathbb{R}^2} e^{i\lambda P(x_1,y_1,x_2,y_2)} f_1(x_1, x_2) f_2(y_1, y_2) f_3(x_1+y_1, x_2+y_2) \eta(x_1, y_1, x_2, y_2) dx_1 dy_1$$

satisfies

$$\begin{aligned} |J_{x_2,y_2}(\lambda P)| \leq & C(1 + \lambda^2 |Q(x_2, y_2)|)^{-\rho} \|f_1(\cdot, x_2)\|_2 \|f_2(\cdot, y_2)\|_2 \|f_3(\cdot, x_2 + y_2)\|_2 \\ = & C(1 + \lambda^2 |Q(x_2, y_2)|)^{-\rho} g_1(x_2) g_2(y_2) g_3(x_2 + y_2), \end{aligned}$$

for some $\rho > 0$ depending only on d, where $g_j(t) = \|f_j(\cdot, t)\|_2$ and $Q(x_2, y_2) = \|P_{(x_2,y_2)}(\cdot)\|^2_{nd}$ is a polynomial of degree $\leq 2d$.

For $\epsilon > 0$, let

$$E_\epsilon := \{(x, y) \in K : |Q(x, y)| < \epsilon\}.$$

A basic sublevel set estimate [11] yields

$$|E_\epsilon| \leq C\|Q\|_{L^\infty(K)}^{-\delta'}\epsilon^{\delta'} \quad \text{for } \delta' = \frac{1}{\deg(Q)}$$

and some absolute constant $C < \infty$, if Q has positive degree. We now split the original integral $I(\lambda P; f_1, f_2, f_3)$ into two pieces and estimate each of them separately. On the one hand, Hölder's and Young's inequalities imply:

$$\begin{aligned}\iint_{E_\epsilon} |J_{x_2,y_2}(\lambda P)| dx_2 dy_2 \leq & C \iint_{E_\epsilon} g_1(x_2) g_2(y_2) g_3(x_2+y_2) dx_2 dy_2 \\ \leq & C|E_\epsilon|^{1/4}\Big(\iint_{\mathbb{R}^2} g_1^{4/3}(x_2) g_2^{4/3}(y_2) g_3^{4/3}(x_2+y_2) dx_2 dy_2\Big)^{3/4} \\ \leq & C|E_\epsilon|^{1/4}\prod_{j=1}^{3}\|g_j^{4/3}\|_{3/2}^{3/4} \\ = & C|E_\epsilon|^{1/4}\prod_{j=1}^{3}\|g_j\|_2 \\ \leq & C\|Q\|_{L^\infty(K)}^{-\delta}\epsilon^{\delta}\prod_{j=1}^{3}\|f_j\|_2.\end{aligned}$$

On the other hand,

$$\begin{aligned}\iint_{\mathbb{R}^2\setminus E_\epsilon} |J_{x_2,y_2}(\lambda P)| dx_2 dy_2 \leq & C(1+\lambda^2\epsilon)^{-\rho}\iint_{\mathbb{R}^2} g_1(x_2) g_2(y_2) g_3(x_2+y_2) dx_2 dy_2 \\ \leq & C(1+\lambda^2\epsilon)^{-\rho}\prod_{j=1}^{3}\|g_j\|_{3/2} \\ \leq & C(1+\lambda^2\epsilon)^{-\rho}\prod_{j=1}^{3}\|f_j\|_2.\end{aligned}$$

If Q has degree zero then the same conclusion is reached more simply, with $\epsilon^{\delta'}$ replaced by 1. Thus

$$|I(\lambda P; f_1, f_2, f_3)| \leq C\Big[\|Q\|_{L^\infty(K)}^{-\delta}\epsilon^{\delta} + (\lambda^2\epsilon)^{-\rho}\Big]\prod_{j=1}^{3}\|f_j\|_2.$$

Since $\|Q\|_{L^\infty(K)} = \sup_{(x_2,y_2)\in K} \|P_{(x_2,y_2)}\|_{nd}^2$, optimizing in ϵ yields an upper bound

$$|I(\lambda P; f_1, f_2, f_3)| \le C\Big(\lambda \sup_{(x_2,y_2)\in K} \|P_{(x_2,y_2)}\|_{nd}\Big)^{-\frac{2\rho\delta}{\rho+\delta}} \prod_{j=1}^{3} \|f_j\|_2.$$

□

2.4 Third reduction

The goal of this step is to show that it is enough to consider functions of the form $f_j(u_1,u_2) = e^{i\phi_j(u_1,u_2)}$, where ϕ_j has polynomial dependence on u_1 of bounded degree and is real-valued.

From the last section, we get the desired decay rate unless P is "almost degenerate" with respect to the projections $(x_1,y_1) \mapsto x_1, y_1, x_1+y_1$ for almost every $(x_2,y_2) \in \mathbb{R}^2$, in the sense that

$$\sup_{(x_2,y_2)\in K} \|P_{(x_2,y_2)}\|_{nd} \lesssim \lambda^{-1+\tau} \text{ for some } \tau > 0.$$

We have the freedom to choose τ arbitrarily small later on in the argument.

In this case, one can decompose

$$P(x,y) = q_1(x_1,x_2,y_2) + q_2(y_1,x_2,y_2) + q_3(x_1+y_1,x_2,y_2) + R(x,y),$$

for some measurable functions q_j and R which are polynomials of degree $\le d$ in x_1 and y_1, and where the remainder R satisfies

$$|R(x,y)| \lesssim \lambda^{-1+\tau} \text{ if } (x,y) \in K'$$

for any fixed compact set $K' \subset \mathbb{R}^4$. To justify this, for each integer $k \ge 0$ choose some Hilbert space norm for the vector space of all homogeneous polynomials in x_1 and y_1 of degree k. Write $P_{(x_2,y_2)}(x_1,y_1) = P(x,y)$. Express $P(x,y) = \sum_{k=0}^{d} P_{k,(x_2,y_2)}(x_1,y_1)$ where $P_{k,(x_2,y_2)}$ is a homogeneous polynomial of degree k in (x_1,y_1), whose coefficients are polynomials in (x_2,y_2). Now we use two facts implicitly shown in [4]. Firstly, if $p = \sum_k p_k$ is a decomposition of a polynomial in (x_1,y_1) into its homogeneous summands of degree k, then $\sum_k \|p_k\|_{nd}$ is comparable to $\|p\|_{nd}$. Secondly, if $p(x_1,y_1)$ is homogeneous of degree k, then for any $d \ge k$, the norm of p in the space of all homogeneous polynomials of degree k modulo polynomials $cx_1^k + c'y_1^k + c''(x_1+y_1)^k$ is comparable to the norm $\|p\|_{nd}$ of p in the space of all polyomials of degrees $\le d$ modulo all degenerate polynomials of degrees $\le d$, where degenerate polynomials are those which are sums of polynomials in x_1, polynomials in y_1, and polynomials in x_1+y_1. Qualitative versions of these two facts were established in [4]; the quantitative versions stated here follow from the equivalence of all norms in any finite-dimensional vector space.

For each k, the projection $Q_{k,(x_2,y_2)}$ of $P_{k,(x_2,y_2)}$ onto the span of $x_1^k, y_1^k, (x_1+y_1)^k$ has polynomial dependence on (x_2, y_2). Moreover, all coefficients of $P_{k,(x_2,y_2)} - Q_{k,(x_2,y_2)}$ are $O(\lambda^{-1+\tau})$ for $(x_2, y_2) \in K$, and therefore for (x_2, y_2) in any fixed bounded set. For $k \geq 2$, $Q_{k,(x_2,y_2)}$ decomposes uniquely as $q_{1,k}(x_2, y_2)x_1^k + q_{2,k}(x_2, y_2)y_1^k + q_{3,k}(x_2, y_2)(x_1+y_1)^k$; these coefficients $q_{i,k}$ continue to have polynomial dependence on (x_2, y_2). For $k = 1$ there is likewise a unique such decomposition, with the additional condition $q_{3,k} \equiv 0$, and for $k = 0$, with two additional conditions $q_{2,k} \equiv q_{3,k} \equiv 0$. Recombining terms gives the claim.

Let us use this to work with J_{x_2,y_2} (a similar calculation occurs in [4, p. 15]):

$$\begin{aligned}
J_{x_2,y_2}(\lambda P) &= \iint_{\mathbb{R}^2} e^{i\lambda P(x,y)} f_1(x) f_2(y) f_3(x+y)\eta(x,y)\,dx_1 dy_1 \\
&= \iint_{\mathbb{R}^2} e^{i\lambda q_1(x_1,x_2,y_2)} f_1(x_1,x_2) e^{i\lambda q_2(y_1,x_2,y_2)} f_2(y_1,y_2) \cdot \\
&\qquad \cdot e^{i\lambda q_3(x_1+y_1,x_2,y_2)} f_3(x_1+y_1, x_2+y_2) e^{i\lambda R}\eta\,dx_1 dy_1 \\
&= \iint_{\mathbb{R}^2} g_1(x_1) g_2(y_1) g_3(x_1+y_1)\zeta(x_1,y_1)\,dx_1 dy_1 \\
&= C \iint \Big(\int \widehat{g}_1(\xi_1) e^{ix_1\xi_1} d\xi_1 \Big) g_2(y_1) g_3(x_1+y_1) \cdot \\
&\qquad \cdot \Big(\iint \widehat{\zeta}(\xi_2,\xi_3) e^{i(x_1,y_1)\cdot(\xi_2,\xi_3)} d\xi_2 d\xi_3 \Big) dx_1 dy_1 \\
&= C \iiint \widehat{g}_1(\xi_1)\widehat{\zeta}(\xi_2,\xi_3) \int g_2(y_1) e^{iy_1\xi_3} \cdot \\
&\qquad \cdot \Big(\int g_3(x_1+y_1) e^{ix_1(\xi_1+\xi_2)} dx_1 \Big) dy_1 d\xi_1 d\xi_2 d\xi_3 \\
&= C \iiint \widehat{g}_1(\xi_1)\widehat{g}_2(\xi_1+\xi_2-\xi_3)\widehat{g}_3(-\xi_1-\xi_2)\widehat{\zeta}(\xi_2,\xi_3)\,d\xi_1 d\xi_2 d\xi_3.
\end{aligned}$$

Implicit in this notation is the dependence of the functions $g_j := e^{i\lambda q_j} f_j$ and $\zeta := e^{i\lambda R}\eta$ on x_2 and y_2.

Since λR is a polynomial in x_1 and y_1 of bounded degree which is $O(\lambda^\tau)$ on $\operatorname{supp}(\eta)$ and the same holds for all its derivatives, we have that

$$|\widehat{\zeta}(\xi)| \leq C_{N,\eta}\lambda^{N\tau}(1+|\xi|)^{-N}, \ \forall \xi \in \mathbb{R}^2, \ \forall N \in \mathbb{N},$$

provided $\eta \in C^N$. In particular, if[1] $\eta \in C_0^3(\mathbb{R}^4)$, then

$$|\widehat{\zeta}(\xi)| \leq C\lambda^{3\tau}(1+|\xi|)^{-3}, \forall \xi \in \mathbb{R}^2.$$

We use this (together with Cauchy-Schwarz and Plancherel) to conclude that

$$\begin{aligned} |J_{x_2,y_2}(\lambda P)| \leq & C\lambda^{3\tau}\|\widehat{g}_1\|_\infty \iiint \frac{|\widehat{g}_2(\xi_1+\xi_2-\xi_3)||\widehat{g}_3(-\xi_1-\xi_2)|}{(1+|(\xi_2,\xi_3)|)^3} d\xi_1 d\xi_2 d\xi_3 \\ \leq & C\lambda^{3\tau}\|\widehat{g}_1\|_\infty \|g_2\|_2 \|g_3\|_2. \end{aligned}$$

Let $\delta > 3\tau$ and consider the set:

$$F := \{(x_2,y_2) \in \mathbb{R}^2 : \|\widehat{g}_1\|_\infty \lesssim \lambda^{-\delta}\}.$$

There are two possibilities:

(i) If $|F^{\complement}| \lesssim \lambda^{-\delta}$, then $|I(\lambda P)| \lesssim \lambda^{-(\delta-3\tau)}$, as is easily seen by splitting the integral

$$I(\lambda P) = \iint_{\mathbb{R}^2} J_{x_2,y_2}(\lambda P)dx_2dy_2$$

into the regions F and $F^{\complement}$.

(ii) If $|F^{\complement}| \gtrsim \lambda^{-\delta}$, we set $E := F^{\complement}$. Note that $\|\widehat{g}_1\|_\infty \gtrsim \lambda^{-\delta}$ for every $(x_2,y_2) \in E$.

Since condition (i) yields the desired decay, we restrict attention henceforth to the case in which condition (ii) holds. Then there exists a measurable subset $E \subset \mathbb{R}^2$ such that $|E| \gtrsim \lambda^{-\delta}$ and $\|\widehat{g}_1\|_\infty \gtrsim \lambda^{-\delta}$ for every $(x_2,y_2) \in E$. We still have the freedom to choose $\delta > 0$ as small as we wish later on in the argument.

Why is this conclusion of interest? Since

$$\lambda^{-\delta} \lesssim \|\widehat{g}_1\|_\infty = \sup_\xi \left| \int_{\mathbb{R}} e^{i\lambda q_1(x_1,x_2,y_2)} f_1(x_1,x_2) e^{-ix_1\xi} dx_1 \right|,$$

we can find measurable functions θ and $\widetilde{\theta}$ such that, for $(x_2,y_2) \in E$,

$$\begin{aligned} \lambda^{-\delta} \lesssim & \left| \int e^{i\lambda q_1(x_1,x_2,y_2)} f_1(x_1,x_2) e^{-ix_1\theta(x_2,y_2)} dx_1 \right| \\ = & e^{-i\widetilde{\theta}(x_2,y_2)} \int f_1(x_1,x_2) e^{i\lambda q_1(x_1,x_2,y_2) - ix_1\theta(x_2,y_2)} dx_1. \end{aligned}$$

[1]As remarked in the previous chapter, we lose no generality in assuming this extra smoothness on η.

Because we are working in a fixed bounded region, there exist a measurable subset $E_1 \subset \mathbb{R}$ such that $|E_1| \gtrsim \lambda^{-\delta}$ and a single number $\overline{y_2} \in \mathbb{R}$ such that for each $x_2 \in E_1$, $(x_2, \overline{y_2}) \in E$. Thus

$$\begin{aligned}\lambda^{-\delta} &\lesssim e^{-i\widetilde{\theta}(x_2,y_2)} \int f_1(x_1,x_2) e^{i\lambda q_1(x_1,x_2,y_2) - i x_1 \theta(x_2,y_2)} dx_1 \\ &= \int f_1(x_1,x_2) e^{i\varphi_1(x_1,x_2)} dx_1 \quad \text{if } x_2 \in E_1\end{aligned}$$

where

$$\varphi_1(x_1,x_2) = \lambda q_1(x_1,x_2,\bar{y}_2) - x_1\theta(x_2,\bar{y}_2) - \tilde{\theta}(x_2,\bar{y}_2)$$

is a real-valued polynomial in x_1 of degree $\leq d$, whose coefficients are measurable functions of x_2.

We would like to use this to conclude that f_1 has reasonably large inner product with $e^{-i\varphi_1}$. While this is not necessarily true, the following *extension argument* will prove sufficient for our purposes: for every $x_2 \in \mathbb{R}$, choose $\theta^*(x_2)$ in a measurable way and such that

$$e^{i\theta^*(x_2)} \int_{\mathbb{R}} f_1(x_1,x_2) e^{i\varphi_1(x_1,x_2)} dx_1 \geq 0.$$

We can guarantee that $\theta^* \equiv 0$ on E_1.

Define $\phi_1(x_1,x_2) := \theta^*(x_2) + \varphi_1(x_1,x_2)$. Then ϕ_1 is likewise a real-valued polynomial in x_1 of degree $\leq d$, whose coefficients are measurable functions of x_2. Now

$$\int_{\mathbb{R}} f_1(x_1,x_2) e^{i\phi_1(x_1,x_2)} dx_1 \geq 0 \quad \text{for every } x_2 \in \mathbb{R}$$

while for any $x_2 \in E_1$,

$$\int_{\mathbb{R}} f_1(x_1,x_2) e^{i\phi_1(x_1,x_2)} dx_1 \gtrsim \lambda^{-\delta}.$$

Therefore since $|E_1| \gtrsim \lambda^{-\delta}$,

$$|\langle f_1, e^{-i\phi_1}\rangle| \gtrsim \lambda^{-2\delta}.$$

Since $\|f_1\|_{L^2} = 1$ and f_1 is supported in a fixed bounded set,

$$\|f_1 - \langle f_1, e^{-i\phi_1}\rangle e^{-i\phi_1}\|_2^2 \leq (1 - C\lambda^{-4\delta}). \tag{2.2}$$

Let $\mathbf{A}(\lambda)$ be the best constant in the inequality

$$|I(\lambda P; f_1, f_2, f_3)| \leq \mathbf{A}(\lambda) \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2}.$$

That $\mathbf{A}(\lambda)$ is finite is an immediate consequence of the dual form of Young's convolution inequality and the fact that, in this context, $L^2 \subset L^{3/2}$. Now (2.2) implies

$$\begin{aligned}|I(\lambda P; f_1, f_2, f_3)| =&|I(\lambda P; f_1 - \langle f_1, e^{-i\phi_1}\rangle e^{-i\phi_1}, f_2, f_3) + I(\lambda P; \langle f_1, e^{-i\phi_1}\rangle e^{-i\phi_1}, f_2, f_3)|\\ \leq&\mathbf{A}(\lambda)\|f_1 - \langle f_1, e^{-i\phi_1}\rangle e^{-i\phi_1}\|_2 \|f_2\|_2 \|f_3\|_2 + |\langle f_1, e^{-i\phi_1}\rangle||I(\lambda P; e^{-i\phi_1}, f_2, f_3)|\\ \leq&\mathbf{A}(\lambda)(1 - C\lambda^{-4\delta})^{1/2} + C|I(\lambda P; e^{-i\phi_1}, f_2, f_3)|,\end{aligned}$$

Therefore

$$\mathbf{A}(\lambda) \leq \mathbf{A}(\lambda)(1 - C\lambda^{-4\delta})^{1/2} + C \sup_{\phi_1, f_2, f_3} |I(\lambda P; e^{-i\phi_1}, f_2, f_3)|$$

where the supremum is taken over all functions f_2, f_3 supported in the specified regions satisfying $\|f_j\|_{L^2} = 1$, and over all real-valued functions $\phi_1(x_1, x_2)$ which are polynomials of degree $\leq d$ with respect to x_1, with coefficients depending measurably on x_2. Since $\mathbf{A}(\lambda) < \infty$, it follows that

$$\mathbf{A}(\lambda) \leq C\lambda^{4\delta} \sup_{\phi_1, f_2, f_3} |I(\lambda P; e^{-i\phi_1}, f_2, f_3)|. \tag{2.3}$$

Therefore it suffices to prove that

$$|I(\lambda P; e^{-i\phi_1}, f_2, f_3)| \leq C\lambda^{-\varepsilon}\|f_2\|_{L^2}\|f_3\|_{L^2}$$

for a certain $\varepsilon > 0$; for δ may then be chosen to equal $\varepsilon/5$.

By repeating the above steps for g_2 and g_3, we conclude that it suffices to prove that there exists $\varepsilon > 0$ such that

$$|I(\lambda P; e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})| \leq C\lambda^{-\varepsilon} \tag{2.4}$$

uniformly for all $\lambda \geq 1$, all polynomials P satisfying $\|P\|_{nd} = 1$, and all real-valued measurable functions $\phi_j(u_1, u_2)$ which are polynomials of degree $\leq d$ with respect to u_1.

2.5 Handling remainders

In the last section we have reduced matters to the case where the f_j are of the special form

$$\begin{cases} f_1(x_1, x_2) = e^{i\phi_1(x_1, x_2)} \\ f_2(y_1, y_2) = e^{i\phi_2(y_1, y_2)} \\ f_3(x_1 + y_1, x_2 + y_2) = e^{i\phi_3(x_1+y_1, x_2+y_2)}, \end{cases}$$

where ϕ_j are partial polynomials in the sense described following (2.4). Express

$$\begin{cases} \phi_1(x_1,x_2) = \sum_{j=0}^{d} \theta_{1,j}(x_2)x_1^j \\ \phi_2(y_1,y_2) = \sum_{k=0}^{d} \theta_{2,k}(y_2)y_1^k \\ \phi_3(x_1+y_1,x_2+y_2) = \sum_{l=0}^{d} \theta_{3,l}(x_2+y_2)(x_1+y_1)^l \end{cases}$$

where $\theta_{1,j}$, $\theta_{2,k}$, $\theta_{3,l}$ are measurable and real-valued. Also express

$$P(x,y) = \sum_{j,k} p_{jk}(x_2,y_2)x_1^j y_1^k.$$

Then

$$\begin{aligned} I(\lambda P; f_1, f_2, f_3) &= \iint e^{i\lambda P} e^{i\phi_1} e^{i\phi_2} e^{i\phi_3} \eta dxdy \\ &= \iint \Big(\iint e^{i\sum_{j,k} \psi_{jk}(x_2,y_2)x_1^j y_1^k} \eta dx_1 dy_1 \Big) dx_2 dy_2, \end{aligned}$$

where $\psi_{jk}(x_2,y_2) =$

$$\begin{Bmatrix} \theta_{1,j}(x_2) & \text{if } k=0 \\ 0 & \text{if } k \neq 0 \end{Bmatrix} + \begin{Bmatrix} \theta_{2,k}(y_2) & \text{if } j=0 \\ 0 & \text{if } j \neq 0 \end{Bmatrix} + \binom{j+k}{k} \theta_{3,j+k}(x_2+y_2) + \lambda p_{jk}(x_2,y_2).$$

The desired bound $|I(\lambda P; e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})| \leq C\lambda^{-\varepsilon}$ follows directly from Theorem 2.1, unless there exists a measurable subset $E \subset \mathbb{R}^2$ of measure $|E| \gtrsim \lambda^{-\delta}$ such that

$$\sum_{(j,k)\neq(0,0)} |\psi_{jk}(x_2,y_2)| \lesssim \lambda^r, \quad \forall (x_2,y_2) \in E. \tag{2.5}$$

We may choose $\delta, r > 0$ to be as small as may be desired for later purposes, at the expense of taking ε sufficiently small in (2.4).

The proof of the following lemma will be given later.

Lemma 2.5. *Let $P : \mathbb{R}^2 \to \mathbb{R}^D$ be a real vector-valued polynomial of degree d, and let $f, g : [0,1] \to \mathbb{R}^D$ be measurable functions. Let $E \subseteq [0,1]^2$ be a measurable subset of the unit square of Lebesgue measure $|E| = \epsilon > 0$. Assume that*

$$|f(x) + g(y) + P(x,y)| \leq 1 \text{ for all } (x,y) \in E. \tag{2.6}$$

Then there exist $\mathbb{R}^D$–valued polynomials Q_1 and Q_2 of degrees $\leq d$ and measurable sets $E_1, E_2 \subseteq [0,1]$ such that

$$\begin{cases} |f(x) - Q_1(x)| \lesssim \epsilon^{-C} \text{ for } x \in E_1 \\ |g(y) - Q_2(y)| \lesssim \epsilon^{-C} \text{ for } y \in E_2 \\ |E_1| \geq c\epsilon, \quad |E_2| \geq c\epsilon. \end{cases}$$

The constants $c, C \in \mathbb{R}^+$ depend only on d.

The phase estimates (2.5), together with Lemma 2.5, allow us to control most of the terms θ_i. Letting $k = 0$, we have that

$$|\psi_{j0}(x_2, y_2)| = |\theta_{1,j}(x_2) + \theta_{3,j}(x_2 + y_2) + \lambda p_{j0}(x_2, y_2)| \lesssim \lambda^r$$

if $1 \leq j \leq d$ and $(x_2, y_2) \in E$. Since $|E| \gtrsim \lambda^{-\delta}$, Lemma 2.5 implies that, for every $1 \leq j \leq d$, there exists a real-valued polynomial $\widetilde{Q}_{1,j}$ of degree $\leq d$ such that

$$|\lambda^{-r}\theta_{1,j}(x_2) - \widetilde{Q}_{1,j}(x_2)| \lesssim (\lambda^{-\delta})^{-C}$$

whenever $x_2 \in E_1$; here, $E_1 \subset \mathbb{R}$ is a measurable subset which does not depend on j and such that $|E_1| \gtrsim \lambda^{-\delta}$. A similar conclusion can be drawn for each of the terms $\theta_{3,l}$ with $1 \leq l \leq d$. Choosing $j = 0$ we control the terms $\theta_{2,k}$ for $1 \leq k \leq d$ in an analogous way.

We conclude that, for every $1 \leq j, k, l \leq d$,

$$\begin{cases} \theta_{1,j}(x_2) & = Q_{1,j}(x_2) + \widetilde{R_{1,j}}(x_2) \\ \theta_{2,k}(y_2) & = Q_{2,k}(y_2) + \widetilde{R_{2,k}}(y_2) \\ \theta_{3,l}(x_2 + y_2) & = Q_{3,l}(x_2 + y_2) + \widetilde{R_{3,l}}(x_2 + y_2) \end{cases}$$

where $Q_{1,j}$, $Q_{2,k}$ and $Q_{3,l}$ are polynomials of degree $\leq d$, and the remainders satisfy

$$\begin{cases} |\widetilde{R_{1,j}}(x_2)| \lesssim \lambda^{\beta} & \text{if } x_2 \in E_1, \\ |\widetilde{R_{2,k}}(y_2)| \lesssim \lambda^{\beta} & \text{if } y_2 \in E_2, \\ |\widetilde{R_{3,l}}(x_2 + y_2)| \lesssim \lambda^{\beta} & \text{if } x_2 + y_2 \in E_3, \end{cases}$$

for certain measurable subsets $E_1, E_2, E_3 \subset \mathbb{R}$ which satisfy

$$|E_i| \gtrsim \lambda^{-\delta}.$$

The parameter $\beta := r + C\delta$ is a function of r, δ and the constant $C = C(d)$ from Lemma 2.5.

We have estimates for the remainders R_i in rather small sets only, but it is possible to reduce to the case in which these estimates hold globally, via an extension argument similar to the one used in the previous section. Set $Q_1(x) = \sum_{j=1}^d Q_{1,j}(x_2)x_1^j$ and $\widetilde{R_1}(x) = \sum_{j=1}^d \widetilde{R_{1,j}}(x_2)x_1^j$. By modifying each $\widetilde{R_{1,j}}$ suitably at each point of the complement of E_1, we produce a function $\Phi_1 = \theta_{1,0} + Q_1 + R_1$ such that $\theta_{1,0}$ is a measurable and real-valued function of x_2, $Q_1(x)$ is a polynomial function of $x \in \mathbb{R}^2$ of degree $\leq d$, $R_1(x_1, x_2)$ is a polynomial in x_1 of degree $\leq d$ whose coefficients are measurable functions of x_2, $|R_1(x)| \lesssim \lambda^\beta$ for every $x \in \mathbb{R}^2$, all functions are real-valued, and

$$\langle e^{i\phi_1}, e^{i\Phi_1}\rangle \gtrsim \lambda^{-\delta}. \tag{2.7}$$

By the same argument used to reduce from general f_j to $e^{i\phi_j}$ in (2.3),

$$\mathbf{A}(\lambda) \lesssim \lambda^{C\delta} \sup_{\Phi_1,\phi_2,\phi_3} |I(\lambda P; e^{i\Phi_1}, e^{i\phi_2}, e^{i\phi_3})| \tag{2.8}$$

where the supremum is taken over all Φ_1, ϕ_2, ϕ_3 of the above form. This argument can be repeated twice more to give

$$\mathbf{A}(\lambda) \lesssim \lambda^{C\delta} \sup_{\Phi_1,\Phi_2,\Phi_3} |I(\lambda P; e^{i\Phi_1}, e^{i\Phi_2}, e^{i\Phi_3})|, \tag{2.9}$$

where each of the functions Φ_i shares the properties indicated above for Φ_1.

Now,

$$I(\lambda P; e^{i\Phi_1}, e^{i\Phi_2}, e^{i\Phi_3}) = \iint e^{i\theta_{1,0}(x_2)} e^{i\theta_{2,0}(y_2)} e^{i\theta_{3,0}(x_2+y_2)}\cdot$$
$$\cdot \Big(\iint e^{i\lambda P(x,y)} e^{iQ_1(x)} e^{iQ_2(y)} e^{iQ_3(x+y)} e^{i(R_1(x)+R_2(y)+R_3(x+y))} \eta\, dx_1 dy_1\Big) dx_2 dy_2.$$

Let $\widetilde{P} := P + \lambda^{-1}Q_1 + \lambda^{-1}Q_2 + \lambda^{-1}Q_3$. Since $[P] = [\widetilde{P}]$, $\|P\|_{nd} = \|\widetilde{P}\|_{nd}$. We are left with:

$$I(\lambda P) = \iint e^{i\theta_{1,0}(x_2)} e^{i\theta_{2,0}(y_2)} e^{i\theta_{3,0}(x_2+y_2)} \Big(\iint e^{i\lambda\widetilde{P}(x,y)} e^{i(R_1(x)+R_2(y)+R_3(x+y))} \eta\, dx_1 dy_1\Big) dx_2 dy_2$$

where all functions in the exponents are real-valued, $\widetilde{P}$ is a polynomial of degree $\le d$ such that $\|\widetilde{P}\|_{nd} = \|P\|_{nd} = 1$, the $\theta_{j,0}$ and the R_j are measurable functions, and the remainders $R_j(u_1, u_2)$ are polynomial functions of u_1 of degrees $\le d$ which satisfy $|R_j(u)| \lesssim \lambda^\beta$ for all $u \in \mathbb{R}^2$.

2.6 The end of the proof

Decompose

$$\widetilde{P} = P_0 + P^* \qquad \text{where} \qquad P_0(x_2, y_2) = \widetilde{P}(0, 0, x_2, y_2) \tag{2.10}$$

and $P^* = \widetilde{P} - P_0$.

Since

$$\psi_{00}(x_2,y_2) = \lambda P_0(x_2,y_2) + \theta_{1,0}(x_2) + \theta_{2,0}(y_2) + \theta_{3,0}(x_2+y_2),$$

we can write

$$I(\lambda P) = \iint e^{i\psi_{00}(x_2,y_2)}\Big(\iint e^{i\lambda P^*(x,y)} e^{i(R_1(x)+R_2(y)+R_3(x+y))}\eta dx_1 dy_1\Big) dx_2 dy_2. \quad (2.11)$$

Our main assumption, namely that P is nondegenerate with respect to the projections $\{\pi_j\}_{j=1}^3$, has not yet come into play. To apply it, we need a lemma:

Lemma 2.6. *For any $d \in \mathbb{N}$ there exists $c > 0$ with the following property. Let $\tilde{P} : \mathbb{R}^4 \to \mathbb{R}$ be any real-valued polynomial of degree $\leq d$. Decompose $\tilde{P} = P_0 + P^*$ as in* (2.10)*. Then*

$$\Big\| \|P^*_{(x_2,y_2)}\|_{nc} \Big\|_{\mathcal{P}(d)} + \|P_0\|_{nd} \geq c\|\tilde{P}\|_{nd}.$$

The expression $\|P_0\|_{nd}$ in this lemma has two natural interpretations, but these define the same quantity since

$$\min_{\deg(p_i)\leq d} \|P_0(x_2,y_2) + p_1(x_1,x_2) + p_2(y_1,y_2) + p_3(x_1+y_1,x_2+y_2)\|_{\mathcal{P}(d)}$$
$$= \min_{\deg(q_i)\leq d} \|P_0(x_2,y_2) + q_1(x_2) + q_2(y_2) + q_3(x_2+y_2)\|_{\mathcal{P}(d)}. \quad (2.12)$$

The two inequalities implicit in this equality are obtained by setting $q_j(t) = p_j(0,t)$, and by setting $p_j(t_1,t_2) = q_j(t_2)$, respectively.

Proof of Lemma 2.6. The left-hand side defines a seminorm on the finite-dimensional vector space of polynomials of degrees $\leq d$ modulo degenerate polynomials, so it suffices to show that if $\Big\| \|P^*_{(x_2,y_2)}\|_{nc} \Big\|_{\mathcal{P}(d)}$ vanishes then P^* is degenerate, and correspondingly for P_0. For P_0 this is a tautology, in view of (2.12). On the other hand, $P^*(x,y) = \sum_{(j,k)\neq(0,0)} q_{j,k}(x_2,y_2)x_1^j y_1^k$ where $q_{j,k}$ are uniquely determined polynomials, and

$$\Big\| \|P^*_{(x_2,y_2)}\|_{nc} \Big\|_{\mathcal{P}(d)} = 0 \iff q_{j,k} \equiv 0 \text{ for each } j,k.$$

Thus $P^* \equiv 0$, so in particular, P^* is degenerate. □

Therefore there exists a constant $c_d > 0$ such that (i) $\Big\| \|P^*_{(x_2,y_2)}\|_{nc} \Big\|_{\mathcal{P}(d)} \geq c_d$ or (ii) $\|P_0\|_{nd} \geq c_d$.

Case (i): $\Big\| \|P^*_{(x_2,y_2)}\|_{nc} \Big\|_{\mathcal{P}(d)} \geq c_d$.

We have that

$$I(\lambda P) = \iint e^{i(\lambda P_0(x_2,y_2)+\theta_{1,0}(x_2)+\theta_{2,0}(y_2)+\theta_{3,0}(x_2+y_2))} \cdot \Big(\iint e^{i\lambda P^*(x,y)} e^{i(R_1(x)+R_2(y)+R_3(x+y))} \eta dx_1 dy_1 \Big) dx_2 dy_2,$$

where by Theorem 2.1, the absolute value of the inner integral is

$$\lesssim (\lambda \|P^*_{(x_2,y_2)} + \lambda^{-1} R_{1,(x_2)} + \lambda^{-1} R_{2,(y_2)} + \lambda^{-1} R_{3,(x_2+y_2)}\|_{nc})^{-1/d}.$$

Since $|R_j| \leq \lambda^{\beta}$ and $\beta < 1$, we have that, if λ is large enough, then

$$\|P^*_{(x_2,y_2)} + \lambda^{-1} R_{1,(x_2)} + \lambda^{-1} R_{2,(y_2)} + \lambda^{-1} R_{3,(x_2+y_2)}\|_{nc} \gtrsim \|P^*_{(x_2,y_2)}\|_{nc}$$

for every $(x_2, y_2) \in \mathbb{R}^2$. We conclude that the absolute value of the inner integral is

$$\lesssim (\lambda \|P^*_{(x_2,y_2)}\|_{nc})^{-1/d}.$$

If $(x_2, y_2) \in \mathbb{R}^2$ is such that

$$\|P^*_{(x_2,y_2)}\|_{nc} \gtrsim \lambda^{-1+\tau} \text{ for some } \tau > 0,$$

we get the desired decay. Otherwise, observe that (i) implies a sublevel set estimate of the form

$$|\{(x_2, y_2) \in \mathbb{R}^2 : \|P^*_{(x_2,y_2)}\|_{nc} < \lambda^{-1+\tau}\}| \lesssim (\lambda^{-1+\tau})^{\delta}.$$

Therefore the contribution of the set of such points (x_2, y_2) to the integral is small, and this concludes the analysis of Case 1.

Case (ii): $\|P_0\|_{nd} \geq c_d$.

By Fubini,

$$I(\lambda P) = \iint \Big(\iint e^{i\lambda[P_0(x_2,y_2)+P^*(x,y)]} \cdot e^{i(\theta_{1,0}+R_{1,(x_1)})(x_2)} e^{i(\theta_{2,0}+R_{2,(y_1)})(y_2)} e^{i(\theta_{3,0}+R_{3,(x_1+y_1)})(x_2+y_2)} \eta dx_2 dy_2 \Big) dx_1 dy_1.$$

By Theorem 2.2, the absolute value of the inner integral in the last expression is $\lesssim (1+\lambda^2\|P_0+P^*_{(x_1,y_1)}\|^2_{nd})^{-\rho}$, for some $\rho=\rho(d)>0$. It follows that

$$|I(\lambda P)| \lesssim \iint (1+\lambda^2\|P_0+P^*_{(x_1,y_1)}\|^2_{nd})^{-\rho} dx_1 dy_1.$$

To handle this integral, note that $P^*(0,0,x_2,y_2)=0$ and hypothesis (ii) together imply that

$$|\{(x_1,y_1)\in\mathbb{R}^2 : \|P_0+P^*_{(x_1,y_1)}\|^2_{nd}<\epsilon\}| \lesssim \epsilon^\delta$$

because $(x_1,y_1)\mapsto \|P_0+P^*_{(x_1,y_1)}\|^2_{nd}$ is a polynomial of degree $\leq 2d$. An argument entirely analogous to the one used to prove Lemma 2.4 concludes the analysis.

2.7 Higher Dimensions and Generalization

So far, we have only discussed the case where the domain of P is $\mathbb{R}^4$, but $\mathbb{R}^{2\kappa}$ for $\kappa>2$ is treated in essentially the same way. Now write $x=(x',x_\kappa)$, $y=(y',y_\kappa)$ where $x',y'\in\mathbb{R}^{\kappa-1}$. The proof proceeds by induction on κ. The only significant change in the proof is that in Case 2 of the final step of the proof, since $\mathbb{R}^{\kappa-1}$ is no longer $\mathbb{R}^1$, Theorem 2.2 does not apply; instead, one simply invokes the induction hypothesis.

Our result may be generalized to include arbitrary smooth phases and not just polynomial ones. The details are a straightforward modification of those in [12] and will therefore not be included.

2.8 Proof of Lemma 2.5

It remains to prove Lemma 2.5. The proof will rely on the following related, but simpler, result. Let X be a normed linear space. We write $|x|$ to denote the norm of a vector in X.

Lemma 2.7. *Let Ω,Ω' be probability spaces with measures μ,μ', and let f,f' be X-valued functions defined on these spaces. Let $0<r<1$ and $R\in(0,\infty)$. Let $E\subset\Omega\times\Omega'$ satisfy $(\mu\times\mu')(E)\geq r$. Suppose that*

$$|f(x)-f'(x')|\leq R \text{ for all } (x,x')\in E.$$

Then there exist $a\in X$ and $G\subset\Omega$, $G'\subset\Omega'$ such that

$$\mu(G)\geq cr$$

$$\mu'(G') \geq cr$$
$$|f(x) - a| \leq CR \text{ for all } x \in G$$
$$|f'(x') - a| \leq CR \text{ for all } x' \in G'.$$

Here c, C are certain absolute constants.

Proof. $|E|$ will denote $(\mu \times \mu')(E)$. Let $\pi_1 : \Omega \times \Omega' \to \Omega$ and $\pi_2 : \Omega \times \Omega' \to \Omega'$ denote the canonical projections. For $x \in \pi_1(E)$ and $x' \in \pi_2(E)$, consider the slices

$$\begin{cases} E^x := \{x' \in \Omega' : (x, x') \in E\} \\ E^{x'} := \{x \in \Omega : (x, x') \in E\}. \end{cases}$$

By Fubini, E^x is μ'-measurable for μ-a.e. x and $E^{x'}$ is μ-measurable for μ'-a.e. x'.

Claim. There exists $(x_0, x_0') \in E$ such that

$$\begin{cases} G := E^{x_0'} \subset \Omega \text{ is } \mu\text{-measurable and } \mu(G) \geq cr \\ G' := E^{x_0} \subset \Omega' \text{ is } \mu'\text{-measurable and } \mu(G') \geq cr. \end{cases}$$

Assuming the claim, we have that:

$$\begin{cases} |f(x) - f'(x_0')| \leq R & \text{for every } x \in G, \\ |f(x_0) - f'(x_0')| \leq R & \\ |f(x_0) - f'(x')| \leq R & \text{for every } x' \in G'. \end{cases}$$

Let $a := \frac{f(x_0)+f'(x_0')}{2}$. Then, for any $x \in G$,

$$|f(x) - a| \leq |f(x) - f'(x_0')| + |f'(x_0') - a| \leq \frac{3}{2}R,$$

and similarly for $x' \in G'$.

To prove the claim, start by assuming that E^x and $E^{x'}$ are measurable for every $(x, x') \in E$. Express E as a disjoint union $E = \mathcal{G} \cup \mathcal{B}$, where

$$\begin{cases} \mathcal{G} = \{(x, x') \in E : \mu'(E^x) \geq \frac{r}{4} \text{ and } \mu(E^{x'}) \geq \frac{r}{4}\} \\ \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 := \{(x, x') \in E : \mu'(E^x) < \frac{r}{4}\} \cup \{(x, x') \in E : \mu(E^{x'}) < \frac{r}{4}\}. \end{cases}$$

We prove the stronger statement $|\mathcal{G}| > 0$. Suppose on the contrary that $|E| = |\mathcal{B}|$. Then

$$r \leq |E| = |\mathcal{B}| = |\mathcal{B}_1 \cup \mathcal{B}_2| \leq |\mathcal{B}_1| + |\mathcal{B}_2|,$$

whence $|\mathcal{B}_1| \geq \frac{r}{2}$ or $|\mathcal{B}_2| \geq \frac{r}{2}$. Without loss of generality assume that the former holds. Then

$$\left|\left\{(x, x') \in E : \mu'(E^x) \geq \frac{r}{4}\right\}\right| \leq |E| - \frac{r}{2}.$$

But then, defining $S_1 := \{x \in \pi_1(E) : \mu'(E) \geq \frac{r}{4}\}$ and $S_2 := \pi_1(E) \setminus S_1$, we have that

$$|E| = \int_{\pi_1(E)} \mu'(E^x) d\mu(x) = \int_{S_1} \mu'(E^x) d\mu(x) + \int_{S_2} \mu'(E^x) d\mu(x)$$
$$\leq \Big(|E| - \frac{r}{2}\Big) + \frac{r}{4},$$

a contradiction. □

Proof of Lemma 2.5. Let A be the norm of P in the quotient space of $\mathbb{R}^D$–valued polynomials of degree $\leq d$ modulo degenerate polynomials with respect to the pair of projections $(x, y) \mapsto x$ and $(x, y) \mapsto y$. It is well known [1] that a decay bound of the form (2.1) holds for these projections, that is, for any d there exists $\rho > 0$ such that for any compact sets $K, K' \subset \mathbb{R}$ there exists $C < \infty$ such that $|\int_{\mathbb{R}^2} e^{iQ(x,y)} f(x)g(y)\, dx, dy| \leq C(1 + \|Q\|_{nd})^{-\rho} \|f\|_2 \|g\|_2$ for all functions f, g supported on K, K' respectively, for all real-valued polynomials Q of degree $\leq d$. This in turn, applied to the individual components of P, implies a sublevel set inequality of the form

$$|E| \leq C_\gamma A^{-\gamma},$$

where $C_\gamma = \frac{C}{1-\gamma}$, C is an absolute constant, and γ depends only on d; see for instance the discussion in [9] for the simple derivation.

So $A \leq C_\gamma \epsilon^{-1/\gamma}$, that is,

$$\inf_{\deg p,q \leq d} \sup_{(x,y) \in [0,1]^2} |P(x, y) - p(x) - q(y)| \leq C_\gamma \epsilon^{-1/\gamma}.$$

The infimum is actually a minimum, so there exist polynomials p and q of degree $\leq d$ such that

$$\sup_{(x,y) \in [0,1]^2} |P(x, y) - p(x) - q(y)| \leq C_\gamma \epsilon^{-1/\gamma}.$$

In particular, for $(x, y) \in E$, we have that

$$|(f(x) + p(x)) + (g(y) + q(y))| \leq |f(x) + g(y) + P(x, y)| + |p(x) + q(y) - P(x, y)|$$
$$\leq 1 + C_\gamma \epsilon^{-1/\gamma} = C'_\gamma \epsilon^{-1/\gamma}.$$

Apply Lemma 2.7 to conclude the existence of $a \in \mathbb{C}$ and $E_1 \subset [0, 1]$ such that $|E_1| \geq \frac{\epsilon}{4}$ and

$$|f(x) + p(x) - a| \leq C_\gamma \epsilon^{-1/\gamma} = C_\gamma |E|^{-1/\gamma} \quad \text{for every } x \in E_1.$$

Proceeding similarly for $g + q$ completes the proof. □

Chapter 3

A class of extremal problems related to the Tomas-Stein inequality

Notation. For the remaining three chapters, if x, y are real numbers, we will write $x = O(y)$ or $x \lesssim y$ if there exists a finite constant C such that $|x| \leq C|y|$, and $x \asymp y$ if $C^{-1}|y| \leq |x| \leq C|y|$ for some finite constant $C \neq 0$. If we want to make explicit the dependence of the constant C on some parameter α, we will write $x = O_\alpha(y)$ or $x \lesssim_\alpha y$. As is customary the constant C is allowed to change from line to line. If $\lambda \in \mathbb{R}$ and $A \subseteq \mathbb{R}^d$, we denote its λ-dilation by $\lambda \cdot A := \{\lambda x : x \in A\}$. The Minkowski sum of A with itself will be denoted by $A + A = \{x + x' : x \in A \text{ and } x' \in A\}$. Sharp constants will always appear in bold face. By $\Re z$ and $\Im z$ we will denote, respectively, the real and imaginary parts of the complex number $z \in \mathbb{C}$.

3.1 The Tomas-Stein inequality

It has long been known that certain geometric properties related to curvature originate decay of the Fourier transform, and this phenomenon is explained by the behavior of oscillatory integrals. Let us illustrate this point by briefly recalling the restriction problem for the sphere. Let $d \geq 2$ and $S^{d-1} \subset \mathbb{R}^d$ be the unit sphere in Euclidean space equipped with surface measure σ. The adjoint restriction operator associated to (S^{d-1}, σ) is defined as

$$\widehat{f\sigma}(x) = \int_{S^{d-1}} f(y) e^{-ix \cdot y} d\sigma(y). \quad (x \in \mathbb{R}^d)$$

Theorem 3.1. (Tomas-Stein inequality, [13]) *Let* $d \geq 2$ *and* $p(d) := 2(d+1)/(d-1)$. *If* $p \geq p(d)$, *then* $\mathbf{S}_{d,p} := \sup_{0 \neq f \in L^2} \|\widehat{f\sigma}\|_p \|f\|_2^{-1} < \infty$. *Consequently the following sharp inequality holds:*

$$\|\widehat{f\sigma}\|_{L^p(\mathbb{R}^d)} \leq \mathbf{S}_{d,p} \|f\|_{L^2(S^{d-1})} \quad \text{for every } f \in L^2(S^{d-1}). \tag{3.1}$$

The range of exponents is best possible for L^2 densities, as is shown by a famous construction due to A. Knapp; see [1, pp. 387-388] or [14, Chapter 7]. Curvature plays an essential role, and the discussion below goes through if S^{d-1} is replaced by any compact hypersurface of nonvanishing Gaussian curvature. Several examples will be given in the next sections.

We describe three possible approaches to proving Theorem 3.1, all of which contain ideas that turned out to be crucial in our work.

First proof

The Gaussian curvature of S^{d-1} never vanishes, and this causes the Fourier transform of the corresponding surface measure σ to decay at a precise rate which can be computed without difficulty:

Lemma 3.2. *Let* $d \geq 2$, *and let* σ *denote surface measure on* S^{d-1}. *Then, for* $x \in \mathbb{R}^d$,

$$|\widehat{\sigma}(x)| \lesssim_d (1+|x|)^{-(d-1)/2}. \tag{3.2}$$

More precisely, $\widehat{\sigma}$ *is a* C^∞ *function with asymptotic behavior*

$$\widehat{\sigma}(x) = c_d \cos(|x| + \omega_d)|x|^{-(d-1)/2} + O(|x|^{-(d+1)/2}) \text{ as } |x| \to \infty$$

for certain constants c_d, ω_d.

Proof. Start by noting that $\widehat{\sigma} \in C^\infty$ because σ has compact support. Estimate (3.2) follows from the method of stationary phase. To see this, write $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$, and let $e_d = (0, \ldots, 0, 1)$. Using a partition of unity and the rotational symmetry of S^{d-1}, it is enough to look at what happens to the absolute value of

$$\widehat{\sigma}(\lambda e_d) = \int_{S^{d-1}} e^{-i\lambda x_d} d\sigma(x) \tag{3.3}$$

as $|\lambda| \to \infty$. The phase function x_d on S^{d-1} has exactly two critical points $x = (\mathbf{0}, \pm 1) \in \mathbb{R}^{d-1} \times \mathbb{R}$, and their contributions to the expression (3.3) are complex conjugates of each other. One of them is given by the oscillatory integral

$$I(\lambda) = \int_{\mathbb{R}^{d-1}} e^{-i\lambda(1-|x'|^2)^{1/2}} (1-|x'|^2)^{-1/2} \eta(x') dx',$$

where $\eta \in C_0^\infty(\mathbb{R}^{d-1})$ is a cutoff function supported near $x' = 0$ and $\equiv 1$ in a smaller such neighborhood. The phase function $(1-|x'|^2)^{1/2} = 1 - \frac{|x'|^2}{2} + O(|x'|^4)$ has a nondegenerate critical point at $x' = 0$, and the result follows. The second assertion follows from well-known asymptotics of Bessel functions: in fact, one has that

$$\widehat{\sigma}(x) = |x|^{\frac{2-d}{2}} J_{(d-2)/2}(|x|),$$

where

$$\begin{aligned} J_{(d-2)/2}(|x|) &= \frac{1}{2\pi}\int_0^{2\pi} e^{i|x|\sin\theta} e^{-i\frac{d-2}{2}\theta} d\theta \\ &= \sqrt{\frac{2}{\pi}}|x|^{-1/2}\cos\Big(|x| - \frac{\pi(d-1)}{4}\Big) + O(|x|^{-3/2}) \text{ as } |x| \to \infty; \end{aligned}$$

see [1, p. 338, 347]. □

As a corollary, $\widehat{\sigma} \in L^r(\mathbb{R}^d)$ if and only if $r > 2d/(d-1)$. From this it follows that if an inequality $\|\widehat{f\sigma}\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^q(S^{d-1})}$ holds, then necessarily $p > 2d/(d-1)$. Indeed, take $f \equiv 1$. Then $f \in L^q(S^{d-1})$ for any $q \in [1, \infty]$, yet $\widehat{f\sigma} \in L^p(\mathbb{R}^d)$ only if $p > 2d/(d-1)$.

The restriction conjecture formulated by E. M. Stein in the late 1960's asserts that

$$\|\widehat{f\sigma}\|_{L^p(\mathbb{R}^d)} \lesssim_{d,p} \|f\|_{L^\infty(S^{d-1})}$$

whenever $p > 2d/(d-1)$. In dimensions $d > 2$, this is a long-standing open problem in harmonic analysis which has inspired seminal work throughout the last decades. For more on the restriction conjecture and some recent progress we recommend the manuscript [15].

Let us go back to the proof of inequality (3.1). We focus on the non-endpoint case $p > p(d)$. Having established this, the inequality for the Tomas-Stein exponent $p = p(d)$ follows by complex interpolation.

By the usual TT^* argument, it is enough to show that

$$\|f * \widehat{\sigma}\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^{p'}(\mathbb{R}^d)} \text{ whenever } p > p(d). \tag{3.4}$$

To establish (3.4), consider a smooth function $\phi \in C_0^\infty(\mathbb{R}^d)$ supported on, say, $\{x \in \mathbb{R}^d : \frac{1}{4} \leq |x| \leq 1\}$ and such that

$$\sum_{j=0}^\infty \phi(2^{-j}x) = 1 \text{ if } |x| \geq 1.$$

Let

$$\widehat{\sigma}(x) = \Big(1 - \sum_{j=0}^{\infty} \phi(2^{-j}x)\Big)\widehat{\sigma}(x) + \sum_{j=0}^{\infty} \phi(2^{-j}x)\widehat{\sigma}(x) =: K_{-\infty}(x) + \sum_{j=0}^{\infty} K_j(x).$$

Since $K_{-\infty} \in C_0^\infty(\mathbb{R}^d)$ and $p > 2$, the estimate

$$\|f * K_{-\infty}\|_p \lesssim \|f\|_{p'}$$

follows from Young's convolution inequality. For finite j, the logic is to interpolate between an $L^1 \to L^\infty$ bound and an $L^2 \to L^2$ bound. The following estimates hold:

$$\|f * K_j\|_\infty \leq \|K_j\|_\infty \|f\|_1 \lesssim 2^{-j\frac{d-1}{2}} \|f\|_1; \tag{3.5}$$

$$\|f * K_j\|_2 \simeq \|\widehat{f} \cdot \widehat{K_j}\|_2 \lesssim \|\widehat{K_j}\|_\infty \|f\|_2 \lesssim 2^j \|f\|_2. \tag{3.6}$$

Estimate (3.5) follows directly from Lemma 3.2, whereas estimate (3.6) is slightly more delicate. Details can be consulted in [13] or [14, Chapter 7]. Interpolating between (3.5) and (3.6) yields

$$\|f * K_j\|_p \lesssim 2^{j\theta} 2^{-j\frac{d-1}{2}(1-\theta)} \|f\|_{p'}$$

for $\frac{1}{p} = \frac{\theta}{2}$. Note that this estimate is summable in j precisely when $p > p(d)$. Unraveling the exponents and invoking Fatou's lemma one concludes the proof of estimate (3.4), and thus of Theorem 3.1.

Second proof

A second proof of the Tomas-Stein inequality consists of introducing a time parameter and treating $\widehat{f\sigma}$ as an evolution operator. This allows us to regard the restriction theorem as part of a more general framework which includes Strichartz estimates for various linear partial differential equations with constant coefficients. Two key ingredients for this proof, which already appeared in the early work of Carleson and Sjölin on a certain class of oscillatory integrals [16], are the Hausdorff-Young inequality and fractional integration in the form of the Hardy-Littlewood-Sobolev inequality. Both will show up in the next chapter, and we postpone the details until then.

Third proof

An alternative route to proving Theorem 3.1 is available whenever the Tomas-Stein exponent $p(d) = 2(d + 1)/(d - 1)$ is an even integer. Such is the case if (and only if) $(d, p) \in \{(2, 6), (3, 4)\}$. The idea is to view the Tomas-Stein inequality as an L^2 estimate for a multilinear form which has a convolution structure that provides some smoothing effects. Let us focus on the case $(d, p) = (3, 4)$ which will be considered again in §3.2 below. Plancherel's theorem implies that

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} = \|\widehat{f\sigma}\cdot\widehat{f\sigma}\|_{L^2(\mathbb{R}^3)}^{1/2} = (2\pi)^{3/4}\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)}^{1/2}.$$

Since $\sigma(\xi) = \delta(1 - |\xi|)d\xi$, the double convolution is given by

$$f\sigma * g\sigma(\xi) = \int_{\mathbb{R}^3} f(\xi - \eta)g(\eta)\delta(1 - |\xi - \eta|)\delta(1 - |\eta|)d\eta.$$

Applying Cauchy-Schwarz with respect to the measure $\delta(1-|\xi-\eta|)\delta(1-|\eta|)d\eta$ yields

$$|f\sigma * g\sigma(\xi)|^2 \leq |f|^2\sigma * |g|^2\sigma(\xi) \cdot \sigma * \sigma(\xi)$$

We can integrate this in the variable ξ to get

$$\|f\sigma * g\sigma\|_{L^2(\mathbb{R}^3)}^2 \lesssim |\|f\|_{L^2(S^2)}^2\|g\|_{L^2(S^2)}^2 \cdot \sup_{\xi\in\mathbb{R}^3} |(\sigma * \sigma)(\xi). \tag{3.7}$$

There is an explicit expression for the convolution of surface measure on S^{d-1} with itself which is valid in all dimensions:

Lemma 3.3. *Let $d \geq 2$. Then there exists a dimensional constant $c_d < \infty$ such that the surface measure σ on S^{d-1} satisfies:*

$$\sigma * \sigma(\xi) = c_d \frac{1}{|\xi|}(4 - |\xi|^2)_+^{\frac{d-3}{2}}.$$

Proof. Compute:

$$\sigma * \sigma(\xi) = \int_{\mathbb{R}^d} \delta(1 - |\xi - \eta|)\delta(1 - |\eta|)d\eta \simeq \int_{|\eta|=1} \delta(1 - |\xi - \eta|^2)d\sigma(\eta) =$$

$$= \int_{|\eta|=1} \delta(|\xi|^2 - 2\xi\cdot\eta)d\sigma(\eta) \simeq \frac{1}{|\xi|}\int_{|\eta|=1} \delta\Big(\frac{|\xi|}{2} - \frac{\xi}{|\xi|}\cdot\eta\Big)d\sigma(\eta).$$

By rotational symmetry we can take $\xi = (0, \ldots, 0, |\xi|)$. Let θ be the angle that the unit vector η makes with the north pole $(0, \ldots, 0, 1)$. Passing to spherical coordinates on S^{d-1}, we can write $d\sigma = d\sigma_{d-1} = (\sin\theta)^{d-2}d\theta d\sigma_{d-2}$. Thus:

$$\sigma * \sigma(\xi) \simeq \frac{1}{|\xi|}\int_0^\pi \delta\Big(\frac{|\xi|}{2} - \cos\theta\Big)(\sin\theta)^{d-2}d\theta =$$

$$= \frac{1}{|\xi|}\int_{-1}^1 \delta\Big(\frac{|\xi|}{2} - u\Big)(1 - u^2)^{\frac{d-3}{2}}du = \frac{1}{|\xi|}\Big(1 - \frac{|\xi|^2}{4}\Big)^{\frac{d-3}{2}}$$

if $|\xi|/2 \in [-1, 1]$. □

In the case we are considering, $d = 3$, we see that $\sigma * \sigma$ has a singularity at the origin $\xi = 0$. However, using a partition of unity, we can assume without loss of generality that f and g are supported in a small neighborhood of a point in S^2. In particular, the supremum in (3.7) can be taken just over all $\xi \in \text{supp}(f) + \text{supp}(g)$, which is a set bounded away from 0. The result follows.

Motivation

In the first part of this thesis we will be primarily concerned with the study of extremizers for certain inequalities related to Fourier restriction phenomena. We make the relevant concepts precise in the endpoint case of inequality (3.1) i.e. when $p = p(d) = 2(d+1)/(d-1)$:

$$\|\widehat{f\sigma}\|_{L^{p(d)}(\mathbb{R}^d)} \leq \mathbf{S}_d \|f\|_{L^2(S^{d-1})} \quad \text{for every } f \in L^2(S^{d-1}). \tag{3.8}$$

Here $\mathbf{S}_d := \mathbf{S}_{d,p(d)}$ denotes the corresponding optimal constant.

Definition 3.4. An *extremizing sequence* for the inequality (3.8) is a sequence $\{f_n\}$ of functions in $L^2(S^{d-1})$ satisfying $\|f_n\|_{L^2(S^{d-1})} \leq 1$ such that $\|\widehat{f_n\sigma}\|_{p(d)} \to \mathbf{S}_d$ as $n \to \infty$. An *extremizer* for the inequality (3.8) is a nonzero function $f \in L^2(S^{d-1})$ which satisfies $\|\widehat{f\sigma}\|_{p(d)} = \mathbf{S}_d \|f\|_{L^2(S^{d-1})}$.

Definition 3.5. A nonzero function $f \in L^2(S^{d-1})$ is said to be a *δ-near extremizer* for the inequality (3.8) if $\|\widehat{f\sigma}\|_{p(d)} \geq (1-\delta)\mathbf{S}_d \|f\|_{L^2(S^{d-1})}$.

Definition 3.6. A set $\mathcal{S} \subset L^2(S^{d-1})$ is *precompact* if any sequence of elements in $\mathcal{S}$ has a subsequence which is Cauchy in $L^2(S^{d-1})$.

The following are natural questions:

(1) Do extremizers for inequality (3.8) exist?

(2) Are extremizing sequences precompact in $L^2(S^{d-1})$ modulo the group of symmetries?

(3) What can be said quantitatively about the structure of δ-near extremizers?

(4) What qualitative properties do extremizers enjoy?

(5) What is the value of the optimal constant $\mathbf{S}_d$?

These problems have been studied in several cases, and the next sections are devoted to describing some of the ones that turned out to be more relevant to our work. For instance, Question 1 has been answered in the affirmative for the paraboloid [17, 18, 19] and one of its nonlinear analogues [20], and for the two-dimensional sphere S^2 [21, 22]; see §3.2. Perhaps surprisingly the same question has been shown to have a negative answer in the case of the two-dimensional hyperboloid [23] and that of a truncated paraboloid [24]; see §3.3. Finally, in §3.4 we briefly describe our own work towards a better understanding of these questions in the case of a certain class of planar convex curves which satisfy a natural geometric assumption.

3.2 Positive results

The case of the paraboloid

The endpoint case

Consider the free Schrödinger equation

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = 0, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ \qquad\qquad u(\cdot, 0) = f \in L^2(\mathbb{R}^d), \end{cases} \tag{3.9}$$

where $d \geq 1$ and $u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ is a complex-valued function. The solution u is given by the Schrödinger evolution operator

$$u(x,t) := e^{i\frac{t}{2}\Delta} f(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi - it\frac{|\xi|^2}{2}} \widehat{f}(\xi) d\xi,$$

where $\widehat{f}$ denotes the spatial Fourier transform of f. This almost coincides with the adjoint restriction operator for the paraboloid $\mathbb{P}^d := \{(\xi, |\xi|^2/2) : \xi \in \mathbb{R}^d\} \subset \mathbb{R}^{d+1}$ equipped with projection measure $d\sigma_{\mathbb{P}} = dy$. In fact, one has that

$$u(x,t) = e^{i\frac{t}{2}\Delta} f(x) = \frac{1}{(2\pi)^d} \widehat{\widehat{f}\sigma_{\mathbb{P}}}(-x,t). \tag{3.10}$$

A particular case of the well-known Strichartz estimates [25] states that there exists a (best) constant $\mathbf{P}_d < \infty$ such that

$$\|e^{i\frac{t}{2}\Delta} f\|_{L^{2+4/d}(\mathbb{R}^{d+1})} \leq \mathbf{P}_d \|f\|_{L^2(\mathbb{R}^d)} \quad \text{for every } f \in L^2(\mathbb{R}^d). \tag{3.11}$$

In view of (3.10) this is nothing but a restatement[1] of the Tomas-Stein inequality for the paraboloid. Kunze [17] proved the existence of an extremizer $f_\star \in L^2(\mathbb{R})$ for

[1]Note that $2 + \frac{4}{d} = p(d+1)$, the Tomas-Stein exponent in dimension $d+1$.

the inequality (3.11) in the special case $d = 1$ and $p = 6$, which means that for the corresponding solution $u_\star$ we have equality $\|u_\star\|_{L^6(\mathbb{R}^2)} = \mathbf{P}_1\|f_\star\|_{L^2(\mathbb{R})}$. His argument relied on concentration compactness techniques and established the precompactness in $L^2(\mathbb{R})$ of arbitrary nonnegative extremizing sequences. The method does not provide an explicit expression for the maximizer, nor does it compute the numerical value of the optimal constant $\mathbf{P}_1$.

Foschi [18] used a more direct approach to explicitly determine the families of extremizers and compute the best constant for estimate (3.11) whenever the Tomas-Stein exponent $p(d+1)$ is an even integer. Here is one of his main results:

Theorem 3.7. ([18]) *In the case $d = 1$ and $p = 6$, we have $\mathbf{P}_1 = 3^{-1/12}$; in the case $d = 2$ and $p = 4$, we have $\mathbf{P}_2 = 2^{-1/4}$. In both cases an example of an extremizer $f_\star \in L^2(\mathbb{R}^d)$ for which we have*

$$\|e^{i\frac{t}{2}\Delta} f_\star\|_{L^{2+4/d}(\mathbb{R}^{d+1})} = \mathbf{P}_d\|f_\star\|_{L^2(\mathbb{R}^d)} \tag{3.12}$$

is provided by the Gaussian function $f_\star(x) = \exp(-|x|^2/2)$.

It turns out that the class of all extremizers can be characterized in terms of the geometric invariant properties of the Schrödinger equation (3.9). To be more precise, denote by $\mathcal{G}$ the Lie group of transformations which preserve the solutions of (3.9) and which leave the ratio $\|u\|_{L^{2+4/d}}/\|u(0)\|_{L^2}$ unchanged. This includes phase invariance, scaling, space-time translation and Galilean transformations. Foschi proved that the set of extremizers for which equality (3.12) holds coincides with the set of initial data of solutions to (3.9) in the orbit of $u_\star$ under the action of $\mathcal{G}$. In particular, all extremizers are given by L^2 functions of the form

$$f_\star(x) = \exp(a|x|^2 + b \cdot x + c),$$

where $a, c \in \mathbb{C}$, $b \in \mathbb{C}^d$ and $\Re(a) < 0$.

The proof of Theorem 3.7 relies on a careful examination of the double (in the case $d = 2$) and triple (in the case $d = 1$) convolution of projection measure on $\mathbb{P}^d$. This leads to a sharp proof of the Strichartz estimates whose steps are later optimized by imposing conditions under which all inequalities become equalities. We will come back to this point in the next two chapters, where we also recall some additional aspects of Foschi's work which will be relevant to our discussion.

In higher dimensions much less is known. Shao [19] applied profile decomposition arguments to establish the existence of extremizers for the Schrödinger equation in every dimension, but determining the nature of these extremizers and computing the value of the optimal constants $\mathbf{P}_d$ in dimensions $d > 2$ remains an open problem.

The non-endpoint case

For each $M > 0$, consider the truncated paraboloid

$$\mathbb{P}^d_M := \{(\xi, |\xi|^2/2) : \xi \in \mathbb{R}^d, |\xi| \le M\}$$

equipped with (the restriction of) projection measure $\sigma_{\mathbb{P}}$ as in the previous section. There is an inequality

$$\|\widehat{f\sigma_{\mathbb{P}}}\|_{L^p(\mathbb{R}^{d+1})} \le \mathbf{P}_{d,p,M}\|f\|_{L^2(\mathbb{P}^d_M)} \quad \text{for every } f \in L^2(\mathbb{P}^d_M), \tag{3.13}$$

as long as $p \ge p(d+1) = 2 + 4/d$; as usual, $\mathbf{P}_{d,p,M}$ denotes the optimal constant. Restricting our attention to the case of finite $M < \infty$, we have the following result which asserts the existence of extremizers for the non-endpoint cases of the restriction inequality (3.13):

Theorem 3.8. ([24]) *Let $d \ge 1$. For every $M < \infty$ and every p satisfying $p(d+1) < p \le \infty$ there exists an extremizer for* (3.13)*. Moreover, for every extremizing sequence $\{f_n\}$ of* (3.13) *there exists a sequence $\{\xi_n\} \subset \mathbb{R}^d$ such that $\{e^{-ix\cdot\xi_n} f_n(x)\}$ is precompact in $L^2(\mathbb{P}^d_M)$.*

This should be contrasted with Theorem 3.21 discussed below.

The key ingredient in establishing Theorem 3.8 is the following result which is of interest in its own right. It is a modification of a well-known result of Brézis and Lieb [26] which does not require the a.e. pointwise convergence of the sequence of functions $\{f_n\}$:

Proposition 3.9 ([24])**.** *Let $\mathcal{H}$ be a Hilbert space and T be a bounded linear operator from $\mathcal{H}$ to $L^p(\mathbb{R}^d)$, for some $p \in (2, \infty)$. Let $\{f_n\} \in \mathcal{H}$ be such that*

(i) $\|f_n\|_{\mathcal{H}} = 1$;

(ii) $\lim_{n\to\infty} \|Tf_n\|_{L^p(\mathbb{R}^d)} = \|T\|$;

(iii) $f_n \rightharpoonup f \neq 0$;

(iv) $Tf_n \to Tf$ *a.e. in* $\mathbb{R}^d$.

Then $f_n \to f$ in $\mathcal{H}$; in particular, $\|f\|_{\mathcal{H}} = 1$ and $\|Tf\|_{L^p(\mathbb{R}^d)} = \|T\|$.

Proposition 3.9 is applied to the adjoint Fourier restriction operator on $\mathbb{P}^d_M$ with $\mathcal{H} = L^2(\mathbb{P}^d_M)$. No generality is lost in assuming that conditions (i) and (ii) are automatically satisfied by any extremizing sequence $\{f_n\}$. Almost everywhere convergence of the sequence $\{f_n\}$ cannot be easily checked in this case, but this is the whole point of the proposition: it suffices to check that the extremizing sequence converges weakly to a *nonzero* limit. In other words, Proposition 3.9 states that the only obstruction to the existence of extremizers is the possibility that every L^2 weak limit of any extremizing sequence be zero. After passing to a subsequence, we may assume that the sequence $\{f_n\}$ converges weakly to some function $f \in L^2(\mathbb{P}^d_M)$ by Alaoglu's theorem. Condition (iv) follows immediately from the compact support of the measure. Finally, the possibility that $f = 0$ a.e. is ruled out by appealing to interpolation and the Arzelá-Ascoli theorem. This establishes Theorem 3.8. As noted in [24], the proof goes through for any compact hypersurface in $\mathbb{R}^{d+1}$ of nonvanishing Gaussian curvature equipped with surface measure.

Maximizers for the Strichartz norm for small solutions of mass-critical NLS

A nonlinear version of the problem discussed in the last subsection has been considered by Duyckaerts, Merle and Roudenko [20]. Consider the L^2-critical nonlinear Schrödinger equation (NLS) in space dimension $d \geq 1$:

$$\begin{cases} i\partial_t u + \frac{1}{2}\,\Delta u + \gamma\,|u|^{\frac{4}{d}} u = 0, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ \qquad\qquad u(\cdot, 0) = f \in L^2(\mathbb{R}^d). \end{cases} \tag{3.14}$$

We allow for both focusing ($\gamma = +1$) and defocusing ($\gamma = -1$) equations.

The Strichartz estimates (3.11) are the key ingredients to establish local wellposedness in L^2 of the Cauchy problem (3.14). For small data, the solution is also globally wellposed and the global $L^{2+4/d}_{x,t}$ norm is finite, which implies that the solution scatters in L^2. For more on this and for an extension to large radial data in the defocusing case, see [20] and the references therein.

Consider the quantity

$$I(\delta) = \sup_{\|f\|_2 = \delta} \iint_{\mathbb{R}^d \times \mathbb{R}} |u(x,t)|^{2+\frac{4}{d}} dxdt,$$

where $\delta > 0$ is small and u is the solution to (3.14). The results mentioned in the last paragraph imply that $I(\delta)$ is finite for small δ. A natural extension of Theorem 3.7 would be to show that the maximum is achieved by a unique solution (up to symmetries) of (3.14) and to give a precise estimate for $I(\delta)$. Here is the main result of [20]:

Theorem 3.10. ([20]) *Fix* $\gamma \in \{-1, +1\}$. *There exists a* $\delta_0 > 0$ *such that for all* $\delta \in (0, \delta_0)$, *the maximum* $I(\delta)$ *is attained: there exists a solution* u_δ *of* (3.14) *with initial condition* f_δ *such that*

$$\|f_\delta\|_{L^2(\mathbb{R}^d)} = \delta, \quad I(\delta) = \iint_{\mathbb{R}^d \times \mathbb{R}} |u_\delta(x,t)|^{2+\frac{4}{d}} dxdt.$$

If $d = 1$ *or* $d = 2$, *then*

$$I(\delta) = (\mathbf{P}_d)^{2+\frac{4}{d}} \delta^{2+\frac{4}{d}} + \gamma C_d \delta^{2+\frac{8}{d}} + O(\delta^{2+\frac{12}{d}}) \text{ as } \delta \to 0. \tag{3.15}$$

Here $C_1 = \frac{1}{\pi} \sum_{k \geq 1} \frac{(2k)!}{k 9^k (k!)^2} \approx 0.0867$ *and* $C_2 = \frac{1}{2\pi} \log \frac{4}{3} \approx 0.0458$.

Let $d \in \{1, 2\}$. A key element in the proof of estimate (3.15) is the nondegeneracy of a certain quadratic form in the orthocomplement of the space of null directions related to the invariances of the equation. To state this precisely, consider the L^2-normalized Gaussian $G_0(x) = \pi^{-d/4} e^{-|x|^2/2}$ and let $G_1(x,t) := e^{i\frac{t}{2}\Delta} G_0(x)$. Consider the mapping from $L^2(\mathbb{R}^d)$ to $[0, \infty)$ given by

$$f \mapsto \Theta(f) := (\mathbf{P}_d)^{2+\frac{4}{d}} \Big(\int |f|^2 dx \Big)^{1+\frac{2}{d}} - \iint |e^{i\frac{t}{2}\Delta} f|^{2+\frac{4}{d}} dxdt.$$

The Strichartz estimate (3.11) implies that $\Theta(G_0 + \varphi) \geq 0$ for every $\varphi \in L^2(\mathbb{R}^d)$. Expanding this inequality and using the fact that G_0 is an extremizer (and therefore a critical point of the functional), we obtain that the linear part vanishes i.e.

$$(\mathbf{P}_d)^{2+\frac{4}{d}} \Re \int G_0 \varphi = \Re \iint |G_1|^{\frac{4}{d}} \overline{G_1} (e^{i\frac{t}{2}\Delta} \varphi) \quad \text{for every } \varphi \in L^2(\mathbb{R}^d).$$

The second order expansion yields $\Theta(G_0 + \varphi) = Q(\varphi) + O(\|\varphi\|_{L^2}^3)$, where Q is the real-valued nonnegative symmetric quadratic form on L^2 given by

$$\begin{aligned} Q(\varphi) =& (\mathbf{P}_d)^{2+\frac{4}{d}} \Big(\frac{d+2}{2} \int |\varphi|^2 + \frac{4(d+2)}{d^2} \Big(\Re \int G_0 \varphi \Big)^2 \Big) \\ & - \frac{(d+2)^2}{d^2} \iint |G_1|^{\frac{4}{d}} \Big| e^{i\frac{t}{2}\Delta} \varphi \Big|^2 - \frac{2(d+2)}{d^2} \Re \iint |G_1|^{\frac{4}{d}-2} \overline{G_1}^2 (e^{i\frac{t}{2}\Delta} \varphi)^2. \end{aligned}$$

The following result establishes the coercivity of Q:

Theorem 3.11. ([20]) *Let* $d \in \{1, 2\}$. *There exists a constant* $c_0 > 0$ *such that if* $\varphi \in L^2(\mathbb{R}^d)$ *satisfies the following orthogonality properties*

$$\int_{\mathbb{R}^d} \varphi G_0 = \int_{\mathbb{R}^d} \varphi |x|^2 G_0 = 0, \quad \int_{\mathbb{R}^d} \varphi x G_0 = \mathbf{0}_{\mathbb{R}^d},$$

then

$$Q(\varphi) \geq c_0 \|\varphi\|^2_{L^2(\mathbb{R}^d)}.$$

One reduces Theorem 3.11 to proving that $Q(\varphi) > 0$ for any eigenfunction φ of the harmonic oscillator $\mathfrak{H} = -\Delta + |x|^2$ which is orthogonal to $F := \text{span}_{\mathbb{C}}\{G_0, x_j G_0, |x|^2 G_0\}$, and this turns out to be a doable calculation. The analysis of a very similar quadratic form will be of crucial importance to us in Chapter 4.

The case of the two-dimensional sphere

Christ and Shao [21, 22] were, to the best of our knowledge, the first ones to study extremizers for the endpoint restriction problem on a compact manifold. They considered inequality (3.1) for $d = 3$ and $p = 4$. Denoting by $\mathbf{S} = \mathbf{S}_{3,4}$ the corresponding optimal constant, this translates into

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \leq \mathbf{S}\|f\|_{L^2(S^2)} \quad \text{for every } f \in L^2(S^2). \tag{3.16}$$

Here is their main result:

Theorem 3.12. ([21]) *Any extremizing sequence of nonnegative functions in $L^2(S^2)$ for the inequality* (3.16) *is precompact. In particular, extremizers exist.*

Nonnegative functions play a special role because

$$\|\widehat{f\sigma}\|_4 = (2\pi)^{3/4}\|f\sigma * f\sigma\|_2^{1/2} \leq (2\pi)^{3/4}\||f|\sigma * |f|\sigma\|_2^{1/2} = \|\widehat{|f|\sigma}\|_4.$$

It follows that if $\{f_n\}$ is an extremizing sequence, so is $\{|f_n|\}$, and if f is an extremizer, so is $|f|$. In the course of the analysis, the authors also established that the quantity $\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)}$ never decreases under L^2 norm preserving symmetrization of f with respect to the antipodal map $x \mapsto -x$. In particular, every extremizer satisfies $|f(-x)| = |f(x)|$ for almost every $x \in S^2$.

Contrary to the previously discussed case of the paraboloid, the sphere lacks exact scaling symmetries. While this turns out to be just a technical obstacle for most of the analysis, it is nonetheless linked with the most essential obstruction, namely, the possibility that the optimal constant might be attained only in a limit where $|f|^2$ tends to a Dirac mass, or to a sum of two Dirac masses. An essential ingredient in ruling out this possibility is to compare the optimal constants for the sphere and the paraboloid. From the above discussion we know that extremizers for the paraboloid $\mathbb{P}^2 = \{x \in \mathbb{R}^3 : x_3 = \frac{1}{2}(x_1^2 + x_2^2)\}$ are Gaussian functions of x_1 and x_2; but these are just restrictions to $\mathbb{P}^2$ of simple exponentials $e_\xi(x) := e^{x\cdot\xi}$, where $\xi \in \mathbb{C}^3$ satisfies

$\Re\xi_3 < 0$. It is therefore natural to study the functional $\Lambda(f) := \|\widehat{f\sigma}\|_4^4/\|f\|_2^4$ for $f = e_\xi$. Denote by $\mathbf{P}$ is the best constant for inequality (3.11) in the form:

$$\|\widehat{f\sigma_{\mathbb{P}}}\|_{L^4(\mathbb{R}^3)} \leq \mathbf{P}\|f\|_{L^2(\mathbb{P}^2)}.$$

One can easily check that $\mathbf{P} = 2\pi \cdot \mathbf{P}_2$, and the following result holds:

Proposition 3.13. ([21]) *For every $\xi \in \mathbb{R}^3$ with $|\xi|$ sufficiently large*

$$\|\widehat{e_\xi\sigma}\|_{L^4(\mathbb{R}^3)} > \mathbf{P}\|e_\xi\|_{L^2(S^2)}.$$

Proposition 3.13 is proved via a perturbative analysis of $\|e_\xi\sigma * e_\xi\sigma\|_2/\|e_\xi\sigma\|_2$ which we describe in some detail. When $\xi = (0, 0, \lambda)$, the family $(e_\xi\sigma)^2/\|e_\xi\sigma\|_2^2$ converges weakly as $\lambda \to +\infty$ to a constant multiple of a Dirac mass at the north pole $(0, 0, 1)$. We will therefore work with functions concentrated primarily in a very small neighborhood of the north pole. A point $z \in S^2$ close to $(0, 0, 1)$ can be written as

$$(y, (1 - |y|^2)^{1/2}) = \Big(y, 1 - \frac{|y|^2}{2} - \frac{|y|^4}{8} + O(|y|^6)\Big). \tag{3.17}$$

Likewise, surface measure σ satisfies

$$d\sigma(y) = (1 + \frac{|y|^2}{2} + O(|y|^4))dy. \tag{3.18}$$

For $\epsilon > 0$, define the L^2-normalized family of trial functions

$$f_\epsilon(z) = \epsilon^{-1/2}e^{(z_3-1)/\epsilon}.$$

Note that $f_\epsilon = \epsilon^{-1/2}e^{-1/\epsilon}e_\xi$ where $\xi = (0, 0, \epsilon^{-1})$. Let

$$u_\epsilon = \widehat{f_\epsilon\sigma} \;\text{ and }\; w_\epsilon = \epsilon^{-1/2}u_\epsilon(-\epsilon^{-1}\cdot, \epsilon^{-1/2}\cdot).$$

One easily checks that $\|u_\epsilon\|_6 = \|w_\epsilon\|_6$. Using (3.17) and (3.18) we see that

$$\|u_\epsilon\|_4^4 = \|v_\epsilon\|_4^4 + O(\epsilon^2) \text{ as } \epsilon \to 0^+,$$

where v_ϵ is the approximation of w_ϵ given by

$$v_\epsilon(x, t) = \int_{\mathbb{R}^2} e^{-ix\cdot y}e^{-(1+it)(\frac{|y|^2}{2}+\epsilon\frac{|y|^4}{8})}(1 + \frac{\epsilon}{2}|y|^2)dy.$$

We similarly approximate f_ϵ by g_ϵ where $g_\epsilon(y) = e^{-\frac{|y|^2}{2}-\epsilon\frac{|y|^4}{8}}$. In order to study the functional

$$\Psi(\epsilon) = \log \Lambda(f_\epsilon) = \log \frac{\|u_\epsilon\|_{L^4(\mathbb{R}^3)}^4}{\|f_\epsilon\|_{L^2(S^2)}^4},$$

it is enough to consider the approximations v_ϵ and g_ϵ of, respectively, u_ϵ and f_ϵ. Even if Ψ is initially only defined for $\epsilon > 0$, it is straightforward to verify that it extends continuously and differentiably to $\epsilon = 0$. One has that

$$\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\Psi(\epsilon) = \frac{\partial_\epsilon \mid_{\epsilon=0} \|v_\epsilon\|_4^4}{\|v_0\|_4^4} - 2\frac{\partial_\epsilon \mid_{\epsilon=0} \|g_\epsilon\|_2^2}{\|g_0\|_2^2}. \tag{3.19}$$

Since $\Psi(0) = \log \mathbf{P}^4$, it is enough to verify that expression (3.19) defines a positive quantity. Explicit calculations which we omit show that $\Psi'(0) = 1/4$. Proposition 3.13 can be derived from this and from radial symmetry. It follows that[2] $\mathbf{S} > \mathbf{P}$. As conjectured in [21], this variational computation is generalizable to manifolds other than S^2. In the next chapter we will present a similar computation involving a second variation instead.

The rest of the proof of Theorem 3.12 serves as very direct inspiration for our work on convex arcs but will not be presented here. Tools that were directly relevant to us include multilinear estimates for convolutions and some ideas of concentration compactness flavor. We refer to [27, Chapter 1] for a brief but enlightening summary of the exposition. Several fundamental questions remain open:

Question 3.14. Are extremizers unique modulo rotations and multiplication by constants? Are constant functions extremizers? What is the value of the optimal constant $\mathbf{S}$?

A companion paper [22] dealing with complex-valued extremizers establishes the following result:

Theorem 3.15. ([22]) *If $\{f_n\}$ is any complex-valued extremizing sequence, then there exists a sequence $\{\xi_n\} \subset \mathbb{R}^3$ such that $\{e^{-ix\cdot\xi_n} f_n(x)\}$ is precompact in $L^2(S^2)$. Moreover, every complex-valued extremizer for the inequality* (3.16) *is of the form*

$$ce^{ix\cdot\xi}F(x) \tag{3.20}$$

where $\xi \in \mathbb{R}^3$, $c \in \mathbb{C}$ and F is a nonnegative extremizer.

A real-valued function $0 \neq f \in L^2(S^2)$ is said to be a critical point of the functional Λ defined above if f satisfies the generalized Euler-Lagrange equation

$$(f\sigma * f\sigma * f\sigma) \mid_{S^2} = \lambda \|f\|_2^2 f, \tag{3.21}$$

and f is an extremum of Λ if and only if this holds with $\lambda = (2\pi)^3\mathbf{S}^4$; see [28] for more details. We have the following:

[2] It actually turns out that $\mathbf{S} > (3/2)^{1/4}\mathbf{P}$, and this is essential to rule out concentration at a *pair* of antipodal points. We omit the details and refer the interested reader to [21, Lemma 2.4].

Theorem 3.16. ([22]) *For any* $\lambda \in \mathbb{C}$*, any solution* $f \in L^2(S^2)$ *of* (3.21) *is* C^∞*.*

This theorem together with (3.20) implies that all complex-valued extremizers are smooth as well.

3.3 Negative results

The case of the two-dimensional hyperboloid

Let $d \geq 2$, and consider the hyperboloid

$$\mathbb{H}^d = \{(y, \sqrt{1+|y|^2}) : y \in \mathbb{R}^d\} \subset \mathbb{R}^{d+1}$$

equipped with the measure

$$\sigma_{\mathbb{H}}(y, y') = \delta(y' - \sqrt{1+|y|^2})\frac{dydy'}{\sqrt{1+|y|^2}}.$$

The adjoint Fourier restriction operator for $\mathbb{H}^d$ is given by

$$\widehat{f\sigma_{\mathbb{H}}}(x,t) = \int_{\mathbb{R}^d} e^{-i(x,t)\cdot(y,\sqrt{1+|y|^2})} f(y)(1+|y|^2)^{-1/2} dy.$$

We have an inequality

$$\|\widehat{f\sigma_{\mathbb{H}}}\|_{L^p(\mathbb{R}^{d+1})} \leq \mathbf{H}_{d,p}\|f\|_{L^2(\mathbb{H}^d)} \tag{3.22}$$

whenever $p(d+1) \leq p \leq p(d)$. As usual, $\mathbf{H}_{d,p}$ denotes the optimal constant. Quilodrán [23] studied inequality (3.22) for the endpoint pairs $(d,p) = (2,4), (2,6)$ and $(3,4)$, but here we shall limit our discussion to the case $(d,p) = (2,6)$. Let $\mathbf{H} := \mathbf{H}_{2,6}$. Here is the main result:

Theorem 3.17. ([23]) *The value of the best constant is* $\mathbf{H} = (2\pi)^{5/6}$*, and extremizers for inequality* (3.22) *when* $(d,p) = (2,6)$ *do not exist.*

The exponent $p = 6$ is an even integer, and so we can write inequality (3.22) in convolution form. This follows from the *equality*

$$\|\widehat{f\sigma_{\mathbb{H}}}\|^3_{L^6(\mathbb{R}^3)} = \|(\widehat{f\sigma_{\mathbb{H}}})^3\|_{L^2(\mathbb{R}^3)} = (2\pi)^{3/2}\|(f\sigma_{\mathbb{H}})^{(*3)}\|_{L^2(\mathbb{R}^3)},$$

where $(f\sigma_{\mathbb{H}})^{(*3)} := f\sigma_{\mathbb{H}} * f\sigma_{\mathbb{H}} * f\sigma_{\mathbb{H}}$. The following result is already implicit in the work of Foschi [18]:

Lemma 3.18. ([23]) *Given $f \in \mathcal{S}(\mathbb{R}^2)$, the triple convolution of $f\sigma_{\mathbb{H}}$ with itself satisfies*

$$\|(f\sigma_{\mathbb{H}})^{(*3)}\|_{L^2(\mathbb{R}^3)} \leq \|\sigma_{\mathbb{H}}^{(*3)}\|_{L^\infty(\mathbb{R}^3)}^{1/2} \|f\|_{L^2(\mathbb{H}^2)}^3. \tag{3.23}$$

Moreover, for $f \neq 0$, for equality to hold in (3.23) *it is necessary that $\sigma_{\mathbb{H}}^{(*3)}(\xi,\tau) = \|\sigma_{\mathbb{H}}^{(*3)}\|_{L^\infty(\mathbb{R}^3)}$ for a.e. (ξ,τ) in the support of $(f\sigma_{\mathbb{H}})^{(*3)}$.*

As a corollary, if $\mathbf{H} = (2\pi)^{1/2}\|\sigma_{\mathbb{H}}^{(*3)}\|_{L^\infty(\mathbb{R}^3)}^{1/6}$ and $\sigma_{\mathbb{H}}^{(*3)}(\xi,\tau) < \|\sigma_{\mathbb{H}}^{(*3)}\|_{L^\infty(\mathbb{R}^3)}$ for a.e. (ξ,τ) in the support of $\sigma_{\mathbb{H}}^{(*3)}$, then extremizers do not exist.

Lemma 3.19. ([23]) *For $(\xi,\tau) \in \mathbb{R}^2 \times \mathbb{R}$,*

$$\sigma_{\mathbb{H}}^{(*3)}(\xi,\tau) = (2\pi)^2\Big(1 - \frac{3}{\sqrt{\tau^2 - |\xi|^2}}\Big)\chi_{\{\tau \geq \sqrt{3^2+|\xi|^2}\}}.$$

In particular, $\|\sigma_{\mathbb{H}}^{(*3)}\|_{L^\infty(\mathbb{R}^3)} = (2\pi)^2$ and $\sigma_{\mathbb{H}}^{(*3)}(\xi,\tau) < \|\sigma_{\mathbb{H}}^{(*3)}\|_{L^\infty(\mathbb{R}^3)}$ for every $(\xi,\tau) \in \mathbb{R}^{2+1}$.

This gives us an upper bound $\mathbf{H} \leq (2\pi)^{5/6}$. For the lower bound one exhibits an explicit extremizing sequence:

Lemma 3.20. ([23]) *For $a > 0$ let $f_a(y) = e^{-a\sqrt{1+|y|^2}}$, $y \in \mathbb{R}^2$. Then:*

$$\lim_{a\to 0^+} \|\widehat{f_a\sigma_{\mathbb{H}}}\|_{L^6(\mathbb{R}^3)}\|f_a\|_{L^2(\mathbb{H}^2)}^{-1} = (2\pi)^{5/6}.$$

This concludes the proof of Theorem 3.17. In Chapter 5 we will follow a similar strategy to show that extremizers do not exist for a certain class of convex curves in the plane.

The endpoint case of a truncated paraboloid

Let $M > 0$ and consider the truncated paraboloid $\mathbb{P}_M^d = \{(\xi, |\xi|^2/2) : \xi \in \mathbb{R}^d, |\xi| \leq M\}$ equipped with projection measure $\sigma_{\mathbb{P}}$ as in §3.2. Consider the endpoint inequality

$$\|\widehat{f\sigma_{\mathbb{P}}}\|_{L^{p(d+1)}(\mathbb{R}^{d+1})} \leq \mathbf{P}_{d,M}\|f\|_{L^2(\mathbb{P}_M^d)} \quad \text{for every } f \in L^2(\mathbb{P}_M^d), \tag{3.24}$$

where $p(d+1) = 2 + 4/d$ and $\mathbf{P}_{d,M}$ denotes as usual the optimal constant. Let us restrict our attention to the lower dimensional cases $d = 1$ and $d = 2$. Theorem 3.7 states in particular that extremizers exist if $M = \infty$. In view of this and of Theorem 3.8, the following result is illuminating:

Theorem 3.21. ([24]) *There are no extremizers for inequality* (3.24) *if* $d = 1$ *or* $d = 2$ *provided* $M < \infty$.

The proof of Theorem 3.21 is completely elementary and relies on a rescaling argument and on the explicit knowledge of the extremizers for inequality (3.11). See [24] for details.

3.4 Our contribution: the case of convex arcs

Consider a smooth, convex curve $\Gamma \subset \mathbb{R}^2$ equipped with arclength measure σ. Assume the curvature κ of Γ to be positive everywhere; equivalently, assume that $\lambda := \min_\Gamma \kappa$ is a positive real number. In this case, the Tomas-Stein inequality states that

$$\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)} \leq \mathbf{C}[\Gamma]\|f\|_{L^2(\sigma)} \quad \text{for every } f \in L^2(\sigma). \tag{3.25}$$

By $\mathbf{C}[\Gamma] = \sup_{0\neq f\in L^2} \|\widehat{f\sigma}\|_6 \|f\|_{L^2(\sigma)}^{-1}$ we mean the optimal constant. Here is our main result whose proof can be found in Chapter 4:

Theorem 3.22. *Let* Γ *be a smooth, convex curve in the plane without points with colinear tangents, equipped with arclength measure* σ. *Assume that the curvature* κ *of* Γ *is a strictly positive function. If the second derivative of the curvature with respect to arclength satisfies*

$$\frac{d^2\kappa}{ds^2}(p_0) < \frac{3}{2}(\kappa(p_0))^3 \tag{3.26}$$

at every $p_0 \in \Gamma$ *which is a global minimum of the curvature, then any extremizing sequence of nonnegative functions in* $L^2(\sigma)$ *for the inequality* (3.25) *is precompact. In particular, extremizers for inequality* (3.25) *exist.*

To understand the significance of the geometric condition (3.26), let us turn to a restricted family of key examples. Given data $r, \lambda, a > 0$, consider the curve $\widetilde{\Gamma} = \widetilde{\Gamma}_{r,\lambda,a}$ parametrized by

$$\begin{aligned} \gamma : [-r, r] &\to \mathbb{R}^2 \\ y &\mapsto \Big(y, \tfrac{\lambda}{2}y^2 + ay^4\Big). \end{aligned}$$

Equip $\widetilde{\Gamma}$ with arclength measure $\widetilde{\sigma} = \widetilde{\sigma}_{r,\lambda,a}$. The Tomas-Stein inequality states that there exists a finite constant $\mathbf{C}[\widetilde{\Gamma}] < \infty$ such that

$$\|\widehat{f\widetilde{\sigma}}\|_{L^6(\mathbb{R}^2)} \leq \mathbf{C}[\widetilde{\Gamma}]\|f\|_{L^2(\widetilde{\sigma})}, \quad \text{for every } f \in L^2(\widetilde{\sigma}), \tag{3.27}$$

where by $\mathbf{C}[\widetilde{\Gamma}] = \sup_{0\neq f\in L^2} \|\widehat{f\widetilde{\sigma}}\|_6\|f\|_2^{-1}$ we mean the optimal constant as usual. In this context, we say that a sequence $\{f_n\}$ of functions in $L^2(\widetilde{\sigma})$ satisfying $\|f_n\|_2 \to 1$ as $n \to \infty$ concentrates at the origin if for every $\epsilon, r > 0$ there exists $N \in \mathbb{N}$ such that, for every $n \geq N$,

$$\int_{|y|\geq r} |f_n(\gamma(y))|^2 \|\gamma'(y)\| dy < \epsilon.$$

Observe that Theorem 3.22 ensures the existence of extremizers for inequality (3.27) *provided* that $a < \frac{3}{2}(\frac{\lambda}{2})^3$. In Chapter 5 we establish the following complementary result:

Theorem 3.23. *Let $r, \lambda, a > 0$ be such that $a > 2(\frac{\lambda}{2})^3$, and consider the curve $(\widetilde{\Gamma}, \widetilde{\sigma})$ parametrized by γ as above. Then there exists $r_0 > 0$ such that for every $r < r_0$, the triple convolution $\widetilde{\sigma} * \widetilde{\sigma} * \widetilde{\sigma}$ attains a strict global maximum at the origin. As a consequence, if $r < r_0$, every extremizing sequence concentrates at the origin. In particular, there are no extremizers for inequality* (3.27).

Chapter 4

Extremizers for Fourier restriction inequalities: convex arcs

4.1 Introduction

Consider a compact arc $\Gamma \subset \mathbb{R}^2$ of a smooth, convex curve equipped with arclength measure σ. Assume the curvature κ of Γ to be positive everywhere; equivalently, assume that $\lambda := \min_\Gamma \kappa$ is a positive real number. Let $\ell := \sigma(\Gamma)$ and parametrize Γ by arclength:

$$\begin{aligned} \gamma : [0,\ell] &\rightarrow \mathbb{R}^2 \\ s &\mapsto \gamma(s) = (x(s), y(s)). \end{aligned}$$

For $s \in [0,\ell]$, let $t(s) = (x'(s), y'(s))$ be the tangent indicatrix and let $\theta(s) \in S^1$ measure the net rotation described by the vector $t(s)$ as we run the curve γ from 0 to s. In other words, if we let $e_1 = (1,0)$, then θ is the unique continuous function satisfying $t(s) = (\cos\theta(s), \sin\theta(s))$ for every $s \in [0,\ell]$, and such that $\theta(0) = \arccos(e_1 \cdot t(0))$. The function θ is related to the curvature κ via

$$\theta(s) = \int_0^s \kappa(t)dt.$$

We further assume that the arc Γ has no points with colinear tangents[1] i.e. points $\gamma(s_0), \gamma(s_1) \in \Gamma$ for which $t(s_0) = -t(s_1)$. By compactness, this means that there exists some constant $\delta_0 > 0$ such that

$$|t(s) + t(s')| \geq \delta_0, \quad \forall s, s' \in [0,\ell]. \tag{4.1}$$

[1] We hope to remove this assumption in a later work.

Certain subsets of Γ will be of special interest to us. A cap $\mathcal{C} \subset \Gamma$ is a set of the form

$$\mathcal{C} = \mathcal{C}(s, r) = \{\gamma(s') \in \Gamma : |s - s'| < r\}$$

for some $s \in [0, \ell]$ and $r > 0$. We will write $|\mathcal{C}| := \sigma(\mathcal{C})$.

The space $L^2(\sigma)$ consists of all functions $f : \Gamma \to \mathbb{C}$ for which the quantity

$$\|f\|_{L^2(\sigma)}^2 := \int_\Gamma |f(z)|^2 d\sigma(z)$$

is finite. Given $f \in L^2(\sigma)$, the Fourier transform of the measure $f\sigma$ is defined as

$$\widehat{f\sigma}(x, t) := \int_\Gamma f(z) e^{-i(x,t)\cdot z} d\sigma(z).$$

The Tomas-Stein inequality [13], whose proof we recall in the next section, states that there exists a finite constant $\mathbf{C}[\Gamma] < \infty$ for which

$$\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)} \leq \mathbf{C}[\Gamma] \|f\|_{L^2(\sigma)} \tag{4.2}$$

for every $f \in L^2(\sigma)$; by $\mathbf{C}[\Gamma]$ we mean the optimal constant defined by

$$\mathbf{C}[\Gamma] := \sup_{0 \neq f \in L^2(\sigma)} \|\widehat{f\sigma}\|_6 \|f\|_{L^2(\sigma)}^{-1}.$$

Definition 4.1. An *extremizing sequence* for the inequality (4.2) is a sequence $\{f_n\}$ of functions in $L^2(\sigma)$ satisfying $\|f_n\|_{L^2(\sigma)} \leq 1$ such that $\|\widehat{f_n\sigma}\|_6 \to \mathbf{C}[\Gamma]$ as $n \to \infty$. An *extremizer* for the inequality (4.2) is a nonzero function $f \in L^2(\sigma)$ which satisfies $\|\widehat{f\sigma}\|_6 = \mathbf{C}[\Gamma] \|f\|_{L^2(\sigma)}$.

Definition 4.2. A nonzero function $f \in L^2(\sigma)$ is said to be a δ-*near extremizer* for the inequality (4.2) if $\|\widehat{f\sigma}\|_6 \geq (1 - \delta)\mathbf{C}[\Gamma]\|f\|_{L^2(\sigma)}$.

A natural question is whether extremizers exist. More generally one can ask if extremizing sequences are precompact in $L^2(\sigma)$. Previous work includes the study of extremizers for Strichartz/Fourier restriction inequalities in [17], [18] and [21]. Kunze [17] proved the existence of extremizers for the parabola in $\mathbb{R}^2$ by showing that any nonnegative extremizing sequence is precompact. Foschi [18], whose work will be recalled in greater detail in §4.8, showed that Gaussians are extremizers for this situation and computed the corresponding optimal constant. The best constant and extremizers for the paraboloid in $\mathbb{R}^3$ were also computed in [18]. The existence of extremizers for the restriction on the sphere S^2 was proved by Christ and Shao in

[21], and this is to the best of our knowledge the only result concerning existence of extremizers for the endpoint restriction problem on a compact manifold.

Other results on (non-)existence of extremizers and/or computation of sharp constants for Fourier restriction operators and Strichartz inequalities can be found in [29, 20, 24, 30, 31, 32, 23].

Here is our main result:

Theorem 4.3. *Let Γ be a compact arc of a smooth, convex curve in the plane without points with colinear tangents, equipped with arclength measure σ. Assume that the curvature κ of Γ is a strictly positive function. If the second derivative of the curvature with respect to arclength satisfies*

$$\frac{d^2\kappa}{ds^2}(p_0) < \frac{3}{2}(\kappa(p_0))^3 \tag{4.3}$$

at every $p_0 \in \Gamma$ which is a global minimum of the curvature, then any extremizing sequence of nonnegative functions in $L^2(\sigma)$ for the inequality (4.2) *is precompact.*

We conclude this section by briefly outlining the structure of this chapter and giving an idea of the proof of Theorem 4.3.

In the next section we follow the classical argument of Carleson and Sjölin [16] to prove the Tomas-Stein inequality (4.2). Using the analysis of a certain bilinear form from [33, 32], we establish the following refinement:

$$\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\sigma)}^{1-\beta/2} \sup_{\mathcal{C}\subset\Gamma} \left(|\mathcal{C}|^{-1/4}\int_{\mathcal{C}} |f|^{3/2} d\sigma\right)^{\beta/3}, \tag{4.4}$$

where the supremum ranges over all caps $\mathcal{C} \subset \Gamma$ and $\beta > 0$ is a small universal constant.

We use estimate (4.4) in §4.3 to describe an iterative procedure which takes a nonnegative function $f \in L^2(\sigma)$ as input and produces a sequence of functions $\{f_n\}$ associated with disjoint caps $\{\mathcal{C}_n\} \subset \Gamma$ for which $f = \sum_n f_n$ in the L^2-sense. This decomposition enjoys certain geometric properties which are described in §4.4. After introducing a suitable metric on the set of all caps, we establish the fact that distant caps interact weakly. Together with the decomposition algorithm, this implies an inequality of geometric nature which is a key step towards gaining control of extremizing sequences.

In §4.5 we prove that any near extremizer satisfies appropriately scaled upper bounds with respect to some cap, and we use this to obtain a result of *concentration compactness* [34] flavor in §4.6. This result basically states that a nonnegative extremizing sequence behaves in one of two possible ways (up to extraction of a subsequence and up to a small L^2 error): it is either uniformly integrable, or it concentrates at a point. Precompactness can be derived in the former case, since the main obstruction pointed out in [24] is easy to rule out: in fact, L^2 weak limits of nonnegative, uniformly integrable sequences of functions are *nonzero*.

The proof is therefore finished once we show that concentration cannot occur. Aiming at a contradiction, we explore some of the properties that an extremizing sequence which concentrates at a point would have to enjoy. In §4.7, we compute a certain limiting operator norm exactly, and in particular show that an extremizing sequence which concentrates must do so at a point of minimal curvature. A second ingredient consists in comparing the constant $\mathbf{C}[\Gamma]$ from inequality (4.2) with the optimal constant for the adjoint Fourier restriction inequality on an appropriately dilated parabola equipped with projection measure, as studied in [18]. We accomplish this in §4.8, postponing some of the more technical estimates to Appendix 1. We derive the desired contradiction in §4.9, and that concludes the proof of Theorem 4.3.

4.2 The cap estimate

Let $f, g \in L^2(\sigma)$. We seek to estimate the L^3 norm of the product

$$\widehat{f\sigma} \cdot \widehat{g\sigma}(x,t) = \int_0^\ell \int_0^\ell f(\gamma(s))g(\gamma(s'))e^{-i(x,t)\cdot(\gamma(s)+\gamma(s'))}dsds'. \tag{4.5}$$

For that purpose, it will be enough to estimate the $L^{3/2}$ norm of the convolution of measures $f\sigma * g\sigma$, which is defined by duality as

$$\langle f\sigma * g\sigma, \varphi\rangle = \int_0^\ell \int_0^\ell f(\gamma(s))g(\gamma(s'))\varphi(\gamma(s)+\gamma(s'))dsds'$$

for any test function $\varphi \in C_0^\infty(\Gamma, \sigma)$.

To analyze the integral (4.5), we make the following change of variables:

$$(s, s') \mapsto (u, v) = (x(s) + x(s'), y(s) + y(s')), \tag{4.6}$$

Splitting

$$f(\gamma(s))g(\gamma(s')) = f(\gamma(s))g(\gamma(s'))(\chi_{\{s>s'\}} + \chi_{\{s<s'\}}) \text{ for a.e. } (s, s')$$

and using the triangle inequality, we lose no generality in assuming that $s > s'$ in the support of $f(\gamma(s))g(\gamma(s'))$. As a consequence, the transformation (4.6) is injective in the support of $f(\gamma(s))g(\gamma(s'))$. It follows that

$$\widehat{f\sigma}\cdot\widehat{g\sigma}(x,t) = \iint_{\Gamma+\Gamma} f(\gamma(s(u,v)))g(\gamma(s'(u,v)))e^{-i(x,t)\cdot(u,v)}J^{-1}dudv,$$

where by $J = J(s(u,v), s'(u,v))$ we denote the Jacobian of the transformation (4.6) on the region $\{s > s'\}$:

$$J(s,s') = \Big|\frac{\partial(u,v)}{\partial(s,s')}\Big| = |x'(s)y'(s') - x'(s')y'(s)| = |\sin(\theta(s) - \theta(s'))|.$$

Note that, for $(u,v) \in \mathbb{R}^2$,

$$f\sigma * g\sigma(u,v) = \begin{cases} f(\gamma(s(u,v)))g(\gamma(s'(u,v)))J^{-1} & \text{if } (u,v) \in \Gamma+\Gamma \\ 0 & \text{otherwise.} \end{cases}$$

The Hausdorff-Young inequality implies that

$$\|\widehat{f\sigma}\cdot\widehat{g\sigma}\|_3 \leq \|f\sigma * g\sigma\|_{3/2} \lesssim \Big(\int_0^\ell\int_0^\ell |f(\gamma(s))|^{3/2}|g(\gamma(s'))|^{3/2}|\sin(\theta(s)-\theta(s'))|^{-1/2}dsds'\Big)^{2/3}. \tag{4.7}$$

Since Γ has no points with colinear tangents (i.e. condition (4.1) holds),

$$|\sin(\theta(s) - \theta(s'))| \geq \min\Big\{\frac{2}{\pi}|\theta(s) - \theta(s')|, \delta_0(1 + O(\delta_0^2))\Big\}$$

for every $s, s' \in [0,\ell]$. On the other hand, since $\lambda = \min_\Gamma \kappa$,

$$|\theta(s) - \theta(s')| = \Big|\int_{s'}^s \kappa(t)dt\Big| \geq \lambda|s - s'|.$$

It follows that

$$\begin{aligned}\|f\sigma * g\sigma\|_{3/2}^{3/2} &\lesssim \int_0^\ell\int_0^\ell |f(\gamma(s))|^{3/2}|g(\gamma(s'))|^{3/2}|\sin(\theta(s) - \theta(s'))|^{-1/2}dsds' \\ &\lesssim_{\lambda,\delta_0} \int_0^\ell\int_0^\ell |f(\gamma(s))|^{3/2}|g(\gamma(s'))|^{3/2}|s - s'|^{-1/2}dsds'.\end{aligned}$$

Note that the implicit constant blows up as $\lambda \downarrow 0^+$, and this is why we assume that Γ has everywhere positive curvature.

For $0 < \alpha < 1$, consider the bilinear form:

$$B_\alpha(F,G) := \iint_{\mathbb{R}^2} F(x)G(x')|x-x'|^{-\alpha} dx dx'.$$

The case $\alpha = 1/2$ is related to the preceding discussion. In fact, setting $F := |f\circ\gamma|^{3/2}$ and $G := |g\circ\gamma|^{3/2}$, we already know that

$$\|f\sigma * g\sigma\|_{3/2}^{3/2} \lesssim B_{1/2}(F,G). \tag{4.8}$$

For $0<\alpha<1$ and $p = 2/(2-\alpha)$, the Hardy-Littlewood-Sobolev inequality implies that $|B_\alpha(F,F)| \lesssim_p \|F\|^2_{L^p(\mathbb{R}^d)}$. In particular, estimate (4.8) combines with the $L^{4/3}$ bound for $B_{1/2}$ to yield the Tomas-Stein inequality (4.2):

$$\|\widehat{f\sigma}\|_6 = \|(\widehat{f\sigma})^2\|_3^{1/2} \lesssim \|f\sigma * f\sigma\|_{3/2}^{1/2} \lesssim B_{1/2}(F,F)^{1/3} \lesssim \|F\|_{4/3}^{2/3} = \|f\|_{L^2(\sigma)}.$$

Following previous work from [35] and [33], Quilodrán proved in [32, Proposition 4.5] that, for the same range of α and value of p, there exists a constant $\beta>0$ such that

$$|B_\alpha(F,F)| \lesssim \|F\|_p^{2-\beta} \sup_I \left(|I|^{-1+1/p}\int_I |F|\right)^\beta \tag{4.9}$$

for every $F \in L^p(\mathbb{R})$. Here the supremum ranges over all compact intervals I of $\mathbb{R}$. If instead of the $L^{4/3}$ bound for $B_{1/2}$ we use the more refined estimate (4.9), then reasoning in a similar way as before leads to the following improved estimate:

Proposition 4.4. (Cap estimate) *There exists $C<\infty$ and $\beta>0$ such that for every $f \in L^2(\sigma)$, the following estimate holds:*

$$\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)} \le C\|f\|_{L^2(\sigma)}^{1-\beta/2} \sup_{\mathcal{C}\subset\Gamma} \left(|\mathcal{C}|^{-1/4}\int_{\mathcal{C}} |f|^{3/2} d\sigma\right)^{\beta/3}. \tag{4.10}$$

Proof. Set, as before, $F(s) := |f(\gamma(s))|^{3/2}$. Then (4.8) and (4.9) imply:

$$\begin{aligned}\|\widehat{f\sigma}\|_6 &\lesssim \|f\sigma * f\sigma\|_{3/2}^{1/2} \lesssim B_{1/2}(F,F)^{1/3} \lesssim \|F\|_{4/3}^{2/3-\beta/3} \sup_I \left(|I|^{-1/4}\int_I |F(s)|ds\right)^{\beta/3}\\ &= \|f\|_{L^2(\sigma)}^{1-\beta/2} \sup_{\mathcal{C}} \left(|\mathcal{C}|^{-1/4}\int_{\mathcal{C}} |f|^{3/2} d\sigma\right)^{\beta/3},\end{aligned}$$

as desired. ∎

4.3 The decomposition algorithm

The cap estimate (4.10) is the only ingredient we need to prove the analog of [21, Lemma 2.6], which establishes a weak connection between functions satisfying modest lower bounds $\|\widehat{f\sigma}\|_6 \gtrsim \delta\|f\|_2$ and characteristic functions of caps:

Lemma 4.5. *For any $\delta > 0$ there exist $C_\delta < \infty$ and $\eta_\delta > 0$ with the following property: if $f \in L^2(\sigma)$ satisfies $\|\widehat{f\sigma}\|_6 \geq \delta\mathbf{C}[\Gamma]\|f\|_{L^2(\sigma)}$, then there exists a decomposition $f = g + h$ and a cap $\mathcal{C} \subset \Gamma$ satisfying*

$$0 \leq |g|, |h| \leq |f|, \tag{4.11}$$

$$g, h \text{ have disjoint supports}, \tag{4.12}$$

$$|g(\gamma(s))| \leq C_\delta\|f\|_2|\mathcal{C}|^{-1/2}\chi_{\mathcal{C}}(\gamma(s)), \text{ for all } s \in [0, \ell], \tag{4.13}$$

$$\|g\|_{L^2(\sigma)} \geq \eta_\delta\|f\|_{L^2(\sigma)}. \tag{4.14}$$

Proof. The proof is analogous to the one of [21, Lemma 2.6] but we reproduce it here for the convenience of the reader. We can, without loss of generality, normalize so that $\|f\|_{L^2(\sigma)} = 1$. By Proposition 4.4 there exists a cap $\mathcal{C}$ such that

$$\int_{\mathcal{C}} |f|^{3/2} d\sigma \geq \frac{1}{2}c(\delta)|\mathcal{C}|^{1/4}.$$

Here $c(\delta) = c_0 \cdot \delta^{3/\beta}$ for some absolute constant $c_0 > 0$ whose exact value is not important for the analysis. Let $R \geq 1$, and define $E := \{\gamma(s) \in \mathcal{C} : |f(\gamma(s))| \leq R\}$. Set $g = f\chi_E$ and $h = f - f\chi_E$. Then g and h have disjoint supports, $g + h = f$, g is supported on $\mathcal{C}$, and $\|g\|_\infty \leq R$. Since $|h(\gamma(s))| \geq R$ for almost every $\gamma(s) \in \mathcal{C}$ for which $h(\gamma(s)) \neq 0$, we have

$$\int_C |h|^{3/2} d\sigma \leq R^{-1/2} \int_{\mathcal{C}} |h|^2 d\sigma \leq R^{-1/2}\|f\|_2^2 = R^{-1/2}.$$

Define R by $R^{-1/2} = \frac{1}{4}c(\delta)|\mathcal{C}|^{1/4}$. Then

$$\int_{\mathcal{C}} |g|^{3/2} d\sigma = \int_{\mathcal{C}} |f|^{3/2} d\sigma - \int_{\mathcal{C}} |h|^{3/2} d\sigma \geq \frac{1}{4}c(\delta)|\mathcal{C}|^{1/4}.$$

By Hölder's inequality, since g is supported on $\mathcal{C}$,

$$\|g\|_2 \geq |\mathcal{C}|^{-1/6}\Big(\int_{\mathcal{C}} |g|^{3/2} d\sigma\Big)^{2/3} \geq c'(\delta) = c'(\delta)\|f\|_2 > 0. \qquad \square$$

Conditions 4.13 and 4.14 easily imply a lower bound on the L^1 norm of g:

Lemma 4.6. *Let $g \in L^2(\sigma)$ satisfy $|g(x)| \leq a|\mathcal{C}|^{-1/2}\chi_{\mathcal{C}}(x)$ and $\|g\|_2 \geq b$ for some $a, b > 0$ and $\mathcal{C} \subset \Gamma$. Then there exists a constant $C = C(a,b) > 0$ such that*

$$\|g\|_{L^1(\sigma)} \geq C|\mathcal{C}|^{1/2}.$$

Proof. Estimate:

$$\|g\|_{L^1(\sigma)} = \int_{\mathcal{C}} |g| d\sigma \geq a^{-1}|\mathcal{C}|^{1/2}\|g\|^2_{L^2(\sigma)} \geq a^{-1}b^2|\mathcal{C}|^{1/2}. \qquad \square$$

In what follows we will restrict our attention to *nonnegative* functions[2]. Indeed, by Plancherel's theorem, inequality (4.2) is equivalent to

$$\|f\sigma * f\sigma * f\sigma\|_{L^2(\mathbb{R}^2)} \leq \frac{\mathbf{C}[\Gamma]^3}{2\pi}\|f\|^3_{L^2(\sigma)}. \tag{4.15}$$

The pointwise inequality $|f\sigma * f\sigma * f\sigma| \leq |f|\sigma * |f|\sigma * |f|\sigma$ then implies that, if f is an extremizer for inequality (4.2), so if $|f|$; similarly, if $\{f_n\}$ is an extremizing sequence, so is $\{|f_n|\}$.

A decomposition algorithm analogous to the one from [21, Step 6A] may be applied to any given nonnegative $f \in L^2(\sigma)$. We describe it precisely:

Decomposition algorithm. Initialize by setting $G_0 = f$ and $\epsilon = 1/2$.

Step n: The inputs for step n are a nonnegative function $G_n \in L^2(\sigma)$ and a positive number ϵ_n. Its outputs are functions f_n, G_{n+1} and nonnegative numbers $\epsilon_n^\star, \epsilon_{n+1}$.

If $\|G_n\sigma * G_n\sigma * G_n\sigma\|_2 = 0$, then $G_n = 0$ almost everywhere. The algorithm then terminates, and we define $\epsilon_n^\star = 0$, $f_n = 0$, and $G_m = f_m = 0$, $\epsilon_m = 0$ for all $m > n$.

If $0 < \|G_n\sigma * G_n\sigma * G_n\sigma\|_2 < \epsilon_n^3(2\pi)^{-1}\mathbf{C}[\Gamma]^3\|f\|_2^3$, then replace ϵ_n by $\epsilon_n/2$; repeat until the first time that $\|G_n\sigma * G_n\sigma * G_n\sigma\|_2 \geq \epsilon_n^3(2\pi)^{-1}\mathbf{C}[\Gamma]^3\|f\|_2^3$. Define $\epsilon_n^\star$ to be this value of ϵ_n. Then

$$(\epsilon_n^\star)^3\frac{\mathbf{C}[\Gamma]^3}{2\pi}\|f\|_2^3 \leq \|G_n\sigma * G_n\sigma * G_n\sigma\|_2 \leq 8(\epsilon_n^\star)^3\frac{\mathbf{C}[\Gamma]^3}{2\pi}\|f\|_2^3. \tag{4.16}$$

Apply Lemma 4.5 to obtain a cap $\mathcal{C}_n$ and a decomposition $G_n = f_n + G_{n+1}$ with disjointly supported nonnegative summands satisfying $f_n \leq C_n\|f\|_2|\mathcal{C}_n|^{-1/2}\chi_{\mathcal{C}_n}$

[2] For much of the analysis this makes no difference, but nonnegativity will play a crucial role in §4.6 when we establish precompactness of uniformly integrable extremizing sequences; see the proof of Lemma 4.23.

and $\|f_n\|_2 \geq \eta_n \|f\|_2$. Here, C_n, η_n are bounded above and below, respectively, by quantities which depend only on $\|G_n\sigma * G_n\sigma * G_n\sigma\|_2^{1/3}/\|G_n\|_2 \gtrsim \epsilon_n^\star$. Define $\epsilon_{n+1} = \epsilon_n^\star$, and move on to step $n+1$. $\square$

The following exact analogs of [21, Lemmas 8.1, 8.3, 8.4] hold:

Lemma 4.7. *Let $f \in L^2(\sigma)$ be a nonnegative function with positive norm. If the decomposition algorithm never terminates for f, then $\epsilon_n^\star \to 0$ as $n \to \infty$, and $\sum_{n=0}^N f_n \to f$ in $L^2(\sigma)$ as $N \to \infty$.*

The proof of Lemma 4.7 is identical to the corresponding one in [21] and therefore is omitted.

This decomposition is in general very inefficient. However, if f nearly extremizes inequality (4.2), then more useful properties hold. Before we turn into these, let us recall a useful fact about near extremizers which already appeared in [27, Lemma 9.2]:

Lemma 4.8. *Let $f = g + h \in L^2(\sigma)$. Suppose that $g \perp h$, $g \neq 0$, and that f is δ-near extremizer for some $\delta \in (0, \frac{1}{4}]$. Then*

$$\frac{\|h\|_2}{\|f\|_2} \leq C \max\Big(\frac{\|\widehat{h\sigma}\|_6}{\|h\|_2}, \delta^{1/2}\Big). \tag{4.17}$$

Here $C < \infty$ is a constant independent of g and h.

The proof is almost identical to that of [21, Lemma 7.1] but we reproduce it here for the convenience of the reader.

Proof. The inequality is invariant under multiplication of f by a positive constant, so we may assume without loss of generality that $\|g\|_2 = 1$. We may assume that $\|h\|_2 > 0$, since otherwise the conclusion is trivial. Define $y = \|h\|_2$ and

$$\eta = \frac{\|\widehat{h\sigma}\|_6}{\mathbf{C}[\Gamma]\|h\|_2}.$$

If $\eta > \frac{1}{2}$ then (4.17) holds trivially with $C = 2/\mathbf{C}[\Gamma]$, for the left-hand side cannot exceed 1 since $f = g + h$ with $g \perp h$.

We also have that

$$(1-\delta)\mathbf{C}[\Gamma]\|f\|_2 \leq \|\widehat{f\sigma}\|_6 \leq \|\widehat{g\sigma}\|_6 + \|\widehat{h\sigma}\|_6 \leq \mathbf{C}[\Gamma](1+\eta y).$$

Since $g \perp h$, $\|f\|_2^2 = 1 + y^2$ and therefore

$$(1-\delta)(1+y^2)^{1/2} \le 1 + \eta y.$$

Squaring gives

$$(1-2\delta)(1+y^2) \le 1 + 2\eta y + \eta^2 y^2.$$

Since $\delta \in (0, \frac{1}{4}]$ and $\eta \le \frac{1}{2}$,

$$\frac{1}{2}y^2 \le 2\delta + 2\eta y + \eta^2 y^2 \le 2\delta + 2\eta y + \frac{1}{4}y^2$$

whence either $y^2 \le 16\delta$ or $y \le 16\eta$.

Substituting the definitions of y, η, and majorizing $\|h\|_2/\|f\|_2$ by $\|h\|_2/\|g\|_2$, yields the stated conclusion. □

Regardless of whether the decomposition algorithm terminates for f, the norms of f_n, G_n enjoy upper bounds independent of f, for all but very large n:

Lemma 4.9. *There exist a sequence of positive constants $\gamma_n \to 0$ and a function $N : (0, \frac{1}{2}] \to \mathbb{N}$ satisfying $N(\delta) \to \infty$ as $\delta \to 0$ with the following property: for any nonnegative δ-near extremizer $f \in L^2(\sigma)$, the quantities $\epsilon_n^\star$ and the functions f_n, G_n obtained when the decomposition algorithm is applied to f satisfy*

$$\epsilon_n^\star \le \gamma_n \text{ for all } n \le N(\delta), \tag{4.18}$$

$$\|G_n\|_2 \le \gamma_n \|f\|_2 \text{ for all } n \le N(\delta), \text{ and} \tag{4.19}$$

$$\|f_n\|_2 \le \gamma_n \|f\|_2 \text{ for all } n \le N(\delta). \tag{4.20}$$

Proof. By (4.15) and (4.16),

$$\frac{\mathbf{C}[\Gamma]^3}{2\pi}\|G_n\|_2^3 \ge \|G_n\sigma * G_n\sigma * G_n\sigma\|_2 \ge (\epsilon_n^\star)^3 \frac{\mathbf{C}[\Gamma]^3}{2\pi}\|f\|_2^3 = \Big(\frac{(\epsilon_n^\star)^3\|f\|_2^3}{\|G_n\|_2^3}\Big)\cdot\frac{\mathbf{C}[\Gamma]^3}{2\pi}\|G_n\|_2^3,$$

so $\epsilon_n^\star \le \|G_n\|_2/\|f\|_2$. Thus the second conclusion implies the first. Since $\|f_n\|_2 \le \|G_n\|_2$, it also implies the third.

We recall two facts. Firstly, Lemma 4.8 applied to $h = G_n$ and $g = f_0 + \ldots + f_{n-1}$ asserts that there are constants $c_0, C_1 \in \mathbb{R}^+$ such that whenever $f \in L^2$ is a δ-near extremizer, either $\|\widehat{G_n\sigma}\|_6 \ge c_0\|G_n\|_2^2\|f\|_2^{-1}$, or $\|G_n\|_2 \le C_1\delta^{1/2}\|f\|_2$. Secondly, according to Lemma 4.5, there exists a nondecreasing function $\rho : (0,\infty) \to (0,\infty)$ satisfying $\rho(t) \to 0$ as $t \to 0$ such that for every nonzero $f \in L^2$ and any n, if $\|\widehat{G_n\sigma}\|_6 \ge t\|G_n\|_2$, then $\|f_n\|_2 \ge \rho(t)\|G_n\|_2$.

Choose a sequence $\{\gamma_n\}$ of positive numbers which tends monotonically to zero, but does so sufficiently slowly to satisfy

$$(n+1)\gamma_n\rho(c_0\gamma_n) > 1 \text{ for all } n.$$

Define $N(\delta)$ to be the largest integer satisfying

$$\gamma_{N(\delta)} \geq C_1\delta^{1/2}.$$

Note that $N(\delta) \to \infty$ as $\delta \to 0$ because $\gamma_n > 0$ for all n.

Let f, δ be given. Suppose that $n \leq N(\delta)$. Aiming at a contradiction, suppose that $\|G_n\|_2 > \gamma_n\|f\|_2$. Then by definition of $N(\delta)$, $\|G_n\|_2 > C_1\delta^{1/2}\|f\|_2$. By the above dichotomy,

$$\|\widehat{G_n\sigma}\|_6 \geq c_0\|G_n\|_2^2\|f\|_2^{-1} \geq c_0\gamma_n\|G_n\|_2.$$

By the second fact reviewed above,

$$\|f_n\|_2 \geq \rho(c_0\gamma_n)\|G_n\|_2 \geq \gamma_n\rho(c_0\gamma_n)\|f\|_2.$$

Since $\|G_m\|_2 \geq \|G_n\|_2$ for every $m \leq n$, the same lower bound follows for $\|f_m\|_2$ for every $m \leq n$. Since the functions f_m are pairwise orthogonal, $\sum_{m\leq n}\|f_m\|_2^2 \leq \|f\|_2^2$, and consequently $(n+1)\gamma_n\rho(c_0\gamma_n) \leq 1$, a contradiction. □

The following lemma is a direct consequence of the decomposition algorithm coupled with Lemma 4.5:

Lemma 4.10. *For every $\epsilon > 0$ there exist $\delta_\epsilon > 0$ and $C_\epsilon < \infty$ such that, if $f \in L^2(\sigma)$ is a nonnegative δ_ϵ-near extremizer, then the functions f_n, G_n associated to f by the decomposition algorithm satisfy, for every $n \in \mathbb{N}$,*

(i) If $\|G_n\|_2 \geq \epsilon\|f\|_2$ then there exists a cap $\mathcal{C}_n \subset \Gamma$ such that

$$f_n \leq C_\epsilon\|f\|_2|\mathcal{C}_n|^{-1/2}\chi_{\mathcal{C}_n}.$$

(ii) If $\|G_n\|_2 \geq \epsilon\|f\|_2$, then $\|f_n\|_2 \geq \delta_\epsilon\|f\|_2$.

4.4 A geometric property of the decomposition

Consider two caps $\mathcal{C}, \mathcal{C}' \subset \Gamma$, and assume without loss of generality that $|\mathcal{C}'| \leq |\mathcal{C}|$. Let $f, g \in L^2(\sigma)$ be such that $\text{supp}(f) \subset \mathcal{C}$ and $\text{supp}(g) \subset \mathcal{C}'$. The following estimate is a direct consequence of (4.7) and Hölder's inequality :

$$\|f\sigma * g\sigma\|_{3/2} \lesssim \Big(\frac{\inf_{s,s'}|\sin(\theta(s)-\theta(s'))|}{|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2}}\Big)^{-1/3}\|f\|_{L^2(\sigma)}\|g\|_{L^2(\sigma)}. \tag{4.21}$$

We can rewrite this estimate in the following way: letting $\ell(\mathcal{C},\mathcal{C}') := \inf_{s\in\mathcal{C},s'\in\mathcal{C}'} |s - s'|$, then (4.8) and Hölder's inequality imply

$$\|f\sigma * g\sigma\|_{3/2} \lesssim \Big(\frac{\ell(\mathcal{C},\mathcal{C}')}{|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2}}\Big)^{-1/3}\|f\|_{L^2(\sigma)}\|g\|_{L^2(\sigma)}. \tag{4.22}$$

For characteristic functions of caps we have the following additional estimate:

Lemma 4.11. *Let $\mathcal{C},\mathcal{C}' \subset \Gamma$ be caps. Then:*

$$\|\chi_{\mathcal{C}}\sigma * \chi_{\mathcal{C}'}\sigma\|_{L^{3/2}(\mathbb{R}^2)} \lesssim \Big(\frac{|\mathcal{C}'|}{|\mathcal{C}|}\Big)^{1/12}|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2}. \tag{4.23}$$

Proof. Without loss of generality we might assume that $10|\mathcal{C}'| \leq |\mathcal{C}|$, otherwise estimate (4.23) is just a consequence of the fundamental inequality

$$\|f\sigma * g\sigma\|_{3/2} \lesssim \|f\|_{L^2(\sigma)}\|g\|_{L^2(\sigma)}. \tag{4.24}$$

Let $\mathcal{C}^*$ be a cap neighborhood of $\mathcal{C}'$ with the same center and of size $|\mathcal{C}^*| = |\mathcal{C}|^{3/4}|\mathcal{C}'|^{1/4}$. Split $\chi_{\mathcal{C}} = \chi_{\mathcal{C}\cap\mathcal{C}^*} + \chi_{\mathcal{C}\setminus\mathcal{C}^*}$. Then

$$\|\chi_{\mathcal{C}}\sigma * \chi_{\mathcal{C}'}\sigma\|_{3/2} \leq \|\chi_{\mathcal{C}\cap\mathcal{C}^*}\sigma * \chi_{\mathcal{C}'}\sigma\|_{3/2} + \|\chi_{\mathcal{C}\setminus\mathcal{C}^*}\sigma * \chi_{\mathcal{C}'}\sigma\|_{3/2}.$$

The first summand can be easily estimated using (4.24). While $\mathcal{C}\setminus\mathcal{C}^*$ is not necessarily a cap, it is the union of at most two caps, say, $\mathcal{C}_1$ and $\mathcal{C}_2$. We can use estimate (4.22) to control the contribution of each of these caps. Noting that

$$\min\{\ell(\mathcal{C}_1,\mathcal{C}'),\ell(\mathcal{C}_2,\mathcal{C}')\} \geq \ell(\mathcal{C}\setminus\mathcal{C}^*,\mathcal{C}') \gtrsim |\mathcal{C}^*|,$$

we have that

$$\begin{aligned}\|\chi_{\mathcal{C}}\sigma * \chi_{\mathcal{C}'}\sigma\|_{3/2} &\lesssim \Big(|\mathcal{C}\cap\mathcal{C}^*|^{1/2} + \Big(\frac{\ell(\mathcal{C}_1,\mathcal{C}')}{|\mathcal{C}_1|^{1/2}|\mathcal{C}'|^{1/2}}\Big)^{-1/3}|\mathcal{C}_1|^{1/2} + \Big(\frac{\ell(\mathcal{C}_2,\mathcal{C}')}{|\mathcal{C}_2|^{1/2}|\mathcal{C}'|^{1/2}}\Big)^{-1/3}|\mathcal{C}_2|^{1/2}\Big)|\mathcal{C}'|^{1/2}\\ &\lesssim \Big(|\mathcal{C}^*|^{1/2} + \Big(\frac{|\mathcal{C}^*|}{|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2}}\Big)^{-1/3}|\mathcal{C}|^{1/2}\Big)|\mathcal{C}'|^{1/2}\\ &\lesssim \Big(\frac{|\mathcal{C}'|}{|\mathcal{C}|}\Big)^{1/12}|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2},\end{aligned}$$

as desired. ∎

The set of all caps can be made into a metric space. We define the distance d from $\mathcal{C} = \mathcal{C}(s, r)$ to $\mathcal{C}' = \mathcal{C}(s', r')$ to be the hyperbolic distance from (s, r) to (s', r') in the upper half plane model. More explicitly, we have that

$$d(\mathcal{C}, \mathcal{C}') := \operatorname{arc\,cosh}\Big(1 + \frac{(s-s')^2 + (r-r')^2}{2rr'}\Big). \tag{4.25}$$

If $s = s'$, then the distance depends only on the ratio of the two radii. When $r = r'$, the distance is $\asymp r^{-1}|s - s'|$ and so this distance has the natural scaling.

We can use estimates (4.22) and (4.23) to prove that the quantity $\|\chi_{\mathcal{C}}\sigma * \chi_{\mathcal{C}'}\sigma\|_{3/2}$ is much smaller than the trivial bound $|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2}$ unless $\mathcal{C}, \mathcal{C}'$ have comparable radii and nearby centers:

Lemma 4.12. *For any $\epsilon > 0$ there exists $\rho < \infty$ such that*

$$\|\chi_{\mathcal{C}}\sigma * \chi_{\mathcal{C}'}\sigma\|_{L^{3/2}(\mathbb{R}^2)} < \epsilon|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2}$$

whenever

$$d(\mathcal{C}, \mathcal{C}') > \rho.$$

Proof. Let $\mathcal{C} = \mathcal{C}(s, r)$ and $\mathcal{C}' = \mathcal{C}'(s', r')$. As before, assume $r' \le r$. We consider three cases.

Start by assuming that $\mathcal{C}$ and $\mathcal{C}'$ have comparable radii: say, $\frac{1}{10}r \le r' \le r$. Then $\mathcal{C}$ and $\mathcal{C}'$ are not far apart unless the corresponding centers are far apart. We may therefore assume that $|s - s'| \ge 10r$, which in turn implies $\ell(\mathcal{C}, \mathcal{C}') \gtrsim |s - s'|$. Using estimate (4.22), we conclude that

$$\|\chi_{\mathcal{C}}\sigma * \chi_{\mathcal{C}'}\sigma\|_{3/2} \lesssim \Big(\frac{|s-s'|}{r}\Big)^{-1/3}|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2} \lesssim (\cosh\rho)^{-1/6}|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2}$$

provided $d(\mathcal{C}, \mathcal{C}') > \rho$.

Assume now that $10r' < r$. If $|s - s'| < 10r$, then we can use Lemma 4.11 to conclude

$$\|\chi_{\mathcal{C}}\sigma * \chi_{\mathcal{C}'}\sigma\|_{3/2} \lesssim \Big(\frac{r'}{r}\Big)^{1/12}|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2}.$$

This quantity is $\lesssim (\cosh\rho)^{-1/12}|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2}$ provided $d(\mathcal{C}, \mathcal{C}') > \rho$, and this concludes the analysis in this case. If on the other hand $|s - s'| \ge 10r$, then $d(\mathcal{C}, \mathcal{C}') > \rho$ implies

$$\frac{(s-s')^2}{rr'} \gtrsim \cosh\rho \quad \text{or} \quad \frac{r}{r'} \gtrsim \cosh\rho.$$

Using, as before, estimate (4.22) in the former case and Lemma 4.11 in the latter, we arrive at the desired conclusion. $\square$

For applications later on, we will need a trilinear version of this lemma which follows immediately from the previous result:

Corollary 4.13. *For any $\epsilon > 0$ there exists $\rho < \infty$ such that*

$$\|\chi_{\mathcal{C}}\sigma * \chi_{\mathcal{C}'}\sigma * \chi_{\mathcal{C}''}\sigma\|_{L^2(\mathbb{R}^2)} < \epsilon|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2}|\mathcal{C}''|^{1/2}$$

whenever

$$\max\{d(\mathcal{C},\mathcal{C}'), d(\mathcal{C}',\mathcal{C}''), d(\mathcal{C}'',\mathcal{C})\} > \rho.$$

Proof. Using Cauchy-Schwarz,

$$\|\chi_{\mathcal{C}}\sigma * \chi_{\mathcal{C}'}\sigma * \chi_{\mathcal{C}''}\sigma\|_2 \leq \|\chi_{\mathcal{C}}\sigma * \chi_{\mathcal{C}}\sigma * \chi_{\mathcal{C}'}\sigma\|_2^{1/2}\|\chi_{\mathcal{C}'}\sigma * \chi_{\mathcal{C}''}\sigma * \chi_{\mathcal{C}''}\sigma\|_2^{1/2}. \tag{4.26}$$

Without loss of generality we may assume that the caps $\mathcal{C}$ and $\mathcal{C}'$ are far apart: in view of Lemma 4.12, we can choose $\rho < \infty$ so that $d(\mathcal{C},\mathcal{C}') > \rho$ implies $\|\chi_{\mathcal{C}}\sigma * \chi_{\mathcal{C}'}\sigma\|_{3/2} < \epsilon^2|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2}$. Bound the first factor appearing in (4.26) as follows:

$$\begin{aligned}\|\chi_{\mathcal{C}}\sigma * \chi_{\mathcal{C}}\sigma * \chi_{\mathcal{C}'}\sigma\|_2 &\simeq \|\widehat{\chi_{\mathcal{C}}\sigma}\cdot\widehat{\chi_{\mathcal{C}}\sigma}\cdot\widehat{\chi_{\mathcal{C}'}\sigma}\|_2 \leq \|\widehat{\chi_{\mathcal{C}}\sigma}\|_6\|\widehat{\chi_{\mathcal{C}}\sigma}\cdot\widehat{\chi_{\mathcal{C}'}\sigma}\|_3\\ &\leq \|\widehat{\chi_{\mathcal{C}}\sigma}\|_6\|\chi_{\mathcal{C}}\sigma * \chi_{\mathcal{C}'}\sigma\|_{3/2} < \epsilon^2|\mathcal{C}||\mathcal{C}'|^{1/2}.\end{aligned}$$

The proof is then complete in view of the trivial estimate $\|\chi_{\mathcal{C}'}\sigma * \chi_{\mathcal{C}''}\sigma * \chi_{\mathcal{C}''}\sigma\|_2 \leq |\mathcal{C}'|^{1/2}|\mathcal{C}''|$. ☐

Corollary 4.13 allows us to establish the following additional inequality of geometric nature, which can be proved in an identical way to [27, Lemma 2.38].

Lemma 4.14. *For any $\epsilon > 0$ there exist $\delta > 0$ and $\lambda < \infty$ such that for any nonnegative $f \in L^2(\sigma)$ which is δ-near extremizer, the summands f_n produced by the decomposition algorithm and the associated caps $\mathcal{C}_n$ satisfy*

$$d(\mathcal{C}_j,\mathcal{C}_k) \leq \lambda \textit{ whenever } \|f_j\|_2 \geq \epsilon\|f\|_2 \textit{ and } \|f_k\|_2 \geq \epsilon\|f\|_2.$$

We provide one proof which follows the proof of [21, Lemma 9.2] more closely:

Proof. It suffices to prove this for all sufficiently small $\epsilon > 0$. Let f be a nonnegative L^2 function which satisfies $\|f\|_2 = 1$ and is δ-near extremizer for a sufficiently small $\delta = \delta(\epsilon)$, and let $\{f_n, G_n\}$ be associated to f via the decomposition algorithm. Set $F = \sum_{n=0}^N f_n$.

Suppose that $\|f_{j_0}\|_2 \geq \epsilon$ and $\|f_{k_0}\|_2 \geq \epsilon$. Let N be the smallest integer such that $\|G_{N+1}\|_2 < \epsilon^3$. Since $\|G_n\|_2$ is a nonincreasing function of n, and since $\|f_n\|_2 \leq \|G_n\|_2$, necessarily $j_0, k_0 \leq N$. Moreover, by Lemma 4.9, there exists $M_\epsilon < \infty$

depending *only* on ϵ such that $N \leq M_\epsilon$. By Lemma 4.10, if δ is chosen to be a sufficiently small function of ϵ then since $\|G_n\|_2 \geq \epsilon^3$ for all $n \leq N$, $f_n \leq \theta(\epsilon)|\mathcal{C}_n|^{-1/2}\chi_{\mathcal{C}_n}$ for all such n, where θ is a continuous, strictly positive function on $(0, 1]$.

Now let $\lambda < \infty$ be a large quantity to be specified. It suffices to show that if $\delta(\epsilon)$ is sufficiently small, an assumption that $d(\mathcal{C}_j, \mathcal{C}_k) > \lambda$ implies an upper bound, which depends only on ϵ, for λ.

As proved in [21, Lemma 9.1], there exists a decomposition $F = F_1 + F_2 = \sum_{n \in S_1} f_n + \sum_{n \in S_2} f_n$ where $[0, N] = S_1 \cup S_2$ is a partition of $[0, N]$, $j_0 \in S_1$, $k_0 \in S_2$, and $d(\mathcal{C}_j, \mathcal{C}_k) \geq \lambda/2N \geq \lambda/2M_\epsilon$ for all $j \in S_1$ and $k \in S_2$. Certainly $\|F_1\|_2 \geq \|f_{j_0}\|_2 \geq \epsilon$, and similarly $\|F_2\|_2 \geq \epsilon$. One of the cross term satisfies

$$\begin{aligned}
\|F_1\sigma * F_1\sigma * F_2\sigma\|_2 &\leq \sum_{i \in S_1} \sum_{j \in S_1} \sum_{k \in S_2} \|f_i\sigma * f_j\sigma * f_k\sigma\|_2 \\
&\leq \theta(\epsilon)^3 \sum_{i \in S_1} \sum_{j \in S_1} \sum_{k \in S_2} |\mathcal{C}_i|^{-1/2}|\mathcal{C}_j|^{-1/2}|\mathcal{C}_k|^{-1/2}\|\chi_{\mathcal{C}_i}\sigma * \chi_{\mathcal{C}_j}\sigma * \chi_{\mathcal{C}_k}\sigma\|_2 \\
&\leq M_\epsilon^3 \gamma(\lambda/2M_\epsilon)\theta(\epsilon)^3
\end{aligned}$$

where $\gamma(\lambda) \to 0$ as $\lambda \to \infty$ by Corollary 4.13. The other cross term $F_1\sigma * F_2\sigma * F_2\sigma$ can be estimated in an identical way. It follows that

$$\begin{aligned}
\|F\sigma * F\sigma * F\sigma\|_2 \leq\ & \|F_1\sigma * F_1\sigma * F_1\sigma\|_2 + \|F_2\sigma * F_2\sigma * F_2\sigma\|_2 + 3\|F_1\sigma * F_1\sigma * F_2\sigma\|_2 + \\
&+ 3\|F_1\sigma * F_2\sigma * F_2\sigma\|_2 \\
\leq\ & (2\pi)^{-1}\mathbf{C}[\Gamma]^3(\|F_1\|_2^3 + \|F_2\|_2^3) + 6M_\epsilon^3\gamma(\lambda/2M_\epsilon)\theta(\epsilon)^3.
\end{aligned}$$

Since F_1 and F_2 have disjoint supports, $\|F_1\|_2^2 + \|F_2\|_2^2 \leq \|f\|_2^2 = 1$ and consequently

$$\|F_1\|_2^3 + \|F_2\|_2^3 \leq \max(\|F_1\|_2, \|F_2\|_2) \cdot (\|F_1\|_2^2 + \|F_2\|_2^2) \leq (1 - \epsilon^2)^{1/2} \cdot 1 \leq (1 - \epsilon^2)^{1/2}.$$

Thus

$$\|F\sigma * F\sigma * F\sigma\|_2 \leq (2\pi)^{-1}\mathbf{C}[\Gamma]^3(1 - \epsilon^2)^{1/2} + 6M_\epsilon^3\gamma(\lambda/2M_\epsilon)\theta(\epsilon)^3.$$

Since

$$\begin{aligned}
(2\pi)^{-1}\mathbf{C}[\Gamma]^3(1 - \delta)^3 \leq \|f\sigma * f\sigma * f\sigma\|_2 &\leq \|F\sigma * F\sigma * F\sigma\|_2 + C\|F\|_2^2\|G_{N+1}\|_2 \\
&\leq \|F\sigma * F\sigma * F\sigma\|_2 + C\epsilon^3,
\end{aligned}$$

by transitivity we have that

$$(2\pi)^{-1}\mathbf{C}[\Gamma]^3(1-\delta)^3 \le C\epsilon^3 + (2\pi)^{-1}\mathbf{C}[\Gamma]^3(1-\epsilon^2)^{1/2} + 6M_\epsilon^3\gamma(\lambda/2M_\epsilon)\theta(\epsilon)^3.$$

Since $\gamma(t) \to 0$ as $t \to \infty$, this implies, for all sufficiently small $\epsilon > 0$, an upper bound for λ which depends only on ϵ, as was to be proved. □

4.5 Upper bounds for extremizing sequences

The decomposition algorithm and Lemma 4.14 allow us to prove that any near extremizer satisfies appropriate scaled upper bounds with respect to some cap. First we need a definition:

Definition 4.15. Let $\Theta : [1, \infty) \to (0, \infty)$ satisfy $\Theta(R) \to 0$ as $R \to \infty$. A function $f \in L^2(\sigma)$ is said to be *upper normalized* (with gauge function Θ) with respect to a cap $\mathcal{C} = \mathcal{C}(\gamma(s_0), r_0) \subset \Gamma$ of radius r_0 and center $\gamma(s_0)$ if

$$\|f\|_{L^2(\sigma)} \le C < \infty, \tag{4.27}$$

$$\int_{\{s:|f(\gamma(s))|\ge Rr_0^{-1/2}\}} |f(\gamma(s))|^2 ds \le \Theta(R), \ \ \forall R \ge 1, \tag{4.28}$$

$$\int_{\{s:|s-s_0|\ge Rr_0\}} |f(\gamma(s))|^2 ds \le \Theta(R), \ \ \forall R \ge 1. \tag{4.29}$$

Proposition 4.16. *There exists a function $\Theta : [1, \infty) \to (0, \infty)$ satisfying $\Theta(R) \to 0$ as $R \to \infty$ with the following property. For every $\epsilon > 0$, there exist a cap $\mathcal{C} \subset \Gamma$ and a threshold $\delta > 0$ such that any nonnegative $f \in L^2(\sigma)$ which is a δ-near extremizer with $\|f\|_2 = 1$ may be decomposed as $f = F + G$, where:*

$$G, F \ge 0 \text{ have disjoint supports}, \tag{4.30}$$

$$\|G\|_2 < \epsilon, \tag{4.31}$$

$$F \text{ is upper normalized with respect to } \mathcal{C}. \tag{4.32}$$

Proposition 4.16 is actually equivalent to the following superficially weaker statement:

Lemma 4.17. *There exists a function $\Theta : [1, \infty) \to (0, \infty)$ satisfying $\Theta(R) \to 0$ as $R \to \infty$ with the following property. For every $\epsilon > 0$ and $\bar{R} \ge 1$, there exist a cap $\mathcal{C} = \mathcal{C}(s_0, r_0)$ and a threshold $\delta > 0$ such that any nonnegative $f \in L^2(\sigma)$ which is a*

δ-near extremizer with $\|f\|_2 = 1$ may be decomposed as $f = F + G$, where:

$$G, F \geq 0 \text{ have disjoint supports,} \tag{4.33}$$
$$\|G\|_2 < \epsilon, \tag{4.34}$$
$$\int_{\{s: F(\gamma(s)) \geq R r_0^{-1/2}\}} F(\gamma(s))^2 ds, \int_{\{s:|s-s_0| \geq R r_0\}} F(\gamma(s))^2 ds \leq \Theta(R), \quad \forall R \in [1, \bar{R}]. \tag{4.35}$$

Proof that Lemma 4.17 implies Proposition 4.16 is the exactly the same as in [21, p. 26] and so we do not include it here.

Proof of Lemma 4.17. Let $\eta : [1, \infty) \to (0, \infty)$ be a function to be chosen below, satisfying $\eta(t) \to 0$ as $t \to \infty$. This function will not depend on the quantity $\bar{R}$.

Let $\bar{R} \geq 1$, $R \in [1, \bar{R}]$, and $\epsilon > 0$ be given. Let $\delta = \delta(\epsilon, \bar{R}) > 0$ be a small quantity to be chosen below. Let $0 \leq f \in L^2(\sigma)$ be δ-near extremizer, with $\|f\|_2 = 1$.

Let $\{f_n\}$ be the sequence of functions obtained by applying the decomposition algorithm to f. Choose $\delta = \delta(\epsilon) > 0$ sufficiently small and $M = M(\epsilon)$ sufficiently large to guarantee that $\|G_{M+1}\|_2 < \epsilon/2$ and that f_n, G_n satisfy all the conclusions of Lemma 4.10 and Lemma 4.9 for $n \leq M$. Set $F = \sum_{n=0}^{M} f_n$. Then $\|f - F\|_2 = \|G_{M+1}\|_2 < \epsilon/2$.

Let $N \in \mathbb{N}_0$ be the minimum of M, and the smallest number such that $\|f_{N+1}\|_2 < \eta$. N is majorized by a quantity which depends only on η. Set $\mathcal{F} = \mathcal{F}_N = \sum_{k=0}^{N} f_k$. It follows from part (ii) of Lemma 4.10 that

$$\|F - \mathcal{F}\|_2 \leq \|G_{N+1}\|_2 < \zeta(\eta) \text{ where } \zeta(\eta) \to 0 \text{ as } \eta \to 0. \tag{4.36}$$

This function ζ is independent of ϵ and $\bar{R}$.

To prove the lemma, we must produce an appropriate cap $\mathcal{C} = \mathcal{C}(s_0, r_0)$, and must establish the existence of Θ. To do the former is simple: to f_0 is associated a cap $\mathcal{C}_0$ such that $f_0 \leq C|\mathcal{C}_0|^{-1/2}\chi_{\mathcal{C}_0}$. Then $\mathcal{C} = \mathcal{C}_0$ is the required cap. Note that, by Lemma 4.6, $\|f_0\|_1 \geq c$ for some positive universal constant c.

Suppose that functions $R \mapsto \eta(R)$ and $R \mapsto \Theta(R)$ are chosen so that

$$\eta(R) \to 0 \text{ as } R \to \infty \tag{4.37}$$
$$\zeta(\eta(R))^2 \leq \Theta(R) \text{ for all } R. \tag{4.38}$$

Then, by (4.36), $F - \mathcal{F}$ already satisfies the desired inequalities in $L^2(\sigma)$, so it suffices to show that $\mathcal{F}(\gamma(s)) \equiv 0$ whenever $|s - s_0| \geq R r_0$, and that $\|\mathcal{F}\|_\infty < R r_0^{-1/2}$.

Each summand of $\mathcal{F}$ satisfies $f_k \le C(\eta)|\mathcal{C}_k|^{-1/2}\chi_{\mathcal{C}_k}$ where $C(\eta) < \infty$ depends only on η, and in particular, f_k is supported in $\mathcal{C}_k$. $\|f_k\|_2 \ge \eta$ for all $k \le N$, by definition of N. Therefore by Lemma 4.14, there exists a function $\eta \mapsto \lambda(\eta) < \infty$, such that if δ is sufficiently small as a function of η then $d(\mathcal{C}_k, \mathcal{C}_0) \le \lambda(\eta)$ for every $k \le N$. This is needed for $\eta = \eta(R)$ for all R in the compact set $[1, \bar{R}]$, so such a δ may be chosen as a function of $\bar{R}$ alone; conditions already imposed on δ above make it a function of both ϵ and $\bar{R}$.

In the region of all $\gamma(s) \in \Gamma$ such that $|s - s_0| \ge Rr_0$, either $f_k \equiv 0$, or $\mathcal{C}_k$ has radius $\ge \frac{1}{2}Rr_0$, or the center $\gamma(s_k)$ of $\mathcal{C}_k$ is such that $|s_k - s_0| \ge \frac{1}{2}Rr_0$. Choose a function $R \mapsto \eta(R)$ which tends to 0 sufficiently slowly that the latter two cases would contradict the inequality $d(\mathcal{C}_k, \mathcal{C}_0) \le \lambda$, and therefore cannot arise. Then $\mathcal{F}(\gamma(s)) \equiv 0$ when $|s - s_0| \ge Rr_0$.

With the function η specified, Θ can be defined by $\Theta(R) := \zeta(\eta(R))^2$. Then

$$\int_{\{s:|s-s_0|\ge Rr_0\}} F(\gamma(s))^2 ds \le \Theta(R), \quad \forall R \in [1, \bar{R}]. \tag{4.39}$$

We claim next that $\|\mathcal{F}\|_\infty < Rr_0^{-1/2}$ if R is sufficiently large as a function of η. Indeed, because the summands f_k have pairwise disjoint supports, it suffices to control $\max_{k\le N} \|f_k\|_\infty$. Again by Lemma 4.10, $\|f_k\|_\infty \le C(\eta)|\mathcal{C}_k|^{-1/2}$. If $\eta(R)$ is chosen to tend to zero sufficiently slowly as $R \to \infty$ to ensure that $2C(\eta(R))\cosh\lambda(\eta(R)) < R$ for all $k \le N$, then

$$\|f_k\|_\infty < R(2\cosh\lambda(\eta(R)))^{-1} r_k^{-1/2} \le Rr_0^{-1/2}$$

since $d(\mathcal{C}_k, \mathcal{C}_0) \le \lambda(\eta(R))$. It follows that

$$\int_{\{s:F(\gamma(s))\ge Rr_0^{-1/2}\}} F(\gamma(s))^2 ds \le \Theta(R), \quad \forall R \in [1, \bar{R}], \tag{4.40}$$

provided that Θ is defined as above.

The final function η must be chosen to tend to zero slowly enough to satisfy the requirements of the proofs of both (4.39) and (4.40). □

4.6 A concentration compactness result

Let us start by making precise the previously mentioned notions of *uniform integrability* and *concentration at a point.*

Definition 4.18. Let $(X, \mathcal{S}, \mu)$ be a measure space and let $p \in [1, \infty)$. A subset $\mathcal{U}$ of $L^p(X)$ is called *uniformly integrable* of order p if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every measurable subset A of X for which $\mu(A) < \delta$,

$$\int_A |f|^p d\mu < \epsilon, \quad \text{for every } f \in \mathcal{U}.$$

If $\mathcal{U}$ is a bounded subset of $L^p(X)$, it is straightforward to check that $\mathcal{U}$ is uniformly integrable of order p if and only if

$$\lim_{R\to\infty} \int_{\{|f|>R\}} |f|^p d\mu = 0$$

uniformly with respect to $f \in \mathcal{U}$.

If the measure space is finite, then it is well-known that uniform integrability coupled with a weaker form of convergence is enough to ensure strong convergence:

Proposition 4.19. *Suppose $\mu(X) < \infty$, and let $p \in [1, \infty)$. Let $\{f_n\}$ be a sequence in $L^p(X)$ and let $f \in L^p(X)$. The sequence $\{f_n\}$ converges to f in L^p if (and only if) the following two conditions are satisfied:*

(i) The sequence $\{f_n\}$ converges in measure to f;

(ii) The family $\{f_n : n \in \mathbb{N}\}$ is uniformly integrable of order p.

Proof. We prove only the direction that will be of use to us (the *if* part). The assumptions, together with the fact that any family consisting of one single function is automatically uniformly integrable, make it clear that the family $\{f_n - f : n \in \mathbb{N}\}$ is also uniformly integrable of order p. Given $\epsilon > 0$,

$$\int_{\{|f-f_n|<\epsilon\}} |f - f_n|^p d\mu \leq \epsilon^p \mu(X).$$

On the other hand, $\mu(\{|f - f_n| \geq \epsilon\}) \to 0$ as $n \to \infty$ because of the convergence in measure, and so

$$\lim_{n\to\infty} \int_{\{|f-f_n|\geq\epsilon\}} |f - f_n|^p d\mu = 0$$

by definition of uniform integrability. The conclusion follows. □

Definition 4.20. For $p = \gamma(s_0) \in \Gamma$, we say that a sequence $\{f_n\}$ of functions in $L^2(\sigma)$ satisfying $\|f_n\|_2 \to 1$ as $n \to \infty$ *concentrates at* p if for every $\epsilon, r > 0$ there exists $N \in \mathbb{N}$ such that, for every $n \geq N$,

$$\int_{|s-s_0|\geq r} |f_n(\gamma(s))|^2 ds < \epsilon.$$

These introductory remarks are relevant in the context of the following concentration-compactness result which is a consequence of Proposition 4.16:

Proposition 4.21. *Let* $\{f_n\}$ *be an extremizing sequence for* (4.2) *of nonnegative functions in* $L^2(\sigma)$. *Then there exists a subsequence, again denoted* $\{f_n\}$, *and a decomposition* $f_n = F_n + G_n$ *where* F_n *and* G_n *are nonnegative with disjoint supports,* $\|G_n\|_2 \to 0$, *and* $\{F_n\}$ *satisfies one of the two possibilities:*

$$\{F_n : n \in \mathbb{N}\} \text{ is uniformly integrable of order } 2. \tag{4.41}$$

$$\{F_n\} \text{ concentrates at a point of } \Gamma. \tag{4.42}$$

Proof. Apply Proposition 4.16 to each element of the sequence $\{f_n\}$ to get a decomposition $f_n = F_n + G_n$ where $F_n, G_n \geq 0$ have disjoint supports and $\|G_n\|_2 \to 0$. For each n, there exists a cap $\mathcal{C}_n = \mathcal{C}(\gamma(s_n), r_n)$ such that F_n is upper normalized with respect to $\mathcal{C}_n$. It is important to note that Proposition 4.16 yields a uniform statement in n i.e. the gauge function Θ in conditions (4.28) and (4.29) can be chosen independently of n. Let $r^* := \limsup_{n\to\infty} r_n$.

If $r^* > 0$, let $\{r_{n_k}\}_k$ be a subsequence which converges to r^*. Renaming, we may assume that $\lim_{n\to\infty} r_n = r^*$. Choosing $N \in \mathbb{N}$ large enough so that $n \geq N$ implies $r_n \geq r^*/4$, we have

$$\int_{\{F_n > R\}} F_n(\gamma(s))^2 ds \leq \int_{\{F_n \geq \frac{R\sqrt{r^*}}{2} r_n^{-1/2}\}} F_n(\gamma(s))^2 ds \leq \Theta\Big(\frac{R\sqrt{r^*}}{2}\Big),$$

which tends to 0 as $R \to \infty$, uniformly in n. In other words, the sequence $\{F_n\}$ is uniformly integrable of order 2.

If $r^* = 0$, choose a subsequence $\{s_{n_k}\}_k$ converging to some $s^* \in [0, \ell]$. Renaming, we may assume that $\lim_{n\to\infty} s_n = s^*$. Let $\epsilon, r > 0$ be given. Start by choosing $N_1 = N_1(r)$ such that $|s_n - s^*| \leq r/2$ if $n > N_1$. Then $|s - s^*| \geq r$ implies $|s - s_n| \geq r/2$ if $n > N_1$. Choose $R = R(\epsilon)$ such that $\Theta(R) < \epsilon$. Finally, choose $N_2 = N_2(\epsilon, r)$ such that

$$r_n \leq \frac{r}{2R} \text{ if } n > N_2.$$

Such N_2 exists since $\lim_{n\to\infty} r_n = 0$. If $n > \max\{N_1, N_2\}$, we then have that

$$\int_{|s-s^*|\geq r} F_n(\gamma(s))^2 ds \leq \int_{|s-s_n|\geq \frac{r}{2}} F_n(\gamma(s))^2 ds \leq \int_{|s-s_n|\geq Rr_n} F_n(\gamma(s))^2 ds \leq \Theta(R) < \epsilon$$

i.e. the sequence $\{F_n\}$ concentrates at $\gamma(s^*)$. $\square$

Fanelli, Vega and Visciglia [24] proved the following interesting modification of a well-known result of Brézis and Lieb [26] which does not require the a.e. pointwise convergence of the sequence of functions $\{h_n\}$.

Proposition 4.22 ([24]). *Let $\mathcal{H}$ be a Hilbert space and T be a bounded linear operator from $\mathcal{H}$ to $L^p(\mathbb{R}^d)$, for some $p \in (2,\infty)$. Let $\{h_n\} \in \mathcal{H}$ be such that*

(i) $\|h_n\|_{\mathcal{H}} = 1$;

(ii) $\lim_{n\to\infty} \|Th_n\|_{L^p(\mathbb{R}^d)} = \|T\|$;

(iii) $h_n \rightharpoonup h \neq 0$;

(iv) $Th_n \to Th$ *a.e. in* $\mathbb{R}^d$.

Then $h_n \to h$ in $\mathcal{H}$; in particular, $\|h\|_{\mathcal{H}} = 1$ and $\|Th\|_{L^p(\mathbb{R}^d)} = \|T\|$.

We will be applying this proposition to the adjoint Fourier restriction operator on Γ with $\mathcal{H} = L^2(\sigma)$. We lose no generality in assuming that conditions (i) and (ii) are automatically satisfied by any extremizing sequence $\{f_n\}$. After passing to a subsequence, we may assume that $\{f_n\}$ converges weakly in $L^2(\sigma)$ by Alaoglu's theorem. If $f_n \rightharpoonup f$ in $L^2(\sigma)$, then condition (iv) follows because σ is compactly supported. Thus Proposition 4.22 states that, for compactly supported measures, the only obstruction to the existence of extremizers is the possibility that every L^2 weak limit of any extremizing sequence be zero.

The advantage of working with *nonnegative* extremizing sequences in this context first appeared in the work of Kunze [17]. The following is the sole step in the analysis which works only for nonnegative extremizing sequences:

Lemma 4.23. *Let $\{f_n\}$ and $\{F_n\}$ be as in Proposition 4.21. Suppose that $\{F_n\}$ satisfies condition* (4.41). *Then $\{f_n\}$ is precompact in $L^2(\sigma)$.*

Proof. By assumption the sequence $\{F_n\}$ consists of nonnegative functions and is uniformly integrable of order 2. Moreover, $\|F_n\|_2 \to 1$ as $n \to \infty$.

We first show that every L^2 weak limit of $\{F_n\}$ is nonzero. The set of L^2 weak limits of $\{F_n\}$ is clearly nonempty. We can assume, possibly after extraction of a subsequence, that $F_n \rightharpoonup F$ for some $F \in L^2(\sigma)$. Suppose that $F = 0$ a.e. on Γ. Then

$$\int_\Gamma F_n d\sigma \to \int_\Gamma F d\sigma = 0 \text{ as } n \to \infty.$$

Since $F_n \geq 0$, this means that the sequence $\{F_n\}$ converges to 0 in $L^1(\sigma)$, and thus $F_n \to 0$ in measure. In view of Proposition 4.19, $F_n \to 0$ in $L^2(\sigma)$, and so $1 = \|F_n\|_2 \to 0$ as $n \to \infty$, a contradiction. Thus $F \neq 0$ as was to be shown.

We use this to prove that the sequence $\{f_n\}$ is precompact. Since $f_n = F_n + G_n$, we have that

$$\|\widehat{F_n\sigma}\|_6 \geq \|\widehat{f_n\sigma}\|_6 - \|\widehat{G_n\sigma}\|_6 \geq \|\widehat{f_n\sigma}\|_6 - \mathbf{C}[\Gamma]\|G_n\|_2.$$

It follows that $\|\widehat{F_n\sigma}\|_6 \to \mathbf{C}[\Gamma]$ as $n \to \infty$. This means that $\{F_n\}$ is itself an extremizing sequence. By the previous paragraph, we can assume, possibly after extraction of a subsequence, that $F_n \rightharpoonup F$ for some nonzero $F \in L^2(\sigma)$. Then an application of Proposition 4.22 (with $d = 2$, $p = 6$, $\mathcal{H} = L^2(\sigma)$ and $T =$ Fourier extension operator on Γ defined by $Tf := \widehat{f\sigma}$) allows us to conclude that $f_n \to F$ in $L^2(\sigma)$ as $n \to \infty$, and so $\{f_n\}$ is precompact. □

We will be done with the proof of Theorem 4.3 once we show that condition (4.42) in Proposition 4.21 cannot happen, and this is the subject of the next two sections.

4.7 Exploring concentration

We start by recalling some aspects of Foschi's work [18]. Consider the parabola

$$\mathbb{P} := \{(y, z) \in \mathbb{R}^2 : z = y^2\}$$

equipped with projection measure[3] $d\sigma_{\mathbb{P}} := dy$ instead of arclength measure. We have an inequality

$$\|\widehat{f\sigma_{\mathbb{P}}}\|_{L^6(\mathbb{R}^2)} \leq \mathbf{C}_F \|f\|_{L^2(\sigma_{\mathbb{P}})}, \tag{4.43}$$

where $\mathbf{C}_F$ denotes again the optimal constant. Foschi showed that extremizers exist for the inequality (4.43) and computed the optimal constant

$$\mathbf{C}_F = \frac{(2\pi)^{1/2}}{12^{1/12}}. \tag{4.44}$$

An example of one such extremizer is given by the Gaussian $G(y) := e^{-y^2}$. Other extremizers are obtained from G by space-time translations, parabolic dilations, space rotations, phase shifts and Galilean transformations.

A straightforward scaling argument shows the following: consider the dilated parabola

$$\mathbb{P}_\mu := \Big\{(y, z) \in \mathbb{R}^2 : z = \frac{\mu y^2}{2}\Big\},$$

[3]See [21] and the references therein for a discussion of why this measure is natural from a geometric point of view.

again equipped with projection measure $d\sigma_{\mathbb{P}_\mu} = dy$. Then the optimal constant in the inequality

$$\|\widehat{f\sigma_{\mathbb{P}_\mu}}\|_{L^6(\mathbb{R}^2)} \leq \mathbf{C}_F[\mu]\|f\|_{L^2(\sigma_{\mathbb{P}_\mu})} \tag{4.45}$$

satisfies $\mathbf{C}_F[\mu] = \mathbf{C}_F[1]\mu^{-1/6}$. In particular, $\mathbf{C}_F[1] = (2\pi)^{1/2}3^{-1/12}$.

Since projection measure can be regarded as a limit of arclength measures, the analysis of extremizers for inequality (4.45) is of significance for our discussion. If an extremizing sequence $\{f_n\}$ for inequality (4.2) concentrates at a point $\gamma(s) \in \Gamma$, then the sequence consisting of certain natural transplantations of f_n to functions $\tilde{f}_n$ (each $\tilde{f}_n$ begin defined on the limiting parabola) will also be an extremizing sequence for (4.45) with $\mu = \kappa(\gamma(s))$. To see why this is the case, denote by $T_{s,r}$ the restriction of the Fourier extension operator to a given cap $\mathcal{C} = \mathcal{C}(s,r) \subset \Gamma$:

$$T_{s,r}f(x,t) = \int_{\mathcal{C}} f(y)e^{-i(x,t)\cdot y}d\sigma(y) = \int_{s-r}^{s+r} f(\gamma(s))e^{-i(x,t)\cdot\gamma(s)}ds.$$

We are interested in the operator norm $\|T_{s,r}\| := \sup_{0\neq f\in L^2(\sigma)} \|T_{s,r}f\|_6\|f\|_{L^2(\sigma)}^{-1}$, and prove the following result:

Proposition 4.24. *For every $s \in (0,\ell)$,*

$$\lim_{r\to 0^+} \|T_{s,r}\| = \mathbf{C}_F[\kappa(\gamma(s))].$$

Proof. Fix $s \in (0,\ell)$ and let $\kappa := \kappa(\gamma(s))$. That the lefthand side is greater than or equal to the righthand side can be easily seen by taking the function $G(y) = e^{-\frac{\kappa y^2}{2}}$ and considering the dilated family $G_\delta(y) = \delta^{-1/2}G(\delta^{-1}y)$ for $\delta > 0$. For details, see §4.8 below.

So we focus on proving the reverse inequality. Let $\sigma_{s,r}$ denote the restriction of arclength measure σ to the cap $\mathcal{C}(s,r)$, and denote the triple convolution of $\sigma_{s,r}$ with itself by $\sigma_{s,r}^{(*3)} := \sigma_{s,r} * \sigma_{s,r} * \sigma_{s,r}$. We have that

$$\begin{aligned}
\|T_{s,r}f\|_6^6 &= (2\pi)^2 \iint_{\mathbb{R}^2} |f\sigma_{s,r} * f\sigma_{s,r} * f\sigma_{s,r}(\xi,\tau)|^2 d\xi d\tau \\
&\leq (2\pi)^2 \iint |f|^2\sigma_{s,r} * |f|^2\sigma_{s,r} * |f|^2\sigma_{s,r}(\xi,\tau) \cdot \sigma_{s,r} * \sigma_{s,r} * \sigma_{s,r}(\xi,\tau) d\xi d\tau \\
&\leq (2\pi)^2 \sup_{(\xi,\tau)\in\ \mathrm{supp}(\sigma_{s,r}^{(*3)})} \sigma_{s,r}^{(*3)}(\xi,\tau) \cdot \iint |f|^2\sigma_{s,r} * |f|^2\sigma_{s,r} * |f|^2\sigma_{s,r}(\xi,\tau) d\xi d\tau \\
&= (2\pi)^2 \|\sigma_{s,r}^{(*3)}\|_{L^\infty(\mathbb{R}^2)} \|f\|_{L^2(\sigma_{s,r})}^6,
\end{aligned}$$

where we used Hölder's inequality twice. It will then be enough to show that

$$\|\sigma_{s,r} * \sigma_{s,r} * \sigma_{s,r}\|_\infty \to \frac{\mathbf{C}_F[\kappa]^6}{(2\pi)^2} \text{ as } r \to 0^+.$$

After applying a rigid motion[4] of $\mathbb{R}^2$ to a cap $\mathcal{C} = \mathcal{C}(s,r)$, we may parametrize it in the following way:

$$\begin{array}{rcl} \widetilde{\gamma}_{s,r} : I_r & \to & \mathbb{R}^2 \\ y & \mapsto & \Big(y, g(y) = \frac{\kappa}{2}y^2 + \phi(y)\Big), \end{array}$$

where I_r is an interval centered at the origin of length $\asymp r$, $\kappa = g''(0)(1+g'(0)^2)^{-3/2}$ is the curvature of Γ at $\gamma(s)$, and ϕ is a real-valued smooth function satisfying $\phi(y) = O(|y|^3)$ as $|y| \to 0$. We also let $\eta_r \in C_0^\infty(\mathbb{R})$ be a mollified version of the characteristic function of the interval I_r: to accomplish this, fix $\eta \in C_0^\infty(\mathbb{R})$ such that $\eta \equiv 1$ on $[-1,1]$ and $\eta(\xi) = 0$ if $|\xi| \geq 2$, and define $\eta_r := \eta(2|I_r|^{-1}\cdot)$.

With these definitions we have that, for $|\xi| \leq |I_r|/2$,

$$\sigma_{s,r}(\xi,\tau) = G_r(\xi)\delta(\tau - g(\xi))d\xi d\tau,$$

where $G_r(\xi) := (1+g'(\xi)^2)^{1/2}\eta_r(\xi)$ is a smooth function supported on $\{\xi \in \mathbb{R} : |\xi| \lesssim r\}$. Observe that G_r (and therefore $\sigma_{s,r}$) depends also on s, even though this is not explicitly indicated by the notation, with uniform bounds on s and appropriately uniform dependence on r after dilations. Following [18] and [23] we compute:

$$\begin{aligned} \sigma_{s,r}^{(*3)}(\xi,\tau) &= \iint G_r(\omega_1)G_r(\omega_2-\omega_1)G_r(\xi-\omega_2)\delta(\tau - g(\omega_1) - g(\omega_2-\omega_1) - g(\xi-\omega_2))d\omega_1 d\omega_2 \\ &= \iiint_{|\omega|\lesssim r} G_r(\omega_1)G_r(\omega_2)G_r(\omega_3)\delta\begin{pmatrix} \tau - g(\omega_1) - g(\omega_2) - g(\omega_3) \\ \xi - \omega_1 - \omega_2 - \omega_3 \end{pmatrix} d\omega_1 d\omega_2 d\omega_3. \end{aligned}$$

Change variables $\omega = O \cdot \zeta$, where $O \in SO(3)$ is the orthogonal matrix

$$O = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}.$$

Under this transformation,

$$\langle(1,1,1),\omega\rangle = \langle(1,1,1), O\cdot\zeta\rangle = \langle O^*\cdot(1,1,1),\zeta\rangle = \sqrt{3}\langle(1,0,0),\zeta\rangle = \sqrt{3}\zeta_1.$$

[4]For a more detailed discussion of this procedure, see §4.8 below.

The integral becomes

$$\sigma_{s,r}^{(*3)}(\xi,\tau) = \iiint_{|\zeta|\lesssim r} G_r(\frac{\zeta_1}{\sqrt{3}} - \frac{\zeta_2}{\sqrt{2}} - \frac{\zeta_3}{\sqrt{6}})G_r(\frac{\zeta_1}{\sqrt{3}} + \frac{\zeta_2}{\sqrt{2}} - \frac{\zeta_3}{\sqrt{6}})G_r(\frac{\zeta_1}{\sqrt{3}} + \frac{2\zeta_3}{\sqrt{6}})\cdot$$
$$\cdot\delta\begin{pmatrix} \tau - g(\frac{\zeta_1}{\sqrt{3}} - \frac{\zeta_2}{\sqrt{2}} - \frac{\zeta_3}{\sqrt{6}}) - g(\frac{\zeta_1}{\sqrt{3}} + \frac{\zeta_2}{\sqrt{2}} - \frac{\zeta_3}{\sqrt{6}}) - g(\frac{\zeta_1}{\sqrt{3}} + \frac{2\zeta_3}{\sqrt{6}}) \\ \xi - \sqrt{3}\zeta_1 \end{pmatrix} d\zeta_1 d\zeta_2 d\zeta_3.$$

Renaming variables and setting

$$\mathfrak{G}_r(\xi;\zeta_2,\zeta_3) = G_r\Big(\frac{\xi}{3} - \frac{\zeta_2}{\sqrt{2}} - \frac{\zeta_3}{\sqrt{6}}\Big)G_r\Big(\frac{\xi}{3} + \frac{\zeta_2}{\sqrt{2}} - \frac{\zeta_3}{\sqrt{6}}\Big)G_r\Big(\frac{\xi}{3} + \frac{2\zeta_3}{\sqrt{6}}\Big),$$

this simplifies to

$$\sigma_{s,r}^{(*3)}(\xi,\tau) = \frac{\chi(|\xi|\lesssim r)}{\sqrt{3}} \iint_{|(\zeta_2,\zeta_3)|\lesssim r} \mathfrak{G}_r(\xi;\zeta_2,\zeta_3)\cdot$$
$$\cdot\,\delta\Big(\tau - \frac{\kappa}{6}\xi^2 - \frac{\kappa}{2}|\zeta|^2 - \phi\Big(\frac{\xi}{3} - \frac{\zeta_2}{\sqrt{2}} - \frac{\zeta_3}{\sqrt{6}}\Big) - \phi\Big(\frac{\xi}{3} + \frac{\zeta_2}{\sqrt{2}} - \frac{\zeta_3}{\sqrt{6}}\Big) - \phi\Big(\frac{\xi}{3} + \frac{2\zeta_3}{\sqrt{6}}\Big)\Big) d\zeta_2 d\zeta_3.$$

Introducing polar coordinates on the (ζ_2,ζ_3)-plane,

$$\sigma_{s,r}^{(*3)}(\xi,\tau) = \frac{\chi(|\xi|\lesssim r)}{\sqrt{3}} \int_0^{2\pi}\int_{0\leq\rho\lesssim r} \widetilde{\mathfrak{G}}_r(\xi;\rho,\theta)\delta\Big(\tau - \frac{\kappa}{6}\xi^2 - 3\phi\Big(\frac{\xi}{3}\Big) - \psi(\xi;\rho,\theta)\Big)\rho d\rho d\theta$$

where

$$\widetilde{\mathfrak{G}}_r(\xi;\rho,\theta) = G_r\Big(\frac{\xi}{3} - \rho\Big(\frac{\cos\theta}{\sqrt{2}} + \frac{\sin\theta}{\sqrt{6}}\Big)\Big)G_r\Big(\frac{\xi}{3} + \rho\Big(\frac{\cos\theta}{\sqrt{2}} - \frac{\sin\theta}{\sqrt{6}}\Big)\Big)G_r\Big(\frac{\xi}{3} + \frac{2}{\sqrt{6}}\rho\sin\theta\Big)$$

and

$$\psi(\xi;\rho,\theta) = \frac{\kappa}{2}\rho^2 - 3\phi\Big(\frac{\xi}{3}\Big) + \phi\Big(\frac{\xi}{3} - \rho\Big(\frac{\cos\theta}{\sqrt{2}} + \frac{\sin\theta}{\sqrt{6}}\Big)\Big) +$$
$$\phi\Big(\frac{\xi}{3} + \rho\Big(\frac{\cos\theta}{\sqrt{2}} - \frac{\sin\theta}{\sqrt{6}}\Big)\Big) + \phi\Big(\frac{\xi}{3} + \frac{2}{\sqrt{6}}\rho\sin\theta\Big). \quad (4.46)$$

We prepare to change variables again. Recall the assumption that $\kappa > 0$. Note that $\psi(\xi;0,\theta) = 0$ and $\psi(\xi;\rho,\theta) > 0$ for every sufficiently small $\rho > 0$. Moreover, a calculation shows that the same thing happens with first derivatives: $\partial_\rho\psi(\xi;0,\theta) = 0$

and $\partial_\rho\psi(\xi;\rho,\theta) > 0$ if $\rho > 0$ is sufficiently small. We also have that $\partial^2_\rho\psi(\xi;0,\theta) = \kappa + \phi''(\xi/3) = g''(\xi/3)$.

For $u \geq 0$, set $\rho = \rho(u) := \psi^{-1}(u)$, and compute:

$$\sigma^{(*3)}_{s,r}(\xi,\tau) = \frac{\chi(|\xi| \lesssim r)}{\sqrt{3}} \int_0^{2\pi} \int_{0\leq u\leq C\psi} \widetilde{\mathfrak{G}}_r(\xi;\rho(u),\theta)\delta\Big(\tau - \frac{\kappa}{6}\xi^2 - 3\phi(\frac{\xi}{3}) - u\Big)\frac{\rho(u)}{\partial_\rho\psi(\rho(u))}dud\theta \tag{4.47}$$

$$= \frac{\chi(|\xi| \lesssim r)}{\sqrt{3}} \int_0^{2\pi} \widetilde{\mathfrak{G}}_r(\xi;\rho(\tau - \frac{\kappa}{6}\xi^2 - 3\phi(\frac{\xi}{3})),\theta)\frac{\rho(\tau - \frac{\kappa}{6}\xi^2 - 3\phi(\frac{\xi}{3}))}{\partial_\rho\psi(\rho(\tau - \frac{\kappa}{6}\xi^2 - 3\phi(\frac{\xi}{3})))}d\theta. \tag{4.48}$$

Note that, for each $\theta \in [0,2\pi]$, the integrand in (4.48) is supported in the region

$$\Big\{(\xi,\tau) \in \mathbb{R}^2 : 0 \leq \tau - \frac{\kappa}{6}\xi^2 - 3\phi(\frac{\xi}{3}) \leq C\psi(\xi;r,\theta)\Big\},$$

where the constant $C < \infty$ is large enough that the restriction $u \leq C\psi(\xi;r,\theta)$ in the inner integral of (4.47) becomes redundant because of support limitations on factors present in its integrand. From expression (4.48) it is clear that the restriction of $\sigma_{s,r} * \sigma_{s,r} * \sigma_{s,r}$ to its support defines a continuous function of (ξ,τ) at $(0,0)$. Indeed, $\widetilde{\mathfrak{G}}_r$ is a smooth function of compact support in the variables ξ,θ and $(\tau - \frac{\kappa}{6}\xi^2 - 3\phi(\frac{\xi}{3}))^{1/2}$. Additionally, as $(\xi,\tau) \to (0,0)$,

$$\frac{\rho(\tau - \frac{\kappa}{6}\xi^2 - 3\phi(\frac{\xi}{3}))}{\partial_\rho\psi(\rho(\tau - \frac{\kappa}{6}\xi^2 - 3\phi(\frac{\xi}{3})))} = \frac{\rho(\tau - \frac{\kappa}{6}\xi^2 - 3\phi(\frac{\xi}{3})) - \rho(0)}{\partial_\rho\psi(\rho(\tau - \frac{\kappa}{6}\xi^2 - 3\phi(\frac{\xi}{3}))) - \partial_\rho\psi(\rho(0))} \to \frac{1}{\partial^2_\rho\psi(\rho(0))} = \frac{1}{g''(0)}.$$

If $(\xi,\tau) \in \operatorname{supp}(\sigma_{r,s} * \sigma_{r,s} * \sigma_{r,s})$ and $r \to 0^+$, then $(\xi,\tau) \to (0,0)$. It follows that

$$\lim_{r\to 0^+} \|\sigma_{r,s} * \sigma_{r,s} * \sigma_{r,s}\|_\infty = \frac{1}{\sqrt{3}} \int_0^{2\pi} \widetilde{\mathfrak{G}}_0(0;\rho(0),\theta)\frac{1}{\partial^2_\rho\psi(0;\rho(0),\theta)}d\theta.$$
$$= \frac{2\pi}{\sqrt{3}}\frac{(1+g'(0)^2)^{3/2}}{g''(0)} = \frac{2\pi}{\sqrt{3}}\frac{1}{\kappa} = \frac{\mathbf{C}_F[\kappa]^6}{(2\pi)^2},$$

as desired.

□

An immediate consequence is that an extremizing sequence which concentrates must do so at a point of minimal curvature:

Corollary 4.25. *Let $\{f_n\} \subset L^2(\sigma)$ be an extremizing sequence of nonnegative functions for inequality* (4.2). *Suppose that $\{f_n\}$ concentrates at a point $\gamma(s) \in \Gamma$. Then $\kappa(\gamma(s)) = \lambda$.*

Corollary 4.26. *Let $\{f_n\}$ and $\{F_n\}$ be as in Proposition 4.21. Suppose that $\{F_n\}$ concentrates at a point $\gamma(s) \in \Gamma$. Then $\kappa(\gamma(s)) = \lambda$.*

4.8 Comparing optimal constants

As was mentioned before, a potential obstruction to the existence of extremizers for inequality (4.2), and certainly to the precompactness of arbitrary nonnegative extremizing sequences, is the possibility that for an extremizing sequence satisfying $\|f_n\|_{L^2(\sigma)} = 1$, $|f_n|^2$ could conceivably converge weakly to a Dirac mass at a point on the curve. Indeed, let $p \in \Gamma$. The osculating parabola of Γ at p is $\mathbb{P}_{\kappa(p)}$. If $\mathbf{C}[\Gamma]$ were equal to $\mathbf{C}_F[\kappa(p)]$, then Foschi's work implies that there would necessarily exist extremizing sequences of the type just described. Therefore an essential step in our analysis is to determine under which conditions one has that

$$\mathbf{C}[\Gamma] > \max_{p \in \Gamma} \mathbf{C}_F[\kappa(p)] = \mathbf{C}_F[\lambda].$$

The main goal of this section is to prove the following:

Proposition 4.27. *Let $\Gamma \subset \mathbb{R}^2$ be an arc satisfying the conditions of Theorem 4.3. Then $\mathbf{C}[\Gamma] > \mathbf{C}_F[\lambda]$.*

Introducing local coordinates

Let $p_0 \in \Gamma$ be a point of minimum curvature i.e. such that $\kappa(p_0) = \lambda$. We will be assuming that p_0 is *not* an endpoint of Γ, and we postpone the discussion of the validity of this assumption until the end of this section. By translating the curve we can assume without loss of generality that $p_0 = (0,0)$. Possibly after a suitable rotation, the arc Γ can be parametrized in a neighborhood of the origin in the following way:

$$\begin{aligned} \widetilde{\gamma} : I &\rightarrow \mathbb{R}^2 \\ y &\mapsto (y, h(y)), \end{aligned}$$

where[5]

$$h(y) = \frac{\lambda y^2}{2} + ay^4 + \psi(y)$$

and ψ is a real-valued smooth function satisfying $\psi(y) = O(|y|^5)$ as $|y| \to 0$. The parameter a, on the other hand, is a function of the second derivative of the curvature with respect to arclength at 0; see formula (4.51) below. We take $I \subseteq \mathbb{R}$ to be an interval centered at the origin which will be chosen as a function of λ, a and ψ later on. Finally, let $\eta_I \in C_0^\infty(\mathbb{R})$ be a mollified version of the characteristic function of I such that $\eta_I \equiv 1$ on I and $\eta_I \equiv 0$ outside $2 \cdot I$. As before, we accomplish this by fixing $\eta \in C_0^\infty(\mathbb{R})$ such that $\eta \equiv 1$ on $[-1, 1]$ and $\eta(y) = 0$ if $|y| \geq 2$, and defining $\eta_I := \eta(2|I|^{-1}\cdot)$.

These data determine a compact arc $\widetilde{\gamma}(I) =: \widetilde{\Gamma} \subset \Gamma$ in the plane, which comes equipped with arclength measure $\widetilde{\sigma}$ given, for $y \in I$, by

$$d\widetilde{\sigma}(y) = (1 + h'(y)^2)^{1/2}\eta_I(y)dy. \tag{4.49}$$

Taylor expanding around $y = 0$, we note that the first two non-zero terms in the expansion of $d\widetilde{\sigma}(y)$ are independent of a and ψ:

$$d\widetilde{\sigma}(y) = \Big(1 + \frac{\lambda^2}{2}y^2 + O(y^4)\Big)\eta_I(y)dy. \tag{4.50}$$

On the other hand, the curvature of $\widetilde{\Gamma}$ at a point $\widetilde{\gamma}(y)$ is given, for small y, by

$$\kappa(y) = \frac{h''(y)}{(1 + h'(y)^2)^{3/2}} = \lambda + (12a - \frac{3\lambda^3}{2})y^2 + O_{\lambda,a,\psi}(y^4).$$

We have that $\kappa(0) = h''(0) = \lambda > 0$. For κ to have a minimum at $y = 0$ it is necessary that $a \geq (\frac{\lambda}{2})^3$. Let s denote, as usual, the arclength parameter for $\widetilde{\Gamma}$. Then, by a straightforward application of the chain rule, we have that

$$\frac{d^2\kappa}{ds^2}(0) = \frac{d^2\kappa}{dy^2}(0) = 24a - 3\lambda^3, \tag{4.51}$$

and so hypothesis (4.3) in Theorem 4.3 is equivalent to $a < \frac{3}{2}(\frac{\lambda}{2})^3$. All in all we have that[6]

$$\Big(\frac{\lambda}{2}\Big)^3 \leq a < \frac{3}{2}\Big(\frac{\lambda}{2}\Big)^3. \tag{4.52}$$

[5]There is no cubic term in the expression for h because by assumption the curvature has a minimum at $\gamma(0) = (0, 0) = p_0$. Constant and linear terms were likewise removed via the affine change of variables described above.

[6]See Chapter 5 for a partial result concerning the case when condition (4.52) fails in a rather strong sense.

In what follows, we will again denote by $\mathbf{C}[\widetilde{\Gamma}] = \mathbf{C}[\widetilde{\Gamma}; \lambda, a, \psi, I]$ the optimal constant in the inequality

$$\|\widehat{f\widetilde{\sigma}}\|_{L^6(\mathbb{R}^2)} \leq \mathbf{C}[\widetilde{\Gamma}]\|f\|_{L^2(\widetilde{\sigma})}. \tag{4.53}$$

The unperturbed case

If $a = \psi = 0$, we are dealing with the unperturbed parabola $\mathbb{P}_\lambda$. As we have mentioned before, the corresponding optimal constant satisfies $\mathbf{C}_F[\lambda] = \mathbf{C}_F[1]\lambda^{-1/6}$, and examples of extremizers are given by Gaussian functions $e^{-\rho y^2/2}$ for $\rho > 0$.

Set $G_0(y) := e^{-\lambda y^2/2}$, and consider functions of the form $f = G_0 + \phi$ with $\phi \in L^2(\sigma_{\mathbb{P}_\lambda})$. Consider the corresponding functional

$$\phi \mapsto \mathbf{C}_F[\lambda]^6\Big(\int_{\mathbb{R}} |G_0 + \phi|^2 dy\Big)^3 - \iint_{\mathbb{R}^2} |G_1 + \phi_1|^6 dxdt \geq 0, \tag{4.54}$$

where $G_1(x,t) := \widehat{G_0\sigma_{\mathbb{P}_\lambda}}(-x,t)$ and $\phi_1(x,t) := \widehat{\phi\sigma_{\mathbb{P}_\lambda}}(-x,t)$. The former can be explicitly computed:

$$\begin{aligned}
G_1(x,t) =& \widehat{G_0\sigma_{\mathbb{P}_\lambda}}(-x,t) = \int_{\mathbb{R}} G_0(y) e^{-it\frac{\lambda y^2}{2}} e^{ixy} dy \\
=& \int_{\mathbb{R}} e^{-(1+it)\frac{\lambda y^2}{2}} e^{ixy} dy \\
=& \Big(\frac{2\pi}{\lambda}\Big)^{1/2} (1+it)^{-1/2} e^{-\frac{x^2}{2\lambda(1+it)}}.
\end{aligned}$$

One readily checks that $G_1 \in L^p_{x,t}(\mathbb{R}^2)$ if and only if $p > 4$, but we are interested in L^6 norms. Since $\widehat{G_0\sigma_{\mathbb{P}_\lambda}}$ is an even function of x,

$$\|G_1\|_{L^6(\mathbb{R}^2)} = \mathbf{C}_F[\lambda]\|G_0\|_{L^2(\sigma_{\mathbb{P}_\lambda})}. \tag{4.55}$$

We follow the work of [20] and expand the functional (4.54) up to second order, collecting the terms which do not depend on ϕ in **I**, the ones which depend linearly on the real and imaginary parts of ϕ in **II**, and the ones which depend quadratically on the real and imaginary parts of ϕ in **III**. This yields:

$$\mathbf{I} = \mathbf{C}_F[\lambda]^6\|G_0\|_2^6 - \|G_1\|_6^6; \tag{4.56}$$

$$\mathbf{II} = 6\mathbf{C}_F[\lambda]^6\|G_0\|_2^4 \Re \int G_0\phi - 6\Re \iint |G_1|^4 \overline{G_1}\phi_1; \tag{4.57}$$

$$\mathbf{III} = 3\mathbf{C}_F[\lambda]^6\|G_0\|_2^4\int|\phi|^2 + 12\mathbf{C}_F[\lambda]^6\|G_0\|_2^2\Big(\Re\int G_0\phi\Big)^2 - 9\iint|G_1|^4|\phi_1|^2 - 6\Re\iint|G_1|^2\overline{G_1}^2\phi_1^2. \quad (4.58)$$

We already know from (4.55) that $\mathbf{I} = 0$. Since G_0 is an extremizer, we have that $\mathbf{II} = 0$ as well. Finally, note that $\mathbf{III}$ is, by definition, a quadratic form in ϕ; denote it by $Q(\phi)$. By the symmetries of the problem (respectively, multiplication by a real number, phase shift, space translation, Galilean invariance, scaling and time translation), we have that

$$Q(G_0) = Q(iG_0) = Q(yG_0) = Q(iyG_0) = Q(y^2G_0) = Q(iy^2G_0) = 0, \quad (4.59)$$

and it is proved in [20] that Q is positive definite in the subspace of L^2 functions which are orthogonal to the functions indicated in (4.59). This non-degeneracy property will not be used here; rather, what is essential for our application is that $Q(\phi) \geq 0$ for every $\phi \in L^2(\mathbb{R})$. This is immediate because $\mathbf{II}$ vanishes for every $\phi \in L^2$ and (4.54) defines a nonnegative quantity.

A variational calculation

In the spirit of the variational calculation in [21, §17], we consider the one-parameter family of trial functions given by

$$(G_0 + \epsilon\varphi)_{0<\epsilon\leq\epsilon_0},$$

for some sufficiently small $\epsilon_0 > 0$, where $G_0(y) = e^{-\lambda y^2/2}$ and $\varphi \in L^2(\mathbb{R})$ will be chosen below, in such a way that

$$\|\varphi\|_2 = 1 \text{ and } \int G_0\varphi = 0. \quad (4.60)$$

For technical reasons that will become apparent soon, we introduce an appropriate dilation of the cut-off η_I which localizes to the region $|y| \lesssim \epsilon\log\frac{1}{\epsilon}$, and define

$$f_\epsilon(y) := \epsilon^{-1/2}(G_0 + \epsilon\varphi)(\epsilon^{-1}y)\eta_I\Big(\frac{1}{\epsilon\log\frac{1}{\epsilon}}y\Big). \quad (4.61)$$

Notice that the family $(f_\epsilon)_{\epsilon>0}$ is L^2-normalized in the sense that

$$\|f_\epsilon\|_{L^2(\widetilde{\sigma})}^2 = \|G_0\|_2^2 + O(\epsilon^2) \text{ as } \epsilon \to 0^+. \quad (4.62)$$

Consider the quantity:

$$\Xi(\epsilon) = \mathbf{C}_F[\lambda]^6 \|f_\epsilon\|^6_{L^2(\widetilde{\sigma})} - \|\widehat{f_\epsilon \widetilde{\sigma}}\|_6^6. \tag{4.63}$$

This is no longer a nonnegative expression by construction like (4.54).

Let

$$\varphi(u) := c_\lambda u e^{-\frac{\lambda u^2}{2}}, \tag{4.64}$$

where the constant $c_\lambda := (\pi/4\lambda^3)^{-1/4}$ is chosen to normalize $\|\varphi\|_2 = 1$. Then φ satisfies conditions (4.60) and $Q(\varphi) = 0$.

With this choice of φ, we claim that the function $\Xi = \Xi(\epsilon)$ has the following property: for every $\lambda > 0$, for every $a \in \mathbb{R}$ satisfying (4.52), and for every real-valued smooth ψ satisfying $\psi(y) = O(|y|^5)$ as $|y| \to 0$, Ξ is a strictly concave function of ϵ in a sufficiently small half-neighborhood of 0, provided the interval I is chosen sufficiently small (as a function of λ, a and ψ). Once we prove this we will be able to conclude that $\mathbf{C}_F[\lambda] < \mathbf{C}[\widetilde{\Gamma}]$, and Proposition 4.27 follows.

Start by noting that $\lim_{\epsilon\to 0}\Xi(\epsilon) = 0$ and $\Xi'(0) = 0$. Indeed, one has that

$$\lim_{\epsilon\to 0}\Xi(\epsilon) = \mathbf{C}_F[\lambda]^6 \lim_{\epsilon\to 0}\|f_\epsilon\|^6_{L^2(\widetilde{\sigma})} - \lim_{\epsilon\to 0}\|\widehat{f_\epsilon\widetilde{\sigma}}\|_6^6 = \mathbf{C}_F[\lambda]^6\|G_0\|_2^6 - \|G_1\|_6^6 = 0$$

for every $a \in \mathbb{R}$ and $\lambda > 0$. On the other hand, explicit computations show that

$$\partial_\epsilon|_{\epsilon=0}\|f_\epsilon\|^6_{L^2(\widetilde{\sigma})} = 6\|G_0\|_2^4 \Re\int G_0\varphi$$

and

$$\partial_\epsilon|_{\epsilon=0}\|\widehat{f_\epsilon\widetilde{\sigma}}\|_6^6 = 6\Re\iint |G_1|^4\overline{G_1}\varphi_1.$$

Since $\mathbf{II} = 0$ and $G_0 \perp \varphi$, we have that

$$\Re\iint |G_1|^4\overline{G_1}\varphi_1 = \mathbf{C}_F[\lambda]\|G_0\|_2^4\Re\int G_0\varphi = 0 \tag{4.65}$$

and it follows that $\Xi'(0) = 0$, as claimed.

Useful information will come from looking at second variations. The strategy will be to compute the second derivatives with respect to ϵ (at $\epsilon = 0$) of good enough approximations to the two terms appearing in the definition of Ξ. We start by analyzing the most involved one.

The term $\|\widehat{f_\epsilon\widetilde{\sigma}}\|_6^6$.

To make the notation less cumbersome, we introduce the following parametrization:

$$\begin{aligned}\widetilde{\gamma}_\epsilon : \epsilon^{-1}\cdot I \;&\to\; \mathbb{R}^2\\ u \;&\mapsto\; \Big(u, \widetilde{h}_\epsilon(u) = \tfrac{\lambda u^2}{2} + a\epsilon^2 u^4 + \epsilon^{-2}\psi(\epsilon u)\Big).\end{aligned}$$

Changing variables $y = \epsilon u$, we have that:

$$\begin{aligned}\widehat{f_\epsilon\widetilde{\sigma}}(x,t) =& \int_{\mathbb{R}} f_\epsilon(y) e^{-i(x,t)\cdot(y,h(y))}(1+h'(y)^2)^{1/2}\eta_I(y)dy\\ =&\epsilon^{1/2}\int_{\mathbb{R}}(G_0+\epsilon\varphi)(u)e^{-i(\epsilon x,\epsilon^2 t)\cdot(u,\widetilde{h}_\epsilon(u))}(1+h'(\epsilon u)^2)^{1/2}\eta_I\Big(\frac{1}{\log\frac{1}{\epsilon}}u\Big)du.\end{aligned}$$

Consider an approximate version of

$$\begin{aligned}w_\epsilon(x,t) :=& \epsilon^{-1/2}\widehat{f_\epsilon\widetilde{\sigma}}(\epsilon^{-1}x,\epsilon^{-2}t)\\ =&\int_{\mathbb{R}}(G_0+\epsilon\varphi)(u)e^{-i(x,t)\cdot(u,\widetilde{h}_\epsilon(u))}(1+h'(\epsilon u)^2)^{1/2}\eta_I\Big(\frac{1}{\log\frac{1}{\epsilon}}u\Big)du\end{aligned}$$

given by Taylor expanding $J_\epsilon(u) := \sqrt{1+h'(\epsilon u)^2} \in C^\infty(I)$ and defining

$$v_\epsilon(x,t) := \int_{\mathbb{R}}(G_0+\epsilon\varphi)(u)e^{-i(x,t)\cdot(u,\widetilde{h}_\epsilon(u))}\Big(1+\frac{\lambda^2}{2}\epsilon^2u^2\Big)\eta_I\Big(\frac{1}{\log\frac{1}{\epsilon}}u\Big)du. \tag{4.66}$$

Also, set

$$g_\epsilon(u) := (G_0+\epsilon\varphi)(u),\quad g_\epsilon^\sharp(u) := u^2(G_0+\epsilon\varphi)(u) \text{ and } d\widetilde{\sigma}_\epsilon(u) := (1+\frac{\lambda^2}{2}\epsilon^2u^2)du.$$

Lemma 4.28. *If I is a sufficiently small interval centered at the origin (chosen as a function of λ, a, ψ but not ϵ), then*

$$\|\widehat{f_\epsilon\widetilde{\sigma}}\|_6^6 = \|w_\epsilon\|_6^6 = \|v_\epsilon\|_6^6 + O(\epsilon^4) \tag{4.67}$$

and

$$\|f_\epsilon\|^2_{L^2(\widetilde{\sigma})} = \|g_\epsilon\|^2_{L^2(\widetilde{\sigma}_\epsilon)} + O(\epsilon^4). \tag{4.68}$$

as $\epsilon \to 0^+$.

Proof. By construction, v_ϵ is just w_ϵ with the term $J_\epsilon = (1+h'(\epsilon\cdot)^2)^{1/2}$ replaced by its Taylor approximation to order 2. For $\epsilon < 1$, set

$$G_\epsilon(u) := (1 + \frac{\lambda^2}{2}\epsilon^2 u^2)(G_0 + \epsilon\varphi)(u)$$

and

$$s_\epsilon(x,t) := -\int_{\mathbb{R}} G_\epsilon(u) e^{-i(x,t)\cdot\widetilde{\gamma}_\epsilon(u)}\Big(\eta_I(\epsilon u) - \eta_I\Big(\frac{1}{\log\frac{1}{\epsilon}}u\Big)\Big)du.$$

It follows that

$$v_\epsilon(x,t) = \widehat{g_\epsilon\sigma_{P,\epsilon}}(x,t) + \frac{\lambda^2\epsilon^2}{2}\widehat{g_\epsilon^\sharp\sigma_{P,\epsilon}}(x,t) + s_\epsilon(x,t), \tag{4.69}$$

where $d\sigma_{P,\epsilon} = \eta_I(\epsilon u)du$. We obtain a uniform estimate for its L^6 norm:

Claim 4.29. There exists a constant $C < \infty$ such that $\|v_\epsilon\|_6 \leq C$, for every sufficiently small $\epsilon > 0$.

The claimed uniformity in ϵ needs to be justified: it follows from undoing the substitutions $y \mapsto \epsilon^{-1}y$ and $(x,t) \mapsto (\epsilon^{-1}x, \epsilon^{-2}t)$. Indeed,

$$\begin{aligned}
\widehat{g_\epsilon\sigma_{P,\epsilon}}(x,t) &= \int g_\epsilon(u) e^{-i(x,t)\cdot(u,\widetilde{h}_\epsilon(u))}\eta_I(\epsilon u)du \\
&= \epsilon^{-1/2}\int \epsilon^{-1/2}(G_0+\epsilon\varphi)(\epsilon^{-1}y)e^{-(\epsilon^{-1}x,\epsilon^{-2}t)\cdot(y,h(y))}\eta_I(y)dy \\
&= \epsilon^{-1/2}\widehat{f_\epsilon\sigma_P}(\epsilon^{-1}x,\epsilon^{-2}t),
\end{aligned}$$

where $d\sigma_P(y) := \eta_I(y)dy$. Since $\|\epsilon^{-1/2}\widehat{f_\epsilon\sigma_P}(\epsilon^{-1}\cdot,\epsilon^{-2}\cdot)\|_6 = \|\widehat{f_\epsilon\sigma_P}\|_6$, we have that

$$\|\widehat{g_\epsilon\sigma_{P,\epsilon}}\|_6 = \|\epsilon^{-1/2}\widehat{f_\epsilon\sigma_P}(\epsilon^{-1}\cdot,\epsilon^{-2}\cdot)\|_6 = \|\widehat{f_\epsilon\sigma_P}\|_6 \lesssim \|f_\epsilon\|_2 \leq C\|G_0\|_2,$$

for some $C < \infty$ independent of ϵ, as claimed. Proceed similarly to get a bound $O(\epsilon^2)$ for the term involving $g_\epsilon^\sharp$. Finally, define

$$g_\epsilon^\flat(u) := -G_\epsilon(u)\Big(\eta_I(\epsilon u) - \eta_I\Big(\frac{1}{\log\frac{1}{\epsilon}}u\Big)\Big)$$

and notice that $s_\epsilon(x,t) = \widehat{g_\epsilon^\flat\sigma_{P,\epsilon}}(x,t)$. Estimate:

$$\begin{aligned}
\|g_\epsilon^\flat\|_2^2 &\asymp \int_{\log\epsilon^{-1}\lesssim|u|\lesssim\epsilon^{-1}} \Big(1 + c_\lambda\epsilon u + \frac{\lambda^2\epsilon^2}{2}u^2 + c_\lambda\frac{\lambda^2\epsilon^3}{2}u^3\Big)^2 e^{-\lambda u^2}du \\
&\lesssim e^{-C(\log\epsilon^{-1})^2} \lesssim_N \epsilon^N, \text{ for every } N \in \mathbb{N}.
\end{aligned}$$

Eliminating the substitutions as before, we conclude that

$$\|s_\epsilon\|_6 \lesssim \|g_\epsilon^\flat\|_2 \lesssim \epsilon^N, \quad \forall N \in \mathbb{N},$$

where the implicit constants are all independent of ϵ. Thus the contribution of the third summand is likewise small, and this concludes the verification of Claim 4.29.

If we choose the interval I small enough (as a function of λ, a and ψ) such that

$$y \in I \Rightarrow |h'(y)| = |\lambda y + 4ay^3 + \psi'(y)| \leq 1,$$

then the remainder

$$r_\epsilon(x,t) := w_\epsilon(x,t) - v_\epsilon(x,t)$$

will satisfy favorable bounds. By Taylor's theorem we have that

$$|r_\epsilon(x,t)| \leq C\epsilon^4 \Big| \int_{\mathbb{R}} J_\epsilon''''(c_0) u^4 (G_0 + \epsilon\varphi)(u) e^{-i(x,t)\cdot\widetilde{\gamma}_\epsilon(u)} \eta_I\Big(\frac{1}{\log\frac{1}{\epsilon}} u\Big) du \Big|, \tag{4.70}$$

for some $c_0 \in (-\epsilon u, \epsilon u)$ and some absolute constant $C < \infty$. An argument analogous to the one used to establish Claim 4.29 yields the following estimate for the remainder term:

Claim 4.30. There exists a constant $C < \infty$ such that $\|r_\epsilon\|_6 \leq C\epsilon^4$, for every sufficiently small $\epsilon > 0$.

To finish the proof of Lemma 4.28, notice that $\|w_\epsilon\|_6^6 = \|v_\epsilon + r_\epsilon\|_6^6 = \|v_\epsilon\|_6^6 +$ 63 terms, all of which are $O(\epsilon^4)$ as $\epsilon \to 0^+$. This is an immediate consequence of Hölder's inequality, together with Claims 4.29 and 4.30: for example,

$$\iint |v_\epsilon|^4 \overline{v_\epsilon} r_\epsilon dx dt \leq \|v_\epsilon\|_6^5 \|r_\epsilon\|_6 \leq C\epsilon^4.$$

All other terms can be dealt with in a similar way, and the result follows. The verification of (4.68) is easier and we omit the details. □

Since we are interested in second variations with respect to ϵ of the L^6 norm $\|\widehat{f_\epsilon\widetilde{\sigma}}\|_6^6$ at $\epsilon = 0$, it will suffice, in light of (4.67), to analyze $\|v_\epsilon\|_6^6$. Start by noting that

$$v_0(x,t) = G_1(x,t) \tag{4.71}$$
$$\partial_\epsilon|_{\epsilon=0} v_\epsilon(x,t) = \varphi_1(x,t) \tag{4.72}$$
$$\partial_\epsilon^2|_{\epsilon=0} v_\epsilon(x,t) = \lambda^2 G_2(x,t) - 2ita G_3(x,t) \tag{4.73}$$

where, as before,

$$G_1(x,t) := \widehat{G_0\sigma_{\mathbb{P}_\lambda}}(-x,t) = \left(\frac{2\pi}{\lambda}\right)^{1/2}(1+it)^{-1/2}e^{-\frac{x^2}{2\lambda(1+it)}}$$

and

$$\varphi_1(x,t) := \widehat{\varphi\sigma_{\mathbb{P}_\lambda}}(-x,t) = \frac{ic_\lambda}{\lambda}\left(\frac{2\pi}{\lambda}\right)^{1/2}(1+it)^{-3/2}xe^{-\frac{x^2}{2\lambda(1+it)}}.$$

Additionally,

$$\begin{aligned}G_2(x,t) &:= [(u^2G_0)\sigma_{\mathbb{P}_\lambda}]^\wedge(-x,t)\\ &= \int_{\mathbb{R}} y^2G_0(y)e^{-it\frac{\lambda y^2}{2}}e^{ixy}dy = 2\lambda^{-1}i\partial_t G_1(x,t)\\ &= \Big(\lambda^{-1}(1+it)^{-1} - \lambda^{-2}x^2(1+it)^{-2}\Big)G_1(x,t)\end{aligned}$$

and

$$\begin{aligned}G_3(x,t) &:= [(u^4G_0)\sigma_{\mathbb{P}_\lambda}]^\wedge(-x,t)\\ &= \int_{\mathbb{R}} y^4G_0(y)e^{-it\frac{\lambda y^2}{2}}e^{ixy}dy = -4\lambda^{-2}\partial_t^2 G_1(x,t)\\ &= \Big(3\lambda^{-2}(1+it)^{-2} - 6\lambda^{-3}x^2(1+it)^{-3} + \lambda^{-4}x^4(1+it)^{-4}\Big)G_1(x,t).\end{aligned}$$

Remark 4.31. The calculations to follow, which lead to formula (4.74) below, are largely formal and need to be justified. In particular, the fact that $\epsilon \mapsto \|v_\epsilon\|_6^6$ is twice differentiable at $\epsilon = 0$ is proved in Appendix 1.

As a first step in the direction of computing $\partial_\epsilon^2|_{\epsilon=0}\|v_\epsilon\|_6^6$, we look at $\partial_\epsilon|v_\epsilon|^6(x,t)$:

$$\begin{aligned}\partial_\epsilon|v_\epsilon|^6 &= \partial_\epsilon(v_\epsilon^3\overline{v_\epsilon}^3)\\ &= 3v_\epsilon^2(\partial_\epsilon v_\epsilon)\overline{v_\epsilon}^3 + 3\overline{v_\epsilon}^2(\partial_\epsilon\overline{v_\epsilon})v_\epsilon^3.\end{aligned}$$

Recalling (4.71)−(4.72), it follows that

$$\partial_\epsilon|_{\epsilon=0}|v_\epsilon|^6 = 6\Re(G_1^2\varphi_1\overline{G_1}^3)$$

and so (4.65) implies

$$\partial_\epsilon|_{\epsilon=0}\|v_\epsilon\|_6^6 = 6\Re\iint |G_1|^4\overline{G_1}\varphi_1 dxdt = 0,$$

which we already knew. We differentiate once again and obtain

$$\partial_\epsilon^2 |v_\epsilon|^6 = 2\Re\Big(6v_\epsilon(\partial_\epsilon v_\epsilon)^2\overline{v_\epsilon}^3 + 3v_\epsilon^2(\partial_\epsilon^2 v_\epsilon)\overline{v_\epsilon}^3 + 9|v_\epsilon|^4|\partial_\epsilon v_\epsilon|^2\Big),$$

which at $\epsilon = 0$ equals (recall (4.71)−(4.73))

$$\begin{aligned}\partial_\epsilon^2|_{\epsilon=0}|v_\epsilon|^6 =& 2\Re\Big(6v_0(\partial_\epsilon|_{\epsilon=0}v_\epsilon)^2\overline{v_0}^3 + 3v_0^2(\partial_\epsilon^2|_{\epsilon=0}v_\epsilon)\overline{v_0}^3 + 9|v_0|^4|\partial_\epsilon|_{\epsilon=0}v_\epsilon|^2\Big)\\ =& 2\Re\Big(6G_1\varphi_1^2\overline{G_1}^3 + 3G_1^2(\lambda^2 G_2 - 2itaG_3)\overline{G_1}^3 + 9|G_1|^4|\varphi_1|^2\Big).\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{2}\partial_\epsilon^2|_{\epsilon=0}\|v_\epsilon\|_6^6 =& 9\iint |G_1|^4|\varphi_1|^2 dxdt + 6\Re\iint |G_1|^2\overline{G_1}^2\varphi_1^2 dxdt\\ &+ 3\lambda^2\Re\iint |G_1|^4\overline{G_1}G_2 dxdt - 6a\iint \Re\Big\{it|G_1|^4\overline{G_1}G_3\Big\}dxdt. \end{aligned}\tag{4.74}$$

The first two summands on the right hand side of (4.74) appear (with opposite signs) in the expression (4.58) for the quadratic form Q. The last two summands, on the other hand, can be explicitly evaluated, and that is our next task. The proofs of the following claims are deferred to Appendix 2:

Claim 4.32.

$$3\lambda^2\Re\iint_{\mathbb{R}^2} |G_1(x,t)|^4\overline{G_1(x,t)}G_2(x,t)dxdt = \frac{3}{2}\pi^{3/2}\lambda^{-1/2}\mathbf{C}_F[\lambda]^6. \tag{4.75}$$

Claim 4.33.

$$-6a\iint_{\mathbb{R}^2}\Re\Big\{it|G_1(x,t)|^4\overline{G_1(x,t)}G_3(x,t)\Big\}dxdt = -4a\pi^{3/2}\lambda^{-7/2}\mathbf{C}_F[\lambda]^6. \tag{4.76}$$

It follows from (4.74), (4.75) and (4.76) that

$$\begin{aligned}\frac{1}{2}\partial_\epsilon^2|_{\epsilon=0}\|v_\epsilon\|_6^6 =& 9\iint |G_1|^4|\varphi_1|^2 dxdt + 6\Re\iint |G_1|^2\overline{G_1}^2\varphi_1^2 dxdt\\ &+ \frac{3}{2}\pi^{3/2}\lambda^{-1/2}\mathbf{C}_F[\lambda]^6 - 4a\pi^{3/2}\lambda^{-7/2}\mathbf{C}_F[\lambda]^6. \end{aligned}\tag{4.77}$$

The term $\|f_\epsilon\|^6_{L^2(\widetilde{\sigma})}$

In view of (4.68), it will be enough to compute the approximate expression $\|g_\epsilon\|^2_{L^2(\widetilde{\sigma}_\epsilon)}$. Since $G_0 \perp \varphi$, we have that

$$\begin{aligned}\|g_\epsilon\|^2_{L^2(\widetilde{\sigma}_\epsilon)} &= \int_{\mathbb{R}} |(G_0+\epsilon\varphi)(u)|^2(1+\frac{\lambda^2}{2}u^2\epsilon^2)du \\ &= \int |G_0(u)|^2du + \Big(\int |\varphi(u)|^2du + \frac{\lambda^2}{2}\int u^2|G_0(u)|^2du\Big)\epsilon^2 \\ &\qquad + \Big(\lambda^2\Re\int u^2G_0(u)\varphi(u)du\Big)\epsilon^3 + \Big(\frac{\lambda^2}{2}\int u^2|\varphi(u)|^2du\Big)\epsilon^4,\end{aligned}$$

which in turn implies $\partial_\epsilon|_{\epsilon=0}\|g_\epsilon\|^2_{L^2(\widetilde{\sigma}_\epsilon)} = 0$ and

$$\partial^2_\epsilon|_{\epsilon=0}\|g_\epsilon\|^6_{L^2(\widetilde{\sigma}_\epsilon)} = 6\|G_0\|_2^4\Big(\int |\varphi(u)|^2du + \frac{\lambda^2}{2}\int u^2|G_0(u)|^2du\Big).$$

One last computation shows that

$$\begin{aligned}6\|G_0\|_2^4\Big(\frac{\lambda^2}{2}\int u^2|G_0(u)|^2du\Big) &= 3\lambda^2\Big(\int_{-\infty}^{\infty} e^{-\lambda u^2}du\Big)^2\Big(\int_{-\infty}^{\infty} u^2e^{-\lambda u^2}du\Big) \\ &= 3\lambda^2\frac{\pi}{\lambda}\frac{\pi^{1/2}}{2\lambda^{3/2}} = \frac{3}{2}\pi^{3/2}\lambda^{-1/2},\end{aligned}$$

and so

$$\frac{1}{2}\partial^2_\epsilon|_{\epsilon=0}\|g_\epsilon\|^6_{L^2(\widetilde{\sigma}_\epsilon)} = 3\|G_0\|_2^4\int |\varphi(u)|^2du + \frac{3}{4}\pi^{3/2}\lambda^{-1/2}. \tag{4.78}$$

Putting it all together

Using the approximations (4.67) and (4.68) given by Lemma 4.28, we get that:

$$\frac{\Xi''(0)}{2} = \mathbf{C}_F[\lambda]^6\frac{1}{2}\partial^2_\epsilon|_{\epsilon=0}\|g_\epsilon\|^6_{L^2(\widetilde{\sigma}_\epsilon)} - \frac{1}{2}\partial^2_\epsilon|_{\epsilon=0}\|v_\epsilon\|_6^6.$$

The main terms in these approximations have been explicitly computed in (4.77) and (4.78). We obtain:

$$\begin{aligned}\frac{\Xi''(0)}{2} &= 3\mathbf{C}_F[\lambda]^6\|G_0\|_2^4\int |\varphi(s)|^2ds + \frac{3}{4}\mathbf{C}_F[\lambda]^6\pi^{3/2}\lambda^{-1/2} - 9\iint |G_1|^4|\varphi_1|^2dxdt \\ &\quad - 6\Re\iint |G_1|^2\overline{G_1}^2\varphi_1^2dxdt - \frac{3}{2}\pi^{3/2}\lambda^{-1/2}\mathbf{C}_F[\lambda]^6 + 4a\pi^{3/2}\lambda^{-7/2}\mathbf{C}_F[\lambda]^6.\end{aligned}$$

Recalling the definition (4.58) of the quadratic form Q and the fact that $G_0 \perp \varphi$,

$$\frac{\Xi''(0)}{2} = Q(\varphi) - \frac{3}{4}\pi^{3/2}\lambda^{-1/2}\mathbf{C}_F[\lambda]^6 + 4a\pi^{3/2}\lambda^{-7/2}\mathbf{C}_F[\lambda]^6.$$

It follows that

$$\Xi(\epsilon) = \Big(Q(\varphi) - \frac{3}{4}\pi^{3/2}\lambda^{-1/2}\mathbf{C}_F[\lambda]^6 + 4a\pi^{3/2}\lambda^{-7/2}\mathbf{C}_F[\lambda]^6\Big)\epsilon^2 + O(\epsilon^3)$$

for sufficiently small $\epsilon > 0$, and so Ξ is strictly concave in a neighborhood of 0 if and only if

$$Q(\varphi) - \frac{3}{4}\pi^{3/2}\lambda^{-1/2}\mathbf{C}_F[\lambda]^6 + 4a\pi^{3/2}\lambda^{-7/2}\mathbf{C}_F[\lambda]^6 < 0,$$

that is, if and only if

$$a < \frac{3}{2}\Big(\frac{\lambda}{2}\Big)^3 - \frac{Q(\varphi)}{4\pi^{3/2}\mathbf{C}_F[\lambda]^6}\lambda^{7/2}.$$

The right hand side of this expression equals $\frac{3}{2}\Big(\frac{\lambda}{2}\Big)^3$ since $Q(\varphi) = 0$.

We have proved that, for the choice of φ given by (4.64), for every $\lambda > 0$, for every $a \in \mathbb{R}$ satisfying (4.52) and for every real-valued smooth ψ satisfying $\psi(y) = O(|y|^5)$ as $|y| \to 0$, Ξ is a strictly concave function of ϵ, for sufficiently small ϵ. In particular, $\Xi(\epsilon) < 0$, which is equivalent to

$$\mathbf{C}_F[\lambda]^6 \|f_\epsilon\|^6_{L^2(\widetilde{\sigma})} < \|\widehat{f_\epsilon\widetilde{\sigma}}\|_6^6.$$

Together with (4.53), this implies

$$\mathbf{C}_F[\lambda] < \mathbf{C}[\widetilde{\Gamma}].$$

Trivially, $\mathbf{C}[\widetilde{\Gamma}] \leq \mathbf{C}[\Gamma]$, and this finishes the proof of Proposition 4.27 modulo the work deferred to the Appendices.

Remark 4.34. We have been working under the additional assumption that the curvature κ of Γ does not attain a global minimum at one of its endpoints. This represents no loss of generality. Indeed, if that were not the case and, say, κ attained a global minimum at $p_0 = \gamma(0)$, then we could perform an identical variational calculation with functions f_ϵ defined in a similar way to (4.61) but supported instead on small neighborhoods $\widetilde{\Gamma}_\epsilon$ of points $p_\epsilon = \gamma(s_\epsilon) \in \Gamma$ with $|s_\epsilon| \asymp \epsilon$. It is straightforward to check that the computation carries through. We omit the details.

4.9 The end of the proof

Recall that by $T_{s,r}$ we mean the adjoint Fourier restriction operator on a cap $\mathcal{C} = \mathcal{C}(s,r) \subset \Gamma$. The following (easy) estimate is the last one we need in order to finish the argument:

Lemma 4.35. *If a sequence $\{f_n\} \subset L^2(\sigma)$ concentrates at a point $\gamma(s) \in \Gamma$, then*

$$\lim_{n\to\infty} \|\widehat{f_n\sigma}\|_6 \leq \lim_{r\to 0^+} \lim_{n\to\infty} \|T_{s,r} f_n\|_6.$$

Proof. Set $Tf := \widehat{f\sigma}$, and let $r > 0$ be arbitrary. The Tomas-Stein inequality implies

$$\|(T - T_{s,r})f_n\|_6 \lesssim \Big(\int_{\Gamma\setminus\mathcal{C}(s,r)} |f_n|^2 d\sigma\Big)^{1/2}.$$

It follows that

$$\begin{aligned}\|Tf_n\|_6 &\leq \|T_{s,r}f_n\|_6 + \|(T - T_{s,r})f_n\|_6 \\ &\leq \|T_{s,r}f_n\|_6 + C\Big(\int_{\Gamma\setminus\mathcal{C}(s,r)} |f_n|^2 d\sigma\Big)^{1/2}.\end{aligned}$$

Since $\{f_n\}$ concentrates at $\gamma(s)$, we have that

$$\lim_{n\to\infty} \|Tf_n\|_6 \leq \lim_{n\to\infty} \|T_{s,r}f_n\|_6 + 0$$

for every $r > 0$. The result follows. □

We can now prove that condition (4.42) in Proposition 4.21 cannot happen i.e. that the sequence $\{F_n\}$ promised by that proposition cannot concentrate. Suppose it did concentrate at some point $\gamma(s) \in \Gamma$, and assume as before that $\|F_n\|_2 \to 1$. By Corollary 4.26 we know that $\kappa(\gamma(s)) = \lambda$. We may again assume that $\gamma(s)$ is not an endpoint of Γ. Then, by Lemma 4.35 and Proposition 4.24,

$$\mathbf{C}[\Gamma] = \lim_{n\to\infty} \|\widehat{F_n\sigma}\|_6 \leq \lim_{r\to 0^+} \lim_{n\to\infty} \|T_{s,r}F_n\|_6 \leq \lim_{r\to 0^+} \|T_{s,r}\| = \mathbf{C}_F[\lambda],$$

a contradiction to Proposition 4.27. This concludes the proof of Theorem 4.3.

4.10 Appendix 1: $\|v_\epsilon\|_6^6$ is twice differentiable at $\epsilon = 0$

Recall that

$$v_\epsilon(x,t) = \int_{\mathbb{R}} (G_0 + \epsilon\varphi)(u) e^{-i(x,t)\cdot(u, \frac{\lambda u^2}{2} + a\epsilon^2 u^4 + \epsilon^{-2}\psi(\epsilon u))} \Big(1 + \frac{\lambda^2}{2}\epsilon^2 u^2\Big) \eta_I\Big(\frac{1}{\log\frac{1}{\epsilon}} u\Big) du,$$

where $0 < \epsilon \le \epsilon_0$, $\lambda = \min_\Gamma \kappa$, $G_0(u) = e^{-\lambda u^2/2}$, $\varphi(u) = c_\lambda u e^{-\lambda u^2/2}$, $a \in [\lambda^3/8, 3\lambda^3/16)$, ψ is a real-valued smooth function satisfying $\psi(y) = O(|y|^5)$ as $|y| \to 0$, and η_I is a mollified version of the characteristic function of the interval I.

The main goal of this appendix is to prove the following:

Proposition 4.36. *The function $\epsilon \mapsto \|v_\epsilon\|^6_{L^6(\mathbb{R}^2)}$ is twice differentiable at $\epsilon = 0$, and*

$$\partial_\epsilon^2|_{\epsilon=0}\|v_\epsilon\|_6^6 = 18\iint |G_1|^4|\varphi_1|^2 dxdt + 12\Re\iint |G_1|^2\overline{G_1}^2\varphi_1^2 dxdt \\ + 3\pi^{3/2}\lambda^{-1/2}\mathbf{C}_F[\lambda]^6 - 8a\pi^{3/2}\lambda^{-7/2}\mathbf{C}_F[\lambda]^6. \quad (4.79)$$

It is enough to show that

$$\iint_{\mathbb{R}^2} \Big|\partial_\epsilon |v_\epsilon(x,t)|^6\Big| dxdt \le C \ \text{ and } \ \iint_{\mathbb{R}^2} \Big|\partial_\epsilon^2 |v_\epsilon(x,t)|^6\Big| dxdt \le C \quad (4.80)$$

for a finite constant C (independent of ϵ) and every $0 < \epsilon \le \epsilon_0$. The existence of $\partial_\epsilon^2|_{\epsilon=0}\|v_\epsilon\|_6^6$ will then follow from standard tools of analysis, and the formal computations from Section 4.8 show that its value is given by formula (4.79).

For $\epsilon < 1$, set, as in the proof of Proposition 4.28,

$$G_\epsilon(u) = \Big(1 + \frac{\lambda^2}{2}\epsilon^2 u^2\Big)(G_0 + \epsilon\varphi)(u).$$

We have the following expression for the oscillatory integral:

$$v_\epsilon(x,t) = \int_{\mathbb{R}} e^{-it\phi_\epsilon(u)} G_\epsilon(u) \eta_I\Big(\frac{1}{\log\frac{1}{\epsilon}} u\Big) du,$$

where

$$\phi_\epsilon(u) := \frac{x}{t}u + \frac{\lambda u^2}{2} + a\epsilon^2 u^4 + \epsilon^{-2}\psi(\epsilon u). \quad (4.81)$$

In order to present the main estimates for v_ϵ, it is convenient to perform a decomposition of the (x,t)-plane which we now describe. Let $\eta_0, \eta \in C_0^\infty(\mathbb{R})$ be even and smooth cut-off functions supported in $[-1,1]$ and $[-2,-1/2]\cup[1/2,2]$ respectively, with the properties that $0 \leq \eta_0 \leq 1$, $0 \leq \eta \leq 1$, and

$$\eta_I\Big(\frac{1}{\log\frac{1}{\epsilon}}u\Big) = \eta_0(u) + \sum_{k=1}^{K(\epsilon)} \eta(2^{-k+1}u) \quad \text{for every } u \in \mathbb{R}. \tag{4.82}$$

This can be accomplished with $2^{K(\epsilon)} \asymp \log\frac{1}{\epsilon}$. We obtain, in particular, a smooth partition of unity in the interval $\frac{1}{2}\log\frac{1}{\epsilon}\cdot I$ subordinate to the dyadic regions $\mathcal{D}_k := \{u \in \mathbb{R} : 2^{k-2} \leq |u| \leq 2^k\}$. This allows us to express v_ϵ as a sum of $\asymp K(\epsilon)$ integrals:

$$v_\epsilon(x,t) = v_{\epsilon,0}(x,t) + \sum_{k=1}^{K(\epsilon)} v_{\epsilon,k}(x,t), \tag{4.83}$$

where

$$v_{\epsilon,0}(x,t) := \int_{\mathbb{R}} e^{-it\phi_\epsilon(u)} G_\epsilon(u)\eta_0(u)du$$

and

$$v_{\epsilon,k}(x,t) := \int_{\mathbb{R}} e^{-it\phi_\epsilon(u)} G_\epsilon(u)\eta(2^{-k+1}u)du \tag{4.84}$$

for $k \in \{1,2,\ldots,K(\epsilon)\}$. Define:

$$\begin{aligned}
\mathfrak{C}_0 &:= \Big\{(x,t) \in \mathbb{R}^2 : \Big|\frac{x}{t}\Big| \leq \lambda\Big\};\\
\mathfrak{C}_k &:= \Big\{(x,t) \in \mathbb{R}^2 : \lambda 2^{k-2} \leq \Big|\frac{x}{t}\Big| \leq \lambda 2^k\Big\}; \quad (k \in \{1,2,\ldots,K(\epsilon)\})\\
\mathfrak{C}_\infty &:= \Big\{(x,t) \in \mathbb{R}^2 : \Big|\frac{x}{t}\Big| \geq \lambda \log\frac{1}{\epsilon}\Big\}.
\end{aligned}$$

This yields a decomposition of the (x,t)-plane as a union of cones

$$\mathbb{R}^2 = \mathfrak{C}_0 \cup \bigcup_{k=1}^{K(\epsilon)} \mathfrak{C}_k \cup \mathfrak{C}_\infty. \tag{4.85}$$

which parallels (4.82). For $k \in \{2,3,\ldots,K(\epsilon)-2\}$, define the "enlarged" cones

$$\mathfrak{C}_k^* := \mathfrak{C}_k \cup \mathfrak{C}_{k\pm1} \cup \mathfrak{C}_{k\pm2}.$$

Additionally, let

$$\mathfrak{C}_0^* := \mathfrak{C}_0 \cup \mathfrak{C}_1 \cup \mathfrak{C}_2;$$
$$\mathfrak{C}_1^* := \mathfrak{C}_0 \cup \mathfrak{C}_1 \cup \mathfrak{C}_2 \cup \mathfrak{C}_3;$$
$$\mathfrak{C}_{K(\epsilon)-1}^* := \mathfrak{C}_{K(\epsilon)-3} \cup \mathfrak{C}_{K(\epsilon)-2} \cup \mathfrak{C}_{K(\epsilon)-1} \cup \mathfrak{C}_{K(\epsilon)} \cup \mathfrak{C}_\infty;$$
$$\mathfrak{C}_{K(\epsilon)}^* := \mathfrak{C}_{K(\epsilon)-2} \cup \mathfrak{C}_{K(\epsilon)-1} \cup \mathfrak{C}_{K(\epsilon)} \cup \mathfrak{C}_\infty;$$

The estimates in the following proposition are an expression of the *stationary phase principle*, which roughly states that the main contribution for an oscillatory expression like (4.84) comes from the information concentrated on neighborhoods of the stationary points of its phase function.

Proposition 4.37. *For $k \in \{1, \dots, K(\epsilon)\}$ and for every sufficiently small $\epsilon > 0$, there exist $a_k \geq 0$ such that*

$$|v_{\epsilon,k}(x,t)| \lesssim a_k \cdot \begin{cases} \langle t \rangle^{-1/2} & \textit{if } (x,t) \in \mathfrak{C}_k^*, \\ \langle t \rangle^{-1} & \textit{if } (x,t) \in \mathfrak{C}_0, \\ \langle x \rangle^{-1} & \textit{if } (x,t) \in \left(\bigcup_{|j-k|>2} \mathfrak{C}_j \right) \cup \mathfrak{C}_\infty. \end{cases}$$

and $\sum_k 2^k a_k < \infty$. If $k = 0$, then

$$|v_{\epsilon,0}(x,t)| \lesssim \begin{cases} \langle t \rangle^{-1/2} & \textit{if } (x,t) \in \mathfrak{C}_0^*, \\ \langle x \rangle^{-1} & \textit{otherwise.} \end{cases}$$

Proof. To make the notation less cumbersome, we will limit our discussion to the case when $k \in \{3, 4, \dots, K(\epsilon) - 2\}$. The other cases follow in a similar way.

Case 1. $(x,t) \in \mathfrak{C}_k^*$

Without loss of generality we may assume that $(x,t) \in \mathfrak{C}_k^*$ is such that the phase ϕ_ϵ has a critical point in the support of the cut-off function $\eta_I \left(\frac{1}{\log \frac{1}{\epsilon}} \cdot \right)$, for otherwise we could integrate (4.84) by parts and obtain a better decay in t. This critical point is necessarily unique. In other words, there exists a unique $u_0 \in 2 \log \frac{1}{\epsilon} \cdot I$ such that

$$\frac{d\phi_\epsilon}{du}(u_0) = \frac{x}{t} + \lambda u_0 + 4a\epsilon^2 u_0^3 + \epsilon^{-1} \psi'(\epsilon u_0) = 0. \tag{4.86}$$

In general one cannot hope to solve this equation explicitly for u_0. However, since $|u_0| \lesssim \log \frac{1}{\epsilon}$, we have that

$$u_0 = -\frac{x}{\lambda t} + O_{a,\psi}(\epsilon).$$

In particular, since $(x,t)\in\mathfrak{C}_k^*$, we have that $|u_0|\asymp 2^k$.

Translating $u\mapsto u+u_0$ and defining $\widetilde{\phi_\epsilon}(u):=\phi_\epsilon(u+u_0)-\phi_\epsilon(u_0)$, we have that

$$v_\epsilon(x,t)=e^{-it\phi_\epsilon(u_0)}\int_{\mathbb{R}}e^{-it\widetilde{\phi_\epsilon}(u)}G_\epsilon(u+u_0)\eta_I\Big(\frac{1}{\log\frac{1}{\epsilon}}(u+u_0)\Big)du.$$

It will suffice to get good estimates on

$$\widetilde{v_\epsilon}(x,t):=e^{it\phi_\epsilon(u_0)}v_\epsilon(x,t)=\int_{\mathbb{R}}e^{-it\widetilde{\phi_\epsilon}(u)}G_\epsilon(u+u_0)\eta_I\Big(\frac{1}{\log\frac{1}{\epsilon}}(u+u_0)\Big)du. \tag{4.87}$$

The new phase function $\widetilde{\phi_\epsilon}$ satisfies

$$\widetilde{\phi_\epsilon}(0)=0=\frac{d\widetilde{\phi_\epsilon}}{du}(0)\ \text{ and }\ \frac{d^2\widetilde{\phi_\epsilon}}{du^2}(0)=\lambda+O(\epsilon).$$

In particular, the origin is its unique nondegenerate critical point. This property is shared by the quadratic function $v\mapsto\frac{v^2}{2}$. Inspired by the proof[7] of the usual method of stationary phase, we will change variables once again.

Recalling definition (4.81) and identity (4.86), and using Taylor's formula, we have that

$$\begin{aligned}\frac{v^2}{2}:=&\widetilde{\phi_\epsilon}(u)=\phi_\epsilon(u+u_0)-\phi_\epsilon(u_0)\\ =&\frac{u^2}{2}\Big((\lambda+12au_0^2\epsilon^2)+8au_0\epsilon^2u+2a\epsilon^2u^2\Big)+\epsilon^{-2}\psi(\epsilon(u+u_0))-\epsilon^{-2}\psi(\epsilon u_0)-u\epsilon^{-1}\psi'(\epsilon u_0)\\ =&\frac{u^2}{2}\Big((\lambda+12au_0^2\epsilon^2)+8au_0\epsilon^2u+2a\epsilon^2u^2+\psi''(\epsilon(u_0+\theta u))\Big)\end{aligned}$$

for some $\theta\in(0,1)$. Taking square roots,

$$v=u\Big((\lambda+12au_0^2\epsilon^2)+8au_0\epsilon^2u+2a\epsilon^2u^2+\psi''(\epsilon(u_0+\theta u))\Big)^{1/2}=:\Phi_\epsilon(u). \tag{4.88}$$

Take $\epsilon>0$ sufficiently small. Then Φ_ϵ is a C^∞ diffeomorphism from $2\log\frac{1}{\epsilon}\cdot I$ onto its image, whose inverse we denote by $\Psi_\epsilon:=\Phi_\epsilon^{-1}$. One can verify directly that $\frac{d\Phi_\epsilon}{du}(u)>0$ for every $u\in 2\log\frac{1}{\epsilon}\cdot I$. As a consequence,

$$\widetilde{v_\epsilon}(x,t)=\int_{\mathbb{R}}e^{-it\frac{v^2}{2}}G_\epsilon(\Psi_\epsilon(v)+u_0)\eta_I\Big(\frac{1}{\log\frac{1}{\epsilon}}(\Psi_\epsilon(v)+u_0)\Big)\frac{d\Psi_\epsilon}{dv}(v)dv. \tag{4.89}$$

[7]See, for instance, [1, pp. 334–337].

Rewriting (4.88) as

$$\Phi_\epsilon(u) = u\Big(\lambda + 2a(2u_0^2 + (2u_0+u)^2)\epsilon^2 + \psi''(\epsilon(u_0+\theta u))\Big)^{1/2} \tag{4.90}$$

and recalling that $\lambda, a > 0$ and $\psi''(\epsilon(u_0+\theta u)) = O(\epsilon^3)$, we see that Φ_ϵ is twice differentiable as a function of ϵ for sufficiently small ϵ. The same holds for $\frac{d\Phi_\epsilon}{du}$, and using the chain rule one can draw similar conclusions about Ψ_ϵ and $\frac{d\Psi_\epsilon}{dv}$.

In what follows, C will be a finite non-zero constant that may change from line to line and depend on the parameters λ, a, the function ψ and the interval I but will always be independent of ϵ, x and t. This uniformity is crucial in our analysis.

We proceed to prove some uniform (in ϵ) bounds for $\frac{d\Psi_\epsilon}{dv}$ and $\frac{d^2\Psi_\epsilon}{dv^2}$. Start by observing that, for every sufficiently small $\epsilon > 0$,

$$C^{-1}|u| \leq |\Phi_\epsilon(u)| \leq C|u|, \quad \forall u \in 2\log\frac{1}{\epsilon}\cdot I.$$

Since $\Phi_\epsilon \circ \Psi_\epsilon = id$ on $\Phi_\epsilon(2\log\frac{1}{\epsilon}\cdot I) \subseteq C\log\frac{1}{\epsilon}\cdot I$, we have the same uniform bounds for Ψ_ϵ:

$$C^{-1}|v| \leq |\Psi_\epsilon(v)| \leq C|v|, \quad \forall v \in C\log\frac{1}{\epsilon}\cdot I. \tag{4.91}$$

We also have the uniform bounds

$$C^{-1} \leq \Big|\frac{d\Phi_\epsilon}{du}(u)\Big| \leq C, \quad \forall u \in 2\log\frac{1}{\epsilon}\cdot I. \tag{4.92}$$

By the inverse function theorem,

$$\frac{d\Psi_\epsilon}{dv}(v) = \frac{1}{\frac{d\Phi_\epsilon}{du}(\Psi_\epsilon(v))}, \tag{4.93}$$

and so (4.92) implies

$$C^{-1} \leq \Big|\frac{d\Psi_\epsilon}{dv}(v)\Big| \leq C, \quad \forall v \in C\log\frac{1}{\epsilon}\cdot I. \tag{4.94}$$

Only the upper bound will be useful to us. In a similar way one can conclude that

$$\Big|\frac{d^2\Psi_\epsilon}{dv^2}(v)\Big| \leq C, \quad \forall v \in C\log\frac{1}{\epsilon}\cdot I. \tag{4.95}$$

We also need to estimate the Gaussian term G_ϵ appearing in (4.89) and some of its derivatives. The following claim, whose proof is straightforward and therefore omitted, provides good enough bounds:

Claim 4.38. The following uniform estimates hold for every sufficiently small $\epsilon > 0$, for every nonnegative integer n and for every $u \in 2\log\frac{1}{\epsilon}\cdot I$:

$$\Big|\frac{d^n}{du^n}G_\epsilon(u)\Big| \lesssim_n \langle u\rangle^n e^{-\frac{\lambda u^2}{2}}.$$

Let us go back to (4.89). Introducing the cut-off functions η and η_0 as before, we can expand

$$\widetilde{v}_\epsilon(x,t) = \widetilde{v}_{\epsilon,0}(x,t) + \sum_{k=1}^{K(\epsilon)} \widetilde{v}_{\epsilon,k}(x,t), \tag{4.96}$$

where

$$\widetilde{v}_{\epsilon,0}(x,t) := \int_{\mathbb{R}} e^{-it\frac{v^2}{2}} G_\epsilon(\Psi_\epsilon(v)+u_0)\frac{d\Psi_\epsilon}{dv}(v)\eta_0(\Psi_\epsilon(v)+u_0)dv$$

and

$$\widetilde{v}_{\epsilon,k}(x,t) := \int_{\mathbb{R}} e^{-it\frac{v^2}{2}} G_\epsilon(\Psi_\epsilon(v)+u_0)\frac{d\Psi_\epsilon}{dv}(v)\eta(2^{-k+1}(\Psi_\epsilon(v)+u_0))dv$$

for $k\in\{1,\ldots,K(\epsilon)\}$. As before, $|\widetilde{v}_{\epsilon,k}| = |v_{\epsilon,k}|$ pointwise.

Define:

$$b_\epsilon(v) := G_\epsilon(\Psi_\epsilon(v)+u_0)\frac{d\Psi_\epsilon}{dv}(v) \quad \text{and} \quad \eta_k(v) := \eta(2^{-k+1}(\Psi_\epsilon(v)+u_0)). \tag{4.97}$$

Notice that $\eta_k \in C_0^\infty(\mathbb{R})$ is supported on

$$\mathcal{E}_k := \{v \in C\log\frac{1}{\epsilon}\cdot I : 2^{k-2} \leq |\Psi_\epsilon(v)+u_0| \leq 2^k\},$$

and that b_ϵ is a Schwartz function on $\mathcal{E}_k$. This justifies the use of Plancherel's Theorem:

$$\widetilde{v}_{\epsilon,k}(x,t) = \int_{\mathbb{R}} e^{-it\frac{v^2}{2}} b_\epsilon(v)\eta_k(v)dv = (-2\pi i)^{1/2}t^{-1/2}\int_{\mathbb{R}} e^{it^{-1}\frac{\xi^2}{2}}\widehat{b_\epsilon\eta_k}(\xi)d\xi.$$

Set $a_k := \|\widehat{b_\epsilon\eta_k}\|_{L^1}$. We will be done analyzing Case 1 once we verify that the sequence $\{a_k\}_k$ decays rapidly enough to force the series $\sum_k 2^k a_k$ to converge.

We start by estimating the L^2 norm of the function $b_\epsilon\eta_k$. Changing back to the original variable $u = \Psi_\epsilon(v)$ and using Hölder's inequality together with estimate (4.94) and Claim 4.38 for $n=0$, one gets that

$$\begin{aligned}\|b_\epsilon\eta_k\|_{L^2}^2 &= \int \Big|G_\epsilon(\Psi_\epsilon(v)+u_0)\frac{d\Psi_\epsilon}{dv}(v)\eta(2^{-k+1}(\Psi_\epsilon(v)+u_0))\Big|^2 dv\\ &\leq \int_{|u+u_0|\asymp 2^k}\Big|G_\epsilon(u+u_0)\eta(2^{-k+1}(u+u_0))\Big|^2\Big|\frac{d\Psi_\epsilon}{dv}(\Phi_\epsilon(u))\Big|du \lesssim 2^k e^{-\lambda 4^{k-1}}.\end{aligned}$$

An analogous argument, using estimate (4.95) instead, yields

$$\Big\|\frac{d}{dv}(b_\epsilon\eta_k)\Big\|_{L^2}^2 \lesssim 2^{3k}e^{-\lambda 4^{k-1}}.$$

Using Cauchy-Schwarz and Plancherel, we see that these two estimates are enough for our purposes:

$$\begin{aligned} a_k = \|\widehat{b_\epsilon\eta_k}\|_{L^1} =& \int_{|\xi|\leq 1}|\widehat{b_\epsilon\eta_k}(\xi)|d\xi + \int_{|\xi|\geq 1}\frac{1}{|\xi|}\Big(|\xi||\widehat{b_\epsilon\eta_k}(\xi)|\Big)d\xi \\ \lesssim& \|\widehat{b_\epsilon\eta_k}\|_{L^2} + \Big(\int_{|\xi|\geq 1}|\xi|^{-2}d\xi\Big)^{1/2}\Big(\int_{|\xi|\geq 1}|\xi|^2|\widehat{b_\epsilon\eta_k}(\xi)|^2 d\xi\Big)^{1/2} \\ \lesssim& \|b_\epsilon\eta_k\|_{L^2} + \Big\|\frac{d}{du}(b_\epsilon\eta_k)\Big\|_{L^2} \lesssim 2^{3k/2}e^{-\lambda 2^{2k-3}}. \end{aligned}$$

This concludes the analysis of Case 1.

Case 2. $(x,t)\in\mathfrak{C}_0$

The crucial observation is that, since $(x,t)\in\mathfrak{C}_0$ and $k>2$, the phase ϕ_ϵ has no critical points in the support of $\eta(2^{-k+1}\cdot)$ i.e. the dyadic region $\mathcal{D}_k=\{u\in\mathbb{R}: 2^{k-2}\leq|u|\leq 2^k\}$. Indeed, since $|\frac{x}{t}|\leq\lambda$, we have that

$$\Big|\frac{d\phi_\epsilon}{du}(u)\Big| = \Big|\frac{x}{t}+\lambda u+O(\epsilon)\Big| \geq \frac{1}{2}\Big(|\lambda u|-\Big|\frac{x}{t}\Big|\Big) \geq \frac{\lambda}{2}(2^{k-2}-1)\geq\frac{\lambda}{2} \tag{4.98}$$

if $\epsilon>0$ is chosen sufficiently small.

Integrating (4.84) by parts, we get

$$\begin{aligned} |v_{\epsilon,k}(x,t)| =& \frac{1}{|t|}\Big|\int_{\mathcal{D}_k} e^{-it\phi_\epsilon(u)}\frac{d}{du}\Big(\frac{G_\epsilon(u)\eta(2^{-k+1}u)}{\frac{d\phi_\epsilon}{du}(u)}\Big)du\Big| \\ \lesssim& \frac{1}{|t|}\int_{\mathcal{D}_k}\Big|\frac{\frac{d}{du}(G_\epsilon(u)\eta(2^{-k+1}u))}{\frac{d\phi_\epsilon}{du}(u)}\Big| + \Big|\frac{G_\epsilon(u)\eta(2^{-k+1}u)\frac{d^2\phi_\epsilon}{du^2}(u)}{(\frac{d\phi_\epsilon}{du}(u))^2}\Big|du. \end{aligned}$$

Hölder's inequality implies

$$|v_{\epsilon,k}(x,t)| \lesssim \frac{2^k}{|t|}\Big(\Big\|\frac{\frac{d}{du}(G_\epsilon\eta(2^{-k+1}\cdot))}{\frac{d\phi_\epsilon}{du}}\Big\|_{L^\infty(\mathcal{D}_k)} + \Big\|\frac{G_\epsilon\eta(2^{-k+1}\cdot)\frac{d^2\phi_\epsilon}{du^2}}{(\frac{d\phi_\epsilon}{du})^2}\Big\|_{L^\infty(\mathcal{D}_k)}\Big),$$

and the desired estimate[8] now follows from (4.98), Claim 4.38 and the fact that $\frac{d^2\phi_\epsilon}{du^2}$ is uniformly bounded on $\mathcal{D}_k$.

[8] One could repeat this argument N times and obtain a bound $|v_{\epsilon,k}(x,t)|\lesssim_{k,N}\langle t\rangle^{-N}$ for $(x,t)\in\mathfrak{C}_0$, but this extra knowledge would be of no significance to our analysis.

Case 3. $(x,t) \in \mathfrak{C}_j$ for some j such that $|k-j|>2$, or $(x,t)\in\mathfrak{C}_\infty$ The proof is identical to that of Case 2 and is therefore omitted. □

Let us go back to the proof of Proposition 4.36. Observe that the estimates from Proposition 4.37 readily imply the following special case of the $L^2 \to L^6$ adjoint restriction inequality:

$$\iint_{\mathbb{R}^2} |v_\epsilon(x,t)|^6 dxdt \leq C. \tag{4.99}$$

Indeed, using the expansion (4.83), we have that

$$\iint_{\mathbb{R}^2} |v_\epsilon(x,t)|^6 dxdt = \sum_{k_1,\dots,k_6} \iint_{\mathbb{R}^2} v_{\epsilon,k_1}\overline{v_{\epsilon,k_2}}v_{\epsilon,k_3}\overline{v_{\epsilon,k_4}}v_{\epsilon,k_5}\overline{v_{\epsilon,k_6}}dxdt$$

where, for each $j \in \{1,\dots,6\}$, the sum is taken over $k_j \in \{0,1,\dots,K(\epsilon)\}$. For a fixed $(k_1,\dots,k_6)$ the corresponding integral can be written as a sum of $\asymp K(\epsilon)$ integrals over the regions given by decomposition (4.85), and using the bounds given by Proposition 4.37 on each of these regions one readily obtains (4.99). Note that

$$\iint_{\mathfrak{C}_k} \langle t\rangle^{-3} dxdt \asymp 2^k,$$

and so it is crucial to know that $\sum_k 2^k a_k < \infty$.

To prove (4.80), it is enough to control the following integrals:

$$\begin{aligned} I_0(\epsilon) &:= \iint_{\mathbb{R}^2} |v_\epsilon(x,t)|^5 |\partial_\epsilon v_\epsilon(x,t)| dxdt;\\ I_1(\epsilon) &:= \iint_{\mathbb{R}^2} |v_\epsilon(x,t)|^5 |\partial_\epsilon^2 v_\epsilon(x,t)| dxdt;\\ I_2(\epsilon) &:= \iint_{\mathbb{R}^2} |v_\epsilon(x,t)|^4 |\partial_\epsilon v_\epsilon(x,t)|^2 dxdt. \end{aligned}$$

The reasoning just described to prove (4.99) can be used to establish bounds for the integrals $I_0(\epsilon)$, $I_1(\epsilon)$ and $I_2(\epsilon)$ which are uniform in ϵ, as long as we have an analogue of Proposition 4.37 for first and second derivatives. As before, bounds for $\partial_\epsilon \widetilde{v}_\epsilon$ and $\partial_\epsilon^2 \widetilde{v}_\epsilon$ will suffice. Let us focus on the more involved case of second derivatives:

Proposition 4.39. *For $k \in \{1,\dots,K(\epsilon)\}$ and for every sufficiently small $\epsilon>0$, there exist $b_k \geq 0$ such that*

$$|\partial_\epsilon^2 \widetilde{v}_{\epsilon,k}(x,t)| \lesssim b_k \cdot \begin{cases} \langle t\rangle^{-1/2} & \text{if } (x,t)\in\mathfrak{C}_k^*,\\ \langle t\rangle^{-1} & \text{if } (x,t)\in\mathfrak{C}_0,\\ \langle x\rangle^{-1} & \text{if } (x,t)\in\left(\bigcup_{|j-k|>2}\mathfrak{C}_j\right)\cup\mathfrak{C}_\infty. \end{cases}$$

and $\sum_k 2^k b_k < \infty$. *If* $k = 0$, *then*

$$|\partial_\epsilon^2 \widetilde{v}_{\epsilon,0}(x,t)| \lesssim \begin{cases} \langle t\rangle^{-1/2} & \textit{if } (x,t) \in \mathfrak{C}_0^*, \\ \langle x\rangle^{-1} & \textit{otherwise.} \end{cases}$$

The proof follows the same steps of Proposition 4.37, with only one difference:

We need appropriate bounds for $\partial_\epsilon\Big(\frac{d^j\Psi_\epsilon}{dv^j}\Big)$ *and* $\partial_\epsilon^2\Big(\frac{d^j\Psi_\epsilon}{dv^j}\Big)$ *for* $j \in \{1,2\}$.

Let us briefly outline how to accomplish this. Using (4.88), we have that

$$\Phi_\epsilon(\Psi_\epsilon(v)) = \Psi_\epsilon(v)\Big(\lambda + (12au_0^2 + 8au_0\Psi_\epsilon(v) + 2a\Psi_\epsilon(v)^2)\epsilon^2 + \psi''(\epsilon(u_0 + \theta\Psi_\epsilon(v)))\Big)^{1/2}.$$

Differentiate both sides of the last identity with respect to ϵ, the left hand side being obviously equal to 0. For the right hand side, we get two kinds of terms, depending on whether or not they contain a factor of the form $\partial_\epsilon \Psi_\epsilon$. Grouping together in one side of the equation all the terms which do contain such a factor, we can estimate:

$$|\partial_\epsilon \Psi_\epsilon(v)| \leq C, \quad \forall v \in C\log\frac{1}{\epsilon}\cdot I. \tag{4.100}$$

It is also elementary to show that

$$\Big|\partial_\epsilon\Big(\frac{d\Phi_\epsilon}{du}\Big)(u)\Big| \leq C, \quad \forall u \in 2\log\frac{1}{\epsilon}\cdot I. \tag{4.101}$$

Differentiating both sides of (4.93) with respect to ϵ, we obtain

$$\partial_\epsilon\Big(\frac{d\Psi_\epsilon}{dv}\Big)(v) = -\frac{\partial_\epsilon\Big(\frac{d\Phi_\epsilon}{du}(\Psi_\epsilon(v))\Big)}{\Big(\frac{d\Phi_\epsilon}{du}(\Psi_\epsilon(v))\Big)^2}.$$

Using estimates (4.92) and (4.100), we similarly conclude that

$$\Big|\partial_\epsilon\Big(\frac{d\Psi_\epsilon}{dv}\Big)(v)\Big| \leq C, \quad \forall v \in C\log\frac{1}{\epsilon}\cdot I. \tag{4.102}$$

Repeating this whole procedure once again, we conclude in an analogous way that

$$\Big|\partial_\epsilon^2\Big(\frac{d\Psi_\epsilon}{dv}\Big)(v)\Big| \leq Cv^2, \quad \forall v \in C\log\frac{1}{\epsilon}\cdot I. \tag{4.103}$$

The terms $\partial_\epsilon\Big(\frac{d^2\Psi_\epsilon}{dv^2}\Big)$ and $\partial_\epsilon^2\Big(\frac{d^2\Psi_\epsilon}{dv^2}\Big)$ can be dealt with in a similar way. Recalling what we already know from (4.94) and (4.95), we arrive at the following lemma:

Lemma 4.40. *The following estimates hold for $j \in \{1, 2\}$, for every sufficiently small $\epsilon > 0$ and for every $v \in C \log \frac{1}{\epsilon} \cdot I$:*

(i) $\left| \frac{d^j \Psi_\epsilon}{dv^j}(v) \right| \leq C$;

(ii) $\left| \partial_\epsilon \left(\frac{d^j \Psi_\epsilon}{dv^j} \right)(v) \right| \leq C$;

(iii) $\left| \partial_\epsilon^2 \left(\frac{d^j \Psi_\epsilon}{dv^j} \right)(v) \right| \leq C v^2$.

Lemma 4.40 can be used together with the estimates from Claim 4.38 to prove Proposition 4.39. We omit the details.

4.11 Appendix 2: Two explicit calculations

Claim 4.32.

$$3\lambda^2 \Re \iint_{\mathbb{R}^2} |G_1(x,t)|^4 \overline{G_1(x,t)} G_2(x,t) dx dt = \frac{3}{2} \pi^{3/2} \lambda^{-1/2} \mathbf{C}_F[\lambda]^6. \tag{4.104}$$

Proof. As first observations, note that

$$\overline{G_1(x,t)} G_2(x,t) = \Big(\lambda^{-1}(1+it)^{-1} - \lambda^{-2} x^2 (1+it)^{-2} \Big) |G_1(x,t)|^2 \tag{4.105}$$

and

$$|G_1(x,t)|^2 = \frac{2\pi}{\lambda} (1+t^2)^{-1/2} e^{-\frac{x^2}{\lambda(1+t^2)}}.$$

It follows that the left hand side in (4.104) equals

$$3\lambda^2 \Big(\frac{2\pi}{\lambda} \Big)^3 \Re \iint \Big(\lambda^{-1}(1+it)^{-1} - \lambda^{-2} x^2 (1+it)^{-2} \Big) (1+t^2)^{-3/2} e^{-\frac{3x^2}{\lambda(1+t^2)}} dx dt.$$

Write $(1+it)^{-1} = (1+t^2)^{-1}(1-it)$, $(1+it)^{-2} = (1+t^2)^{-2}(1-it)^2$, and change variables $y = \frac{x}{(1+t^2)^{1/2}}$ to compute

$$\begin{aligned} \mathbf{I} := \Re \iint (1+t^2)^{-1}(1-it)(1+t^2)^{-3/2} e^{-\frac{3x^2}{\lambda(1+t^2)}} dx dt &= \iint (1+t^2)^{-5/2} e^{-\frac{3x^2}{\lambda(1+t^2)}} dx dt \\ &= \Big(\int_{-\infty}^{\infty} \frac{1}{(1+t^2)^2} dt \Big) \cdot \Big(\int_{-\infty}^{\infty} e^{-\frac{3y^2}{\lambda}} dy \Big) \\ &= \frac{\pi}{2} \cdot \Big(\frac{\pi}{3/\lambda} \Big)^{1/2} = \frac{\pi^{3/2}}{2\sqrt{3}} \lambda^{1/2} \end{aligned}$$

and

$$\begin{aligned}\mathbf{II} :=&\Re \iint x^2(1+t^2)^{-2}(1-it)^2(1+t^2)^{-3/2}e^{-\frac{3x^2}{\lambda(1+t^2)}}dxdt\\ =&\iint (1+t^2)^{-7/2}(1-t^2)x^2e^{-\frac{3x^2}{\lambda(1+t^2)}}dxdt\\ =&\Big(\int_{-\infty}^{\infty}\frac{1-t^2}{(1+t^2)^2}dt\Big)\Big(\int_{-\infty}^{\infty}y^2e^{-\frac{3y^2}{\lambda}}dy\Big)=0.\end{aligned}$$

All in all we have that

$$\begin{aligned}3\lambda^2\Re\iint |G_1(x,t)|^4\overline{G_1(x,t)}G_2(x,t)dxdt &= 3\lambda^2\Big(\frac{2\pi}{\lambda}\Big)^3(\lambda^{-1}\mathbf{I}-\lambda^{-2}\mathbf{II}) = 3\lambda^{-2}(2\pi)^3\mathbf{I}+0\\ &=\frac{3}{2}\pi^{3/2}\lambda^{-1/2}\frac{(2\pi)^3}{\sqrt{3}}\lambda^{-1} = \frac{3}{2}\pi^{3/2}\lambda^{-1/2}\mathbf{C}_F[\lambda]^6.\square\end{aligned}$$

Claim 4.33.

$$-6a\iint_{\mathbb{R}^2}\Re\Big\{it|G_1(x,t)|^4\overline{G_1(x,t)}G_3(x,t)\Big\}dxdt = -4a\pi^{3/2}\lambda^{-7/2}\mathbf{C}_F[\lambda]^6. \quad (4.106)$$

Proof. As with (4.105), we have that

$$\overline{G_1(x,t)}G_3(x,t) = \Big(3\lambda^{-2}(1+it)^{-2} - 6\lambda^{-3}x^2(1+it)^{-3}+\lambda^{-4}x^4(1+it)^{-4}\Big)|G_1(x,t)|^2,$$

and so the left hand side in (4.106) equals

$$\begin{aligned}-6a\Big(\frac{2\pi}{\lambda}\Big)^3\iint\Re\Big\{it\Big(3\lambda^{-2}(1+it)^{-2}-6\lambda^{-3}x^2(1+it)^{-3}+\lambda^{-4}x^4(1+it)^{-4}\Big)\cdot\\ \cdot(1+t^2)^{-3/2}e^{-\frac{3x^2}{\lambda(1+t^2)}}\Big\}dxdt.\end{aligned}$$

Note that $\Re\{i(1+it)^{-2}\} = 2t(1+t^2)^{-2}$, $\Re\{i(1+it)^{-3}\} = (3t-t^3)(1+t^2)^{-3}$ and $\Re\{i(1+it)^{-4}\} = (4t-4t^3)(1+t^2)^{-4}$. Change variables $y=\frac{x}{(1+t^2)^{1/2}}$ to compute

$$\begin{aligned}\mathbf{I} := \iint 2t^2(1+t^2)^{-2}(1+t^2)^{-3/2}e^{-\frac{3x^2}{\lambda(1+t^2)}}dxdt &= \iint 2t^2(1+t^2)^{-7/2}e^{-\frac{3x^2}{\lambda(1+t^2)}}dxdt\\ &=\Big(\int_{-\infty}^{\infty}\frac{2t^2}{(1+t^2)^3}dt\Big)\cdot\Big(\int_{-\infty}^{\infty}e^{-\frac{3y^2}{\lambda}}dy\Big)\\ &=\frac{\pi}{4}\cdot\Big(\frac{\pi}{3/\lambda}\Big)^{1/2} = \frac{\pi^{3/2}}{4\sqrt{3}}\lambda^{1/2},\end{aligned}$$

$$\begin{aligned}
\mathbf{II} :=& \iint x^2(3t^2 - t^4)(1+t^2)^{-3}(1+t^2)^{-3/2} e^{-\frac{3x^2}{\lambda(1+t^2)}} dxdt \\
=& \iint (3t^2 - t^4)(1+t^2)^{-9/2} x^2 e^{-\frac{3x^2}{\lambda(1+t^2)}} dxdt \\
=& \Big(\int_{-\infty}^{\infty} \frac{3t^2 - t^4}{(1+t^2)^3} dt \Big) \Big(\int_{-\infty}^{\infty} y^2 e^{-\frac{3y^2}{\lambda}} dy \Big) = 0
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{III} :=& \iint x^4(4t^2 - 4t^4)(1+t^2)^{-4}(1+t^2)^{-3/2} e^{-\frac{3x^2}{\lambda(1+t^2)}} dxdt \\
=& \iint (4t^2 - 4t^4)(1+t^2)^{-11/2} x^4 e^{-\frac{3x^2}{\lambda(1+t^2)}} dxdt \\
=& \Big(\int_{-\infty}^{\infty} \frac{4t^2 - 4t^4}{(1+t^2)^3} dt \Big) \cdot \Big(\int_{-\infty}^{\infty} y^4 e^{-\frac{3y^2}{\lambda}} dy \Big) \\
=& -\pi \cdot \Big(\frac{1}{12} \Big(\frac{\pi}{3/\lambda} \Big)^{1/2} \Big) = -\frac{\pi^{3/2}}{12\sqrt{3}} \lambda^{5/2}.
\end{aligned}$$

All in all we have that

$$\begin{aligned}
-6a \iint \Re\Big\{ it|G_1(x,t)|^4 \overline{G_1(x,t)} G_3(x,t) \Big\} dxdt &= -6a \Big(\frac{2\pi}{\lambda} \Big)^3 (3\lambda^{-2}\mathbf{I} - 6\lambda^{-3}\mathbf{II} + \lambda^{-4}\mathbf{III}) \\
&= -a(18\lambda^{-2}\mathbf{I} + 0 + 6\lambda^{-4}\mathbf{III}) \Big(\frac{2\pi}{\lambda} \Big)^3 \\
&= -\Big(\frac{18}{4} - \frac{6}{12} \Big) a\pi^{3/2} \lambda^{-7/2} \frac{(2\pi)^3}{\sqrt{3}} \lambda^{-1} \\
&= -4a\pi^{3/2}\lambda^{-7/2} \mathbf{C}_F[\lambda]^6,
\end{aligned}$$

as claimed. □

Chapter 5

Nonexistence of extremizers for certain convex curves

In the last chapter we considered a certain class of convex curves (Γ, σ) in the plane whose curvature κ satisfies[1]

$$(\kappa'' \cdot \kappa^{-3})(p) < \frac{3}{2} \tag{5.1}$$

for all points $p \in \Gamma$ at which κ attains a global minimum. In this case, we proved in particular that extremizers for the corresponding Fourier restriction inequality exist. It is natural to ask about the significance of the geometric condition (5.1). For instance, if it is not satisfied, does this mean that extremizers fail to exist?

While we are unable to provide a complete answer to this question, in this chapter we turn our attention to a restricted family of key examples for the same problem, and prove a complementary (negative) result along these lines.

Let us start by setting up the problem precisely. Given data $r, \lambda, a > 0$, consider the curve $\Gamma = \Gamma_{r,\lambda,a}$ parametrized by

$$\begin{aligned} \gamma = \gamma_{r,\lambda,a} : [-r, r] &\to \mathbb{R}^2 \\ y &\mapsto \Big(y, g(y) = \tfrac{\lambda}{2} y^2 + a y^4\Big). \end{aligned}$$

Equip Γ with arclength measure $\sigma = \sigma_{r,\lambda,a}$. The curvature κ of Γ at a point $(y, g(y)) \in \Gamma$ is given by

$$\kappa(y) = \frac{g''(y)}{(1 + g'(y)^2)^{3/2}} = \frac{\lambda + 12 a y^2}{(1 + \lambda^2 y^2 + 8 a \lambda y^4 + 16 a^2 y^6)^{3/2}},$$

[1]Derivatives in (5.1) are taken with respect to the natural arclength parameter.

and one can easily check that κ attains a minimum at the origin $(0,0) \in \Gamma$ if and only if $a \geq (\frac{\lambda}{2})^3$. The Tomas-Stein inequality states that there exists a finite constant $\mathbf{C} = \mathbf{C}_{r,\lambda,a} < \infty$ such that

$$\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)} \leq \mathbf{C}\|f\|_{L^2(\sigma)}, \quad \text{for every } f \in L^2(\sigma); \tag{5.2}$$

by $\mathbf{C} = \sup_{0\neq f \in L^2} \|\widehat{f\sigma}\|_6 \|f\|_2^{-1}$ we mean the optimal constant as usual. Here is our main result:

Theorem 5.1. *Let $r, \lambda, a > 0$ be such that $a > 2(\frac{\lambda}{2})^3$, and consider the curve (Γ, σ) parametrized by γ as above. Then there exists $r_0 > 0$ such that for every $r < r_0$, the triple convolution $\sigma * \sigma * \sigma$ attains a strict global maximum at the origin. As a consequence, if $r < r_0$, every extremizing sequence concentrates at the origin. In particular, there are no extremizers for inequality* (5.2).

The rest of this chapter is devoted to the proof of Theorem 5.1. The strategy will be to analyze the explicit formula for the triple convolution $\sigma * \sigma * \sigma$ obtained in §4.7 and use it to show that a suitable modification thereof is sufficiently regular inside its support[2]. We compute its (one-sided) partial derivatives of order 2 at the critical point $(0,0)$ in order to apply the second derivative test. The last assertions of the theorem follow via an argument based on ideas of Foschi [18] and Quilodrán [23].

Proof of Theorem 5.1. Let $\eta_r \in C_0^\infty(\mathbb{R})$ denote the mollified version of the characteristic function of the interval $I_r := [-r, r]$ described in the proof of Proposition 4.24. If $|\xi| \leq r$, we have that

$$\sigma(\xi,\tau) = G_r(\xi)\delta(\tau - g(\xi))d\xi d\tau,$$

where $G_r(y) := (1 + g'(y)^2)^{1/2}\eta_r(y)$ is a smooth function inside its support $\{y \in \mathbb{R} : |y| \lesssim r\}$.

For $\theta \in [0, 2\pi]$, consider the quantities

$$\alpha(\theta) = -\frac{\cos\theta}{\sqrt{2}} - \frac{\sin\theta}{\sqrt{6}},\ \beta(\theta) = \frac{\cos\theta}{\sqrt{2}} - \frac{\sin\theta}{\sqrt{6}} \text{ and } \gamma(\theta) = 2\frac{\sin\theta}{\sqrt{6}},$$

which satisfy the relations

$$\alpha(\theta) + \beta(\theta) + \gamma(\theta) = 0 \text{ and} \tag{5.3}$$
$$\alpha(\theta)^2 + \beta(\theta)^2 + \gamma(\theta)^2 = 1. \tag{5.4}$$

[2]For an analogous reasoning in the bilinear setting, see [36] and the references therein, especially [37].

Let $|\xi| \leq r$ and $0 \leq \rho \lesssim r$. The following functions will appear in the expression (5.7) for $\sigma * \sigma * \sigma$ below:

$$\widetilde{\mathfrak{G}}_r(\xi;\rho,\theta) := G_r\Big(\frac{\xi}{3}+\alpha(\theta)\rho\Big)G_r\Big(\frac{\xi}{3}+\beta(\theta)\rho\Big)G_r\Big(\frac{\xi}{3}+\gamma(\theta)\rho\Big); \tag{5.5}$$

$$\psi(\xi;\rho,\theta) := \frac{\lambda}{2}\rho^2 + \frac{2}{3}a\xi^2\rho^2 - \Big(\frac{2}{3}\Big)^{3/2} a\xi\sin(3\theta)\rho^3 + \frac{a}{2}\rho^4. \tag{5.6}$$

The function $\widetilde{\mathfrak{G}}_r$ is smooth inside its support, which is compact in all three variables. It will not give us much trouble. On the other hand, the function ψ defines a smooth function in all its variables, but we will need to somehow invert it. This is a somewhat delicate procedure, which already showed up in Appendix 1 of Chapter 4.

From the proof of Proposition 4.24, we have that the triple convolution $\sigma^{(*3)} := \sigma * \sigma * \sigma$ is given by

$$\sigma^{(*3)}(\xi,\tau) = \frac{\chi(|\xi|\lesssim r)}{\sqrt{3}} \int_0^{2\pi}\int_{0\leq\rho\lesssim r} \widetilde{\mathfrak{G}}_r(\xi;\rho,\theta)\delta\Big(\tau - 3g(\xi/3) - \psi(\xi;\rho,\theta)\Big)\rho d\rho d\theta. \tag{5.7}$$

We introduce a change of variables which is related but not identical to the one described in the proof of Proposition 4.24. Start by noting that, for $v \geq 0$, the equation $\psi(\xi;\rho,\theta) = v^2$ can be uniquely solved for $\rho = \rho_{\xi,\theta}(v) \geq 0$ in a sufficiently small right half-neighborhood of the origin. Write the equation in the equivalent form

$$v = v_{\xi,\theta}(\rho) = \rho\Big(\frac{\lambda}{2} + \frac{2}{3}a\xi^2 - (\frac{2}{3})^{3/2}a\xi\sin(3\theta)\rho + \frac{a}{2}\rho^2\Big)^{1/2}.$$

One sees that v defines a C^∞ diffeomorphism from $[0,r]$ onto its image, provided $r > 0$ is chosen sufficiently small (as a function of λ and a). Moreover, r can be chosen to ensure

$$\frac{\partial v}{\partial \rho}(\rho) > 0 \text{ if } \rho \in [0,r].$$

By the inverse function theorem, $\rho = \rho_{\xi,\theta}(v)$ will then define a smooth function of v on $v([0,r]) \subset [0,\infty)$, and $\partial_v \rho(v) > 0$ on the same interval. In particular,

$$\rho'_{\xi,\theta}(0) = \frac{1}{v'_{\xi,\theta}(0)} = \frac{1}{(\lambda/2 + 2/3a\xi^2)^{1/2}} > 0. \tag{5.8}$$

Changing variables $\rho = \rho_{\xi,\theta}(v)$ in (5.7), we have that

$$\sigma^{(*3)}(\xi,\tau) = \frac{\chi(|\xi|\lesssim r)}{\sqrt{3}} \int_0^{2\pi}\int_{0\leq v\leq C\psi^{1/2}} \widetilde{\mathfrak{G}}_r(\xi;\rho_{\xi,\theta}(v),\theta)\cdot \\ \cdot\,\delta\Big(\tau - 3g(\xi/3) - v^2\Big)\frac{\rho_{\xi,\theta}(v)}{\partial_\rho\psi(\xi;\rho_{\xi,\theta}(v),\theta)} 2v dv d\theta, \tag{5.9}$$

and so

$$\sigma^{(*3)}(\xi,\tau) = \frac{\chi(|\xi| \lesssim r)}{\sqrt{3}} \int_0^{2\pi} \widetilde{\mathfrak{G}}_r(\xi;\rho_{\xi,\theta}(\sqrt{\tau - 3g(\xi/3)}),\theta) \frac{\rho_{\xi,\theta}(\sqrt{\tau - 3g(\xi/3)})}{\partial_\rho \psi(\xi;\rho_{\xi,\theta}(\sqrt{\tau - 3g(\xi/3)}),\theta)} d\theta. \tag{5.10}$$

For $\theta \in [0, 2\pi]$, the integrand in (5.10) is supported in the region

$$\{(\xi,\tau) \in \mathbb{R}^2 : 0 \le \tau - 3g(\xi/3) \le C^2\psi(\xi;r,\theta)\},$$

where the constant $C < \infty$ is large enough that the restriction $v \le C\psi(\xi;r,\theta)^{1/2}$ in the inner integral of (5.9) becomes redundant because of support limitations on factors present in the integrand.

As we have seen in Chapter 4, expression (5.10) defines a continuous function of the variables ξ, τ. However, it is *not* a differentiable function of τ at $\tau = 0$. To remedy this, introduce a new parameter $\epsilon \ge 0$ defined by the equation

$$\epsilon^2 := \tau - 3g(\xi/3) = \tau - \frac{\lambda}{6}\xi^2 - \frac{a}{27}\xi^4.$$

The geometric significance of ϵ is clear: for $|\xi| \le 3r$ and $\tau \ge 3g(\xi/3)$, it measures the (square root of the) vertical distance from the point $(\xi,\tau) \in \mathbb{R}^2$ to the "lower boundary" of the support of the triple convolution $\sigma^{(*3)}$, given by

$$\Big\{(\xi,\tau) \in \mathbb{R}^2 : \tau = 3g(\xi/3)\Big\}.$$

For $\epsilon \ge 0$, define the function

$$F(\xi,\epsilon) := \frac{\chi(|\xi| \lesssim r)}{\sqrt{3}} \int_0^{2\pi} \widetilde{\mathfrak{G}}_r(\xi;\rho_{\xi,\theta}(\epsilon),\theta) \frac{\rho_{\xi,\theta}(\epsilon)}{\partial_\rho \psi(\xi;\rho_{\xi,\theta}(\epsilon),\theta)} d\theta. \tag{5.11}$$

In view of (5.10), a sufficient condition for the convolution $\sigma * \sigma * \sigma$ to have a strict local maximum at $(\xi,\tau) = (0,0)$ is that the function F has a strict local maximum at $(\xi,\epsilon) = (0,0)$. The advantage of considering (5.11) instead of (5.10) lies in its extra regularity, which is needed to justify the computations that will follow:

Claim 5.2. There exists $r_1 > 0$ such that expression (5.11) defines a C^∞ function of both ξ and ϵ in the rectangle

$$\{(\xi,\epsilon) \in \mathbb{R}^2 : |\xi| \le r_1, 0 \le \epsilon \le r_1\}.$$

Proof of Claim 5.2. All the work has basically been done. Since ρ is a smooth function of ϵ, the function $\widetilde{\mathfrak{G}}_r$ is smooth in the variables ξ, ϵ and θ; it is also compactly supported in all its variables. Compute:

$$\frac{\rho_{\xi,\theta}(\epsilon)}{\partial_\rho \psi(\xi; \rho_{\xi,\theta}(\epsilon), \theta)} = \frac{1}{\lambda + 4/3a\xi^2 - 2\sqrt{2}/\sqrt{3}a\xi \sin(3\theta)\rho_{\xi,\theta}(\epsilon) + 2a\rho_{\xi,\theta}(\epsilon)^2}. \tag{5.12}$$

If $r_1 > 0$ is chosen sufficiently small, then $\rho/\partial_\rho\psi(\rho)$ defines a positive, smooth function of ξ (trivial) and ϵ which is bounded above uniformly by $1/\lambda$. Indeed, the denominator in (5.12) is never zero as long as ξ, ϵ are small enough, and moreover we have that

$$\frac{4}{3}a\xi^2 + 2a\rho_{\xi,\theta}(\epsilon)^2 \geq 2\frac{2\sqrt{2}}{\sqrt{3}}a|\xi|\rho_{\xi,\theta}(\epsilon) \geq \frac{2\sqrt{2}}{\sqrt{3}}a|\xi|\sin(3\theta)\rho_{\xi,\theta}(\epsilon). \qquad \square$$

We now compute the Hessian of the function F at the critical point $(\xi, \epsilon) = (0, 0)$.

Consider the case $\xi = 0$. Then $\psi(0; \rho, \theta)$ does not depend on θ and can be explicitly inverted[3]. In fact, $\psi(0; \rho, \theta) = \psi(\rho) = \frac{\lambda}{2}\rho^2 + \frac{a}{2}\rho^4$ and $\partial_\rho\psi(\rho) = \lambda\rho + 2a\rho^3$. Since $\psi(\rho) = \epsilon^2$ and $\rho \geq 0$,

$$\rho(\epsilon) = \sqrt{-\frac{\lambda}{2a} + \frac{\sqrt{\lambda^2 + 8a\epsilon^2}}{2a}}. \tag{5.13}$$

It follows that

$$\frac{\rho(\epsilon)}{\partial_\rho\psi(\rho(\epsilon))} = \frac{1}{\lambda + 2a\rho(\epsilon)^2} = \frac{1}{\sqrt{\lambda^2 + 8a\epsilon^2}}; \tag{5.14}$$

as noticed before, this defines a smooth, positive function of ϵ which is bounded above uniformly by $1/\lambda$. In particular, for $\epsilon \leq C\psi(r)^{1/2}$, we have from (5.11) and (5.14) that

$$F(0, \epsilon) = \frac{1}{\sqrt{3}} \int_0^{2\pi} \widetilde{\mathfrak{G}}_r(0; \rho(\epsilon), \theta) \frac{\rho(\epsilon)}{\partial_\rho\psi(\rho(\epsilon))} d\theta = \frac{1}{\sqrt{3}\sqrt{\lambda^2 + 8a\epsilon^2}} \int_0^{2\pi} \widetilde{\mathfrak{G}}_r(0; \rho(\epsilon), \theta) d\theta. \tag{5.15}$$

To the best of our knowledge, the integral in this last expression cannot be explicitly evaluated. We can, however, Taylor expand it. For $|\zeta| \leq r$, we have that $\eta_r(\zeta) \equiv 1$ and

$$G_r(\zeta) = G(\zeta) = (1 + g'(\zeta)^2)^{1/2} = (1 + (\lambda\zeta + 4a\zeta^3)^2)^{1/2} = 1 + \frac{\lambda^2}{2}\zeta^2 + O(\zeta^4). \tag{5.16}$$

[3]This algebraic trick simplifies matters greatly and is not available if, say, the function g contains a cubic term of the form by^3.

By (5.5), we have that

$$\widetilde{\mathfrak{G}}_r(0;\rho(\epsilon),\theta) = G_r(\alpha(\theta)\rho(\epsilon)) \cdot G_r(\beta(\theta)\rho(\epsilon)) \cdot G_r(\gamma(\theta)\rho(\epsilon)).$$

Using the approximation given by (5.16) on each of these three factors, recalling the relation (5.4), and integrating with respect to the angular variable θ, we get that

$$\int_0^{2\pi} \widetilde{\mathfrak{G}}_r(0;\rho(\epsilon),\theta)d\theta = 2\pi\Big(1+\frac{\lambda^2}{2}\rho(\epsilon)^2 + O(\rho(\epsilon)^4)\Big)$$

if $\epsilon \le C\psi(r)^{1/2}$ and $r > 0$ is sufficiently small. Plugging this into (5.15) and noting that $\rho(\epsilon) = O(\epsilon)$, one gets that

$$F(0,\epsilon) = \frac{2\pi}{\sqrt{3}} \cdot \frac{1}{\sqrt{\lambda^2 + 8a\epsilon^2}}\Big(1+\frac{\lambda^2}{2}\rho(\epsilon)^2\Big) + O(\epsilon^4) \quad \text{if } \epsilon \le C\psi(r)^{1/2}. \tag{5.17}$$

We have an explicit expression for $\rho(\epsilon)$ given by (5.13). Using it, one finally obtains

$$\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0^+} F(0,\epsilon) = 0 \tag{5.18}$$

and

$$\frac{\partial^2}{\partial\epsilon^2}\Big|_{\epsilon=0^+} F(0,\epsilon) = 2\frac{2\pi}{\sqrt{3}}\Big(1-\frac{4a}{\lambda^3}\Big). \tag{5.19}$$

Expression (5.19) defines a negative quantity if and only if $a > 2(\frac{\lambda}{2})^3$. This turns out to be a necessary and sufficient condition for the function F to have a strict local maximum at the origin $(0,0)$, provided $r > 0$ is sufficiently small. Let us verify this in detail, thus establishing the first part of Theorem 5.1.

Let $a > 2(\frac{\lambda}{2})^3$. We start by noting that (5.18) is valid also when $\xi \neq 0$ is sufficiently small. Using expression (5.11), one can easily check that

$$\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0^+} F(\xi,\epsilon) = 0 \quad \text{if } |\xi| \lesssim r. \tag{5.20}$$

Indeed, the derivative in ϵ may fall in one of two factors. There is no contribution from the function $\widetilde{\mathfrak{G}}_r$ because equations (5.3) and (5.8) imply

$$\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0^+} \widetilde{\mathfrak{G}}_r(\xi;\rho_{\xi,\theta}(\epsilon),\theta) = \rho'_{\xi,\theta}(0)(\alpha(\theta)+\beta(\theta)+\gamma(\theta))G'_r(\xi/3)G_r(\xi/3)^2 = 0.$$

The factor containing the quotient $\rho/\partial_\rho\psi(\rho)$ likewise does not contribute: we can use (5.12), (5.8), recall that $\rho_{\xi,\theta}(0) = 0$, and conclude that

$$\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0^+}\frac{\rho_{\xi,\theta}(\epsilon)}{\partial_\rho\psi(\rho_{\xi,\theta}(\epsilon))}=\frac{4}{\sqrt{3}}a\xi\Big(\lambda+\frac{4}{3}a\xi^2\Big)^{-5/2}\cdot\sin 3\theta.$$

The integral of the function $\sin 3\theta$ over the interval $[0,2\pi]$ vanishes, and this concludes the verification of (5.20). As a consequence,

$$\frac{\partial^2}{\partial\xi\partial\epsilon}\Big|_{(\xi,\epsilon)=(0,0^+)}F(\xi,\epsilon)=0. \tag{5.21}$$

At the lower boundary of the support of $\sigma^{(*3)}$ (i.e. when $|\xi|\lesssim r$ and $\epsilon=0$), we have that

$$\begin{aligned}F(\xi,0)=\frac{1}{\sqrt{3}}\int_0^{2\pi}\widetilde{\mathfrak{G}}_r(\xi;\rho_{\xi,\theta}(0),\theta)\frac{\rho_{\xi,\theta}(0)}{\partial_\rho\psi(\xi;\rho_{\xi,\theta}(0),\theta)}d\theta\\ =\frac{2\pi}{\sqrt{3}}\frac{(1+g'(\xi/3)^2)^{3/2}}{g''(\xi/3)}=\frac{2\pi}{\sqrt{3}}\frac{1}{\kappa(\xi/3)}.\end{aligned} \tag{5.22}$$

Since $a\geq(\frac{\lambda}{2})^3$, the curvature κ attains a local minimum at 0, and so $F(\cdot,0)$ attains a local maximum along the lower boundary of its support. This actually defines a concave function of ξ in a neighborhood of the origin since

$$\frac{\partial}{\partial\xi}\Big|_{\xi=0}F(\xi,0)=0 \tag{5.23}$$

and

$$\frac{\partial^2}{\partial\xi^2}\Big|_{\xi=0}F(\xi,0)=\frac{2\pi}{\sqrt{3}}\frac{d^2}{d\xi^2}\Big|_{\xi=0}\frac{1}{\kappa(\xi/3)}=\frac{2\pi}{\sqrt{3}}\Big(\frac{\lambda}{3}-\frac{8a}{3\lambda^2}\Big)<0. \tag{5.24}$$

We may now use the second partial derivative test to conclude from (5.19), (5.21) and (5.24) that the function F attains a strict local maximum at the origin. As discussed before, it follows that, working as we are under the assumption $a>2(\frac{\lambda}{2})^3$, the triple convolution $\sigma*\sigma*\sigma$ also has a strict local maximum at the origin. Choosing $r>0$ to be sufficiently small, one can further ensure this maximum to be a *global* one.

This is the crucial ingredient in showing that extremizers for inequality (5.2) do not exist for sufficiently small caps $[-r,r]$. The proof goes along the lines of what was done in [18, 23], and we recall it here. Let $Tf:=\widehat{f\sigma}$. As in the previous chapter,

$$\begin{aligned}
\|Tf\|_6^6 &= (2\pi)^2 \iint_{\mathbb{R}^2} |f\sigma * f\sigma * f\sigma(\xi,\tau)|^2 d\xi d\tau \qquad (5.25)\\
&\leq (2\pi)^2 \iint |f|^2\sigma * |f|^2\sigma * |f|^2\sigma(\xi,\tau)\cdot \sigma*\sigma*\sigma(\xi,\tau) d\xi d\tau\\
&\leq (2\pi)^2 \sup_{(\xi,\tau)\in \operatorname{supp}(\sigma^{(*3)})} \sigma^{(*3)}(\xi,\tau)\cdot \iint |f|^2\sigma * |f|^2\sigma * |f|^2\sigma(\xi,\tau) d\xi d\tau\\
&= (2\pi)^2 \|\sigma^{(*3)}\|_{L^\infty(\mathbb{R}^2)} \|f\|_{L^2(\sigma)}^6,
\end{aligned}$$

where we used Hölder's inequality twice. All the inequalities become equalities if and only if $\sigma^{(*3)}(\xi,\tau) = \|\sigma^{(*3)}\|_{L^\infty(\mathbb{R}^2)}$ for a.e. $(\xi,\tau)\in \operatorname{supp}(\sigma^{(*3)})$.

As before, assume that $a > 2(\frac{\lambda}{2})^3$ and take $r_0 > 0$ sufficiently small to ensure that $\sigma^{(*3)}$ attains a strict *global* maximum at the origin $(0,0)$ if $r < r_0$. Then

$$\|\sigma^{(*3)}\|_{L^\infty(\mathbb{R}^2)} = \sigma^{(*3)}(0,0) = \frac{2\pi}{\sqrt{3}}\frac{1}{\lambda} = \frac{\mathbf{C}_F[\lambda]^6}{(2\pi)^2}. \qquad (5.26)$$

As in [23], the lower bound $\|T\| \geq \mathbf{C}_F[\lambda]$ can be easily verified with the help of an explicit extremizing sequence. For that purpose, consider the scaled Gaussian $G(y) = e^{-\lambda y^2/2}$ and its evolution via the Schrödinger flow

$$G_1(x,t) = \int_{\mathbb{R}} G(y) e^{-it\frac{\lambda y^2}{2}} e^{ixy} dy.$$

For $\delta > 0$, consider the family of trial functions $f_\delta(y) = \delta^{-1/2} G(\delta^{-1}y) \in L^2(\sigma)$. It is shown in the course of the variational calculation of Chapter 4 that

$$\lim_{\delta\to 0^+} \|Tf_\delta\|_{L^6(\mathbb{R}^2)} \|f_\delta\|_{L^2(\sigma)}^{-1} = \|G_1\|_6 \|G\|_2^{-1} = \mathbf{C}_F[\lambda].$$

Aiming at a contradiction, let $0 \neq f \in L^2(\sigma)$ be an extremizer for inequality (5.2). In particular, $\|Tf\|_6 = \|T\|\cdot\|f\|_{L^2(\sigma)}$. Using the lower bound just described, and the upper bound given by the chain of inequalities (5.25), we get that

$$\mathbf{C}_F[\lambda]\|f\|_{L^2(\sigma)} \leq \|T\|\cdot\|f\|_{L^2(\sigma)} = \|Tf\|_6 \leq (2\pi)^{1/3}\|\sigma^{(*3)}\|_{L^\infty(\mathbb{R}^2)}^{1/6}\|f\|_{L^2(\sigma)}. \qquad (5.27)$$

The L^∞ norm of the triple convolution $\sigma^{(*3)}$ has been computed in (5.26). Using it, we see that all inequalities in (5.27) are equalities. As mentioned before, this forces $\sigma^{(*3)}(\xi,\tau) = \|\sigma^{(*3)}\|_{L^\infty(\mathbb{R}^2)}$ for a.e. $(\xi,\tau)\in \operatorname{supp}(\sigma^{(*3)})$. But this cannot happen since $\sigma^{(*3)}$ has a *strict* local maximum at the origin. The contradiction shows that extremizers for inequality (5.2) cannot exist.

To finish the proof of Theorem 5.1, let $\{f_n\}$ be an extremizing sequence for inequality (5.2). Then $\{|f_n|\}$ is an extremizing sequence of nonnegative functions for the same inequality. As a consequence of the dichotomy established in Proposition 4.21 and the conclusion of Lemma 4.23, the sequence $\{|f_n|\}$ must concentrate at some point $(y, g(y)) \in \Gamma$, and so $\{f_n\}$ concentrates at the same point. From Corollary 4.25 it follows that $y = 0$. $\square$

Compare with what we had already obtained in Chapter 4: the functional Ξ is a strictly concave function in a neighborhood of 0 if and only if $a < \frac{3}{2}(\frac{\lambda}{2})^3$. It follows that $\mathbf{C}_F[\lambda] < \mathbf{C}[\Gamma]$, and so extremizing sequences cannot concentrate. Precompactness is ensured in this case. The natural question is then:

Question 5.3. Do extremizers exist when the parameter a lies in the interval $[\frac{3}{2}(\frac{\lambda}{2})^3, 2(\frac{\lambda}{2})^3]$?

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