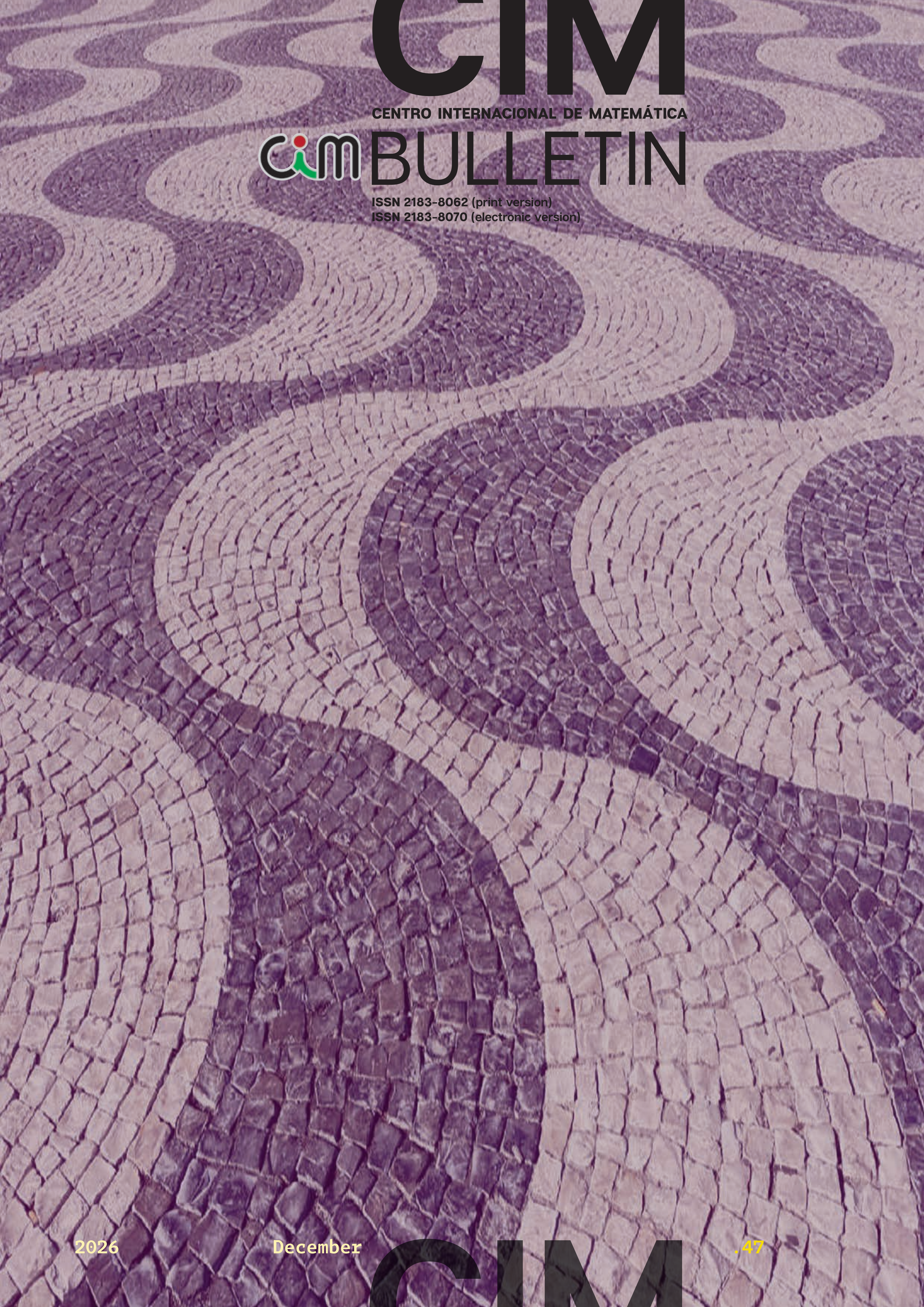

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Contents

Editorial

In this issue, we include three articles on topical research subjects.

One of them presents an overview of geometric and Lie theoretic aspects behind Poisson brackets and mentions some recent applications to quantization and geometric numerical methods.

Another article introduces the basics of Deep Neural Networks, presents results on their large width limit, and discusses the implications of such results for supervised machine learning.

The last article explores hyperbolic groups, presenting their geometric foundations, highlighting the rich interplay between group theory, geometry, topology, and dynamics, and emphasizing their algorithmic properties and applications to the dynamics of endomorphisms.

Inserted in the cycle of historical articles, we feature an article that reviews the origins and development of the National Seminar on the History of Mathematics (SNHM) since its creation in 1988. It highlights its role in promoting research and dissemination in the history of Mathematics—particularly the history of Portuguese mathematics—through regular meetings, international collaboration, and academic initiatives over nearly four decades.

In January 2025, the Academy of Sciences of Lisbon, in celebration of the International Mathematics Day on March 14, launched a challenge to students in grades 9 to 12 from schools across the country under the inspiring theme *Mathematics, Art and Creativity*. The challenge invited them to create proposals for the five missing symmetry patterns in Lisbon's *calçada*, with the aim of completing the set of 24 flat symmetries in its streets and squares.

We publish an article describing this beautiful initiative.

We publish two interviews. We interviewed Michael Christ and Hugo Duminil-Copin, who were the distinguished mathematicians invited to deliver this year's Pedro Nunes' lectures, which is an emblematic initiative of CIM with the support of SPM.

As usual, we publish several summaries and reports regarding the activities partially supported by CIM during the last year.

We recall that the Bulletin continues to welcome the submission of review, feature, outreach and research articles in Mathematics and its applications.

Maria Joana Torres

Centro de Matemática and Departamento de Matemática, Universidade do Minho

Overview of Lie theory for Poisson brackets

by **Alejandro Cabrera***

Abstract.—Poisson brackets are a central notion in a variety of topics related to mathematical-physics. In this note, we provide an overview of geometric and Lie theoretic aspects behind Poisson brackets and mention some recent ramifications and applications. In particular, we describe the notion of symplectic groupoids viewed as integrations of Poisson manifolds, explicit constructions of these integrations, the general theory of multiplicative structures on Lie groupoids, and applications to quantization and geometric numerical methods.

1 Introduction

Poisson brackets became a central notion in a variety of areas motivated by mathematical-physics (see e.g. the textbooks [30, 39] and more references therein). Their initial role in the description of mechanical systems (see [1, 4]) was latter complemented by their relevance in the connection between classical and quantum mechanics (see [5, 6, 39]), spreading into the corresponding modern mathematical theories. Presently, the applications of Poisson brackets range from more classical ones in Hamiltonian dynamics and integrable systems to symbol calculus and non-commutative algebra (e.g. [35, 47, 42, 45]), passing through topics like Lie theory and quantum groups (e.g. [38, 44, 46]).

This short overview is intended to provide a brief description of the *Lie theory associated with Poisson brackets* (see main results in [28, 29]) as well as to highlight some recent developments in the area, applications and ramifications. The central topics grow from the idea that, just as Lie algebras correspond to infinitesimal version of Lie groups, Poisson brackets correspond to the infinitesimal version of so-called *(local) symplectic groupoids*. This viewpoint is largely motivated by both the general study of the *Poisson geometry* behind these brackets and by the study of their *quantization*. We thus intend to summarize the main ideas related to this generalization of classical Lie theory and to mention some of their recent applications.

The structure of this paper is as follows. In Subsection 1.1 we recall the main definitions and features of Poisson brackets and of the underlying Poisson geometry. In Section 2, we provide an overview of the corresponding Lie theory in which the role played by transformation groups in classical Lie theory is now played by symplectic groupoids. We also include a summary of the Lie theory for general groupoids and algebroids (Section 2.1) as well as a description of explicit integration procedures (Section 2.3). Finally, we review some applications and ramifications in Section 3. In particular, we describe the general theory of multiplicative structures on Lie groupoids (Section 3.1), the link to the quantization of Poisson brackets (Section 3.2) and mention some recent work applying the general Lie theory to the construction of geometric numerical methods (Section 3.3).

1.1 Poisson brackets

Let us first give the basic definition and then comment on the underlying dynamical and geometric fea-

* Instituto de Matemática, Universidade Federal do Rio de Janeiro

tures. We observe that such general Poisson brackets can arise from more standard ones via quotient or reduction by symmetries, as well as by the presence of dynamical constraints, and that the axioms capture their most characteristic properties. (See more details in [1, 30].)

Definition 1.— Let M be a smooth manifold. A *Poisson bracket* on M is a $\mathbb{R}$-bilinear operation

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

such that: for any $f_1, f_2, f_3 \in C^\infty(M)$,

1. it is a derivation in each argument,
$$\{f_1, f_2 f_3\} = \{f_1, f_2\} f_3 + f_2\{f_1, f_3\},$$
and analogous for $\{f_1 f_2, f_3\}$;
2. it defines a *Lie bracket*, namely, it is skew symmetric $\{f_1, f_2\} = -\{f_2, f_1\}$ and satisfies the Jacobi identity
$$\{f_1, \{f_2, f_3\}\} = \{\{f_1, f_2\}, f_3\} + \{f_2, \{f_1, f_3\}\}.$$

A *Poisson map* $\varphi : (M_1, \{,\}_1) \to (M_2, \{,\}_2)$ is a smooth map that preserves brackets as follows,

$$\varphi^*\{f_1, f_2\}_2 = \{\varphi^* f_1, \varphi^* f_2\}_1, \ \forall f_1, f_2 \in C^\infty(M_2).$$

We now describe the significance of these axioms, starting with their role in Hamiltonian dynamics. The derivation property allows one to promote any function $H \in C^\infty(M)$, called the *Hamiltonian*, to an ODE on M given by the corresponding *Hamiltonian vector field* $X_H \in \mathfrak{X}(M)$. Seen as a derivation, X_H is defined by

$$X_H(f) = \{H, f\}.$$

It then follows that the evolution law for any *observable* given by a function $f \in C^\infty(M)$ along the flow φ_t^H of X_H is determined by the Poisson brackets,

$$\dot{f} := \frac{d}{dt}(f \circ \varphi_t^H) = \{H, f\} \circ \varphi_t^H.$$

The skew symmetry axiom thus implies the so-called *conservation of energy*, $\dot{H} = \{H, H\} = 0$, so that the solutions restrict to the level sets of H. The Jacobi identity implies that φ_t^H defines a Poisson autormorphism of $(M, \{,\})$ for any Hamiltonian $H \in C^\infty(M)$.

We end this section by discussing the underlying *Poisson geometry* of such $(M, \{,\})$. The derivation axiom together with skewsymmetry are equivalent to the fact that the brackets are determined by a tensor called *Poisson bivector*, $\pi \in \Gamma\Lambda^2 TM$, via

$$\{f_1, f_2\}|_x = \pi_x(df_1|_x, df_2|_x),$$

for $x \in M$ and $f_1, f_2 \in C^\infty(M)$.

One can think of π_x as a skew-symmetric bilinear form on each cotangent space T_x^*M, smoothly depending on $x \in M$. Hamiltonian vector fields are obtained as $X_H = \pi(dH, \cdot)$ so that they are in the range of the bilinear form π. The Jacobi identity is equivalent to a quadratic PDE for π which can be written as $[\pi, \pi] = 0$, for $[,]$ the Schoutens bracket extension of the Lie bracket of vector fields to the exterior algebra $\Gamma\Lambda^\bullet TM$. We shall call (M, π) a *Poisson manifold*.

Remark 1.— (The symplectic case) When the bilinear form π_x is non-degenerate for every $x \in M$, the Poisson structure is called *symplectic*. In this case, the inverse bilinear form at each point defines a 2-form $\omega = \pi^{-1} \in \Omega^2(M)$ which is non-degenerate and closed, $d\omega = 0$. In this way, one recovers the usual description of symplectic structures as a particular case of Poisson structures. One important example is $M = \mathbb{R}^n \times \mathbb{R}^n \ni (q, p)$ with[1] $\omega = dq^i \wedge dp_i$. In this case, the Hamiltonian vector field X_H of $H(q, p) = |p|^2/2m + V(q)$ yields the standard Hamilton's equations in mechanics. This example generalizes to cotangent bundles $M = T^*Q$ of any manifold Q endowed with a *canonical symplectic structure* $\omega_c \in \Omega^2(T^*Q)$. (See more in [1, 4].)

A fundamental structural result in Poisson geometry is *Weinstein's splitting theorem*, [43]. It asserts that, locally around any point $x_0 \in M$, the Poisson structure is isomorphic to a product $(M_1, \pi_1) \times (M_2, \pi_2)$ with π_1 being non-degenerate (i.e. symplectic, as in the remark above) and with $\pi_2|_{x_0} = 0$. One consequence is that every Poisson manifold (M, π) has an underlying partition $M = \coprod L$ into (immersed) symplectic submanifolds (L, ω_L) corresponding to the symplectic factor in the local model. This determines a singular foliation with leaves of varying dimension (given by the rank of π at each point) which is called *the symplectic foliation* of (M, π). This foliation also admits a dynamical characterization: the leaf through $x \in M$ is given by the all the points that can be reached from x following a fininte number of Hamiltonian flows $\varphi_{t_j}^{H_j}$.

We finish by noting that there are more geometric structures induced by (M, π), see [30, 43]. The linearization of the degenerate local factor π_2 in Weinstein's splitting gives rise to a Lie algebra $\mathfrak{g}_x$ for each $x \in M$ called *isotropy Lie algebra*. These are isomorphic for points on the same symplectic leaf. Similarly,

[1] Sum over repeated indices is understood throughout this paper.

higher order terms of π_2 define a *transverse structure* to the symplectic foliation. In this way, Poisson geometry can be seen as the branch of differential geometry which studies such rich set of structures defined by (M, π). Some natural problems in this context are: linearization of the structure around a leaf and stability of leaves under deformation.

Example 1.— (A non-symplectic Poisson structure from rigid body dynamics) Let us consider $M = \mathbb{R}^3$ with the bracket $\{f_1, f_2\}|_x = (\nabla f_1|_x \times \nabla f_2|_x) \cdot x$, where the formula indicates the cross product of the gradients followed by inner product with $x \in \mathbb{R}^3$. By chosing $H(x)$ a suitable quadratic form (the "inertia tensor"), the corresponding Hamiltonian vector field X_H yields Euler's equation for the angular momentum of a free rigid body (see [1]). The symplectic foliation $M = \coprod S^2_r$ is given by spheres S^2_r with center at the origin $x = 0$ and radius r. Note that the origin itself is a singular leaf of dimension zero, while the rest are leaves of dimension 2 for $r > 0$. The symplectic structure ω_r on S^2_r is given by $1/r$ times the standard area form. The isotropy Lie algebra $\mathfrak{g}_0$ at the origin is given by infinitesimal rotations $\mathfrak{so}(3)$ while $\mathfrak{g}_x$ is one-dimensional for $x \neq 0$. From this geometric description, it follows that typical trajectories of Euler's equation X_H are given by the intersection of a sphere S^2_r and the ellipsoid $H^{-1}(H(x(t_0)))$. This Poisson structure generalizes to $M = \mathfrak{g}^*$, for any Lie algebra $\mathfrak{g}$, endowed with a naturally induced *linear Poisson structure* $\pi_x = (1/2)x([e^i, e^j])\partial_{x^i} \wedge \partial_{x^j}$, for (e^i) a basis for $\mathfrak{g}$ and $x \in \mathfrak{g}^*$ with dual linear coordinates (x^i). The present example corresponds to the three dimensional $\mathfrak{g} = \mathfrak{so}(3)$.

Other interesting examples come from *Poisson-Lie groups* (G, π) in which a Lie group G is endowed with a compatible Poisson structure. These arise as classical limits of so-called quantum groups (see [39, 44] and references therein). Other examples include: duals of Lie algebroids, Poisson homogeneous spaces, polynomial Poisson structures on vector spaces and Poisson groupoids, among others.

2 Symplectic groupoids as integrations

We now move in the direction of describing the Lie theory behind Poisson brackets, see [27, 38, 44]. Let us first recall some facts from classical Lie theory for latter comparison, see [32] for more details. If $G \subset Gl(n)$ is a matrix Lie group and g_t is a smooth curve through the identity $1 \in G$ at $t = 0$, then

$$g_t = 1 + t\xi + O(t^2)$$

with the possible first order parameters $\xi \in \mathfrak{g} \subset \mathfrak{gl}(n)$ spanning the underlying Lie algebra. These ideas define the differentiation functor Lie from Lie groups to Lie algebras. Conversely, the main results in classical Lie theory say that the infinitesimal parameters $\mathfrak{g}$ define a connected Lie group $G \in Lie^{-1}(\mathfrak{g})$ uniquely up to covering and isomorphism; when G is 1-connected, Lie algebra morphisms $\mathfrak{g} \to Lie(G')$ integrate uniquely to Lie group morphisms $G \to G'$. Lastly, the restriction of a Lie group to a neighborhood $U \subset G$ of the identity can be intrinsically characterized as a *local* Lie group. The Lie functor factors through any such restriction and the structure of U can be thought of as arising from small enough parameters $\xi \sim 0 \in \mathfrak{g}$. Given a Lie algebra $\mathfrak{g}$, one can construct a natural local Lie group structure integrating $\mathfrak{g}$ on a neighborhood $U \subset \mathfrak{g}$ of 0 via the Baker-Campbell-Hausdorff formula.

2.1 Lie groupoids and algebroids

We now review the general notions of Lie groupoids and algebroids, see [41]. A *groupoid* is an algebraic structure, denoted $\mathcal{G} \rightrightarrows M$, which can be described as the data of a category, with all morphisms given by a set $\mathcal{G}$ and all objects given by a set M, and with the special property that every morphism is invertible. The two arrows in the notation correspond to the source and target maps $s, t : \mathcal{G} \to M$, respectively, so that we think of $g \in \mathcal{G}$ as an arrow between the corresponding objects in M,

$$y = t(g) \xleftarrow{g} x = s(g).$$

We adopt the following notational convention for composition of morphisms or "groupoid multiplication" in $\mathcal{G}$,

$$m : \mathcal{G}^{(2)} = \{(g_1, g_2) : s(g_1) = t(g_2)\} \to \mathcal{G},$$

$$(g_1, g_2) \mapsto m(g_1, g_2) \equiv g_1 g_2.$$

In this way, $\mathcal{G} \rightrightarrows M$ can be seen as an equivalence relation $y \sim x$ between the objects $y, x \in M$ which has been enhanced by the information of a set $\mathcal{G}_{yx} = t^{-1}(y) \cap s^{-1}(x)$ parameterizing different possible ways in which y is equivalent to x. The underlying equivalence class L_x of x is called the *orbit* through $x \in M$, and autoequivalences $\mathcal{G}_{xx}$ is called

[2]To incorporate interesting examples coming from foliations, it is customary to allow $\mathcal{G}$ to be a non-Hausdorff manifold but with s-fibers being

the *isotropy* at x.

A *Lie groupoid* is an algebraic groupoid $\mathcal{G} \rightrightarrows M$ together with given smooth manifold structures[2] on $\mathcal{G}$ and M such that s, t are smooth submersions (so that $\mathcal{G}^{(2)} \subset \mathcal{G} \times \mathcal{G}$ has a natural manifold structure) and that all the other category structure maps are smooth. A *morphism of Lie groupoids* is a functor $F : (\mathcal{G}_1 \rightrightarrows M_1) \to (\mathcal{G}_2 \rightrightarrows M_2)$ which is also a smooth map $\mathcal{G}_1 \to \mathcal{G}_2$. Orbits of $\mathcal{G} \rightrightarrows M$ define a singular foliation $M = \coprod L$ by (immersed) submanifolds of varying dimensions and isotropies $\mathcal{G}_{xx}$ define a Lie group for each $x \in M$.

Some typical examples of Lie groupoids are the following. The pair groupoid $\mathcal{G} = M \times M \rightrightarrows M$ in which arrows $g = (y, x)$ are completely determined by their source and target, and composition is defined in the obvious way for the underlying equivalence relation to be transitive. A Lie group G can be seen as a Lie groupoid $\mathcal{G} = G \rightrightarrows *$ over an abstract 1-point set; this groupoid is pure isotropy. The fundamental groupoid $\mathcal{G} = \Pi_1(M) \rightrightarrows M$ is given by paths in the manifold M modulo homotopies with fixed endpoints and composition is given by composition of paths. The orbits are connected components of M and isotropies $\mathcal{G}_{xx} = \pi_1(M, x)$ are the fundamental groups at different points. Other examples include: holonomy groupoid of a foliation in M, total spaces of vector bundles, the ”gauge” groupoid associated to a principal bundle $P \to M$; among others.

We now describe briefly the infinitesimal version of a Lie groupoid $\mathcal{G} \rightrightarrows M$, namely, its underlying *Lie algebroid*. Following what we recalled from classical Lie theory for groups, if we consider a smooth curve $g_t \in \mathcal{G}$ through an identity arrow 1_x, $x \in M$, and with fixed source $s(g_t) = x$, we will have

$$g_t = 1_x + t\xi_x + O(t^2) \tag{1}$$

from which we could identify the possible first order parameters ξ_x in a linear space $A_x \subset T_{1(x)}\mathcal{G}$ normal to $1(M)$. By analogy with the classical case, suitably linearizing the structure of $\mathcal{G}$ one should arrive to the relevant Lie algebroid structure on $A = \coprod_x A_x$. Indeed, this procedure works as expected and one is led to the following definition: a Lie algebroid $(A \to M, \rho, [,])$ is a vector bundle $A \to M$ together with a vector bundle map $\rho : A \to TM$, called the *anchor*, and a Lie bracket $[,]$ on the space of sections ΓA such that the following derivation property holds

$$[a, fb] = f[a, b] + (L_{\rho(a)}f)b, \; a, b \in \Gamma A, f \in C^\infty(M).$$

Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, the corresponding Lie algebroid $Lie(\mathcal{G}) = (A \to M, \rho, [,])$ is defined by $A = Ker(Ds)|_{1(M)}$, $\rho = Dt$ and $[,]$ determined by the identification of sections of A with the Lie algebra of right-invariant vector fields on $\mathcal{G}$. When $Lie(\mathcal{G}) \simeq A$, we say that $\mathcal{G} \rightrightarrows M$ *integrates* the Lie algebroid A. Two of the key features of classical Lie theory generalize directly to this setting: when $\mathcal{G} \rightrightarrows M$ has connected source fibers, the algebroid $A = Lie(\mathcal{G})$ determines $\mathcal{G} \rightrightarrows M$ uniquely up to covering of its source-fibers and isomorphism; when the source-fibers of $\mathcal{G}$ are 1-connected, algebroid morphisms $A \to Lie(\mathcal{G}')$ integrate uniquely to Lie groupoid morphisms $\mathcal{G} \to \mathcal{G}'$. The generalized theory deviates, though, from the classical one in terms of integrability, as remarked next.

Remark 2.— (Integrability and local Lie groupoids) As opposed to classical Lie theory, not every Lie algebroid comes from some Lie groupoid. When it does, we say that A is *integrable*. The general theory of obstructions to integrability was developped in [28]. On the other hand, every Lie algebroid can be integrated by a *local* Lie groupoid. Local Lie groupoids correspond to the intrinsic structure inherited by neighborhoods $U \subset \mathcal{G}$ of the identities $1(M)$, just as local Lie groups are to neighborhoods of $1 \in G$. This can be formalized (see [18, 27]) by considering that each structure map has a domain given by a neighborhood containing the underlying identities and each axiom also has an open domain where it holds. Given a Lie algebroid A, its integrability admits the following alternative characterizations: suitable discreteness of underlying monodromy groups ([28]), by the existence of complete actions ([28, 2]), and by complete associativity properties of the associated germ of local Lie groupoids ([34]).

We mention some examples: $Lie(M \times M \rightrightarrows M) = Lie(\Pi_1(M) \rightrightarrows M)$ is given by the *tangent algebroid* $A = TM$ with $\rho = id$ and $[,]$ the standard Lie bracket of vector fields; when $\mathcal{G} = G \rightrightarrows *$ is a Lie group then $Lie(\mathcal{G}) = Lie(G) = \mathfrak{g}$ is the standard Lie algebra seen as a bundle over an abstract point. Other examples include an involutive regular distribution $D \subset TM$ and the gauge (or ”Atiyah”) algebroid TP/G associated with a principal G-bundle P. We end this subsection with the most relevant case for our purposes: every Poisson manifold $(M, \{,\} \equiv \pi)$ determines a *cotangent Lie algebroid* structure de-

Hausdorff, see [30, 41].

noted $T^*_\pi M$ on $A = T^*M \to M$. The anchor map is determined by $\rho(dH) = X_H$, the Hamiltonian vector field, and the bracket is uniquely defined by $[df_1, df_2] = d\{f_1, f_2\}$.

2.2 Symplectic groupoids and Poisson brackets

In this subsection, we shall see that Poisson brackets can be seen as the infinitesimal structure behind so-called symplectic groupoids, see [27, 38, 41, 44, 46]. Indeed, we will also see that these are the integration of the Lie algebroid $T^*_\pi M$ associated with a Poisson manifold $(M, \{,\} \equiv \pi)$.

Definition 2.— A *symplectic groupoid* $(\mathcal{G} \rightrightarrows M, \omega)$ is defined by a Lie groupoid $\mathcal{G} \rightrightarrows M$ and a symplectic form $\omega \in \Omega^2(\mathcal{G})$ satisfying the following compatibility condition,

$$m^*\omega = pr_1^*\omega + pr_2^*\omega, \qquad (2)$$

where $m, pr_1, pr_2 : \mathcal{G}^{(2)} \to \mathcal{G}$ denote the multiplication map and the two projections $(g_1, g_2) \mapsto g_j$, for $j = 1, 2$, respectivelly.

The compatibility condition can be understood by saying that ω defines a 1-cocycle in a suitable complex and any form $\omega \in \Omega^k(\mathcal{G})$ satisfying the above equation is called *multiplicative* (see [7]). Moreover, the above equation for a symplectic 2-form turns out to be equivalent to the fact that the graph of the multiplication map m, given by triples $(g_1, g_2, m(g_1, g_2))$ with $(g_1, g_2) \in \mathcal{G}^{(2)}$, defines a Lagrangian submanifold[3] in $(\mathcal{G}, -\omega) \times (\mathcal{G}, -\omega) \times (\mathcal{G}, \omega)$, see [27].

Let us summarize some key properties of a symplectic groupoid $(\mathcal{G} \rightrightarrows M, \omega)$. The identities

$$1 : M \hookrightarrow (\mathcal{G}, \omega)$$

define a Lagrangian submanifold and, thus, $\dim(\mathcal{G}) = 2\dim(M)$. The inversion map $\mathrm{inv} : \mathcal{G} \to \mathcal{G}$, given by $g \mapsto g^{-1}$ is anti-symplectic, $\mathrm{inv}^*\omega = -\omega$.

More importantly, there exists a unique Poisson structure $\{,\} \equiv \pi$ on M such that the source map $s : (\mathcal{G}, \omega) \to (M, \pi)$ is a Poisson morphism. We say that $(\mathcal{G} \rightrightarrows M, \omega)$ *integrates* the underlying Poisson manifold (M, π) and that s defines a *symplectic realization* of (M, π). We also observe that the target map is automatically anti-Poisson, namely, a Poisson map of the form $t : (\mathcal{G}, \omega) \to (M, -\pi)$.

In terms of the Lie theory for groupoids and algebroid recalled in Section 2.1, one can show that when $(\mathcal{G} \rightrightarrows M, \omega)$ integrates (M, π) in the sense above, then $\mathcal{G} \rightrightarrows M$ integrates the cotangent Lie algebroid $T^*_\pi M$ associated with (M, π), namely, $Lie(\mathcal{G}) \simeq T^*_\pi M$ naturally as Lie algebroids. Conversely, given (M, π) Poisson, if a Lie groupoid $\mathcal{G} \rightrightarrows M$ integrates $T^*_\pi M$ and its source fibers are 1-connected, there exists a unique symplectic $\omega \in \Omega^2(\mathcal{G})$ turning $(\mathcal{G} \rightrightarrows M, \omega)$ into a symplectic groupoid integrating (M, π) (see also Section 3.1 below). In this way, the two notions of integration introduced above coincide.

Let us also observe that the symplectic geometry and the groupoid structure are deeply interconnected. For any $f \in C^\infty(M)$ the symplectic Hamiltonian flow $\phi_t^{s^*f}$ of s^*f in $(\mathcal{G}, \omega)$ is left-invariant for m,

$$\phi_t^{s^*f}(g) = m(g, \phi_t^{s^*f}(1(s(g))).$$

In particular, one can deduce that the *symplectic realization data* given by $(\mathcal{G}, \omega, s, 1)$ completely determines the germ of the other groupoid structure maps around units [27]. In terms of families of arrows $g_t \in \mathcal{G}$ of eq. 1 above, we have that the first order generating parameters are, in this case, covectors $\xi_x \in T^*_x M$ on M. We can understand these as a point-based, finite-dimensional version of the 1-forms dH which integrate to the flows φ_t^H of the Hamiltonian vector fields $X_H = \pi(dH, \cdot)$ and which generate a complicated infinite dimensional group. Thus, working with group*oids* instead of with groups can be seen as a structural price paid in exchange of a finite dimensional treatment of Hamiltonian flows.

Some examples are given as follows. When $\pi = 0$, then $\mathcal{G} = T^*M$ with $s = t = q : T^*M \to M$ the bundle projection and $m(\alpha_1, \alpha_2) = \alpha_1 + \alpha_2$ defines a Lie groupoid which, together with the standard symplectic structure $\omega_c \in \Omega^2(T^*M)$, is such that $(T^*M \rightrightarrows M, \omega_c)$ defines a symplectic groupoid integrating $(M, \pi = 0)$. If $(M, \pi = \omega_M^{-1})$ is already symplectic then the pair groupoid $\mathcal{G} = M \times M \rightrightarrows M$ endowed with $\omega = (-\omega_M) \times \omega_M$ defines a symplectic groupoid integrating (M, ω_M^{-1}). The Poisson manifold $(M = \mathbb{R}^3, \pi)$ of Example 1 is integrated by a symplectic groupoid $(\mathcal{G} = T^*SO(3) \rightrightarrows \mathfrak{so}(3)^* \simeq \mathbb{R}^3, \omega_c)$ where ω_c is the canonical symplectic structure and the underlying groupoid $T^*SO(3) \rightrightarrows \mathfrak{so}(3)^*$ is given by a so-called *cotangent lift* of the Lie group structure on $SO(3)$, see [41]. This example generalizes to $(T^*G \rightrightarrows \mathfrak{g}^*, \omega_c)$ integrating the linear Poisson manifold $(M = \mathfrak{g}^*, \pi)$ of that example associated with any Lie group G. For Poisson-Lie groups (G, π), the integration is given by so-called Drinfeld double symplec-

[3]Let us recall that a Lagrangian submanifold $i : \Sigma \hookrightarrow (S, \omega)$ in a symplectic manifold (S, ω) is a submanifold such that $i^*\omega = 0$ and satisfies the maximality condition $dim(\Sigma) = dim(S)/2$.

tic groupoids (see e.g. [41]) generalizing the previous case.

Remark 3.— (Integrability and local integrations in the Poisson case) Not every Poisson manifold is integrable by a symplectic groupoid, see the specialized theory of integrability in [29]. A simple example is given by a minor modification of Example 1 on $M = \mathbb{R}^3$, $\{f_1, f_2\}^a|_x := a(|x|)(\nabla f_1|_x \times \nabla f_2|_x) \cdot x$ where the original bracket has been multiplied by a function $a(r) > 0$ of the radius $r = |x|$. Simple choices of a lead to non-integrable brackets $\{,\}^a$, see [29, Sec. 7.2]. In general, one can verify that (M, π) is integrable by a symplectic groupoid iff the cotangent algebroid $T^*_\pi M$ is integrable by a Lie groupoid (see Section 3.1 below). On the other hand, every Poisson manifold is integrable to a *local* symplectic groupoid, see [19, 27]. Namely, a local Lie groupoid $U \rightrightarrows M$ as in Remark 2 together with a symplectic form $\omega \in \Omega^2(U)$ which satisfies the multiplicativity condition locally around units $(1(x), 1(x)) \in U^{(2)}$. The main properties of symplectic groupoids still hold in the local case and, in particular, such a local symplectic groupoid $(U \rightrightarrows M, \omega)$ induces a unique Poisson structure on M which is then said to be integrated by the former.

2.3 Explicit constructions

We finish this section by outlining explicit constructions of (local) symplectic groupoids $(U \rightrightarrows M, \omega)$ for a given Poisson manifold (M, π). The more concrete ones produce local integrations in the sense of Remark 3 above.

We first mention a more abstract yet universal and global construction based on hommotopy classes of adapted paths, see [28, 29, 24]. A *cotangent path* is a path $\alpha : I = [0, 1] \to T^*M$ over $x : I \to M$ such that

$$\dot{x}(t) = \pi(\alpha(t), \cdot)|_{x(t)} \in T_{x(t)}M.$$

One can understand this equation as the map $TI \to T^*_\pi M, (t, \partial_t) \mapsto \alpha(t)$ defining a Lie algebroid morphism from the tangent algebroid of the interval I. The idea is that the covector $\alpha(t)$ guides the base path $x(t)$ through the Poisson structure. A *cotangent homotopy* with fixed endpoints is defined as a Lie algebroid morphism $T(I \times I) \to T^*_\pi M$ with appropriate boundary conditions. Then, one considers $W(M) = \{\alpha : TI \to T^*_\pi M\}/\sim \rightrightarrows M$ the space of cotangent paths modulo cotangent homotopies, which inherits a groupoid structure from composition of paths, called the *Weinstein groupoid* associated with (M, π). In general, it is just a topological groupoid but one can show the highly non-trivial result that (M, π) is integrable iff $W(M) \rightrightarrows M$ inherits a *Lie* groupoid structure, see [28, 29]. In this case, $W(M)$ has 1-connected source fibers. Moreover, it inherits a symplectic structure $\omega \in \Omega^2(W(M))$ via symplectic reduction from the canonical one on the space of all paths $PT^*M := \{I \to T^*M\}$ (see [29, 24]) and, endowed with it, $(W(M) \rightrightarrows M, \omega)$ integrates the original Poisson manifold (M, π). This construction provides a sort of universal answer to the integration problem but it is rather abstract and hard to construct in concrete cases.

We now move on to more explicit constructions intended to yield *local* symplectic groupoids integrating a given (M, π). These can be understood as *gauge fixing* slices $U \subset T^*M \hookrightarrow PT^*M$ which select particular cotangent paths $\alpha(t)$ in the corresponding homotopy classes via an auxiliary choice (of a *spray*, see below). The resulting structure is only local since these choices only work for *small* paths and break down for *large* ones.

The key auxiliary piece of data is called *Poisson spray* (see [31, 19]). It is given by an ODE on the total space of T^*M, namely, a vector field $V \in \mathfrak{X}(T^*M)$ such that

$$Dq(V|_\alpha) = \pi(\alpha, \cdot) \in T_{q(\alpha)}M, \ \alpha \in T^*M,$$

with $q : T^*M \to M$ the bundle projection, and that it is homogeneous of degree 1, $m^*_\lambda V = \lambda V$ for $m_\lambda : T^*M \to T^*M$ fiber multiplication by $\lambda \in \mathbb{R}_+$. Note that the first condition is the infinitesimal version of the cotangent path equation, so that trajectories of V are indeed cotangent paths. The 1-homogeneous condition implies that, in a chart of canonical coordinates (x^j, p_j), we have $V = \pi^{ij}(x)p_i\partial_{x^j} + \Gamma^{ij}_k(x)p_ip_j\partial_{p_k}$ with $\Gamma^{ij}_k(x)$ defining the Christoffel symbols of a *contravariant connection* ∇ for $T^*_\pi M$ on T^*M without torsion (see [33]). In these coordinates, the ODE defined by V for $\alpha = (x, p)$ imply that $p(t)$ is ∇-contravariantly constant along $x(t) \in M$. One can think of V as a contravariant analogue of a Riemannian spray in $\mathfrak{X}(TM)$ whose trajectories are the underlying Riemannian geodesics.

Given a Poisson spray V for (M, π) there is an associated local symplectic groupoid $(\mathcal{G}_V \rightrightarrows M, \omega_V)$ called *spray groupoid* integrating (M, π), [18, 19]. The symplectic structure was introduced in [31] and is given by an average along the flow of V of the canonical $\omega_c \in \Omega^2(T^*M)$,

$$\omega_V = \int_0^1 (\phi^V_t)^* \omega_c \, dt.$$

It is well defined and symplectic on a neighborhood $U_\omega \subset T^*M$ of 0_M. We now describe the structure maps of $\mathcal{G}_V \rightrightarrows M$ recalling that, for local groupoids, each structure map has a domain of definition including the corresponding identity arrows. The identity for $x \in M$ is given by the zero covector $0_x \in T^*M$ and, hence, the structure of $\mathcal{G}_V$ will be supported in appropriate opens in T^*M. The source map $s = q : T^*M \to M$ is the bundle projection while the target map is $t = q \circ \phi^V_{t=1} : U_{tar} \subset T^*M \to M$ with ϕ^V_t being the flow of the spray V on T^*M. Notice that, since V vanishes on the zero section $0_M \subset T^*M$, there exists a neighborhood $U_1 \subset T^*M$ of 0_M on which the flow is defined up to $t = 1$ and which we can shrink as needed for the domain of other structure maps. The inverse of the local groupoid is given by $inv(\alpha) = \phi^V_{t=-1}(-\alpha)$ for α close enough to zero. Finally, the multiplication map is defined by $m(\alpha_1, \alpha_2) = \beta(t = 1)$, where $\beta(t) \in T^*M$ is the solution to the ODE with initial value:

$$\omega_V(\dot{\beta}(t), \cdot) = (q \circ \phi^V_{t=1})^* \phi^V_t(\alpha_1), \ \beta(0) = \alpha_2.$$

The map m is defined for α_1, α_2 satisfying the composability condition $s(\alpha_1) = t(\alpha_2)$ and close enough to the zero section, for which the above ODE has solution defined up to $t = 1$. The fact that these maps satisfy the axioms of a local groupoid is [18, Thm. 3.8] and the fact that it integrates (M, π) can be found in [31]. More details in the present Poisson case can be found in [19].

When M is a coordinate domain, one can provide even more explicit formulas for its integration.

Remark 4.— (Coordinate M and a generating function) Let $M \subset \mathbb{R}^n$ be an open subset with coordinates x^j. We denote $M^* = (\mathbb{R}^n)^*$ the corresponding dual vector space with dual coordinates p_j. Let us consider the flat contravariant connection ∇ so that $\Gamma^{ij}_k = 0$ in the formula for the spray V. One can show (see [11, 13]) that $(\mathcal{G}_V \rightrightarrows M, \omega_V)$ is isomorphic to a local symplectic groupoid $(\mathcal{G}_\pi \rightrightarrows M, \omega_c)$ with the same units $1_x = 0_x = (x, 0) \in T^*M = M \times M^*$ but with canonical symplectic ω_c. Moreover, all the other structure maps are determined by a *generating function* (see [11, 21])

$$S : U_S \subset M^* \times M^* \times M \to \mathbb{R}, (p_1, p_2, x) \mapsto S(p_1, p_2, x)$$

where U_S is a neighborhood of $\{(p_1 = 0, p_2 = 0, x) : x \in M\}$. The source and target maps are

$$s(x, p) = \partial_{p_2} S(p, 0, x), \ t(x, p) = \partial_{p_1} S(0, p, x)$$

the inverse is $inv(x, p) = (x, -p)$, and the multiplication is defined by the relation

$$m((x_1, p_1), (x_2, p_2)) =$$
$$= (x_3, p_3 = \partial_x S(p_1, p_2, x_3)),$$
$$\text{with } x_j = \partial_{p_j} S(p_1, p_2, x_3), \ j = 1, 2.$$

This relation can be understood as the fact that the Lagrangian $graph(m) \hookrightarrow T^*M \times T^*M \times T^*M \simeq T^*M^3$ is suitably given as the graph of the exact 1-form dS with respect to the projection $T^*M^3 \to M^* \times M^* \times M$. Moreover, one can give formulas for $S(p_1, p_2, x)$ in terms of Hamiltonian flows in T^*M, see [11, Sec. 3.4]. We observe that for the linear Poisson manifold $M = \mathfrak{g}^*$ of Example 1 then S recovers the Baker-Campbell-Hausdorff formula for any Lie algebra $\mathfrak{g}$; we then think of S as a non-linear analogue of this classical formula. For the source map, one can also provide the following implicit definition due to Karasev (see [38, 39])

$$\int_0^1 \phi^V_t(s(x, p), p) \, dt = x$$

for all $p \sim 0$ small enough and with V being the flat spray above. From this, one can deduce (see [13]) a universal Butcher-type Taylor expansion in terms of rooted trees which defines a universal *symplectic realization* $s : (U_s \subset T^*M, \omega_c) \to (M, \pi)$ for any such coordinate Poisson manifold (M, π).

3 Applications and ramifications

In this last section, we outline some relatively recent developments in research lines which grow out of the general Poisson geometry realm of the previous sections.

3.1 Lie theory for multiplicative structures

We have seen that the Lie theoretic integration of Poisson brackets on M consist of a Lie groupoid $\mathcal{G} \rightrightarrows M$ together with a compatible symplectic structure $\omega \in \Omega^2(\mathcal{G})$ in its space of arrows. Remarkably, this situation generalizes to other types of geometric structures on M which are integrated by a Lie groupoid $\mathcal{G} \rightrightarrows M$ together with a compatible and non-degenerate version of the same geometry. In general, this integration $\mathcal{G} \rightrightarrows M$ serves as a powerful tool to study the original geometry on M. Here, we shall review some results of the general formalism behind this phenomenon via the description of *Lie theory for multiplicative forms on Lie groupoids* following [7].

As already mentioned, a *multiplicative k-form* $\omega \in \Omega^k(\mathcal{G})$ on a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a form which satisfies the cocycle condition in eq. 2. It will be interesting shortly to observe that eq. 2 is equivalent to the fact that the induced map

$$\overline{\omega} : \underbrace{T\mathcal{G} \oplus_{\mathcal{G}} T\mathcal{G} \oplus_{\mathcal{G}} \cdots \oplus_{\mathcal{G}} T\mathcal{G}}_{k} \to \mathbb{R},$$

$$(X_1, \dots, X_k) \mapsto \omega(X_1, \dots, X_k)$$

defines a Lie groupoid morphism with respect to the direct sum of the *tangent lifted Lie groupoid structure* $T\mathcal{G} \rightrightarrows TM$ (see [41]) and to $\mathbb{R} \rightrightarrows *$ seen as a group. We denote $\Omega^k_M(\mathcal{G})$ the space of multiplicative k-forms.

Let $A \to M$ be the Lie algebroid of $\mathcal{G} \rightrightarrows M$. The infinitesimal analogues of multiplicative forms are called *infinitesimally multiplicative (IM) k-forms* and admit the following two descriptions. First, an IM k-form on A can be described as a k-form $\omega_A \in \Omega^k(A)$ on the total space of the vector bundle A such that the induced map

$$\overline{\omega}_A : \underbrace{TA \oplus_A TA \oplus_A \cdots \oplus_A TA}_{k} \to \mathbb{R},$$

$$(V_1, \dots, V_k) \mapsto \omega_A(V_1, \dots, V_k),$$

defines a Lie algebroid morphism with respect to the natural direct sum of the *tangent lifted Lie algebroid structure* on $TA \to TM$ (see [41]) and to $\mathbb{R} \to *$ seen as a Lie algebra. We denote $\Omega^k_{IM}(A)$ the space of such IM k-forms. One can then show ([7, Sec. 4.1]) that, given a multiplicative $\omega \in \Omega^k_M(\mathcal{G})$, the Lie-functor applied to the morphism $\overline{\omega}$ will yield an IM-form $\omega_A \in \Omega^k_{IM}(A)$ via $Lie(\overline{\omega}) = \overline{\omega}_A$. One can also provide a second, more minimalistic description of IM forms, as follows. There is a $1-1$ correspondence (see [7, Secs. 2.2 and 3.3]) between $\omega_A \in \Omega^k_{IM}(A)$ and pairs (μ, ν) of vector bundle maps $\mu : A \to \Lambda^{k-1}T^*M$ and $\nu : A \to \Lambda^k T^*M$ over id_M satisfying the following *IM equations*: for all $u, v \in \Gamma(A)$,

$$i_{\rho(u)}\mu(v) = -i_{\rho(v)}\mu(u),$$
$$\mu([u,v]) = L_{\rho(u)}\mu(v) - i_{\rho(v)}d\mu(u) - i_{\rho(v)}\nu(u),$$
$$\nu([u,v]) = L_{\rho(u)}\nu(v) - i_{\rho(v)}d\nu(u).$$

The main theorem [7, Thm. 2] which provides a complete Lie-theoretic description for such forms says that the correspondance

$$\Omega^k_M(\mathcal{G}) \to \Omega^k_{IM}(A), \ \omega \mapsto \omega_A \equiv (\mu, \nu)$$

is $1:1$ when $\mathcal{G} \rightrightarrows M$ has 1-connected source fibers. Moreover, this correspondence commutes with de Rham differential.

Let us now describe some applications to Lie-theoretic integration of geometric structures. In the case of a Poisson structure (M, π) outlined in the previous sections, we have an obvious map $\mu = id : A = T^*_\pi M \to T^*M$ defining an IM 2-form with $\nu = 0$. It corresponds to the canonical symplectic 2-form $\omega_A = \omega_c \in \Omega^2(T^*M)$ on the total space of $A = T^*M$. When $\mathcal{G} \rightrightarrows M$ is a Lie groupoid integrating $T^*_\pi M$ which is source 1-connected, then this IM 2-form integrates to a closed 2-form $\omega \in \Omega^2(\mathcal{G})$. Moreover, since ω_c is non-degenerate, one can show that ω is non-degenerate as well, thus defining a multiplicative symplectic structure on $\mathcal{G} \rightrightarrows M$. This provides a Lie theoretic explanation of the additional "symplectic" nature of the integration when M is Poisson. Another example is given by *twisted Dirac structures* (L, H) on M, see [9]. These are strucutres motivated by both constraints in mechanical systems and field-theoretic applications. It consists of a subbundle $L \subset TM \oplus T^*M$ which is maximally isotropic with respect to the natural symmetric pairing on $TM \oplus T^*M$ and also its sections ΓL are closed under the Courant bracket $[,]_H$ twisted by a background closed 3-form $H \in \Omega^3(M)$. In such a structure, $A = L \to M$ defines a Lie algebroid and the map $\mu : L \to T^*M$ given by the projection onto the covector component can be shown to define an IM 2-form together with $\nu(v \oplus \alpha) = i_v H \in \Lambda^2 TM$, $v \oplus \alpha \in TM \oplus T^*M$. This corresponds to the pullback $\omega_A = \rho^*\omega_c \in \Omega^2(L)$ along the anchor $\rho(v \oplus \alpha) = v \in TM$. Applying the Lie theoretic integration to a source 1-connected $\mathcal{G} \rightrightarrows M$ integrating $A = L$, we obtain a multiplicative $\omega \in \Omega^2_M(\mathcal{G})$ which is not closed by satisfies $d\omega = s^*H - t^*H$ and is not non-degenerate but satisfies the weaker condition $Ker(Ds) \cap Ker(Dt) \cap Ker(\omega) = 0$ at identity arrows. Such a structure $(\mathcal{G} \rightrightarrows M, \omega)$ is called *twisted presymplectic* groupoid as identified in [9] to be the integration of Dirac structures.

The Lie theory described here for k-forms can be extended to *multiplicative multivectors* $\mathfrak{X}^k_M(\mathcal{G})$ ([7, Sec. 6]) and, also, to more general *multiplicative tensors* on $\mathcal{G}$ ([10]). The former includes the important examples of *Poisson-Lie groups and groupoids* while the latter include multiplicative complex structures among other types of multiplicative geometries. Other related ramifications include: higher Dirac structures, reduced Poisson and Dirac structures ([20]); vector bundles and representations up to homotopy ([8]); among others.

3.2 Quantization

As mentioned earlier, the idea of *quantization* of Poisson brackets $\{,\}$ to make contact with the formalism for quantum mechanics has been a powerful driving force in the field, largely motivating the study of Lie theory for $\{,\}$. Here we outilne some of the main notions and some recent results connecting quantization back to the Lie theoretic aspects of Poisson brackets. Some general references for the interested reader are [5, 39, 6, 14, 22, 36, 42, 45].

The quantization of a Poisson $\{,\} \equiv \pi$ on M is given by a *star product*, see [6]. This is a binary operation $\star_\hbar$ motivated by important examples of symbol calculus (see [35, 47]) in which there is a "quantization map" $f \mapsto Q_\hbar(f)$ promoting a classical observable $f \in C^\infty(M)$ to a quantum observable given by a suitable operator on a Hilbert space of states and where one encounters the relation $Q_\hbar(f_1)\circ Q_\hbar(f_2) = Q_\hbar(f_1 \star_\hbar f_2)$. Here $\hbar > 0$ represents Planck's constant seen as a scale parameter, taking different values at different scales, with $\hbar \to 0$ representing the *classical limit*. The intrinsic description of a star product is as a 1-parameter family $\hbar \mapsto \star_\hbar$ of binary operations on $C^\infty(M)$ satisfying three key axioms:

Q1) Deformation:

$$f_1 \star_\hbar f_2 = f_1 f_2 + O(\hbar);$$

Q2) Correspondence:

$$\frac{1}{i\hbar}(f_1 \star_\hbar f_2 - f_2 \star_\hbar f_1) = \{f_1, f_2\} + O(\hbar);$$

Q3) Associativity:

$$(f_1 \star_\hbar f_2) \star_\hbar f_3 = f_1 \star_\hbar (f_2 \star_\hbar f_3) + O(\hbar^\infty).$$

The correspondence axiom (Q2) fixes the underlying Poisson brackets and, in this case, we say that $\star_\hbar$ *quantizes* $(M, \pi \equiv \{,\})$. Often, one focuses on asymptotic expansions and interprets $\hbar$ as a formal parameter so that $f_1 \star_\hbar f_2 \in C^\infty(M)[[\hbar]]$ and $O(\hbar^\infty)$ terms are quotiented to zero. In such case, $\star_\hbar$ defines a formal 1-parameter family of (non-commutative if $\{,\} \neq 0$) associative algebras deforming $C^\infty(M)$ with the pointwise multiplication.

The problem of finding a quantization $\star_\hbar$ of an arbitrary Poisson (M, π) is notably difficult and was finally solved by Kontsevich [40] for formal $\hbar$. Before this solution, a Lie theoretic path going through symplectic groupoids was a well-known program (see [22] and references therein) to try to solve this quantization problem. In the rest of this subsection, we outline some recent connections between Kontsevich's solution and the symplectic groupoid formalism following [11].

Let us consider, for simplicity, $(M \subset \mathbb{R}^n, \pi)$ a coordinate Poisson manifold as in Remark 4. Denoting $p_1, p_2 : M \to \mathbb{R}$ linear functions and $x \in M$, Kontsevich's star product $\star_\hbar$ satisfies the following structural relation (see [21])

$$e^{\frac{i}{\hbar}p_1} \star_\hbar e^{\frac{i}{\hbar}p_2}|_x = \bar{a}_\hbar(p_1, p_2, x) e^{\frac{i}{\hbar}\bar{S}(p_1,p_2,x)}, \qquad (3)$$

where $\bar{a}_\hbar = \bar{a}_0 + \hbar\bar{a}_1 + \hbar^2\bar{a}_2 + \ldots$ and with $\bar{S}, \bar{a}_n \in C^\infty(M)[[p_1, p_2]]$ formal power series on $p_1, p_2 \in M^*$. These formal power series admit a formula in terms of Kontsevich graphs and complicated Kontsevich weights, see [40]. On the other hand, one of the main results in [11] says that the leading exponent $\bar{S}(p_1, p_2, x)$ coincides with the formal Taylor expansion at $t = 0$ along $t \mapsto t\pi$ of the (non-formal) generating function $S_{t\pi}(p_1, p_2, x)$ for the underlying local symplectic groupoid $(\mathcal{G}_{t\pi} \rightrightarrows M, \omega_c)$, as described in Remark 4. Moreover, the principal symbol of the right multiplication operator $\cdot \star_\hbar x^j$ by a coordinate function $x^j : M \to \mathbb{R}$ is given by the j-th coordinate of source map $s^j(x, p)$ described in the same remark (see also [37]). In [13], it was moreover shown that the Bucher series expansion of $s^j(x, p)$ mentioned in Remark 4 coincides with the expansion in terms of Kontsevich tree graphs and weights coming from $\star_\hbar$ (see also [21]). Altogether, these results establish an explicit correspondence between concrete Lie theoretic constructions for Poisson geometry and Kontsevich's formulas for $\star_\hbar$ in the quantization context. These results are also extended to the field-theoretic context of the *Poisson sigma model* [23] in [11, Sec. 5].

To finish this subsection, we mention some recent related work. The relation 3 above can be understood in the context of semi-classical Fourier integral operators (FIOs, see [35]). The coefficient $\bar{a}_0(p_1, p_2, x)$ in that expansion corresponds to a half-density along the graph of the multiplication map $graph(m) \simeq \mathcal{G}^{(2)}$ of the underlying local symplectic groupoid $(\mathcal{G}_\pi \rightrightarrows M, \omega_c)$, which is subject to an associativity condition. In this way, we observe that, behind a star product, there is an underlying *enhancement* of the symplectic groupoid structure (see [17]). Non-formal versions of star products $\star_\hbar$, $\hbar > 0$ can be defined in the context of FIOs (see [14]) and recent work in progress ([15]) indicates that non-integrability of the underlying (M, π) must manifest itself in a failure of associativity for large strings $f_1 \star_\hbar \cdots \star_\hbar f_N$, $N >> 1$.

3.3 Numerical integrators

In this final subsection, we briefly mention a more recent branch of applications to numerical methods adapted to the underlying geometry [25, 26, 12, 16, 3]. The general idea is that, as explained in Section 2, a symplectic groupoid $(\mathcal{G} \rightrightarrows M, \omega)$ integrating (M, π) is meant to provide a finite dimensional description of Hamiltonian flows and it thus turns out to be very useful also in their approximation.

Let (M, π) be a Poisson manifold and $H \in C^\infty(M)$ a given Hamiltonian function. The idea is to provide an approximation scheme for the flow φ_t^H of the corresponding Hamiltonian vector field $X_H = \pi(dH, \cdot) \in \mathfrak{X}(M)$ which retains its key geometric features. Moreover, we want this method of approximation to be implementable numerically and computationally. Thinking of an atlas for M, most of the discussion can be restricted to the coordinate case $(M \subset \mathbb{R}^n, \pi)$ with an arbitrary Poisson structure $\pi = (1/2)\pi^{ij}(x)\partial_{x^i} \wedge \partial_{x^j}$, although the constructions are geometric in nature.

The first approach to this problem was introduced in [25]. It is based on the underlying symplectic realization $s : (\mathcal{G}, \omega) \to (M, \pi)$ together with the Lagrangian section $1 : M \hookrightarrow (\mathcal{G}, \omega)$. Let $L \hookrightarrow (\mathcal{G}, \omega)$ be another deformed Lagrangian which is a *bisection* in the sense that both $s|_L$ and $t|_L$ are (local) diffeomorphisms $L \to M$. Then, there is an induced (locally defined) map $\varphi_L : M \to M$ given by

$$\varphi_L(x) := s(t|_L^{-1}(x)).$$

The key fact (see [27]) is that, as a consequence of L being Lagrangian in $(\mathcal{G}, \omega)$, then φ_L is a Poisson automorphism of (M, π). In particular, it preserves the geometry underlying π, just as Hamiltonian flows φ_t^H do. Moreover, given H and for small enough time $t \sim 0$, there always exists a family of Lagrangians L_t^H such that $\varphi_{L_t^H} = \varphi_t^H$ reproduces the desired Hamiltonian flow.

Since finding the exact L_t^H can be as hard as solving for φ_t^H, the idea introduced in [25] is to consider an *approximate Lagrangian bisection* $\hat{L}_t^H \hookrightarrow (\mathcal{G}, \omega)$ which is close to the exact one L_t^H in an appropriate sense and, thus, defines an approximation $\varphi_{\hat{L}_t^H}$ for φ_t^H. By construction, this approximation $\varphi_{\hat{L}_t^H}$ preserves the underlying Poisson geometry since $\hat{L}_t^H$ is Lagrangian. On appropiate Darboux charts around $1(M) \subset \mathcal{G}$, one can provide a concrete, computationally amenable ansatz for this approximating Lagrangian $\hat{L}_t^H$ in terms of a truncated solution of the Hamilton-Jacobi equation for an underlying generating function, see [25, 26].

In this way, the Lie theoretic integration $(\mathcal{G} \rightrightarrows M, \omega)$ can be used to provide geometric numerical approximations to Hamiltonian ODEs on the underlying Poisson manifold (M, π). We finish by mentioning recent related work. In [12], the authors explore the idea of also approximating the realization $s : \mathcal{G} \to M$ itself, thus enlarging the domain of aplicability to all cases in which the realization geometry is not known exactly. Truncation of the explicit formulas mentioned in Remark 4 for the realization map s can be used in this context. In [16], the authors explore the role of groupoid multiplication $m : \mathcal{G}^{(2)} \to \mathcal{G}$ in these methods which, so far, has not been used. In particular, preliminary results show that the universal generating function S for the groupoid multiplication of Remark 4 can be used as a tool for refining the numerical methods. Analogous methods for other types of geometries on M are explored in [3].

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93rd edition of the Séminaire Lotharingien de Combinatoire (SLC 93)

23–26 March 2025

by António Malheiro*, Samuel Lopes** and Olga Azenhas***

From March 23 to 26, 2025, the 93rd edition of the Séminaire Lotharingien de Combinatoire (SLC 93) took place at the High-Performance Rowing Centre (Centro de Alto Rendimento de Remo) in Pocinho, Portugal. The event was organized with the financial support of the Centro Internacional de Matemática (CIM), to whom we express our sincere gratitude.

The Séminaire Lotharingien de Combinatoire is a long-standing international workshop in enumerative and algebraic combinatorics, known for its informal yet research-intensive atmosphere.

This 93rd edition brought together 42 researchers from different countries and institutions, including both senior mathematicians and early-career participants, who engaged in lively scientific exchanges throughout the four-day meeting.

* NOVA Math, NOVA FCT
** CMUP, FCUP
*** CMUC, UC

The programme featured two invited mini-courses:

Arvind Ayyer (Indian Institute of Science, Bangalore), who delivered a dynamic series of lectures on *The Combinatorics of Multispecies Asymmetric Simple Exclusion Processes*;

Nathan Williams (University of Texas at Dallas), who prepared a course on *Coxeter-Catalan Combinatorics*, which was presented in his absence by Drew Armstrong and Christian Stump.

In addition to the mini-courses, there were 15 contributed talks, covering a wide range of topics in combinatorics and related fields, including: *enumeration*, *cyclic sieving of multisets*, *combinatorial commutative algebra*, *hyperplane arrangements*, *partition identities*, *Rogers-Ramanujan identities*, *quantum groups*, *quantum positroids*, *Lucas analogues*, *Stokes phenomenon*, *matroid theory*, *commuting graphs*, *Robbins polynomials*, *Schubert polynomials*, *Young-Fibonacci Kronecker coefficients*, and much more.

The setting in Pocinho offered an inspiring environment for research interaction, located in the heart of the Douro Valley—a UNESCO World Heritage region known for its cultural landscape and wine tradition. Social activities included a train journey along the scenic Douro railway line, a Port wine tasting session, a group visit to a local wine cellar, and a dinner near the renowned Côa Museum.

We are grateful to CIM for their essential support in making this meeting possible. The success of the event, both scientifically and socially, reflects the importance of such initiatives in fostering international collaboration and advancing research in mathematics.

Further details of the meeting are available at the event's webpage: https://www.mat.univie.ac.at/~slc/.

The organizers:

António Malheiro, *NOVA University Lisbon*
Samuel Lopes, *University of Porto*
Olga Azenhas, *University of Coimbra*
Christian Krattenthaler, *University of Vienna*
André Carvalho, *University of Porto*
Inês Rodrigues, *NOVA University Lisbon*
Diogo Soares, *University of Coimbra*
Gonçalo Varejão, *University of Coimbra*

Michael Christ

by Diogo Oliveira e Silva*

To start, how did your interest in mathematics first take shape?

As a child, I had a sense that somewhere in the world there were deep-thinking people who nobly worked to understand the world, and that all of humanity admired this as a matter of course. I'm not sure how this idea originated; mine was not an intellectual or science-oriented family.

Arithmetic problems were fun, but math was another school subject like all the rest until a turning point at the start of high school. I was 14, at a large public school in the suburbs of Milwaukee, Wisconsin, with students from a wide spectrum of backgrounds. The school selected a small group for an algebra class with an extraordinary teacher. I don't recall volunteering, or being asked whether I wanted to be in that class. We learned logic before algebra, and about cardinality of infinite sets before learning to complete the square for a quadratic polynomial. A few months after first learning arithmetic with negative numbers, I was asked to prove that adjoining the square root of two to the rationals produced a field. I guess it was the conjunction of predisposition with the right environment at the right age.

How did your undergraduate years shape you, and what changed when you moved into graduate school?

I attended Harvey Mudd College in southern California, a very small, focused place—the only major subjects offered at that time were engineering, chemistry, physics, and math. There were 88 students in my graduating class. The faculty

* Center for Mathematical Analysis, Geometry and Dynamical Systems & Departamento de Matemática, Instituto Superior Técnico

excelled as undergraduate teachers, rather than researchers. For someone from Wisconsin lacking sophisticated preparation, it was an excellent environment for mastering the basics of undergraduate mathematics. I took advantage through plenty of formal coursework. For a while no one much noticed me. Then I tried the Putnam exam in my third year, and suddenly I seemed to be someone. Becoming a member of a college faculty, perhaps doing a bit of research, emerged as a possible career path, one of which I had been only dimly aware.

Harvey Mudd didn't offer much exposure to research, but there were glimpses—especially Robert James (of Claremont Graduate School), who worked in Banach space theory. Sandy Grabiner taught an exciting course at neighboring Pomona College. John Greever ran a formative course on point set topology in the Moore style, in which students engaged with definitions and learned by doing. There's a template: get started, become engaged, and allow the learning to follow.

For graduate school I chose the University of Chicago, not through shrewd advice, but because the catalogue listed names already encountered: Herstein, Mac Lane, Kaplansky, Zygmund, Swan. When I visited, people were welcoming, especially Bill Beckner. Berkeley was also on my radar—Henry Helson kindly showed me around—but the place felt enormous and impenetrable. Chicago felt more personal.

What was the analysis scene like at Chicago when you arrived?

One felt a department in transition, with some of those great names nearing the later stages of their careers. In my very first term I chanced upon a wonderful course offered by Robert Fefferman, based on Stein's book on singular integrals. On Bob's recommendation, I subsequently attended Calderón's lectures. Professor Calderón spoke with remarkable deliberation, so much so that initially it was a challenge to pay attention. Years later, he explained that he consciously avoided preparing his lectures because he wanted the audience to think with him, word by word, and he had learned from experience that preparation led him to go too fast. He usually lectured on his own work, recreating it step by step at the board. Once or twice a semester he would become stuck—really stuck—and the audience, postdocs and graduate students, would try to sort it out together. Professor Calderón was a proud man, whose willingness to subject himself to this occasional embarrassment testified to his commitment to meeting his audience on a basis of equality. I tried to absorb not only the mathematical content, but also his way of thinking at a measured pace and analyzing step by step.

A remarkable aspect of the program was the procession of talented young people passing through, including Bob Fefferman, Beckner, Zimmer, Jones, Wolff, Kenig, Phong, Jerison, Uchiyama, Janson. Sometimes a course would be offered by one of them for as few as two students—Once or twice those two were my friend David Barrett and myself. A gift.

The leap from being a good learner to doing research was a struggle. There's a real gap. Calderón first proposed to me a monumental problem, boundedness for the Cauchy integral operator on Lipschitz curves, which had been at the heart of his own research since the early 1960s. This was both generous, and wildly overambitious. I was far too naïve, and made no progress. When I thought I had something, Peter Jones gently, kindly, and devastatingly disabused me of that notion. The actual start came later, almost casually: immediately after an expository seminar talk, Calderón suggested I try for a generalization. A few days later, I had it. Someone else obtained the result simultaneously or slightly earlier, so there was no publication, but it got me moving. Once I began working on something of my own, questions and problems began to flow.

After Chicago you spent time at Princeton. How did that change your trajectory, particularly in relation to Eli Stein and Joe Kohn?

Much of my time in Chicago was invested in observing the work of extraordinary researchers. At Princeton, it was time to produce. Stein, Charles Fefferman, and Kohn were my models there, while my fellow postdocs were also important influences and friends.

Stein's influence was profound, both through his writing,

Michael Christ at UW-Madison with former PhD students (from left to right) Malabika Pramanik, Kevin O'Neill, Michael Goldberg, Diogo Oliveira e Silva, Betsy Stovall, Taryn Flock, A. Martina Neuman, Dominique Maldague, Loukas Grafakos.

and in person. The most helpful thing he did, though simple, was decisive: On our very first meeting, within days of my arrival on campus, he proposed a specific problem related to the Radon transform—hand-picked for me, modest in its initial scope, but also a bit open-ended. It wasn't a monumental problem that had been open for decades; it wasn't a blank wall; it was a pool I could dive into. The problem connected to Kakeya-type questions via the k-plane transform, offering rich terrain with room for both success, and productive failure. The crucial point was that he got me started.

Joe Kohn's influence was also pivotal. As a graduate student I'd studied his work in several complex variables with Narasimhan, who admired it deeply. At Princeton, Joe buttonholed me with an idea about a problem that turned out to be connected to many topics I'd been studying: pseudodifferential operators, PDE, several complex variables, singular integrals, nilpotent groups, and parametrix constructions. Suddenly those strands came together. Over a period of years, that interaction spawned several projects, including L^p estimates for solutions of the $\bar\partial$ and $\bar\partial_b$ equations, analyticity, global and local regularity of the Bergman projection, and weighted inequalities for $\bar\partial$. I might never have pursued those directions without his encouragement, and without his pointing me towards a specific goal. He was incredibly generous in sharing a specific idea along with more general thoughts about microlocal analysis and $\bar\partial_b$. To see so senior a figure so passionate about his research, in his undemonstrative way, was itself a lesson.

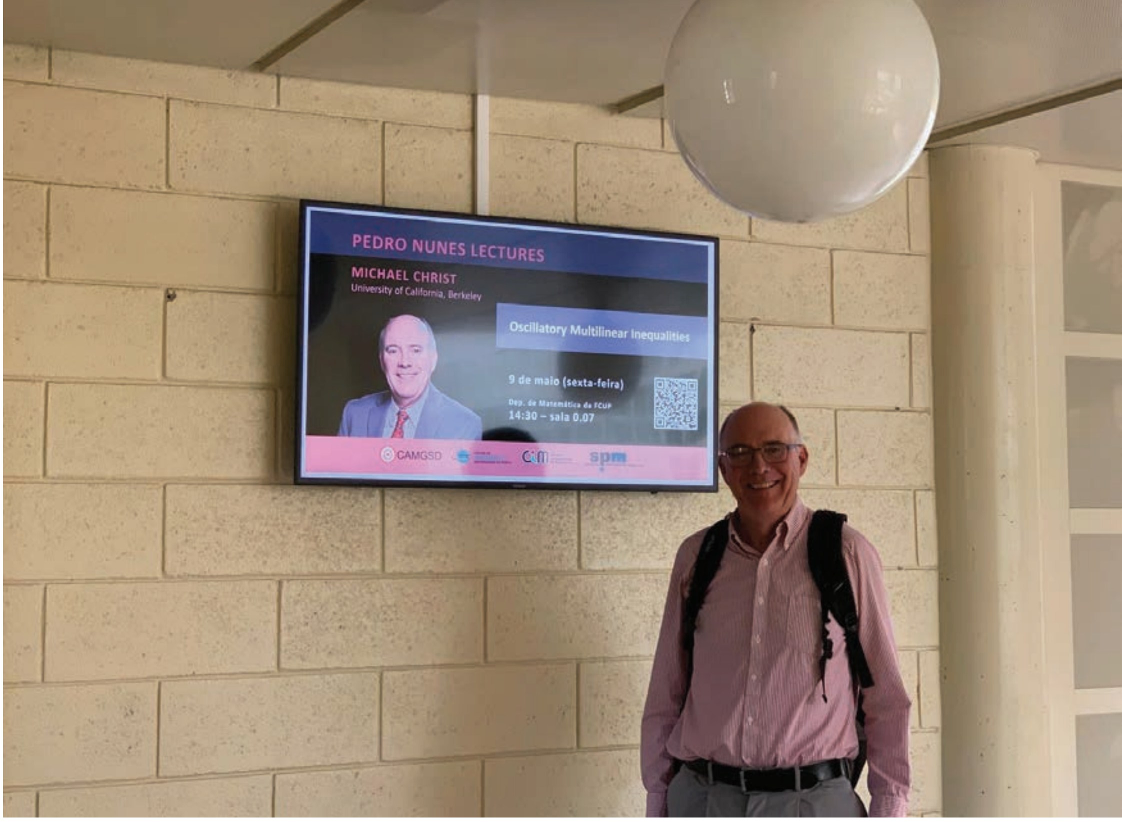
Why harmonic analysis? Was it a plan, or happenstance?

Largely luck. I felt that the two most interesting things I'd studied as an undergraduate were algebraic topology, and Cohen's work on the continuum hypothesis—both quite far from where I ended up. Early on at Chicago I sensed unknown depths in Bob Fefferman's wonderful course on singular integrals. I learned that my understanding was superficial; I knew proofs without grasping underlying principles. The challenge of understanding more deeply, together with the example of mathematicians whose passion was so palpable, drew me in.

You've done influential work on sharp inequalities. Where did that interest come from?

Partly from early exposure to Beckner's work on the Hausdorff-Young and Young convolution inequalities—beautiful theorems about familiar objects, viewed from a less familiar perspective. A lecture by Burkholder also impressed. Constants matter in applied settings, but also in pure contexts, where they can encode structure. Those exposures lingered in the back of my mind for years. Eventually points of contact emerged with topics I knew—for instance, with Radon-type transforms—and I followed those connections. It wasn't a grand strategy; more a matter of recognizing a seam of exposed ore, and digging into it.

Recent developments have pushed parts of harmonic analysis into the spotlight. Do you have predictions about what comes next?



No. In my own work, I try to develop threads that emerge organically. I focus on problems on which I feel I might make headway.

When you're stuck on a problem—which, for most mathematicians, is most of the time—what do you actually do?

Yes, I guess I'm always stuck. The joy is in doing. I like to make progress on specific questions. It's difficult to discern when persistence becomes obstinacy. It's easy to become attached to an idea that had produced a smidgen of progress, and to keep pushing long after it's ceased to bear fruit. One needs to step back, to look around, to change approach, sometimes radically, without prematurely abandoning a promising line. I try to keep a few problems in play so as to shift perspective. But most essential is to get started and to keep going.

How have collaborations figured in your work?

I've had many collaborations of different flavors, though I also work alone quite a bit. Some collaborations involve intense back-and-forth. In others, two people bring complementary skills or perspectives, exchange what's needed, and the project is a merger of distinct contributions from both. I've had particularly fruitful interactions with people whose backgrounds were very different from mine; that contrast can be productive and fun. Michael Weinstein and Sasha Kiselev are good examples.

My interaction with Joe Kohn, though not a formal co-authorship, had an enormous impact. I've enjoyed successful collaborations with Tony Carbery, Daryl Geller, Detlef Müller, Terry Tao, Daniel Tataru, Christoph Thiele, Steve Wainger, Jim Wright, and others. There were too brief but memorable collaborations with people whose time was cut short—Jose Luis Rubio de Francia and Jean-Lin Journé.

I've collaborated formally with several of my PhD students; in the latter half of my career I consciously tried to do a joint project with each.

Collaboration brings energy and ideas one wouldn't reach on one's own. Solo work carries its own satisfactions. Both are great.

Is there a result you're especially proud of—perhaps for the idea behind it or the way it came together?

I'm particularly happy with my work on global irregularity of the Bergman projection, which took twists and turns. The question was whether the Bergman projection maps functions C^∞ up to the boundary (of a smoothly bounded pseudoconvex domain in higher-dimensional complex space) to functions likewise C^∞ up to the boundary, or whether it can manufacture singularities out of nothing. I showed the latter.

Important works of Kohn, of Fefferman, and of Bell and Ligocka had elevated this question about complex analysis in several variables to a position of intense interest. David Barrett, my fellow student for five years in Chicago and fellow postdoc in Princeton for four more, had made a striking yet inconclusive advance in the negative direction. He had shown that the projection does not map Sobolev spaces H^s to H^s for large s for particular examples, the so-called worm domains, a lovely family of domains invented earlier for a quite different purpose by Diederich and Fornaess.

I set out to establish an affirmative result, for those same worm domains, by proving a weaker Sobolev inequality, with H^s mapped to H^{s-1}. A certain density lemma would also have been needed to conclude the analysis, but as density lemmas are typically the easier step, and as the desired affirmative conclusion would imply the validity of such a lemma, the density step was initially relegated to the back burner.

This strategy may not have been wise, but it worked too well. In the end, I realized that I had an *a priori* inequality for every *s* with no loss of derivatives at all, directly contradicting Barrett—provided that the density lemma held. Therefore the density lemma had to be false; therefore the affirmative conclusion could not hold.

I had made an embarrassing error in an earlier attack on this problem, so getting it right was especially satisfying. I also have affection for some smaller, more obscure works. One loves all of one's children.

What problems are on your desk now?

I'd like to understand what I call implicitly oscillatory multilinear integrals. This year's Pedro Nunes lectures are in part an introduction to this topic. It's a modest-looking, easily formulated question, which I believe addresses a natural foundational issue in multilinear analysis, even while the significance of its applications remains uncertain. I also have a backlog of unpublished but largely completed projects which I'd like to shepherd through publication while I still can.

You've supervised many students. Do the benefits outweigh the costs?

It's not a cost-benefit calculation. It's a part of the job, an opportunity, a privilege, to help people begin to do research for themselves. I've tried to do for others what helped me, especially by finding something on which success is within reach. Early success builds momentum and one begins to feel that a subject is one's own.

Motivation is a black box. Human psychology is a mystery. I don't know how to inculcate a drive to solve problems. What I can do is model engagement, exemplify dedication to teaching at both the undergraduate and graduate levels, and try to pass along ideas.

Who are some mathematicians whom you especially admire?

Many are already mentioned above. There are too many to name, but any response must include Lennart Carleson, Jean Bourgain, Guy David, and three Berkeley colleagues alongside whom it has been a privilege to work for decades: Craig Evans, Daniel Tataru, and Maciej Zworski.

Would you like to share a personal anecdote?

My name is a result of circular reasoning, an inauspicious beginning for a mathematician. My grandfather, a child of immigrants, changed his legal name in mid-life from that of his parents, to Christ. The process required a witness, who attested to his identity. His witness was identified on the legal documents as Sarah Christ, his wife, who was born

Sarah Gorman but had taken her husband's surname, Christ, upon marrying him—twenty years earlier. The name thereby presupposed itself.

We've talked about research, teaching, mentoring—and life beyond mathematics. What's the trick to keeping it all going?

I try to keep going. That's all.

Is there anything else that you'd like to add? Is there a question we should have asked?

I was delighted to have been able to refer to the 16th century mathematical work of Pedro Nunes for an example in the 2025 Pedro Nunes lectures!

I can't think of any questions that you missed. In truth, I might have preferred no questions at all.

We may decide later whether to print that. Thank you so much, Mike.

Thank you. It's been a special privilege to have known, taught, and collaborated with you, Diogo.

Nonlinear PDEs in Guimarães

21–25 July 2025

by **Fernando Miranda*** and **Lisa Santos***

This conference was a joint initiative of CMAT (*Centre of Mathematics*) of the University of Minho, which hosted the event, CAMGSD (*Centre for Mathematical Analysis, Geometry and Dynamical Systems*) and CEMS.UL (*Centre for Mathematical Studies*), both from the University of Lisbon, CMUC (*Centre for Mathematics*) of the University of Coimbra, and CNRS (*Centre National de la Recherche Scientifique*) together with the *Institute of Mathematics* of the University of Toulouse. The conference also received financial support from CIM (*Centro Internacional de Matemática*) and FLAD (*Federação Luso-Americana para o Desenvolvimento*). This conference followed a previous one, *Nonlinear PDEs in Braga*, which took place at the Congregados Building of the University of Minho in Braga from June 7 to 9, 2019.

The main goal of the conference was to bring together the large community working on and/or interested in nonlinear PDEs, including elliptic, parabolic, dispersive, and coagulation–fragmentation equations.

Many problems in this field require a multidisciplinary approach, and the organizing committee aimed to bring together experts and early-career mathematicians to foster the exchange of ideas and to forge new collaborations and perspectives that may benefit the entire field of nonlinear PDEs. Lively and stimulating discussions took place during the breaks between talks and over coffee breaks, further enriching the scientific exchange.

The chosen venue, the Azurém Campus of the University of Minho in Guimarães, proved to be an excellent choice, as all participants were able to enjoy the unique character of the city—known as the birthplace of Portugal—and its historic center, a UNESCO World Heritage, during their free time.

* CMAT, University of Minho

THE ASYMPTOTIC STRUCTURE OF DEEP NEURAL NETWORKS

by **João L. Costa***

ABSTRACT.—Deep Neural Networks (DNNs) are the main concept at the center of the artificial intelligence revolution we are experiencing. However, some of the reasons behind their effectiveness (for instance, why do they seem to provide "good" solutions, determined by simple optimization algorithms?), as well as the causes of their limitations (for instance, why are they so parameter and data expensive?), remain somewhat unclear. Therefore, a theoretical/mathematical clarification of these issues would be welcomed and, in principle, might help us in the construction of a new generation of interpretable, safer, sustainable and, consequently, more reliable AI models.

With that in mind, a mathematical approach that has provided some relevant insights is the study of the asymptotic structure of DNNs.

In this article, we will start by introducing the basics of DNNs, followed by a presentation of some results concerning the study of the large width limit of these models and a discussion of the implications that such results have in our understanding of supervised machine learning with DNNs.

1 Introduction

The artificial intelligence revolution we are experiencing, with remarkable breakthroughs in natural language processing, computer vision, natural sciences, among others, is being fueled by the use of unprecedented computational power and data consumption, but, at the core of such developments, is the concept of artificial Deep Neural Networks (DNNs). Mathematically, DNNs correspond to specific parameterized families of functions (explicit examples will be provided below) that were originally designed as (extremely simplified) models of biological brains.

DNNs are trained by tuning their parameters in order to minimize a chosen loss function that measures the ability of a network to reproduce a given relation between training inputs and labels/targets [1]; for instance, in the famous MNIST dataset, the inputs are 28×28 matrices of integers between 0 and 255, encoding a low resolution gray scale image of handwritten digits, and the labels are integers between 0 and 9, identifying the corresponding digit.

It should be stressed, from the beginning, that the final goal of the described machine learning procedure goes beyond the simple "memorization" of the training set, meaning attaining near zero loss in the training set; in fact, this could be achieved by a simple dictionary. The quality of the trained model should be evaluated in terms of its generalization capabilities, meaning its ability to perform well (i.e., have a small loss), in test sets, composed of data that hasn't been used in the training/optimization procedure; for instance, a model trained on MNIST generalizes well if it correctly classifies a high percentage of handwritten digits in images that it hasn't "seen before".

Recently, the increase in the size of deep neural networks, both in the number of trainable parameters and the amount of training data resources, has been proportional to the astonishing success of using DNNs in practical applications [2]. This motivates the study of the asymptotic structure of DNNs, when the number of parameters tends to infinity. As we

[1] Strictly speaking, what we are describing is a problem in the subfield of supervised learning, where a labeled data set, consisting of pairs of training inputs and labels/targets, known as training set, is provided a priori. DNNs are also relevant in self-supervised and reinforcement learning problems.

[2] One should however note, that this trend, plenty of times extrapolated into optimistic "scaling laws", as started to show signs of slowing down [20].

* Departamento de Matemática (ISCTE-IUL) and
Center for Mathematical Analysis, Geometry and Dynamical Syatems (IST-UL)

will see, there are two basic dimensions in a DNN: its width, meaning the number of neurons per layer, and its depth, encoding the number of layers. Here we will be interested in the large width limit of DNNs, which allows us, for instance, to shed some light into the following fundamental questions:

Q1: (*Memorization capacity*) why is it that the training of DNNs, via gradient descent (a standard optimization technique to be described below), is, in various applications, able to achieve (nearly) zero loss, even though the loss function is highly non-convex?

Q2: (*Generalization performance*) why is it that some trained networks demonstrate good generalization performance? Moreover, how does this happen even in the overparametrized regime, where the number of parameters is much larger than the amount of data, in blatant contrast with the expectations of classical statistical learning theory?

Q3: (*Scaling laws*): do we really expect the performance of DNNs to keep on improving with the increase in the number of parameters?

Q4: (*Hyper-parameters tuning*) contrary to some oversimplifying narratives that create unreasonable expectations, in general, it isn't easy to train a DNN. In fact, typically, a successful training process involves, beyond the data pre-processing, an overwhelming number of choices, that go from the choice of architecture, including the number of layers and the number of neurons per layer, to the choice of parameter initialization, among many others. Even though the process has become increasingly more robust, the referred choices often follow educated guesses, based on previous experience and heuristic theoretical arguments, and a strenuous and expensive process of trial and error. It would be relevant to be able to obtain some guidance for these procedures in the form of mathematical results.

Still regarding the challenges facing the training of DNNs we haven't yet mentioned, but are not forgetting, what is arguably the most important one: fighting the tendency of large models to overfit, specially in a context of data scarcity. Overfitting corresponds to the situation when training leads to a "small" loss in the training set, but a "large" loss in the test set, i.e., a situation where the model memorizes the training set, without generalizing well to the test set. Various techniques [9, Chapter 7] have been devised to tame the tendency for overfitting, which range from adapting classical ideas, such as the inclusion of a regularization term in the loss, to the somewhat crude and self-explanatory idea of early-stopping, and a surprisingly clever and efficient strategy known as *dropout*, whose trademark feature is the freezing of randomly chosen components of the model in each step of training. Unfortunately, do to spacetime constraints, we won't be discussing these important topics any further here.

2 Deep Learning: the theoretical minimum

In this section we will describe some examples of artificial Neural Network models and will take the opportunity to clarify what is meant by learning through gradient descent.

2.1 Multilayer Perceptrons

We will start with the first historical example of an artificial neuron [14], which was originally designed by McCulloch and Pitts with the grand goal of being the fundamental unit for neural computations and, later, named accordingly as the Perceptron, by Rosenblatt [16]. This corresponds simply to the parameterized family of functions $f_\theta : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f_\theta(x) = \phi\left(\sum_{i=1}^{n} w_i x_i + b\right) , \qquad (1)$$

where the model parameters, known as weights w_i and bias b, are collectively stored in $\theta = \{w_1, \cdots, w_n, b\}$. The function ϕ is known as the activation function or non-linearity; originally it was chosen to be the Heaviside function $\phi(z) = 1_{[0,+\infty[}(z)$, but since this function has an almost everywhere vanishing derivative it renders the model untrainable in a context of gradient based optimization, that we will discuss soon. Other examples of commonly used activations are the sigmoid function $\phi(z) = 1/(1+e^{-z})$ and the ReLU (Rectified Linear Unit) $\phi(z) = \max\{0, z\}$.

Rosenblat also provided a clever and geometrically transparent learning algorithm that allows Perceptrons to classify linearly separable datasets. Unfortunately, this is all Perceptrons can do. Nonetheless, we can easily extend this class of functions into a far more expressive one: i) first we stack Perceptrons on top of each other creating a Dense Layer, which is a function $\Phi : \mathbb{R}^n \to \mathbb{R}^m$, defined by

$$\Phi(x) = \phi\left(\frac{1}{\sqrt{n}} W x + b\right) ,$$

with weight matrix $W \in \mathbb{R}^{m\times n}$, bias vector $b \in \mathbb{R}^m$, and where the activation function acts entrywise, i.e. $\phi([z_1 \cdots z_m]^t) := [\phi(z_1) \cdots \phi(z_m)]^t$; ii) then we can compose several of these layers to obtain a Multilayer

Perceptron (MLP)[3] which is the class of functions

$$f_\theta : \mathbb{R}^{n_0} \to \mathbb{R}^{n_{\ell+1}} , \tag{2}$$

defined recursively by:

$$\begin{aligned}
y^{(0)} &= x \\
y^{(1)} &= \phi\left(\frac{1}{\sqrt{n_0}} W^{(1)} y^{(0)} + b^{(1)}\right) \\
y^{(2)} &= \phi\left(\frac{1}{\sqrt{n_1}} W^{(2)} y^{(1)} + b^{(2)}\right) \\
&\vdots \\
y^{(\ell+1)} &= \frac{1}{\sqrt{n_\ell}} W^{(\ell+1)} y^{(\ell)} + b^{(\ell+1)} .
\end{aligned}$$

The final layer, layer $\ell+1$, is the output layer, the previous layers, if they exist, are called hidden layers. For each layer we have a matrix of weights $W^{(i)} \in \mathbb{R}^{n_{i+1}\times n_i}$ and a vector of biases $b^{(i)} \in \mathbb{R}^{n_{i+1}}$. As before, we collect all learnable parameters in $\theta = \{W^{(i)}, b^{(i)}\}_{i=1,\cdots,\ell+1}$. In some applications it is also important to apply a non-linearity ϕ_{out} to the output layer, which is typically different from the inner activations ϕ; here we'll be mostly focused on the case $\phi_{out}(z) = z$, as described. We also say that each layer has width n_i, which is sometimes referred as the number of neurons/units in the layer, and say that the network has depth $\ell+1$; so, the "deep" in "deep learning" is simply a reference to the number of consecutive layers that are composed in a given DNN.

The normalization factors $n_\kappa^{-1/2}$ are used, in particular, to tame overflow issues when $\sum_\kappa n_\kappa \gg 1$. The way they were introduced here is not standard in the machine learning literature, where normalization is carried out in the initialization of the parameters, but has the advantage of making the infinite width limits, $n_\kappa \to \infty$, more transparent.

This construction immediately pays off. While Perceptrons (or a single Dense Layer) were highly restricted in their expressiveness, sufficiently wide MLPs, with at least one hidden layer ($\ell \geq 1$), can approximate any function, in relevant classes. Results formalizing this kind of statement are known as Universal Approximation Theorems. An example of such a statement is the following

Theorem 1.— Let $\phi \in C(\mathbb{R})$. The set

$$\operatorname{span}\{\phi(w\cdot x + b) \mid w \in \mathbb{R}^n , b \in \mathbb{R}\}$$

is dense in $C(\mathbb{R}^n)$ if and only if ϕ is not a polynomial.

If a single hidden layer is enough to render DNNs universal approximators why go any deeper? This is a deep question (pun not intended). From a practical standpoint, it is clear that deepness is indispensable to solve complex problems [4], and the typical heuristic explanation for this goes something like this: deeper layers ($\ell \gg 1$) learn more abstract features [5], which are built on top of earlier, more tangible features, and this level of abstraction is required to handle sophisticated problems such as natural language processing and computer vision [6].

At a mathematical level the importance of deepness can be connected to notions of efficiency and expressivity. A simple observation along this lines is the following: consider the function $f(x) = 2\max\{0,x\} - 4\max\{0, x-1/2\}$; in the language of DNNs, this model has a single hidden layer, with 2 neurons and ReLU activation. In a different dialect, it defines a piecewise linear function with 2 affine linear pieces. The ℓ-fold composition of f with itself, $f_\ell = f \circ \cdots \circ f$, is a depth $\ell+1$ MLP with the capacity to produce 2^ℓ affine linear pieces, using only 2ℓ (hidden) neurons; achieving the same with a single hidden layer ReLU network would require $2^{\ell-1}$ neurons [17].

2.2 Training

In supervised machine learning there are three main ingredients: a training set $\{(X_i, Y_i)\}_{i=1}^N$, consisting of inputs X_i and targets/labels Y_i; a model class M, here we are interested in DNNs so we consider $M = \{f_\theta \mid \theta \in \mathbb{R}^P\}$, with, for instance, the f_θ as defined in (2); and a loss function $\mathscr{L} : \mathbb{R}^P \to \mathbb{R}_0^+$, where P is the number of parameters of the chosen class of DNNs; for simplicity let us assume that the loss is the squared error loss, given by

$$\mathscr{L}(\theta) := \frac{1}{2}\sum_{i=1}^N \left(Y_i - f_\theta(X_i)\right)^2 . \tag{3}$$

In this context, training corresponds to minimizing the loss, i.e. solving the problem $\theta^* = \operatorname{argmin} \mathscr{L}(\theta)$.

[3] This architecture goes by various other names, such as *dense feedforward* or *fully conencted* neural networks.
[4] For instance, according to itself, the GPT-3 architecture has 96 transformer layers.
[5] This is a somewhat vague notion, that, in some contexts, can be defined as the component functions of each hidden layer. Also, it is the data that has "features" and, in that regard, we should talk about "feature-detectors", or something along those lines, when talking about models.
[6] Interesting empirical evidences in favor of this argument can be obtained by probing the responses of each layer in trained DNNs [7, Section 9.4.2].

When training DNNs, the typical minimization procedures used are based on gradient descent, meaning that the training trajectory, in parameter space, is determined by

$$\dot{\theta}(t) = -\nabla_\theta \mathscr{L}(\theta(t)), \tag{4}$$

with parameter initialization $\theta_0 = \theta(0)$ chosen according to some fixed probability distribution. One should note that, in practice, one uses discretized versions of gradient descent, with distinct degrees of sophistication: these can include batch learning, second order information and dynamic normalization factors, designed to handle memory issues, to allow convergence speedup and promote training stability.

It is well known that under quite general assumptions, gradient descent converges to a local minimum. Nonetheless, since the loss landscape is highly nonconvex and can often include a large number of local minima, it is not at all clear, a priori, why the described method should lead to "good" solutions as measure by the smallness of the final loss.

Gradient descent also requires the computation of the derivatives of the loss with respect to the model parameters. This isn't done using a numerical computational scheme, such as finite differences. In fact, this is achieved through the explicit computation of recursive formulas, in terms of elementary functions, obtained by using the chain rule; in retrospect, the most natural of tools in view of the compositional nature of DNNs. This procedure is known as backpropagation, since the referred recursion starts from the output layer, layer $\ell + 1$, and then propagates the information to the earlier layers, in reversed order.

2.3 A brief overview of other architectures.

MLPs still play a key role in deep learning, but they are not the only available choice of neural network architecture. In practice, we should think of them as a useful and widely used component in a plentiful menu of architectures. More generally, we can define DNNs as functions $\Psi : \mathbb{D}_1 \to \mathbb{D}_2$ which are the composition of several *layers* $\Phi^{(\kappa)} : \mathbb{D}_{\kappa_1} \to \mathbb{D}_{\kappa_2}$, resulting in

$$\Psi = \Phi^{(L)} \circ \cdots \circ \Phi^{(1)} .$$

The domains $\mathbb{D}$ are typically real vector spaces, including $\mathbb{R}^n$, matrix spaces or higher order tensor spaces, but can also be integer based spaces such as $\mathbb{Z}_k^n$, among others.

In this setting, we can construct new DNN architectures, by defining new layers and choosing appropriate ways to combine them. Here are a couple of famous examples:

A convolutional layer is, as the name suggests, obtained by applying convolution operations; in a somewhat schematical way $\Phi(x) = W * x$, where W is a learnable convolution kernel, that can, for instance, be used to learn how to detect translational invariant features of the input x. A convolutional neural network (CNN), is typically the result of composing several convolutional layers, intertwined with another type of layers known collectively as *pooling layers*, designed to downsample the size of intermediate outputs, and, to finish, an MLP. CNNs have had a tremendous impact in computer vision (see [9, Chapter 9] and references therein, for more information).

An embedding layer is, in some contexts, a map from the integers, encoding, for instance, the position of words (or, more generally, tokens) in a dictionary, to a real vector space $\mathbb{R}^{d_e}$. A self-attention layer, picks up a collection of these embeddings, compiled as columns of a matrix E, and computes an attention score through the now famous formula

$$\text{Attention}(E; W_Q, W_K, W_V) = \\ \text{softmax}\left(\frac{(W_Q E)^t W_K E}{\sqrt{d_e}}\right) W_V E , \tag{5}$$

where the W's are matrices of learnable parameters and $\text{softmax}_j(z_1, \ldots, z_m) := e^{z_j} / \sum_i e^{z_i}$ is used column-wise. The famous *transformer* architecture [18], the "T" in GPT, is an intricate combination of embedding layers, attention layers, normalization layers and, once again, MLPs.

3 The large width limit

3.1 Shallow networks

It is instructive to start by analyzing the simplest model class, within MLPs (2): shallow networks ($\ell = 1$), with a single input ($n_0 = 1$) and a single output ($n_2 = 1$). In such case, MLPs reduce to the class of functions $f_\theta : \mathbb{R} \to \mathbb{R}$ of the form

$$f_\theta(x) = \\ \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left(W_j^{(2)} \, \phi \left(W_j^{(1)} x + b_j^{(1)} \right) + b_j^{(2)} \right) . \tag{6}$$

Notice that here we didn't follow the definition 2 to the letter. Instead we've added extra output bias terms, the $b_j^{(2)}$, one for each hidden neuron. This is sometimes convenient since it allows us to write our network as a sum of perceptrons

$$f_\theta(x) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} p_j(x) . \tag{7}$$

An immediate consequence of this form is the following: if we initialize each set of parameters in a independent and identically distributed (iid) fashion, with finite variance and zero mean [7], then, for each $x \in \mathbb{R}$, the $p_j(x)$ are also iid random variables with zero mean and finite variance, and, therefore, the Central Limit Theorem guarantees that, in the infinite width limit, $n \to \infty$, $f_\theta(x)$ converges, in distribution, to a random variable with Gaussian distribution $N(0, \Sigma_{xx})$, where $\Sigma_{xx} := \mathbb{E}[p^2(x)]$. This provides a clear description of the measure, in the function space $\mathrm{Map}(\mathbb{R}, \mathbb{R})$, induced by a standard parameter initialization via the map $\mathbb{R}^P \ni \theta \mapsto f_\theta \in \mathrm{Map}(\mathbb{R}, \mathbb{R})$; nonetheless, this was considered to be "disappointing", from the time of its discovery [15], since wide neural networks were being "forced" into a well known Gaussian regime, instead of fulfilling the expectation of being flexible enough to develop a wider range of data driven features.

The previous result concerns the expressivity of wide (but shallow) neural networks at initialization. Let's now see what happens during training, via gradient descent (4). Instead of studying the training dynamics in parameter space directly, a clearer path turns out to be traced by the evolution of outputs $f_{\theta(t)}(x)$, which is determined by (4) together with

$$\partial_t f_{\theta(t)}(x) = \sum_{l=1}^{N} \Theta_n(x, X_l; \theta(t)) \left(Y_l - f_{\theta(t)}(X_l)\right) , \tag{8}$$

where, after ordering the parameters $\theta = \{\theta_1, \dots, \theta_{4n}\}$,

$$\Theta_n(z_1, z_2; \theta) := \sum_{i=1}^{4n} \frac{\partial f_\theta}{\partial \theta_i}(z_1) \frac{\partial f_\theta}{\partial \theta_i}(z_2) , \\ = \frac{1}{n} \left(\sum_{j=1}^{n} \sum_{i=1}^{4n} \frac{\partial p_j}{\partial \theta_i}(z_1) \frac{\partial p_j}{\partial \theta_i}(z_2) \right),$$

is known as the (empirical) Neural Tangent Kernel (NTK) [11]. At this point, it is the Law of Large Numbers that controls the asymptotics by ensuring that, as $n \to +\infty$,

$$\Theta_n(z_1, z_2; \theta) \to \Theta_\infty(z_1, z_2) := \mathbb{E}\left[K(z_1, z_2)\right] ,$$

almost surely (a.s.), where $K(z_1, z_2)$ is the random variable

$$K(z_1, z_2) := \sum_j \frac{\partial p_1}{\partial \theta_j}(z_1) \frac{\partial p_1}{\partial \theta_j}(z_2) .$$

So we see that Θ_∞, the infinite width NTK, is a.s. constant in parameter space. This is a fundamental observation, first made explicitly in [11], with remarkable consequences. Loosely speaking, it implies that the "infinite width network" follows a linear learning evolution, obtained by formally taking the $n \to \infty$ limit in (8). In rigor, we can't define the infinite width network as a function, by taking the limit, $n \to \infty$, in (6), since this limit diverges [8] with probability one. To circumvent this difficulty, we take inspiration from the referred formal limit in (8), and define the *infinite width network*, at training time $t \geq 0$, as the solution to the initial value problem

$$\begin{cases} \partial_t f_\infty(x, t) = \\ \qquad - \sum_{j=1}^{N} \Theta_\infty(x, X_j)(f_\infty(X_j, t) - Y_j) \\ f_\infty(x, 0) = f_{\theta(0)}(x) . \end{cases} \tag{9}$$

Is is important to stress that, strictly speaking, the infinite width network is not a neural network and it isn't even a parametric model — "parameters are washed/integrated away by the limit theorems". In view of the constancy of the NTK in parameter space, (9) is a closed linear system of Ordinary Differential Equations (ODEs), that can be integrated to show that training, in the infinite width regime, converges to

$$\lim_{t \to +\infty} f_\infty(x, t) = f_\infty(x, 0) - \sum_{l,m=1}^{N} \Theta_\infty(x, X_l)(\Theta_\infty)^{-1}_{lm}(f_\infty(X_m, 0) - Y_m), \tag{10}$$

where $(\Theta_\infty)^{-1}_{lm}$ is the (l, m) component of the inverse of the matrix $\Theta_\infty(X_i, X_j)$. Formula (10) is an instance of Kernel Regression, another well known technique in machine learning; consequently, the previous result is sometimes summarized by saying that, in the infinite width limit, neural networks enter a kernel regime.

We can now return to finite width neural networks. By using a bootstrap/continuity argument it can be shown that, for sufficiently wide networks, $n \gg 1$, the non-linear learning dynamics defined by both (4) and (8), can be approximated by the infinite width linear evolution. More precisely we can show

[7] This conditions can be easily relaxed.
[8] Has divergent superior and inferior pointwise limits.

that

$$\left|\Theta_n(z_1, z_2; \theta(t)) - \Theta_\infty(z_1, z_2)\right| \leq C \frac{(\log n)^{1/2+\epsilon}}{\sqrt{n}} , \tag{11}$$

and that

$$\left|f_{\theta(t)}(x) - f_\infty(x, t)\right| \leq C \frac{(\log n)^{1/2+\epsilon}}{\sqrt{n}} , \tag{12}$$

for all $t \geq 0$ and all $|x|, |z_1|, |z_2| \leq 1$. The previous, in particular, imply that, for sufficiently large n, the training outputs converge exponentially to their labels:

$$|f_{\theta(t)}(X_l) - Y_l| \leq C e^{-(\lambda_{\min}-\epsilon)t} , \tag{13}$$

where $\lambda_{\min}$ is the minimum eigenvalue of the matrix $\Theta_\infty(X_i, X_j)$, which, for now, we are assuming to be positive.

We should stop for a minute to contemplate what was achieved, even if restricted to the shallow network case: First, it should be clear that (13) implies that, for sufficiently wide networks, the loss (3) converges to zero, a global minimum; this corresponds to the memorization problem discussed in the introduction (see **Q1**). Secondly, Equations (12) and (13) provide control over the trained model in terms of an explicit Kernel Regression, that in turn depends only on the data and initialization; in particular, the generalization performance (see **Q2**) of the learned model is also codified in these and can be further understood by relying on the learning theory for Kernel Regression (see for instance [2, 13]).

As the more cautious reader surely has noticed, some of the presented results are conditional on the assumption that the matrix $\Theta_\infty(X_i, X_j)$ is (strictly) positive definite. This turn out to be the case under very general assumptions: basically one just needs to assume that all training inputs X_i are distinct and that the activation function is continuous, almost everywhere differentiable and non-polynomial [4].

3.2 Deep networks

It turns out that most of the results discussed in the previous section generalize to the context of DNNs, so, hopefully, the pedagogical incursion through the framework of shallow networks was fruitful. In this section we will briefly sketch how this generalization proceeds by focusing of the construction of the NKT in the context of DNNs.

With that in mind, let us drop the restrictions made in the previous section and consider neural networks, more precisely MLPs, of arbitrary depth, and any number of inputs and outputs, as defined in (2).

To see what happens at initilization, we fix $\ell \geq 1$ and assume the network is initialized with i.i.d parameters satisfying $W^{(\kappa)} \sim \mathcal{N}(0,1)$ and $b^{(\kappa)} \sim \mathcal{N}(0,1)$. Then, in the (sequential) limit (when the number of all hidden neurons goes to infinity) $n_1, \ldots, n_\ell \to \infty$, the output component functions $f^{(\ell+1)}_{\theta,\mu} : \mathbb{R}^{n_0} \to \mathbb{R}$, $\mu = 1, \ldots, n_{\ell+1}$, converge in law to i.i.d. centered Gaussian processes $f^{(\ell+1)}_{\infty,\mu} : \mathbb{R}^{n_0} \to \mathbb{R}$ with covariance defined recursively by

$$\Sigma^{(1)}(x, y) = \frac{1}{\sqrt{n_0}} x^t y + 1 , \tag{14}$$

$$\Sigma^{(\kappa+1)}(x, y) = \tag{15}$$
$$\mathbb{E}_{f \sim \mathcal{N}(0, \Sigma^{(\kappa)}(x,y))} [\phi(f(x))\phi(f(y))] + 1 .$$

In this context, the Neural Tangent Kernel (NTK) is the matrix valued Kernel whose components $\Theta^{(\ell+1)}_{\mu\nu} : \mathbb{R}^{n_0} \times \mathbb{R}^{n_0} \to \mathbb{R}$ are defined by

$$\Theta^{(\ell+1)}_{\mu\nu}(x, y) = \sum_{\vartheta \in \theta} \frac{\partial f^{(\ell+1)}_{\theta,\mu}}{\partial \vartheta}(x) \frac{\partial f^{(\ell+1)}_{\theta,\nu}}{\partial \vartheta}(y) , \tag{16}$$

with $\theta = \{W^{(\kappa)}_{ij}, b^{(\kappa)}_k\}$ the set of all parameters. The fundamental observation by Jacot et al [11], discussed in the previous section, was in fact made for deep networks: if we initialize the parameters in a i.i.d fashion according to the law $\theta \sim \mathcal{N}(0,1)$ then, as $n_1, \ldots, n_{L-1} \to \infty$, the NTK converges, in law, to a deterministic kernel

$$\Theta^{(\ell+1)}_{\mu\nu} \to \Theta^{(\ell+1)}_{\infty,\mu\nu} = \Theta^{(\ell+1)}_\infty \delta_{\mu\nu} , \tag{17}$$

with the scalar kernel $\Theta^{(\ell+1)}_\infty : \mathbb{R}^{n_0} \times \mathbb{R}^{n_0} \to \mathbb{R}$ defined recursively by

$$\Theta^{(1)}_\infty(x, y) = \frac{1}{n_0} x^\intercal y + 1 , \tag{18}$$

$$\Theta^{(\kappa+1)}_\infty(x, y) = \Theta^{(\kappa)}_\infty(x, y)\, \dot{\Sigma}^{(\kappa+1)}(x, y) + \Sigma^{(\kappa+1)}(x, y) , \tag{19}$$

where, for $\kappa \geq 1$,

$$\dot{\Sigma}^{(\kappa+1)}(x, y) := \mathbb{E}_{f \sim \mathcal{N}(0, \Sigma^{(\kappa)}(x,y))} \left[\dot{\phi}(f(x))\, \dot{\phi}(f(y))\right] . \tag{20}$$

Similarly to what happens in the shallow case, for deep MLPs the constancy of the infinite width NTK $\Theta^{(\ell+1)}_\infty$, in parameter space, allows us to control the non-linear learning dynamics of large, but finite, width networks in terms of linear systems of ODEs [8, 12]. From this we can, once again, extract results concerning the memorization and generalization capacities of wide networks.

Moreover, in the deep network case, the eigenvalues of the NTK matrix $\Theta^{(\ell+1)}_\infty(X_i, X_j)$ are now functions of the depth (and other hyper-parameters) and relevant notions of stability are linked to these quantities. For instance, it should be clear from (13) that $\lambda_{\min}$, the smallest eigenvalue of this matrix, controls the training convergence speed. So, if $\lambda_{\min}$ converges to zero, when the depth of the network goes to infinity, this means that the training will become arbitrarily slow with the increase of depth. On the other hand, if $\lambda_{\min}$ becomes arbitrarily large with the depth, then this means that the discretization of gradient descent requires, in practice, that we must keep on decreasing the learning rate (the quantity controlling the discretization size), when increasing the depth, to avoid large oscillations or divergences. In conclusion, when designing a network one would like to have $\lambda_{\min} \approx 1$, as a function of depth and other hyper-parameters, in order to have a robust learning setting. This criterium can be used to obtain some mathematical guidelines in fixing the hyper-parameters [6], in relation to question **Q4** raised in the introduction.

3.3 In conclusion: the two sides of the NTK

The NTK theory has opened a window into the learning dynamics of wide neural networks [9], that led to remarkable progress concerning the theoretical understanding of their performance. Nonetheless, the fact that learning with sufficiently wide MLPs is, in essence, restricted to kernel regression is, in a sense, even more disappointing than the Gaussian behavior of MLPs at initialization. To justify this emotional reaction, consider that entering a kernel regime implies, in particular, that all features are fixed at initialization and learning becomes restricted to finding appropriate linear combinations of these features. This situation is sometimes referred to as *lazy training*. This inability to perform data-driven *feature learning* has other unfortunate consequences: for instance, it prevents the use of powerful techniques such as *transfer learning*, where features learned in a "big" data set are later reused in fitting a new model to a smaller dataset [10]. Such results show that simply scaling up the number of units per layer can lead to undesirable scenarios, which relates back to **Q3** raised in the introduction. Nonetheless, several proposals [5, 19] have been set forth to circumvent these difficulties and retain *feature learning*, even in the large width regime.

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REPORT

Interaction of Poisson Geometry, Lie Theory and Symmetry

30 June–4 July 2025

by João Nuno Mestre*

The conference *Interactions of Poisson Geometry, Lie Theory and Symmetry* took place in Instituto Superior Técnico, Lisbon, from June 30 to July 4, organized by the Centre for Mathematical Analysis, Geometry, and Dynamical Systems (CAMGSD) and by the Centre for Mathematics of the University of Coimbra (CMUC).

Poisson geometry and Lie theory are related branches of differential geometry, which are both deeply motivated by their applications to classical and quantum mechanics. A unifying feature is the important role symmetry plays in the study of such systems.

* Departamento de Matemática da Universidade de Coimbra.
Email: jnmestre@mat.uc.pt

The conference explored the many beautiful connections between Poisson geometry, Lie theory and symmetry, themes chosen in honor of the 60th birthday of Professor Rui Loja Fernandes, who has made great contributions to these topics throughout his career.

The event had 115 participants, based in 22 different countries. Of those, 32 participants were students, and 30 participants were early-career researchers (at most 10 years after their PhD). The scientific program included 22 talks from invited speakers, and 25 Poster presentations divided among 5 separate poster sessions. The slides of the talks and most of the posters are available in the webpage of the conference (https://sites.google.com/view/ipgls2025).

The conference had financial support from the research centres CAMGSD and CMUC, FCT (Fundação para a Ciência e a Tecnologia), NSF (US National Science Foundation), EMS (European Mathematical Society), and CIM (Centro Internacional de Matemática). The funding from CIM was used to support the lodging of 3 participants (1 PhD Student and 2 Postdocs) during the conference.

4th Women in Mathematics Meeting in Portugal

14–16 July 2025

by Ana Paula Dias*, Liliana Garrido*, Célia Moreira* and Ana Jacinta Soares**

The 4th edition of the *Women in Mathematics Meeting in Portugal* (WM25) was held at the Department of Mathematics, Faculty of Sciences, University of Porto (Portugal), from 14 to 16 July 2025, with around 60 participants from a diverse range of countries and institutions. See https://sites.google.com/view/fcup-wm25/

The meeting aimed to promote scientific excellence, visibility, and inclusion of women in mathematics, while fostering interaction across career stages and different research areas of Mathematics. This marked an important moment for strengthening the national and international women-in-mathematics community.

The programme of this edition featured invited lectures by Karin Baur (Ruhr-University Bochum, Germany), Célia Borlido (University of Coimbra), Isabel Labouriau (University of Porto), Cláudia Nunes (IST, University of Lisbon), Lisa Santos (University of Minho), and Rita Teixeira da Costa (University of Cambridge, UK). The meeting also included a special session on gender equality led by Mónica Lopes (University of Coimbra), a panel discussion on the role of research centres in gender equality, contributed talks and posters, and the exhibition *Women in Mathematics - History and Portraits*.

The high level of the scientific programme, the strong engagement of participants, and the interesting discussions throughout the meeting clearly reflected the success of the WM25 event, after the first edition in Lisbon, at UNL, in 2019, followed by a short online session in 2021, and the second edition at the University of Minho later in 2021, and the 3rd edition hosted by the University of Coimbra in 2023. Together, these editions have progressively strengthened the visibility, continuity,

* Centro de Matemática da Universidade do Porto (CMUP)
** Centro de Matemática da Universidade do Minho (CMAT)

and impact of this initiative within the Portuguese mathematical community.

The meeting was organized by Ana Paula Dias (CMUP), Liliana Garrido (CMUP), Célia Moreira (CMUP), and Ana Jacinta Soares (CMAT), with the valuable advisory support of the Scientific Committee composed of Sílvia Barbeiro (CMUC, Univ. Coimbra), Sofia Castro (CMUP), Margarida Mendes Lopes (CAMGSD, IST-Univ. Lisbon). It received the financial support from *Centro de Matemática da Universidade do Porto* (CMUP), *Centro de Matemática da Universidade do Minho* (CMAT), *Centro de Matemática da Universidade de Coimbra* (CMUC), *Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, Universidade de Lisboa* (CAMGSD), *Departamento de Matemática da Faculdade de Ciências da Universidade do Porto*, *Faculdade de Ciências da Universidade do Porto*, and *Centro Internacional de Matemática* (CIM).

Going hyperbolic

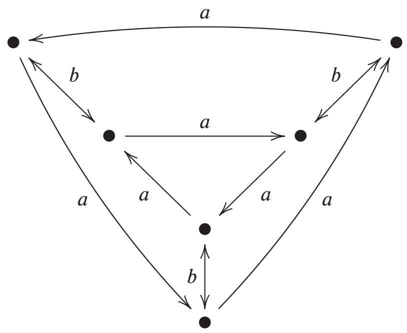
by **Pedro V. Silva***

1 Introduction

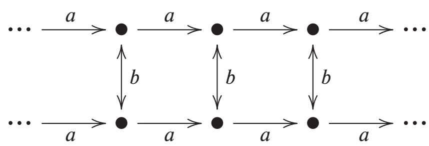
The theories of finite and infinite groups have developed different approaches over the years. On the one hand, finite groups can be viewed as subgroups of finite symmetric groups and relate naturally to combinatorics. On the other hand, infinite groups tend to be viewed as quotients of free groups (through generators and relators) and have strong connections to geometry.

Indeed, the seminal work of the geometer Max Dehn [9] is very much at the source of what is nowadays known as geometric group theory, when he performed an algorithmic study of the so called *surface groups* (fundamental groups of surfaces). His ideas were later generalized by Martin Greendlinger with his small cancellation theory [10].

Cayley graphs turned out to be an important tool in this context, which we now proceed to define. Given a group G generated by a set A, the *Cayley graph* $Cay_A(G)$ has vertex set G and labelled directed edges of the form $g \xrightarrow{a} ga$ for all $g \in G$ and $a \in A^{\pm 1}$ (and the A-labelled edges determine the A^{-1}-labelled ones). For instance, for the symmetric group S_3, generated by the permutations $a = (123)$ and $b = (12)$, we get the Cayley graph



For the direct product $\mathbb{Z} \times \mathbb{Z}_2$, generated by $a = (1, 0)$ and $b = (0, 1)$, we get the Cayley graph



Note that there is no need to identify the vertices since Cayley graphs have a transitive automorphism group.

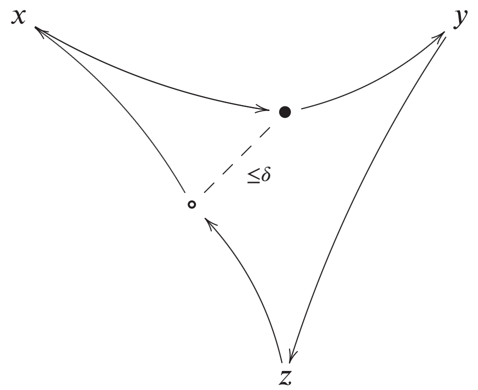
In the eighties, Mikhail Gromov developed a brilliant new idea [11]: if we consider $Cay_A(G)$ as a (geodesic) metric space for some group G finitely generated by A, its geometric properties may imply good algorithmic properties, and hyperbolic geometry means excellent news! This thought is somewhat disturbing… we haven't established yet what is the geometry of *our* universe: is it hyperbolic (negative curvature), spherical (positive curvature) or euclidean (flat)? It is absolutely irrelevant for our daily life, but we are not used to think that one alternative may be better than the other… surprise, surprise: that is precisely what happens in the realm of finitely generated groups. What does this mean exactly?

Defining a metric d_A on $G = \langle A \rangle$ is easy: since $Cay_A(G)$ is connected, we define $d_A(g, h)$ as the length of the shortest path from g to h, and any such path is called a *geodesic*. If we actually imagine the edges of the Cayley graph as real lines (as we often do in our mind when we think about graphs), then every geodesic of length n becomes isometric to the interval $[0, n] \subset \mathbb{R}$, and $Cay_A(G)$ becomes what is known as a *geodesic metric space*.

A popular way of defining hyperbolic geometry in this context is through *geodesic triangles* (a collection of 3 geodesics $\{[x, y], [y, z], [z, x]\}$ connecting 3 ver-

* Centro de Matemática, Faculdade de Ciências, Universidade do Porto

tices x, y, z). Given $\delta \geq 0$, we say that this triangle is *δ-thin* if every point in one of the geodesics is at distance $\leq \delta$ from some point in one of the other two geodesics.



The geodesic metric space is *hyperbolic* if there exists some $\delta \geq 0$ such that every geodesic triangle is δ-thin.

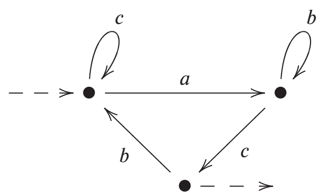
Algebraic structures have not earned a reputation of robustness, to say the least… a slight deformation on an abelian group and oops, it is not abelian anymore. On the contrary, hyperbolic geometry is certainly robust: within geodesic metric spaces, hyperbolicity is preserved through quasi-isometry. An *isometry* $\varphi : (X, d) \to (X', d')$ is a bijection preserving distance, i.e. $d'(\varphi(x), \varphi(y)) = d(x, y)$ for all $x, y \in X$. In a *quasi-isometry*, all these notions are relaxed within constant bounds, for instance through inequalities of the form $d'(\varphi(x), \varphi(y)) \leq Kd(x, y) + L$ and so on. A suggestive image is that quasi-isometric spaces look the same if we watch them from very far away…

Now if A and B are two alternative finite generating sets for a group G, then $Cay_A(G)$ and $Cay_B(G)$ are quasi-isometric, thus we can define G to be a *hyperbolic group* if $Cay_A(G)$ is hyperbolic for some (every) finite generating set A. Using closure under quasi-isometry, one can also prove that if H is a finite index subgroup of G, then G is hyperbolic if and only if H is hyperbolic.

But how abundant in *nature* are hyperbolic groups anyway? Finite groups are trivially hyperbolic, and so are free groups of finite rank (since their Cayley graph with respect to a basis is a tree). This implies that finitely generated *virtually free* groups (characterized by having a free subgroup of finite index and finite rank) are also hyperbolic. And so are fundamental groups of compact riemannian manifolds with negative (not necessarily constant) sectional curvature. On the other hand, no group containing $\mathbb{Z} \times \mathbb{Z}$ (the fundamental group of the torus) as a subgroup is hyperbolic. This might lead us to suspect that hyperbolicity is not such a common property… is it?

A group G is *finitely presented* if it can be defined by finitely many generators and relators, say $\langle A \mid R \rangle$. This means that G is isomorphic to the quotient of the free group F_A on the finite set A by the normal subgroup generated by the finite subset R of F_A. Most group theory is about finitely presented groups, really. It happens that Alexander Ol'shanskii proved in 1992 [12] (but Gromov announced this fact previously) that the probability of a finitely presented group being hyperbolic tends to 1 (under reasonable assumptions)! So we do not know whether our universe is hyperbolic, but the universe of finitely presented groups certainly is…

A surprising feature of hyperbolic groups are the amazing algorithmic properties they satisfy, making computations easy with respect to other groups. Let $Geo_A(G)$ consist of all words labelling geodesics in $Cay_A(G)$ starting at the identity 1. Then $Geo_A(G)$ turns out to be a *rational language*, that is, it is the language of a *finite automaton*. Given a finite automaton A, the language $L(A)$ is the set of all words which can be read in the automaton. For instance, the language of the automaton



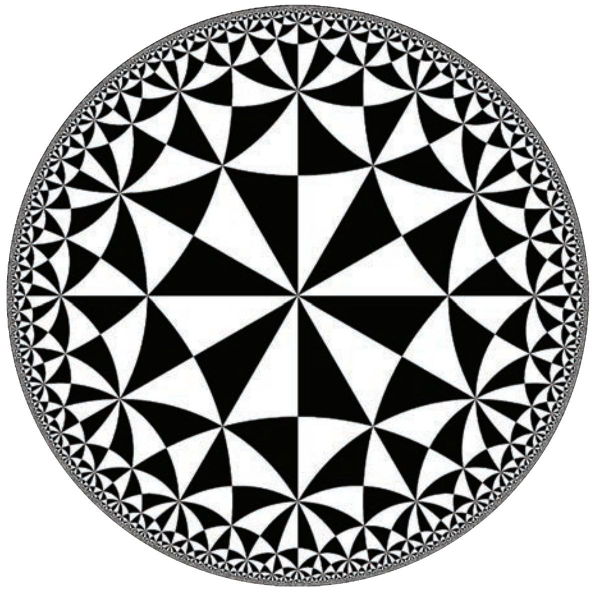
is $\{c, ab^*cb\}^*ab^*c$, where X^* denotes the submonoid generated by X. The fact of $Geo_A(G)$ being rational does not ensure in itself the algorithmic properties we have been boasting about, but the fact is that there exist also finite automata which encode somehow the action by right and left multiplication of each letter on $Geo_A(G)$. It follows that hyperbolic groups are actually *biautomatic* groups, which have solvable word problem and solvable conjugacy problem (indeed they are the only biautomatic groups where $Geo_A(G)$ has these properties). Thus there exist efficient algorithms for deciding whether two arbitrary words on $A^{\pm 1}$ represent the same element or conjugate elements of G. And hyperbolic groups are finitely presentable. We should note that all these problems were actually solved my Max Dehn for surface groups!

Another unexpected theorem arises by considering isoperimetric functions. Consider a presentation $\langle A \mid R \rangle$ of a group G and $w \in F_A$. If $w = 1$ in G, then w is the product of conjugates of elements of $R^{\pm 1}$. Let $\lambda(w)$ denote the minimum number of such factors in such a product. An *isoperimetric function* for $\langle A \mid R \rangle$ is a function $f : \mathbb{N} \to \mathbb{R}^+$ such that $\lambda(w) \leq f(|w|)$ for every $w \in F_A$ representing the identity in G (where $|w|$ denotes the length of w in reduced form). Gromov proved that a finitely presented group is hyperbolic if and only if it admits a linear isoperimetric function (and if and only if it admits a subquadratic isoperimetric function). Therefore we have a theorem that establishes an equivalence between a geometric property and computational complexity.

We already know that hyperbolic groups are deeply tied to geometry, what about topology? Given $G = \langle A \rangle$, the topology of the metric space (G, d_A) is discrete and would not seem very promising. But the concept of boundary introduced by Gromov (for any hyperbolic geodesic metric space, actually) changed the whole game. Assume that $G = \langle A \rangle$ is hyperbolic. A *geodesic ray* is an infinite path in $Cay_A(G)$ starting at the identity such that every finite subpath is a geodesic. Two geodesic rays are equivalent if the Hausdorff distance between their sets of vertices is finite (i.e., every vertex of one of them is at bounded distance from some vertex in the other ray). The *Gromov boundary* ∂G is the set of all equivalence classes of geodesic rays. Gromov defined a topology T on $\overline{G} = G \cup \partial G$ with several important properties:

- both $\overline{G}$ and ∂G are compact for T and its restriction;
- T is metrizable for a family of metrics d on G called *visual metrics*;
- the completion of any visual metric on G induces the topology T on $\overline{G}$;
- ∂G is invariant under quasi-isometry, so we do not need to specify the finite generating set of G.

The boundary is particularly simple to describe in the case of a free group since it consists of all (right) infinite reduced words $a_1 a_2 a_3 \ldots$ (and is a Cantor set). If we consider the *hyperbolic plane* $\mathbb{H}^2$ via the Poincaré disk model,



its boundary is the outlining circumference. We should note that $\mathbb{H}^2$ plays an important role in the theory of hyperbolic groups due to the following alternative theorem due to Bonk and Kleiner [3]: every hyperbolic group is either virtually free or there exists a quasi-isometric embedding of $\mathbb{H}^2$ into its Cayley graph.

2 Hyperbolicity in Porto

I started working with free groups around 20 years ago and later on I moved on to virtually free groups. My approach was essentially automata-theoretic and involved also *transducers* (automata with output). For instance, in [14] I used these ideas to study the dynamics of continuous extensions of endomorphisms to the boundary of a virtually free group. In fact, it is the boundary which becomes interesting from the dynamical viewpoint (recall that the topology of the group itself is discrete). An endomorphism φ of a finitely generated virtually free group G admits a continuous extension Φ to ∂G if and only if it is uniformly continuous for some (any) visual metric. I proved that this happens if and only if φ is *virtually injective* (that is, it has finite kernel). Now the fixed points of Φ are divided into two disjoint subclasses: singular and regular. The *singular* fixed points arise as the topological closure of the fixed points of φ, which constitute a finitely generated subgroup $Fix(\varphi)$ of G (I provided an automata-theoretic proof in [14]). The remaining fixed points are called *regular*, and I proved that the set $Reg(\Phi)$ of regular fixed points is in some sense finitely generated: if we consider the natural action of $Fix(\varphi)$ on $Reg(\Phi)$, it turns out that it has only finitely many orbits. In the automorphism case, it was shown that the regular fixed points are either exponentially stable attractors or exponentially stable repellers.

During my stay at Salvador da Bahia, I worked with Vítor Araújo (UFBA, formerly UP) in problems related to hyperbolic geometry and hyperbolic groups.

One of the motivations for our work was the possibility of defining new pseudometrics on the automorphism group of an arbitrary hyperbolic group. This led us to consider Hölder conditions in [1]. A mapping $\varphi : (X, d) \to (X', d')$ between metric spaces satisfies a *Hölder condition* of exponent $r > 0$ if there exists a constant $K > 0$ such that

$$d'(\varphi(x), \varphi(y)) \leq K(d(x, y))^r$$

holds for all $x, y \in X$. Such a condition clearly implies uniform continuity.

As a preliminary result, we showed that all visual metrics on a hyperbolic group are Hölder equivalent. Our main theorem establishes several equivalent conditions for a nontrivial endomorphism φ of a hyperbolic group $G = \langle A \rangle$ and a visual metric d on G:

- φ satisfies a Hölder condition with respect to d;
- φ admits a continuous extension to ∂G satisfying a Hölder condition with respect to the natural extension of d;
- φ is a quasi-isometric embedding of (G, d_A) into itself (where d_A denotes de geodesic metric);
- φ is virtually injective and $\varphi(G)$ is a quasiconvex subgroup of G.

We should note that quasiconvex subgroups play a major role in the theory of hyperbolic groups. We say that $H \leq G$ is *quasiconvex* if every point in a geodesic of $Cay_A(G)$ with endpoints in H lies at bounded geodesic distance from some vertex of H. It is known that every quasiconvex subgroup of a hyperbolic groups is hyperbolic (but the converse implication does not hold).

In [1], we also proved that if the hyperbolic group G is either virtually free or torsion-free co-hopfian, then φ is uniformly continuous if and only if it satisfies a Hölder condition if and only if it is virtually injective.

Hyperbolic groups are defined upon the concept of hyperbolic geodesic metric space, which admits several equivalent definitions. In this text, we only considered so far thin geodesic triangles, but other important characterizations involve the concept of mesh or the Gromov product. Another goal of ours was to develop a theory for an appropriate subclass of hyperbolic geodesic metric spaces which would play a similar role for the subclass of finitely generated virtually free groups. In [2], we introduced four equivalent geometric conditions on a geodesic metric space such that a finitely generated group $G = \langle A \rangle$ is virtually free if and only $Cay_A(G)$ satisfies any one of these geometric conditions. Since these conditions are preserved through quasi-isometry, this does not depend on the finite generating set considered.

In the condition which is the analogue of thinness of geodesic triangles, we replaced triangles by arbitrary polygons. Consider a *geodesic polygon* $\{[x_0, x_1], [x_1, x_2], \dots [x_{n-1}, x_n], [x_n, x_0]\}$. Given $\delta \geq 0$, we say that this polygon is δ-thin if every point in one of the geodesics it at distance $\leq \delta$ from some point in one of the other geodesics of the polygon. The geodesic metric space is *polygon hyperbolic* if there exists some $\delta \geq 0$ such that every geodesic polygon is δ-thin.

A few years later, my former PhD student André Carvalho developed also an interest on hyperbolic groups. A major achievement consisted on generalizing the *bounded reduction property* (BRP) of free groups to hyperbolic groups [5]. In the free group case, this is a quite useful property: if $\varphi : F_A \to F_A$ is an injective homomorphism, then for every reduced product $uv \in F_A$, there is a bounded number of letters cancelled in the reduction of $\varphi(u)\varphi(v)$. Carvalho presented some equivalent geometric characterizations of the BRP and showed that an endomorphism of a hyperbolic group satisfies the BRP if and only if it preserves a *coarse median* (a certain type of ternary operation defined by Brian Bowditch for metric spaces [4]). He also proved that uniformly continuous endomorphisms of a hyperbolic group (for a visual metric) always satisfy the BRP. As a consequence, he generalized a theorem, proved by Frédéric Paulin for automorphisms [13], showing that every uniformly continuous endomorphism of a hyperbolic group has a finitely generated subgroup of fixed points. And he showed that every uniformly continuous endomorphism of a hyperbolic group satisfies a Hölder condition, solving an open problem proposed on [1].

It should be noted that Carvalho extended these ideas beyond the realm of hyperbolic groups, proving theorems on the BRP for automatic groups [8], and the dynamics of continuous extensions to the boundary for endomorphisms of certain classes of graph groups such as $F_n \times \mathbb{Z}^m$ and $F_n \times F_m$ [6, 7]. In both these cases, the Gromov completion is appropriately replaced by the *Roller completion*.

Hyperbolic groups have been generalized over the years in different directions: automatic groups, semi-hyperbolic groups, (weakly) relatively hyperbolic groups hyperbolic semigroups… however, the exquisite harmony we can find in the theory of hyperbolic groups, where ideas from completely different areas converge to produce unexpected results, has yet to be replicated in these more general settings. If the great Japanese writer Yukio Mishima were a mathematician, he would probably be fond of hyperbolic groups!

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NTQO2025
9th edition of the workshop series NTQO—New Trends in Quaternions and Octonions

12–13 September 2025

by Isabel Cação*, Milton Ferreira** and Nelson Vieira*

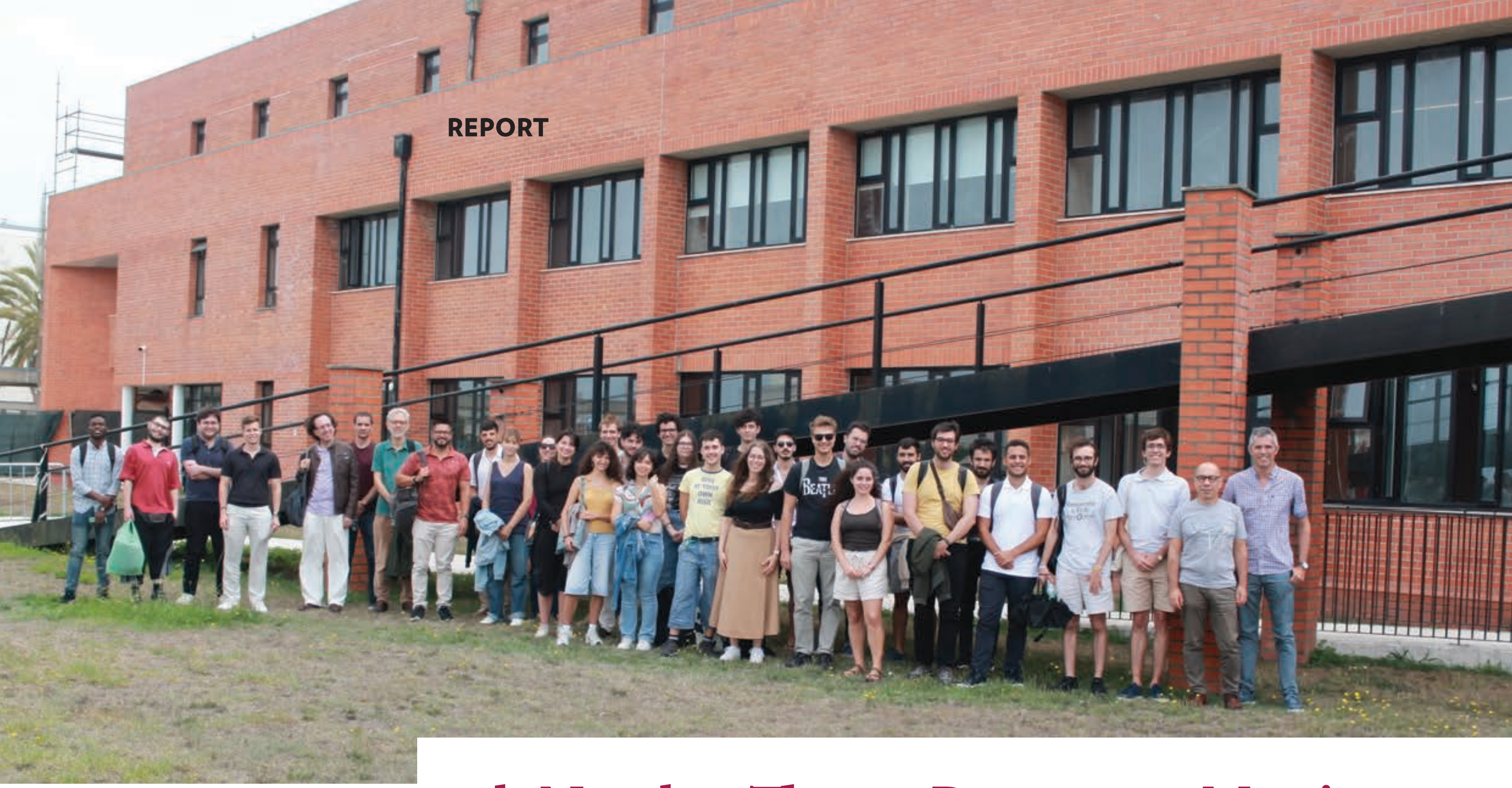
The ninth edition of the workshop series NTQO was held at the University of Aveiro, on 12–13 September 2025. The event had 43 participants from several countries, held in a hybrid format to accommodate remote participation. The workshop brought together scientists from pure and applied mathematics, physics, scientific computing, engineering, and related disciplines. The program included four invited lectures by Anna Kit Ian Kou (University of Macau), Cristina Diogo (ISCTE-University Institute of Lisbon), Soeren Kraußhar (University of Erfurt), and Vladimir V. Kisil (University of Leeds), as well as several short communications.

This edition was particularly significant, as it marked several anniversaries. It celebrated the 10th anniversary of the NTQO workshop series, which began at the University of Aveiro in 2015. The event also honoured the 75th birthday of Helmuth Malonek, one of the NTQO series' founders, and the 145th birth and 75th death anniversaries of Rudolph Fueter, a seminal figure in Quaternionic Analysis.

The NTQO2025 was jointly organized by the universities of Aveiro, Minho and Beira Interior and the participation on the event was free of charge. The organizing committee acknowledges the financial support from the following institutions: *Center for Research and Development in Mathematics of the University of Aveiro* (CIDMA-UA), *Center of Mathematics of the University of Minho* (CMAT-UMinho), *Center of Mathematics and Applications of the University of Beira Interior* (CMA-UBI), and *International Center for Mathematics* (CIM).

Further details can be found at https://sites.google.com/view/ntqo-2025/home

* University of Aveiro • Emails: isabel.cacao@ua.pt and nvieira@ua.pt
** Polytechnic of Leiria • Email: milton.ferreira@ipleiria.pt



4th Number Theory Portuguese Meeting
Aveiro Summer School 2025

by Ariel Pacetti* and António Machiavelo**

Number Theory is one of the oldest areas of mathematics, but is still, unfortunately, an underrepresented research area in Portugal. During the previous three years a series of anual meetings have been organized as a concrete action for reverting this problem, aiming in the long run to make Number Theory an active and productive research area in Portugal.

The overall goal of the present Summer School was to gather together young students interested in the area, mostly from the Iberian Peninsula, and provide an introduction to some fundaments of modern Number Theory.

The Summer School was held at the University of Aveiro, on 1–5 of September, 2025, and was organized around 5 courses:

- Local Fields, by **Rachel Newton** (King's College)
- p-adic L-functions, by **Oscar Rivero** (University of Santiago de Compostela)
- Galois Representations, by **Luis Dieulefait** (University of Barcelona)
- Arithmetic Statistics, by **Alex Bartel** (Glasgow University)
- Expanders Graphs, with perspectives to the Theory of Numbers, by **Harald Helfgott** (CNRS—Institut Mathématique de Jussieu)

Each course consisted of 3 lectures of 90 minutes long each, and included a list of problems that were discussed with the attending students.

Thirty applications were received for financial support from different countries. We funded all the applicants whose mathematical interests were related to Number Theory, including all female students who applied to the school.

There were 25 students attending the school: 11 from Spain, 2 from Italy, 2 from Germany, and 10 from Portugal, with a gender balance of 8 female students and 17 male students. There were another 14 faculty attending the school, including speakers, organizers and other young professors. The lodging costs of all students (but 3 local ones) were cover by the school.

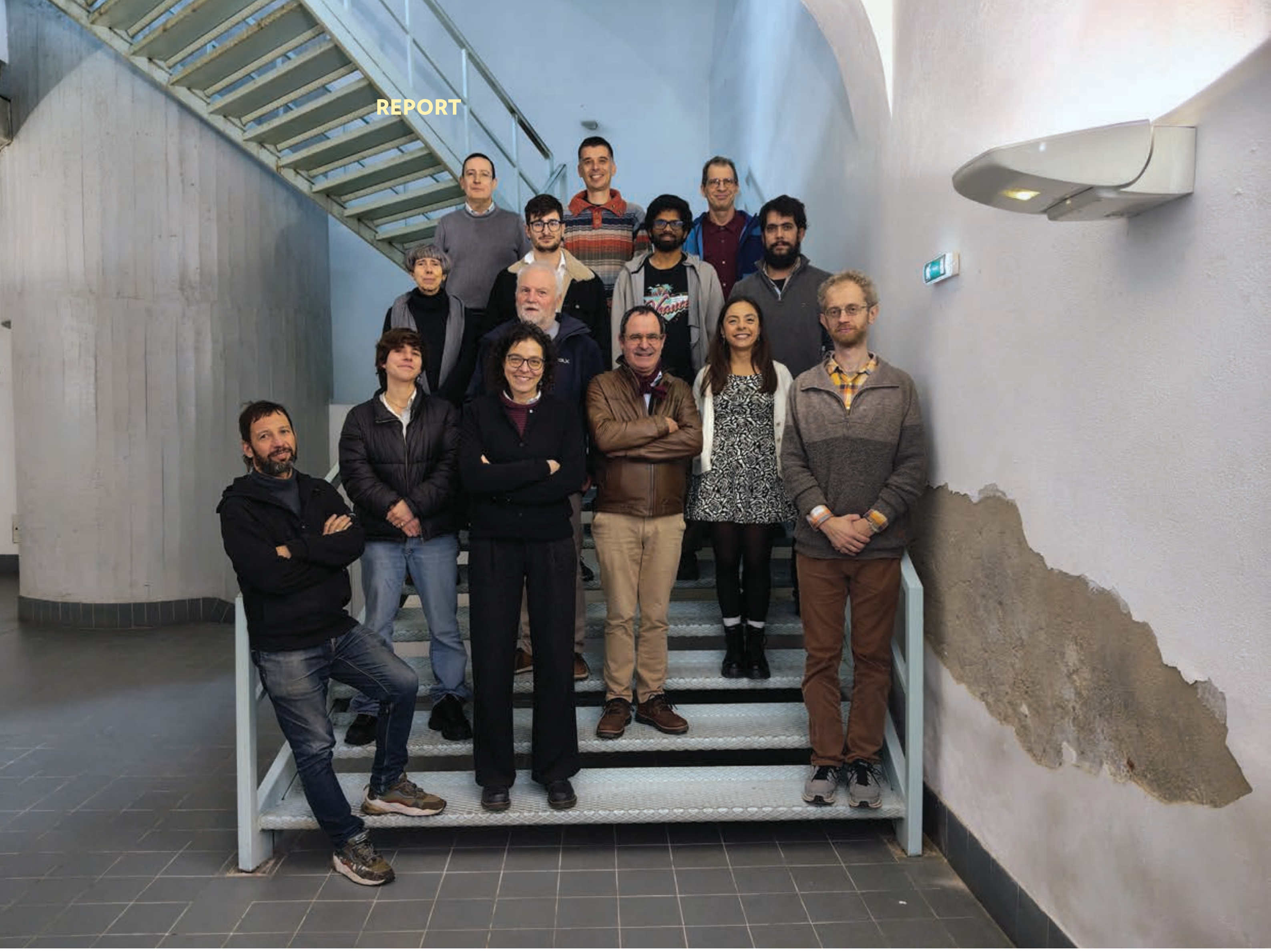
The Summer School received fundings from Foundation Compositio Mathematica, the Number Theory Foundation, the Centro Internacional de Matemática, CIDMA and the FCT (Fundação para a Ciência e a Tecnologia).

More information can be found at the web page https://sweet.ua.pt/apacetti/EPTN/2025.

Organizers

Paulo Almeida (University of Aveiro)
Nuno Freitas (ICMAT, Madrid)
António Machiavelo (University of Porto)
Ariel Pacetti (University of Aveiro)

* Universidade de Aveiro
** Universidade do Porto



The 15th Combinatorics Days
University of Évora, December 5–6, 2025

by Manuel Branco*, João Dias*, Olga Azenhas** and Samuel Lopes***

The *Combinatorics Days* is an itinerant annual conference series that brings together mathematicians working in Combinatorics, widely interpreted, and related fields such as Algebra, Geometry, Probability, Computer Science or Physics.

The 15th edition of Combinatorics Days was hosted by Universidade de Évora, Colégio Luís António Verney, December 5 and 6, 2025. The programme consisted of two mini courses, one by Maria Bras-Amorós (Universitat Politècnica de Catalunya), on "Numerical semigroups, conjectures, algorithms, and applications", and another by Evgeny Feigin (Tel Aviv University), on "Cauchy identities: combinatorics and representation theory". Additionally, there was a plenary lecture by Serguei Popov (CMUP), on "Conditioned simple random walk in two dimensions and some of its surprising properties", and nine diverse thirty-minute oral presentations. About twenty two participants joined to this event.

For more information see:
https://www.mat.uc.pt/~combdays/15thcombday.html

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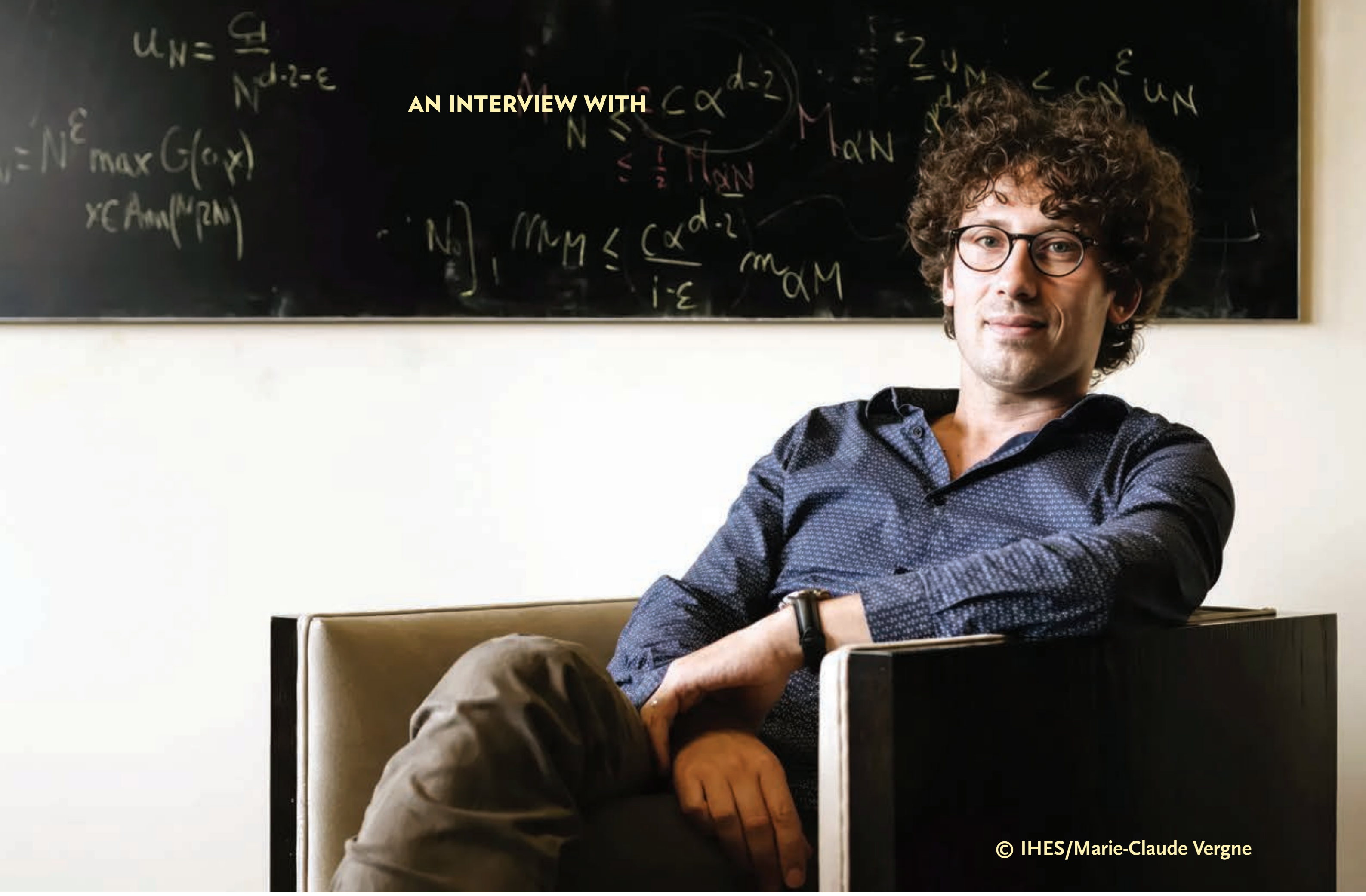


Hugo Duminil-Copin

by Jorge Milhazes de Freitas*

I read that you were keen on sports, and particularly handball. So my first question is: how close were we to having a world champion handball player rather than a Fields Medalist?

Well, you know, at the time I didn't really know what the chances were on either side when I made the decision. I chose quite early, when I was 15. I basically had to decide between going to a special sports and studies school or going to Lycée Louis le Grand, which was a special school for mathematics, in some sense. So I didn't know what the expectations were.

Later on, though, I stopped growing. So I'm pretty sure I would never have made it into the national team, which would have made becoming a world champion rather difficult. At the time, France was the leading nation in handball—it was the generation of Karabatić and company. They were Olympic champions and so on. Making it into the national team was essentially the key step, and with my physical strength I would never have gone that far. So, in retrospect, it was a good choice.

But you also don't go into mathematics thinking you're going to get a Fields Medal. There is so much randomness. Very, very few people could reasonably presume it. Maybe for someone like Terence Tao, one could guess early on that there was a good chance. Personally, I didn't even know about the Fields Medal at the time.

* Departamento de Matemática, Faculdade de Ciências da Universidade do Porto
Email: jmfreita@fc.up.pt

When did you realise that you wanted to do mathematical research for a living, or that you might have a future in mathematics?

It happened in a very strange way. I always loved physics and mathematics, even though I was never as good at physics as at maths. When I arrived at the École Normale Supérieure, the level of the people around me was really intimidating. I ended up dropping physics and, seeing the level of others, I thought I would probably end up teaching—which was something I wanted anyway. I come from a family of teachers, so I was perfectly happy with that.

Then I took the agrégation, the national exam to become a teacher, and it went extremely well for me. I even remember how people told me my ranking: I finished second. They said, "We have the results, and nobody can believe it—you were second." Not because they thought I should have been first, but because they simply never imagined I would be second.

That gave me a bit of confidence. The agrégation is officially an exam for teaching, but in practice it is really an exam in mathematical training. After that, I did a one-year master's program, which included a long stay in Vancouver. There, I had a lot of time, and in some sense some of my former weaknesses became strengths: working for a very long time on a problem, becoming totally obsessed, trying many different approaches because I have a hard time focusing on just one thing.

By the end of that year, I realised that this way of working was actually very effective. I ended up with, in some sense, two research papers, and it became clear that I could start a PhD. But it really came out of the blue. I didn't go there thinking, "Let's become a researcher." I went because it seemed like a good experience and I needed it for my master's degree.

It worked so well that I almost became a mathematician by accident. When I came back, I was already one, without having planned it. After that, I was hooked. I realised that my way of thinking was well suited to research, and from then on I never really doubted it.

And what about your choice of probability theory?

That was a bit simpler, and it is probably related to my inclination toward physics. At ENS, I had a teacher, Jean-François Le Gall, who is famously an outstanding lecturer. His courses were extremely clear and well organised. And yet, for the first two months, his class had a catastrophic effect on me.

There was this notion of a random variable. It was explained very well, but in a very formal way, and I simply didn't get it. I kept thinking: it's just a function—why do they call it a random variable? A function from one space to another.

At some point, I finally understood the subtlety: what information you want to extract, and how randomness is encoded. Then I was completely hooked. I went from total confusion to a global understanding, and suddenly this beautiful course became crystal clear. At first it had felt too perfect, too foreign to my way of thinking; then it became very natural.

That pushed me to enrol in a very focused master's program in probability. I then had fantastic courses, including one by Wendelin Werner. That's where I learned about percolation theory and the Ising model. It became clear to me that this was mathematics deeply connected to physics—I felt like I was bringing physics back into my mathematical life. From then on, it was obvious that this was my field.

It also helped that I was good at it. You can like something and still be bad at it. In my case, it really matched my intuition, and things worked very well.

Do you have a role model in mathematics—someone you particularly admire or who inspired you?

Interestingly, I didn't really have any role models until quite late. When I was young, I didn't even know that being a mathematician was a real job. I wasn't the kind of student who read Thurston's essays or Feynman's lectures.

Later on, I was surrounded by people I deeply admired, such as Wendelin Werner and my PhD advisor, Stanislav Smirnov. More generally, I was inspired by a certain way of doing mathematics.

In that sense, I would single out Harry Kesten. He was not necessarily flashy, but he had an extraordinary ability to drill through problems where nobody else could make progress. His proofs were often not elegant at first—his papers are, frankly, horrible to read. They are extremely technical. But the core ideas are almost always the right ones. After years of polishing by the community, they turn into incredibly powerful tools. I have infinite respect for that ability.

One of the best moments of my mathematical life was meeting him. I met him twice: once at the ICM in 2010, where he gave the laudation for Stanislav Smirnov, and once when I gave a lecture for his birthday. Even though he was already ill at the ICM, he discussed recent work of Stanislav's with me. To have someone I considered a kind of god talk seriously about the work of a young mathematician was incredibly impressive. He is probably the person who impressed me the most, mostly through his mathematics.

We are now in an era in which artificial intelligence is exploding. It is everywhere and changing our habits and the way we interact with computers. Do you feel an impact on your research? Do you use computer-aided proofs or other AI tools? And do you think AI will have a lasting impact on mathematical research?

Yes—I was expecting that question. Let me start with the first part: do I use it?

The daily life of a mathematician is not limited to research. For everything else, I use AI quite a lot, as a very powerful assistant. I always dreamed of having an assistant to handle tasks I am not particularly good at. Nowadays, it is extremely useful for polishing writing. I am still a very basic user, and I don't trust AI at all for mathematical content, but for form and style it is already very helpful.

This year, we have started to hear about AI being able to help prove easy lemmas and similar things. It is still borderline, and in my area—where arguments are often less formal—the impact remains limited. For PDEs and very formalised areas, it seems to be becoming quite efficient. For percolation, where you work a lot with drawings and geometric intuition, it is essentially impossible at the moment. But what is impossible today may become easy in one or two years.

As an assistant for mathematics—just as it already is for writing—I think this will arrive very soon. I try to stay open-minded and not reject it outright, because it is clearly going to be part of our lives. One thing that already works remarkably well is turning handwritten notes into polished LaTeX. For teaching, I still like to write things by hand when I prepare a course, but now you can convert notes into a rough LaTeX file very quickly and then use AI tools to clean it up. That is already a reality, and with some practice it can save a huge amount of time.

As for AI helping to prove things, I am still waiting—but I suspect it will come soon. Whether AI will generate genuinely new ideas is a different question, and that leads to a deeper issue: what is mathematics, really?

If the goal is simply to prove that something equals zero, then I have no doubt that an AI will eventually produce such proofs. But if the proof is 1,000 pages long and unreadable, it does not give me what I want, which is understanding. I am also deeply interested in the journey of proving something, not just the final statement. So I expect that AI will very quickly become better than us. There may not even be a long intermediate phase in which we can enjoy discussing mathematics with AI as equals. We will move rapidly from thinking that AI is a bad mathematician to realising that we are the bad mathematicians.

This forces us to rethink our role. What do we bring to science? Already today, much of modern mathematics is too complicated for non-mathematicians. Our role may increasingly be to prepare conceptual ground, to clarify ideas, and perhaps to act more like philosophers of abstract thought.

For the next series of questions, feel free not to reveal anything you don't want to or can't disclose. Regarding the Fields Medal, did you have a sense at some point that it might happen? Did you know you were on some kind of shortlist—assuming such a list exists? Were you anxious?

I can be quite open about this, because I actually think the way the process works isn't very considerate of the feelings of people who are clearly on some kind of shortlist—let me be generous, but I don't think that's an exaggeration.

Roughly speaking, you can identify maybe the top twenty mathematicians of a generation. Even though that notion of "best" is strange, of course. Sometimes there's one person who stands out so clearly that everyone knows they're going to get it. I'm pretty sure Peter Scholze, for example, had zero doubt that he would get the Fields Medal—though he probably didn't care much. There are people like that.

But beyond those obvious cases, there's usually a group of about twenty people, and within that group maybe three or four really stand out. So pretending that you don't know you're among those twenty is a bit disingenuous. In my case, I knew I was in that group.

I also knew that, even within probability, there were probably four or five people who could reasonably get it. Any of those choices would have made sense. So yes, you live with that. That part is fine. What you don't know is when—or whether—you'll be told.

Not having a fixed date is actually a bit perverse. People who haven't heard yet are waiting. That's human. It's not so much that you want to have it; you just want to know.

I was lucky because I was told quite early. But I think about others—people I respect enormously, and who I believe were at least as deserving as I was—who were probably waiting, thinking there was still a chance. That feels a bit indelicate. I really think there should be a fixed date: say, "We tell people on January 8."

Then, if you don't get the call on January 8, you know. And your life will be perfectly fine. I can tell you: the Fields Medal is wonderful in some respects, but it's also a heavy load. You can absolutely be very happy without it.

This is something that isn't said often enough. The process isn't considerate enough of people's feelings. I remember a conversation in April 2014 with a mathematician—I won't name them—who said, "You know, I don't have it." And I could see the relief in saying that. The list had become too short; at least things were clear. In some sense, knowing helped them make peace with it. I really think this part of the process should change.

Either do it like the Nobel Prize, or keep the secrecy—but at least give a date. By the way, the secrecy itself can be quite funny. I have a few good stories about keeping a secret for six months.

In my case, things were a bit more complicated. Out of kindness, some people—not on the jury, but you know how information circulates—told me back in 2018 that I was on the shortlist. That was actually not easy to navigate.

In 2018, it didn't even cross my mind that I might be shortlisted. I went to the ICM to give my sectional talk, very happy, not thinking about the medal at all.

But once someone tells you that kind of information, you can't un-know it. Of course, it's not the same jury, not the same four years—it's a long process. I had to tell myself: be serious, keep doing mathematics the way you like to do it, not in a way that optimizes your chances.

There's a temptation to go on an "advertising tour," visiting all the top universities. But what I really like is doing math and working on hard problems. Still, it required some mental strength not to think about it too much.

One funny consequence is that, in 2018, I told my wife and my parents—because, well, I'm human. And they became stressed. They're not in control, right?

You're not supposed to tell your family—your spouse is allowed, but not your parents. So in February 2022, they knew I should hear something around February or March.

My father kept saying, "Hugo, you look tired. You look sad. You can tell us if you didn't get it. You know we love you, we don't care."

And I was like, "No, I couldn't be happier." Inside, I was thinking, I look sad? I feel great. But my father was convinced my face looked tired and miserable. I was completely confused.

Eventually, I told them—mostly because they were so stressed that I was sad, even though I wasn't. I was stressed, yes, but not sad. It was funny to see them so worried about my well-being that they didn't realise I had actually won it.

And then, after I told them, they said, "Oh no, that's not what we meant. You just look tired for other reasons." And I was like, "So you think I look tired all the time?" That was also kind of funny.

So how far in advance are you told?

I found out in January.

That's quite early—the ICM is only in July. It's not useless, though. For instance, I could quietly read work by other medalists.

France is also peculiar in this respect. Someone like Cédric Villani really embraced public life—he was brilliant with the media, and the media loved him. There was a clear "Villani effect," and I knew there would be something similar: the Fields Medal is much more visible in France than in many other countries.

So I knew I would get significant media coverage, and that I'd have to deal with it—especially given the comparison with Cédric's very polished public presence. That worried me a bit. I probably should have done media training, but because I wasn't allowed to tell anyone, I couldn't even get advice from my university.

At least I could read and prepare things quietly—my ICM proceedings, for example. That wasn't so bad.

A funny thing about secrecy: I made my entire group read my proceedings without them realising they were the proceedings. I told them, "Some journal asked me to write a review," and they completely bought it. It was ridiculous, and at some point I was afraid they'd catch on.

Your daughter was one year old at the time, right? So she didn't understand…

Well, she still doesn't.

We actually had a math argument the other day about triangles. She showed me a triangle with her fingers and said, "This is a triangle, Dad." I was very proud. Then she turned it upside down and said, "This is not a triangle."

I tried to explain that orientation doesn't matter. She said, "No, it's not." After ten minutes, she concluded, "My teacher said it's not a triangle. So it's not." And she walked away.

So I lost a math argument to my three-year-old daughter—who certainly lied about what the teacher said.

She has absolutely no idea about the Fields Medal. It will probably be funny when she does, but that will be much later. In fact, this is a good summary of my life overall. Among people I know, I'm not really a "Fields medalist." My status hasn't changed much. Even within my research area, colleagues don't treat me differently, which is good—it means the mathematics comes first.

What did change is media attention, politics, and to some extent interactions with students—especially in large lectures, where behaviour can be different.

I had a shocking experience at Harvard. It was a conference in honor of Elliott Lieb's 90th birthday. Elliott Lieb is a god in mathematical physics—his influence is everywhere, especially close to my field. I have infinite respect for him.

During a break, I was talking with Elliott, and a young guy came up and asked for a selfie with me, almost pushing Elliott aside to take the photo. I thought: "You have no idea you're standing next to one of the most accomplished mathematical physicists of the 20th century". I would love my work to shine even a fraction as much as his.

Instead, it was like: "Please move, we want the picture with the Fields medalist." That's a completely ridiculous side of this prize.

My next question was whether it had an impact on your life and whether it changed things significantly—but I think you've already answered that.

What definitely changed is the media exposure.

I feel like I'm constantly being asked to defend mathematics. You suddenly become an ambassador for the field, and that can be overwhelming. Of course, you can choose not to engage too much, but I often feel compelled to give something back. And that comes at a price. It's a very heavy load.

Sometimes it's absolutely exhausting. You're pushed far outside your comfort zone in many ways, and I really think institutions could provide more support for that. It requires a lot of energy, and in some sense you're quite alone. This is something that genuinely changed my life: there are many things I now do because I feel it's my responsibility—even though, honestly, I would sometimes prefer not to do them.

That said, it's demanding but it also has very positive aspects. In 2023, for instance, I gave about a hundred public lectures. That was completely crazy. At the same time, speaking in front of 500 kids is incredibly cool—there's so much energy. You're exhausted, you can't really do much research, but there are real upsides.

You feel useful, in a way. I think I felt useful for the first time in my life when I started interacting with the public. There's much more to be done in that direction. As a community, we should be far more involved in public engagement, because people really enjoy it. A large part of the public actually loves mathematics.

What advice would you give to a young researcher who isn't sure whether to pursue a career in mathematics? What would you say to someone in that situation?

It's a difficult question, but an important one. For me, mathematics has never been easy. I'm not someone who learns it effortlessly. Even when I grasp an idea, translating it into a formal proof can be really hard. I've always had to find ways to keep going, even when things were tough.

One approach that worked for me—and that I'd recommend—is to work on multiple problems at the same time. I always try to keep at least three in mind. First, I like to have a problem that I know I can solve—something below my level. This is my safety net. On days when I feel drained or stuck, working on that problem helps me regain confidence and energy.

Second, I need a problem that's roughly at my level. This is the one that balances effort and reward: it challenges me enough to grow and also contributes meaningfully to the community. Ultimately, we are judged on our output—not just papers, but also talks, ideas, and how we reuse knowledge. Writing ten easy papers isn't going to get you anywhere. You need something at the right level.

And third, it's crucial to have problems that seem too hard—problems that you dream about solving, even if they seem out of reach. In my experience, the most difficult problems I've tackled were ones I initially thought I couldn't solve. If I had only worked on problems that felt achievable, I'd never have attempted them. By balancing the too-easy, right-level, and too-hard problems, the very difficult ones often become approachable over time.

I've tried focusing entirely on a single problem once, and it was frustrating. I felt my energy wasn't being used effectively. Having multiple problems prevents that kind of burnout.

Another key point: mathematics is hard, and it's normal not to understand everything right away. Success comes only if you are truly passionate about the area you're working in. Don't chase what looks shiny or glamorous; follow what excites you. That passion is the only way you can accept, and work through, periods of confusion and difficulty.

Finally, choose your advisor for human reasons, not just academic prestige. The relationship with your PhD advisor—or with your students, if you're supervising—is crucial. We learn far more by observing and mimicking behaviour than by following instructions. A strong human connection allows for that kind of learning. Thurston expressed this even better than I can, but the essence is that direct, personal interaction matters more than formal guidance.

So when picking an advisor, look for someone you feel a genuine connection with. The same goes for students: one of my main concerns when taking on a PhD student is ensuring there's good rapport. Without it, the work will suffer for both sides.

So, last question: have you ever found any real-world application for any of your results?

Honestly, I'm not really interested in the applications of my results, even in areas where I could have some. For example, percolation theory does have applications in other fields. In principle, if you impact theoretical physics, it can ripple into applied physics and engineering. But that has never been my motivation.

I don't even aim for problems with potential applications. I accepted that early in my career. My drive comes from aesthetic purposes—a kind of pure mathematical coherence and harmony. I think it's important that some people are motivated this way, because it complements those who are driven by applications. You don't have to be a superhero, able to produce highly original mathematical ideas and simultaneously find their practical applications. Society benefits from people with different strengths. My strength is trying to produce ideas.

Very often, I work on a problem and get an idea that doesn't quite fit. Because I'm driven by the idea itself, not just the problem, I'll explore whether the idea can work elsewhere. If it resonates with me, I'll try to find another problem for which it's useful. My career is basically built around that approach.

For example, early in my career, with my PhD advisor, I solved a problem proving that the number of self-avoiding walks on the hexagonal lattice grows at a specific rate—a very elegant result. It's a readable and satisfying paper even if you're not into probability. People often asked us, "Where did that original idea come from?"

Here's the story: Stas, my advisor, had an idea in a completely different context. We tried to apply it to self-avoiding walks, and it didn't work. Later, I realised a variant of the idea could be applied to a percolation problem, so we pursued it there. It evolved, leading to two papers in that area—but still didn't work for self-avoiding walks. Eventually, the idea stalled in percolation; six months of effort went nowhere.

Then one day, I realised that version 2.0 of the idea actually fit perfectly back in the original problem—self-avoiding walks. And in two minutes, I had the full proof. Zero doubt. It was one of those rare moments where you see how ideas can travel, evolve, and eventually land in exactly the right place.

21st Computability in Europe Conference

14–18 July 2025

by **Gilda Ferreira*** and **Isabel Oitavem****

The Faculty of Sciences of the University of Lisbon (FCUL) hosted the 21st edition of the international conference *Computability in Europe* (CiE), held from July 14 to 18, 2025.

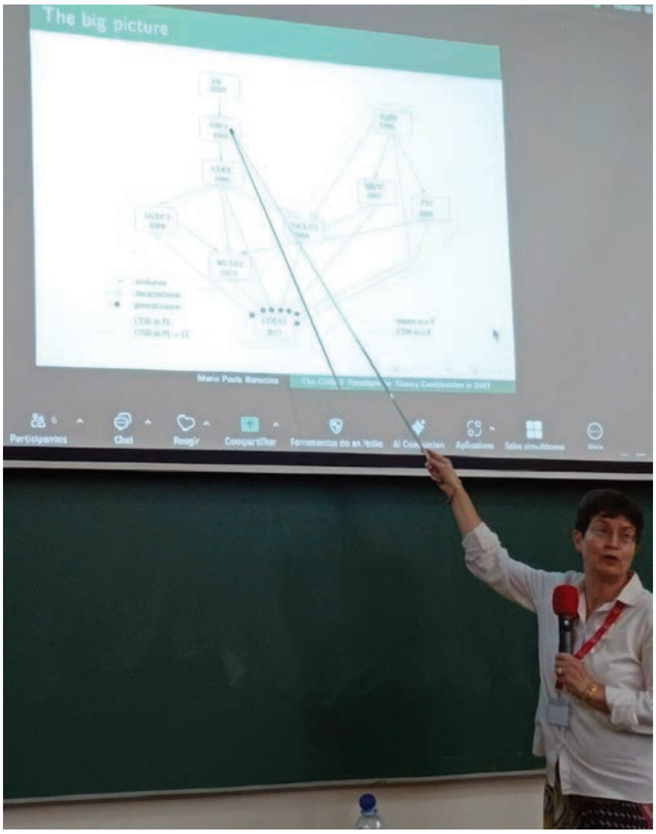
This conference is part of the annual series promoted by the Association CiE, which aims to advance and disseminate both fundamental and applied research in areas related to computability — including mathematics, computer science, logic, computational biology, as well as the history and philosophy of computing.

This year's edition brought together 138 researchers from 29 countries and fostered a stimulating environment for the exchange and discussion of advanced scientific ideas.

The conference featured 6 plenary talks, 2 plenary tutorials, 6 special sessions (each with 4 talks), and 55 contributed talks.

There were 49 papers submitted for publication in the conference proceedings, of which 27 were accepted. The proceedings were published by Springer in a

* Universidade Aberta / Center for Mathematical Studies (CEMS.UL)
** Center for Mathematics and Applications (NOVA Math) and Department of Mathematics, NOVA School of Science and Technology (NOVA FCT)



Lecture Notes in Computer Science volume: https://doi.org/10.1007/978-3-031-95908-0. In addition to the accepted contributed papers, the volume also includes 8 invited papers.

Springer generously funded a Best Paper Award. The winner of this award was the paper *On the Computational Power of C-Random Strings* by Alexey Milovanov.

Since 2016, the conference has included a special interest group named *Women in Computability* (WiC). The WiC program features a workshop, the annual WiC dinner, a mentorship program, and a grant program for young women researchers. The *Women in Computability* workshop continued in 2025, under the coordination of Johanna Franklin, with invited speakers Ana Sokolova (Universitaet Salzburg) and Maria Paola Bonacina (Università degli Studi di Verona).

The social program of the event included a welcome drink in the garden of the Old Canteen, a guided visit to the City Museum at the Pimenta Palace, and a conference dinner at the D'Bacalhau restaurant in Parque das Nações.

The event was supported by several Portuguese research centers, including CEMS.UL, LASIGE, CMAT, NovaMath, CIDMA, and IT; by the *Association for Symbolic Logic*; the *Department of Mathematics of the Faculty of Sciences of the University of Lisbon*; the *Fundação para a Ciência e a Tecnologia*; the *Portuguese Republic*; CiM; *Turismo de Lisboa*; and by the *International Union of History and Philosophy of Science* – Division of Logic, Methodology and Philosophy of Science. The conference was also endorsed by the *Sociedade Portuguesa de Matemática*, the *Sociedade Portuguesa de Lógica*, and the *Kurt Gödel Society*.

Plenary speakers:

Ugo Dal Lago *University of Bologna, Italy*
Daniel Graça *University of Algarve, Portugal*
Ekaterina Komendantskaya *University of Southampton, UK*
Ng Keng Meng *Nanyang Technological University, Singapore*
Paulo Oliva *Queen Mary University of London, UK*
Ana Sokolova *University of Salzburg, Austria*

Tutorial speakers:

Maria Paola Bonacina *University of Verona, Italy*
Igor Carboni Oliveira *University of Warwick, UK*

Special sessions:

Computable Analysis and Topology

Organizers: Takayuki Kihara (Nagoya University, Japan) and Elvira Mayordomo (University of Zaragoza, Spain)

Speakers: Djamel Amir (Paris-Saclay University, France), Emmanuel Rauzy (University of the Bundeswehr Munich, Germany), Holger Thies (Kyoto University, Japan) and Cécilia Pradic (Swansea University, UK).

Computability Aspects of Descriptive Set Theory

Organizers: Luca San Mauro (University of Bari, Italy) and Dino Rossegger (Vienna University of Technology, Austria)
Speakers: Antonio Montalban (University of California, Berkeley, USA), Noah Schweber (Proof School, San Francisco, USA), Dan Turetsky (Victoria University of Wellington, New Zealand) and Benjamin Siskind (Vienna University of Technology, Austria).

Human-Centered AI: Foundational, Historical, and Computational Perspectives

Organizers: Melissa Antonelli (University of Helsinki, Finland), Jean-Baptiste Joinet (Jean Moulin Lyon 3 University, France) and Mattia Petrolo (University of Lisbon, Portugal)
Speakers: Juan Manuel Duran (Delft University of Technology, The Netherlands), Paolo Pistone (Claude Bernard University Lyon 1, France), Carina Prunkl (Utrecht University, The Netherlands) and João Marques-Silva (ICREA, Spain).

Proof Complexity and SAT

Organizers: Maria Luisa Bonet (Polytechnic University of Catalonia, Spain) and Pavel Pudlák (Czech Academy of Sciences, Czech Republic)
Speakers: Albert Atserias (Technical University of Catalonia, Spain), Paul Beame (University of Washington, USA), Sam Buss (University of California, San Diego, USA) and Robert Robere (McGill University, Canada).

Proof Theory: Pure and Applied

Organizers: Anton Freund (University of Würzburg, Germany) and Pedro Pinto (Technical University of Darmstadt, Germany)
Speakers: Horatiu Cheval (University of Bucharest, Romania), Azza Gaysin (University of Passau, Germany), Morenikeji Neri (University of Bath, UK) and Shuwei Wang (University of Leeds, UK).

Quantum Computing

Organizers: Georg Moser (University of Innsbruck) and Romain Péchoux (University of Lorraine, France)
Speakers: Alejandro Diaz-Caro (Inria, France & National University of Quilmes, Argentine), Robin Kaarsgaard (University of Southern Denmark, Denmark), Subhasree Patro (Eindhoven University of Technology, The Netherlands) and Simon Perdrix (Inria, France).

Program Committee Chairs:
Arnold Beckmann (University of Swansea)
Isabel Oitavem (Universidade Nova de Lisboa)

Organizing Committee Chair:
Gilda Ferreira (Universidade Aberta/CEMS.UL)

For more details about the meeting, including the list of the 55 contributed speakers not included in this report, please visit the conference webpage at https://bit.ly/CIE2025

XVI Non-Associative Day in Azores

28 March 2025

by Ana Paula Garrão* and Margarida Raposo**

The XVI Non-Associative Day in Azores took place at the University of the Azores, Ponta Delgada campus, on 28 March 2025, with twelve participants. For full details and programme, see

https://sites.google.com/view/nonassociativedayazores2025/home

This event is the 16th in a series of one-day meetings dedicated to non-associative algebra, broadly interpreted. Previous editions have been held in cities such as Mulhouse (France), Lisbon, Coimbra, Funchal (Portugal), Cáceres (Spain), and Tashkent (Uzbekistan), among others.

These meetings bring together researchers working in various areas, including C*-algebras, representation theory, Lie and Poisson algebras, integrable systems, and related fields.

The XVI edition was supported by the Centro Internacional de Matemática, the Centro de Matemática da Universidade do Porto, the Universidade dos Açores, and the Centro de Matemática e Aplicações da Universidade da Beira Interior.

Organizers: Ivan Kaygorodov, Ana Paula Garrão, Margarida Raposo, and Samuel Lopes.

* NICA - Núcleo Interdisciplinar da Criança e do Adolescente
** FCT-UAc - Faculdade Ciências e Tecnologia da Universidade dos Açores

WODCA 2025

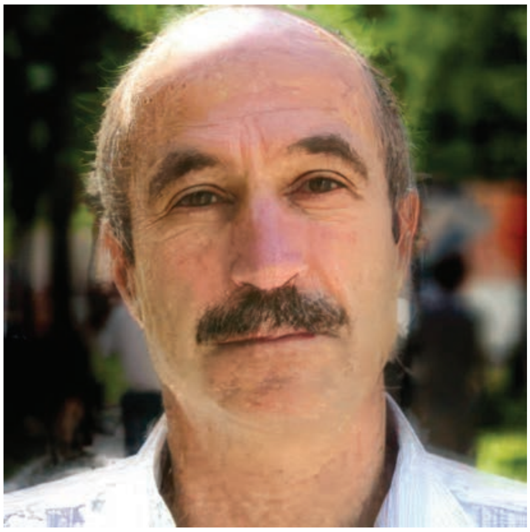
June 11–14, 2025

by **Tatiana Tchemisova***, **Delfim F.M. Torres***, and **João Pedro Cruz***

The international conference Workshop on Optimization, Dynamics, and Convex Analysis (WODCA 2025) took place from June 11 to 14, 2025, at the University of Aveiro, Portugal.

The event was jointly organized by members of the Local Organizing Committee from the University of Aveiro, the University of Algarve, and the Center for Research and Development in Mathematics and Applications (CIDMA), in honour of Professor Alexander Plakhov, marking his 65th birthday.

Professor Plakhov, an associate professor with habilitation at the University of Aveiro, has played a pivotal role in the academic life of the institution for over 25 years. Since 2018, he has led the CIDMA research group on Optimization, Graph Theory, and Combinatorics. His prolific contributions—highly aligned with the themes of the Workshop—include over 100 publications of inter-



Alexander Plakhov

* Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro

Local Organizing Committee

Alexander Plakhov and Keynote and Invited speakers

national relevance, encompassing both scholarly articles and monographs.

WODCA 2025 aimed to gather leading researchers in Billiards Theory, Dynamical Systems, Optimal Mass Transfer, and Optimization, offering a platform for rich academic exchange and collaboration.

The scientific program featured a wide range of topics, such as Billiards, Optimal Mass Transport, the Kakeya Needle Problem, Convex Geometry, Calculus of Variations, Mathematical Applications in Mechanics, Dynamical Systems, Optimal Control, Mathematical Modelling, Mathematical Programming, Convex Analysis, Convex Optimization, Integer Programming, among others.

The Workshop attracted over eighty participants from twenty-two countries. The program consisted of eighteen keynote and invited lectures, fifteen contributed sessions, and a poster session. The overall success of the event owes much to the dedication and professionalism of the International Scientific Committee, to whom we extend our sincere appreciation.

The workshop featured plenary lectures by eighteen distinguished keynote and invited speakers, namely: Sergei Tabachnikov, Pennsylvania State University, USA; Leonid Bunimovich, Georgia Institute of Technology, USA; Michael Bialy, Tel Aviv University, Israel; Giuseppe Buttazzo, University of Pisa, Italy; Alexey Glutsyuk, HSE University, Russia & CNRS, UMPA, ENS de Lyon, France; Konstantin Khanin, Beijing Institute of Mathematical Sciences and Applications, China; Sergey Kryzhevich, Gdansk University of Technology, Poland; Mark Levi, Pennsylvania State University, USA; Lev Lokutsievskiy, Lomonosov Moscow State University, Russia; Robert McCann, University of Toronto, Canada; Gennady Mishuris, Aberystwyth University, UK; Vera Roshchina, UNSW, Sydney, Australia; Vladimir Protasov, University of L'Aquila, Italy; Delfim F. M. Torres, University of Aveiro, Portugal; Pavel Zatitskii, University of Cincinnati, USA; Arseniy Akopyan, USA; and Alexey Davydov, Vladimir State University and Moscow State University, Russia.

The success of WODCA 2025 was made possible by the concerted efforts of the International Scientific Committee, the local Organizing Committee from the University of Aveiro and the University of Algarve, and the Center for Research and Development in Mathematics and Applications (CIDMA).

The Department of Mathematics of the University of Aveiro and the Universities of Aveiro and Algarve provided the Workshop with their institutional support.

The organizers also acknowledge the valuable support of CIM–International Center for Mathematics, and the dissemination support from APDIO–the Portuguese Association of Operational Research.

Additional information is available at: https://sites.google.com/view/wodca2025/

REPORT

Summer JIM Days

7–11 July 2025

by **Carla Rizzo***

The Summer Jovens Investigadores em Matemática Days (Summer JIM Days) were held at the Department of Mathematics of the University of Coimbra from 7 to 11 July 2025. The event was jointly organized by the Centre for Mathematics of the University of Coimbra (CMUC), the Centre for Mathematical Studies of the University of Lisbon (CEMS.UL), the Mathematics Center of the University of Porto (CMUP), the Centre for Research and Development in Mathematics and Applications of the University of Aveiro (CIDMA), and the Research Centre in Mathematics and Applications of the University of Évora (CIMA).

The event brought together a total of 74 participants, including early-career researchers (within 10 years of the PhD), PhD students, and master's students from a wide range of institutions. Designed as a vibrant platform for scientific exchange and community building, the Summer JIM Days were founded on the belief that scientific progress thrives through effective communication, meaningful connections, and dynamic networking. These guiding principles shaped both the scientific and social components of the programme.

The scientific programme was comprehensive and balanced, combining advanced lectures with opportunities for active participation and included a total of 20 invited speakers. It comprised 2 minicourses, 5 plenary lectures, 21 contributed talks of 25 minutes each distributed across thematic parallel sessions in algebra, analy-

* Centro de Matemática da Universidade de Coimbra

sis, geometry, mathematical physics, and applied mathematics, a poster session featuring seven contributions, and 5 short presentations in a five-minute lightning-talk format.

Complementing the lecture-based programme, the conference also included a panel discussion on career opportunities for mathematicians, featuring researchers from diverse professional contexts who addressed topics such as transitions between academic and non-academic sectors, strategies for developing competitive applications, and considerations regarding work–life balance.

Furthermore, three thematic roundtables were organized to encourage open dialogue on issues of particular relevance to early-career researchers. These focused respectively on Women in Mathematics, exploring equity, diversity, and the creation of inclusive academic environments; Science Communication, reflecting on the role of communication in enhancing the visibility and societal impact of mathematical research; and Mental Health in Academic Careers, promoting awareness, destigmatization, and the development of supportive academic cultures. These interactive sessions fostered a constructive exchange of experiences and perspectives, complementing the more formal components of the programme.

A structured social programme was integrated into the event to reinforce its community-building aims and to allow participants to continue scientific discussions in an informal setting. Activities included a speed-networking session enabling participants to meet fellow young mathematicians efficiently and expand their professional networks, a board-games evening providing a relaxed environment for informal connection, a guided tour of Coimbra highlighting the historical and cultural richness of the city, and a social gathering at Parque Verde, which offered space for informal conversations and friendly sporting activities.

The Summer JIM Days received financial support from the research centres CMUC, CEMS.UL, CMUP, and CIDMA, as well as from FCT (Fundação para a Ciência e a Tecnologia) and CIM (Centro Internacional de Matemática). Support from CIM was particularly significant in expanding the conference's opportunities for networking and collaboration by funding the social activities. This contribution also ensured the event's full accessibility, as no registration fees were charged, thereby enabling broad participation in all activities.

Feedback collected through evaluation forms sent after the event indicated a very high level of participant satisfaction, with an overall average score of 4.75 out of 5. This evaluation highlights the strong scientific quality of the programme, the relevance of the thematic discussions, and the effectiveness of the event in fostering an inclusive and stimulating research environment.

The competition *Symmetries of Lisbon's* calçada on π *Day in 2025*‡

by **José Francisco Rodrigues***

At the proposal and initiative of the International Mathematical Union (IMU), the UNESCO General Conference, at its 16th plenary meeting on 25 November 2019, proclaimed the 14th of March of each year as the *International Mathematics Day*.

Among the considerations underlying this decision is the recognition that "global awareness of, and enhanced education in, mathematical sciences are vital to addressing challenges in areas such as artificial intelligence, climate change, energy and sustainable development, and to improving the quality of life in both the developed and the developing worlds" and that "the applications of mathematical sciences are vital for advances in all types of engineering and computer science, while responding to the growing needs of automation and providing access to information via the Internet (the World Wide Web) for the wellbeing of society", recognising "the importance of creating conditions conducive to gender equality in mathematical sciences by promoting successful female role models in science, from Hypatia of Alexandria to Maryam Mirzakhani, not forgetting Emmy Noether, Sophie Germain or Mary Winston Jackson."

Thus, in the third month of each year the 14th day was chosen for International Mathematics Day, also known as *π Day*, since 3.14 is an approximation of that irrational number, already known to Archimedes, the ancient mathematician who demonstrated that the number π is, in any circle, the ratio of the perimeter to the diameter and the ratio of the area to the square of the radius.

In 2025, the theme associated with mathematics for the celebration of π Day was *Art and Creativity*, announced in nine languages (English, Arabic, Chinese, French, German, Korean, Portuguese, Spanish and Turkish — https://www.idm314.org/) and implemented in 922 locations across all continents. In Portugal alone, there were 58 initiatives, of which the one held at the Academy of Sciences was one of four registered in Lisbon.

On the 14th of March 2025, the Academy of Sciences celebrated this symbolic date for the first time in its Noble Hall with two hybrid events: a conference [S] promoted by the Young Scientists Seminar (SJC), where academic Maria Ivette Gomes gave a lecture entitled *Statistics, the science of data*, followed by two presentations by members from that seminar, Emmanuel Cruzeiro with his presentation *Mathematical challenges in quantum foundations* and João Cancela with his presentation entitled *Between the illusion of accuracy and the fear of quantifying: the role of mathematics in the social sciences*; and the [P] ceremony for the presentation of the awards for the *Competition Symmetries of Lisbon's* Calçada.

In January 2025, the Academy of Sciences launched a challenge to students in grades 9 to 12 (corresponding to ages 14 to 18, approximately) from schools across the country to create proposals for the five missing symmetry patterns in Lisbon's *calçada*,[1] with the aim of completing the set of 24 flat symmetries in its streets and squares. A prize of one thousand euros to each of the five winners was announced.

[1] *Calçada* is a Portuguese cobblestone pavement, generally for pedestrian use, made from natural, small stones, varying in shape. (https://calcadaportuguesa.org/en/o-que-e-a-calcada-portuguesa-2/)

‡ Original publication in Portuguese, published by the Academy of Sciences of Lisbon in April 2025
(DOI: https://doi.org/10.58164/4jry-wh91)

* **Academy of Sciences of Lisbon**

Figure 1. The symmetry of the Mar Largo pattern in a contemporary photograph of Lisbon [1][2]

Portuguese pavement is a form of public art and an integral part of the national cultural heritage, which has spread throughout the country and around the world, from Brazil to Macau, passing through Africa. Its symmetries began in Lisbon in 1849 with the *Mar Largo* (Wide Sea) of Rossio geometric pattern, an expression from *Os Lusíadas*, from the eighth verse of stanza 66 of canto IV, regarding the succession of D. João II by D. Manuel, who "Took on the conquest of the wide sea".

The *Mar Largo* pattern, which still exists in Rossio to this day, despite being interrupted between 1919 and 1975, has one symmetry by reflection, like a mirror, and two symmetries by rotation, with 180º turns around two points. It is one of the 17 possible symmetries of crystallographic patterns of the plane, identified by the Russian mineralogist Evgraf Fedorov and German mathematician Artur Schönflies in the late 19th century. Added to these symmetries are the seven symmetries of the friezes, which can be characterised by human footprints, respectively, with steps in normal gait, with feet together, with both feet landing alternately facing forwards and backwards, with feet sideways, with both feet sideways and rotating 180º, with a hop on one foot and with a hop on one foot rotating 180º.

The fact that there can only be exactly 24 flat symmetries, and no more, is a mathematical theorem. In 2017, the city of Lisbon completed all seven types of friezes, but so far only 12 patterns have been identified, meaning that five symmetries are still needed for it to become the first city in the world to materialise this mathematical theorem in its streets and squares.

The Competition *Symmetries of Lisbon's* Calçada aimed to reward the best five proposals for the five missing symmetry patterns, each proposal being a unique motif, consisting of a distinct decorative ornament, the repetition of which forms one of these five patterns, namely the patterns with the symmetries *333, 333, 632, 22X and 0, which in crystallographic notation are represented, respectively, by p3m1, p3, p6, p2gg and p1 and, for the purposes of the competition, labelled symmetry A, B, C, D and E.

The initiative received immediate support from Ciência Viva, the Directorate-General for Education, Lisbon City Council, the Portuguese Calçada Association, Lisbon Tourism and the Ludus associations, the Association of Mathematics Teachers and the Portuguese Mathematical Society.

The jury, chaired by José Francisco Rodrigues (Academy of Sciences of Lisbon), was composed of Rosalia Vargas (Ciência Viva — National Agency for Scientific

Figure 2. The Rossio Square in Lisboa with the *Calçada Mar Largo* in a postcard circa 1900.

Culture), Ana Silva Dias (Cultural Heritage Protection Division of Lisbon City Council), Ana Cannas da Silva (Department of Mathematics, ETH Zurich), António Prôa (Portuguese *Calçada* Association), Pedro Macias Marques (Directorate-General for Education) and Ana Margarida Rodrigues (Portuguese Mathematical Society).

Although the competition period was relatively short, in about six weeks, 127 proposals were received from students across the country, some with multiple proposals, distributed as follows: 35 for symmetry A; 15 for symmetry B; 26 for symmetry C; 22 for symmetry D and 29 for symmetry E.

Assessing the mathematical correctness, artistic quality and feasibility of implementation on the calçada of each of the 127 proposals, after two meetings, the jury decided on the following allocation of the five prizes and seven honourable mentions:

Symmetry Award A

Eduardo Manuel Fontoura Rebelo Rodrigues, from *Escola Profissional de Recuperação do Património de Sintra*

Symmetry Award B

Mila Luana de Gouveia Loureiro, from *Escola Básica e Secundária da Cidadela, Cascais*

Symmetry Award C

Vladyslav Kravets, from *Escola Profissional de Recuperação do Património de Sintra*

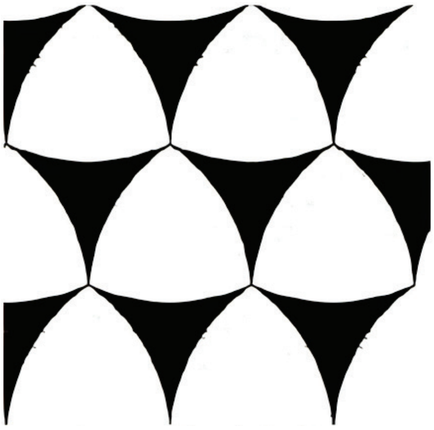
Symmetry Award D

Francisco Miguel Ledo Nazareth Pais Costa, from *Agrupamento de Escolas D. Filipa de Lencastre, in Lisbon*

Symmetry Award E

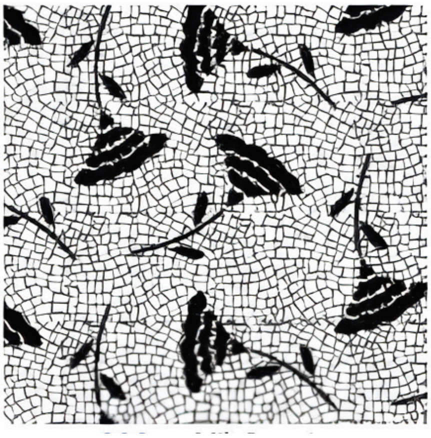
Sebastião José Pacheco Mendes, from *Escola Secundária de Alcácer do Sal*

A



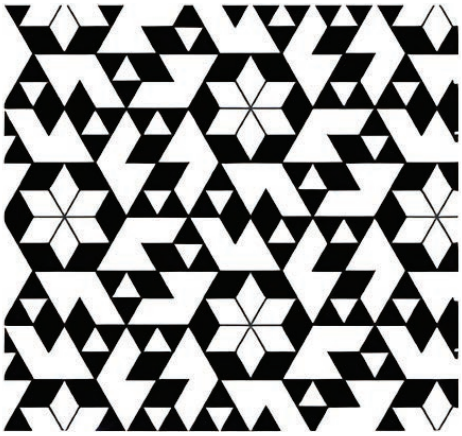
*** 3 3 3** por Eduardo Rodrigues
Escola Profissional de Recuperação do Património de Sintra
Sintra

B



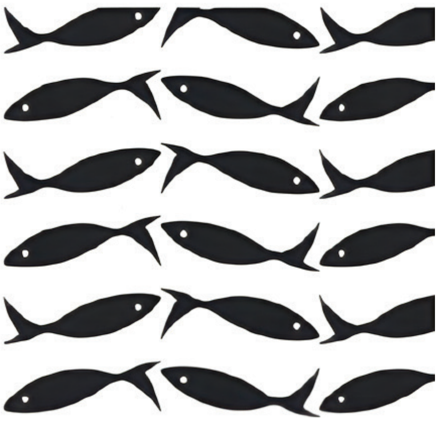
3 3 3 por Mila Loureiro
Escola Básica e Secundária da Cidadela
Cascais

C



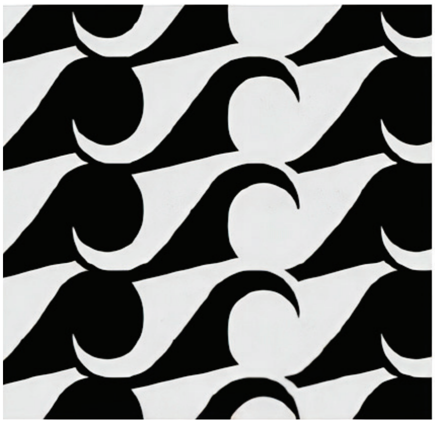
6 3 2 por Vladyslav Kravets
Escola Profissional de Recuperação do Património de Sintra
Sintra

D



2 2 X por Francisco Costa
Escola D. Filipa de Lencastre
Lisboa

E



0 por Sebastião Mendes
Escola Secundária de Alcácer do Sal
Alcácer do Sal

Honourable mention for symmetry A

Yixuan Wu, from *Escola Secundária D. Afonso Sanches, Vila do Conde*, and Vladyslav Kravets from *Escola Profissional de Recuperação do Património de Sintra*

Honourable mention for symmetry B

Hugo Cabrita Henriques, from *Externato Marista de Lisboa*

Honourable mention for symmetry C

Mila Luana de Gouveia Loureiro, from *Escola Básica e Secundária da Cidadela, in Cascais*

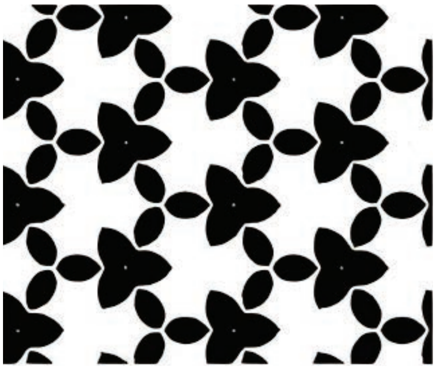
Honourable mention for symmetry D

Patrícia Martins Pessoa, from *Escola Básica e Secundária Artur Gonçalves, in Torres Vedras*

Honourable mention for symmetry E

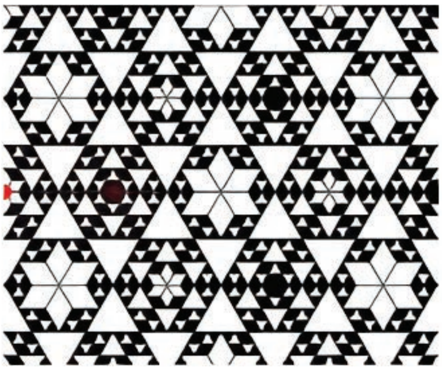
Vasco José Catarino Neves, from *Escola Básica e Secundária da Anadia* and Hugo Cabrita Henriques from *Externato Marista de Lisboa*.

A



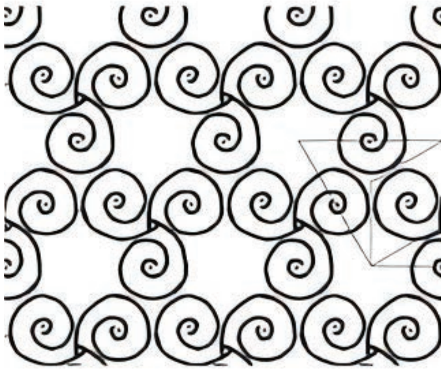
***3 3 3** por Yixuan Wu
Escola Secundária D. Afonso Sanches
Vila do Conde

A



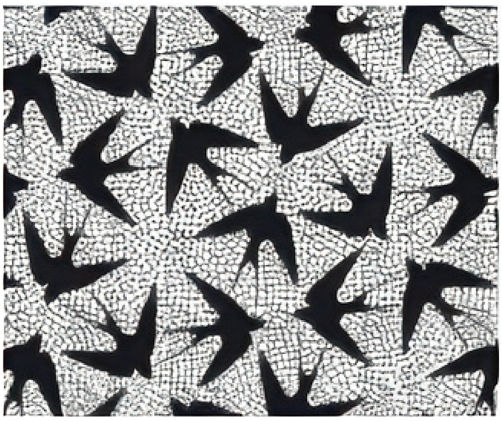
***3 3 3** por Vladyslav Kravets
Escola Profissional de Recuperação do Património de Sintra
Sintra

B



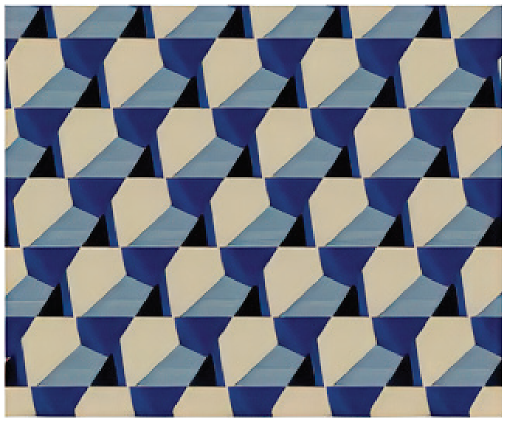
3 3 3 por Hugo Henriques
Extrenato Marista de Lisboa
Lisboa

C



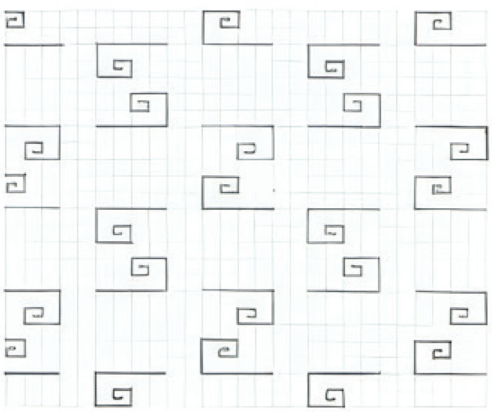
6 3 2 por Mila Loureiro
Escola Básica e Secundária da Cidadela
Cascais

E



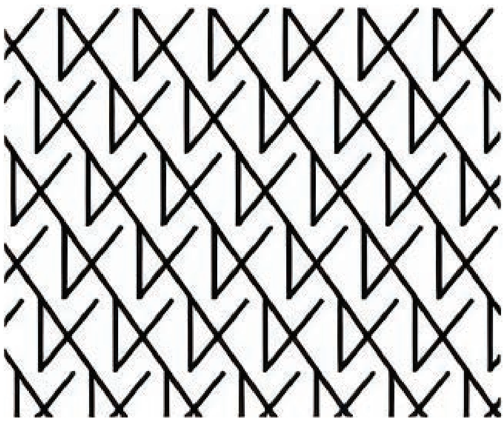
0 por Vasco Neves
Escola Básica e Secundária da Anadia
Anadia

D



2 2 X por Patrícia Martins Pessoa
Escola Básica e Secundária Artur Gonçalves
Torres Novas

E



0 por Hugo Henriques
Externato Marista de Lisboa
Lisboa

Figure 3. The winner of the C -Pattern Award, congratulated by the President of the Academy

This extraordinary collection of twelve proposals poses an enormous challenge for the city of Lisbon to complete the symmetries in its streets and squares and become the first city in the world to complete the 24 flat symmetries in its pavements. They also encourage the expansion to all regions of Portugal, where Portuguese *calçada* already exists or may come to exist, the dissemination and exploration of the algorithms of plane symmetry, whether through school activities or art competitions.

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[S} https://www.youtube.com/watch?v=k4hGCUtBG1E

[P] https://www.youtube.com/watch?v=XgJggEU5jV4

LxDS Spring School 2025

28–30 May 2025

by **Telmo Peixe***

The *LxDS Spring School 2025* was held from 28 to 30 May 2025 at *ISEG–Lisbon School of Economics & Management*, Universidade de Lisboa. Organised by the Lisbon Dynamical Systems Group (LxDS), in partnership with CEMAPRE and CEMS.UL, the event focused on key developments in dynamical systems theory. It provided a platform for early-career researchers and students to deepen their understanding through advanced lectures delivered by leading international specialists.

Over the course of three days, participants engaged in three in-depth mini-courses led by distinguished invited speakers:

Daniel Peralta-Salas (ICMAT) presented *An Introduction to 3D Euler Flows*, exploring stationary solutions of the 3D Euler equations through a dynamical systems lens. The course revisited Arnold's structure theorem and analysed Beltrami flows—an especially rich family of steady-state solutions.

Joel Moreira (University of Warwick) gave a course on *Ergodic Ramsey Theory*, introducing the audience to Furstenberg's correspondence principle and its far-reaching implications in additive combinatorics. The lectures demonstrated how dynamical systems methods

* Department of Mathematics, ISEG–UL (On behalf of the organizing comittee)

THE ORGANIZING COMMITTEE

João Lopes Dias
Universidade de Lisboa, ISEG, CEMAPRE

Pedro Miguel Duarte
Universidade de Lisboa, FCUL, CEMS.UL

José Pedro Gaivão
Universidade de Lisboa, ISEG, CEMAPRE

Telmo Peixe
Universidade de Lisboa, ISEG, CEMAPRE

Alexandre Rodrigues
Universidade de Lisboa, ISEG, CEMAPRE

Pedro Soares
Universidade de Lisboa, ISEG, CEMAPRE

can resolve long-standing combinatorial problems, using accessible proofs that required no prior expertise in the subject.

Santiago Ibáñez (Universidad de Oviedo) delivered the course *Unfolding Chaos: Singularities of Vector Fields*, which focused on how certain singularities—especially of Hopf-Zero and nilpotent type—can underlie chaotic dynamics. The lectures provided both theoretical foundations and examples of global structures (e.g. Shilnikov homoclinic orbits) responsible for the onset of chaos.

The school welcomed approximately 25 participants, including MSc and PhD students, researchers, and academic staff from various Portuguese institutions, all sharing a common interest in dynamical systems.

Complementing the main courses, the school included a session of short research talks. The following abstracts were presented:

Bruno Gonçalves (Universities of Aveiro, Porto and Minho) described bifurcation phenomena and canard dynamics in FitzHugh-Nagumo systems, including evidence of mixed-mode and chaotic mixed-mode oscillations in coupled models.

Ana Rodrigues (University of Évora) presented recent results on the existence and renormalisation of invariant curves in piecewise isometries.

Davide Azevedo (University of Minho) discussed generic ergodicity of Sobolev homeomorphisms, proving that volume-preserving maps in certain Sobolev spaces generically exhibit ergodic and topologically transitive behaviour.

Kush Kinra (FCT Nova) investigated the optimal control of stochastic third-grade fluids, establishing well-posedness and optimality conditions via a reformulation of the original stochastic system.

Each abstract tackled different aspects of dynamical systems, from analytical models of excitable media and fluid dynamics to the ergodic properties of nonlinear transformations and renormalisation techniques in piecewise dynamics.

Generous funding from the *Centro Internacional de Matemática* (CIM) allowed the school to fully support travel, accommodation, and meals for PhD student participants. This financial backing was vital in ensuring their involvement and active participation throughout the event.

In summary, the *LxDS Spring School 2025* successfully brought together a vibrant group of students, researchers, and experts. The combination of advanced courses, original research presentations, and informal discussion fostered a dynamic and collaborative atmosphere. The event significantly contributed to the dissemination and development of current research in dynamical systems, reinforcing the importance of such initiatives within the mathematical community.

More information about the event can be found at https://sites.google.com/view/lxds-ss-2025/

The National Seminar on the History of Mathematics

Thirty eight years of research and dissemination in the History of Mathematics

by **Luis Saraiva***

1 Introduction[1]

The National Seminar on the History of Mathematics (SNHM) was founded in January 1988 at a meeting held at the Department of Mathematics of the University of Coimbra, in which researchers from the Mathematics Departments of the Universities of Coimbra, Lisbon, Minho and Porto played a fundamental role.[2]

The need to create a national institution with the aim of researching and disseminating the history of Mathematics — and in particular the history of Portuguese Mathematics — originated in the commemorations of the Portuguese mathematician José Anastácio da Cunha (1744–1787)[3] on the bicentenary of his death in 1987. Colloquia were organised in Coimbra, Évora and Lisbon, with the publication of Proceedings from all three events.

The University of Coimbra published facsimile editions of da Cunha's most important work, the *Principios Mathematicos*, originally published in 1790 — three years after the author's death [Cunha, 1987a]— and of its French translation by João Manuel de Abreu (1757–1815), published in Bordeaux in 1811 by the publisher André Racle[4] [Cunha, 1987b].

Also in Lisbon, the National Library hosted an exhibition offering a broad view of da Cunha's work, including his literary output, as suggested by its title, *José Anastácio da Cunha (1744–1787) — Mathematician and Poet*. A catalogue was published, including articles on the central aspects of his mathematical and poetic work, together with a small appendix containing some of his poems [Catálogo, 1987].

The main event of this commemorative programme took place in Lisbon: the international colloquium *Anastácio da Cunha (1744–1787) — The Mathematician and the Poet*, held from 8 to 10 October 1987 in Forum Picoas. It brought to Portugal distinguished specialists in the History of Mathematics: Ivor Grattan-Guinness (1841–2014), one of the great historians of the twentieth century and a Fellow of the Royal Society of London, whose extensive body of work includes the landmark three-volume study *Convolutions in French Mathematics, 1800–1840* (approximately 1,600 pages); Ubiratan d'Ambrosio (1932–2021) of the University of Campinas (UNICAMP), the principal founder of Ethnomathematics; Enrico Giusti (1940–2024) of the University of Florence; Jean Mawhin (1942–) of the Catholic University of Louvain; and Jesus Hernandez

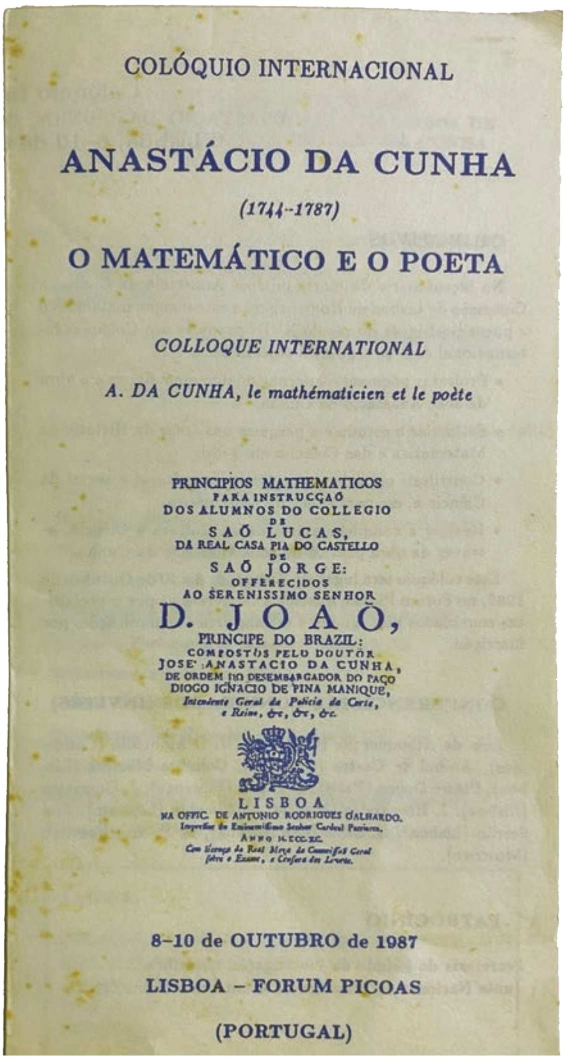
[1] A more extensive and comprehensive paper on the SNHM has been submitted for publication in the *Memórias da Academia das Ciências de Lisboa*. This text aims only to provide a general overview of the foundation and development of the Seminar.

[2] Other researchers were also present, but only those from the universities mentioned above actively contributed to the creation and functioning of the SNHM.

[3] A mathematician praised in articles by Portuguese mathematicians as in [Gonçalves, 1940] and internationally known, mainly since the publishing of the paper [Youschkevitch, 1973].

[4] In 1816, unsold copies of the Bordeaux edition were offered for sale in Paris by the publisher Veuve Courcier, with only the title pages replaced [Giusti, 1990, p. 46, note 3].

* CIUHCT, Department of Mathematics, FCUL
Financed by Fundação para a Ciência e a Tecnologia, I.P./MCTES through national funds UID/00286/2025.



Flyer for the Internacional
Proceedings of the Meeting on da Cunha
Lisbon, 1987

Proceedings of the Lisbon Meeting
Lisbon, 1990

(1944–), Professor at the Autonomous University of Madrid.

This event provided Portuguese mathematicians with the opportunity not only to attend research presentations in the history of Mathematics by leading specialists, but also to engage directly with these scholars. Over the course of the colloquium, the idea gradually emerged that, given the longstanding gap in Portugal in the field of the History of Mathematics — where only sporadic articles by mathematicians on aspects of Portuguese mathematical history had occasionally appeared — a national institution should be created with the central aim of developing research in this area, placing the study of Portuguese mathematics at the heart of its activities, and fostering continuous exchanges among the emerging community of Portuguese historians of mathematics.

The Proceedings of the Lisbon meeting [Anastácio, 1990] already reflected a concern to integrate documents from Portugal's mathematical history and make them available to interested readers. In addition to the texts of the communications, an extensive appendix of important materials was included: two writings by José Anastácio da Cunha — the *Carta Physico-Mathematica sobre a Theorica da Polvora em Geral e a determinação do melhor comprimento das peças em particular* and the *Ensaio sobre os Princípios de Mechanica*; a letter by João Manuel de Abreu (1757–1815) mentioning a list of unpublished works;[5] the texts of the polemic with José Monteiro da Rocha (1734–1819), an import-

[5] Many of these, though not all, were subsequently found in the decades following the Colloquium by researchers associated with the SNHM

Summer School on the History of Mathematics
Évora, July 1990

Some of the participants:
Front row, from the right: Jean Dhombres, first, Sergio Nobre, fourth
Back row, from the right: Ahmed Djebbar, second, Enrico Giusti, fourth; Giorgio Israel, fifth; Luis Saraiva, ninth; António Leal, tenth; Carlos Sá, eleventh; Maria Fernanda Estrada, twelfth. The last four were among the founders of the SNHM.

ant mathematician and colleague of Anastácio at the University of Coimbra who was instrumental in the crucial university reform of 1772; and the texts relating to the controversy surrounding the publication of the French translation of the *Principios Mathematicos*, including the then-known reviews published in foreign journals.[6] Of the 506 pages of the Proceedings volume, approximately one third is taken up by this appendix.

At the close of the Colloquium, some of the Portuguese participants met with Ivor Grattan-Guinness and sought his advice on the general direction the future SNHM should take. He emphasised that the Seminar should hold regular meetings where members could present and discuss their work, and that the presence of researchers from other countries should always be ensured — an essential factor not only for the progress of the SNHM but also for establishing its connection with the international network of historians of Mathematics. It was equally important for Portuguese researchers to participate in international events, particularly colloquia and workshops.

2 The Early Years of the SNHM (1988–1990)

The SNHM was founded in January 1988 with the prospect that the new institution would consolidate and develop, to fill the gap that had long existed in Portugal in the field of the History of Mathematics, both in terms of research and dissemination. Given that this was a new beginning, and making use of the work carried out in the preceding months during the Anastácio da Cunha commemorations, this mathematician, his work and his era were taken as the preferred theme for 1988. As the Seminar developed, the history of mathematics education was also incorporated among its research and dissemination themes in later years.

The SNHM started without institutional support, subsisting solely on contributions from its members. Meetings were therefore held in rooms provided by the Mathematics Departments of its members, so that no venue rental was required. For foreign speakers, researchers who happened to be in Portugal for

[6] In 2011, João Caramalho Domingues, in a paper published in issue 65 of the SPM Bulletin, added to the three known reviews published in foreign journals a fourth, made in 1816 to the volume published in Paris [Caramalho, 2011].

José Anastácio da Cunha Colloquium
Braga, 2006

Seven of the founding members of the SNHM with Professors Grattan-Guinness and Jean Mawhin.

From left to right:
Front row: Luis Saraiva, Maria Fernanda Estrada, António Leal, Jaime Carvalho e Silva and Ivor Grattan-Guinness. *Back row*: João Queiró, José Francisco Rodrigues, Jean Mawhin and Carlos Sá

some reason were approached: on occasion they had been invited to give lectures at one of our faculties, meaning their travel costs were covered by the host departments, so no additional travel expenses were necessary. Once in Portugal, either SNHM members transported them in their own vehicles, or the necessary train or coach tickets were purchased.

The first Seminar meeting took place in Braga in April 1988, at the *University of Minho*, with foreign guest Professor Ubiratan D'Ambrosio, one of the participants at the Lisbon Colloquium, who was in Portugal at the time.

The second meeting took place in Lisbon in November 1988, at the *Institute of Physics and Mathematics of the University of Lisbon*, with Professor Christian Houzel of the University of Paris XIII as guest, taking advantage of his visit to Lisbon to participate in an *Autumn School* organised by the *Portuguese Mathematical Society* (SPM) at the *Instituto Franco-Português*. A member of the Seminar had met Professor Houzel in September of that year at a History of Mathematics meeting in Cetrano, Italy (*Giornate di Storia della Matematica*), and knowing of his upcoming visit to Lisbon, had invited him to participate in the SNHM meeting. Due to an unexpected problem arising during the SPM event, Professor Houzel's lecture, originally planned for the SNHM meeting, had to be delivered during the *Autumn School* instead.

The third meeting took place at the *Faculty of Sciences of the University of Lisbon* in March 1989, with Professor Jean Dhombres, then of the University of Nantes, as the foreign guest — again making use of his visit to Lisbon to give a lecture at the Department of Mathematics of that faculty.

The aims of the SNHM are in accordance with the fundamental principles and objectives of the SPM, and it was therefore natural that the SNHM should seek integration into the SPM's structure. This took place on 16 November of that year, when it became an autonomous section of the SPM.

A fourth meeting took place in April 1990, with the participation of Professor Ivor Grattan-Guinness, at the Department of Mathematics of the *University of Coimbra*. Professor Grattan-Guinness came to Portugal with grants from the *Lisbon Academy of Sciences* and the *London Royal Society*. In addition to his participation in the SNHM meeting, he delivered a series of 14 lectures in Lisbon, Coimbra, Évora and Braga, in collaboration with various departments of the Universities of Lisbon, Coimbra, Évora and Minho.

The founders of the SNHM were all holders of doctorates in mathematics but, for the most part, lacked specific training in the literary and historical analysis of topics in the History of Mathematics. At the colloquia held during the Anastácio da Cunha commemorations, they had been able to observe how different historical research and analysis was from the mathematical work to which they were accustomed. The need was felt for training provided by specialists in this area. To this end, Professor Jean Dhombres was approached, and he offered to put together a small team of specialists to come to Portugal that year (1990) to hold a *Summer School in the History of Mathematics*.

Together with Professor Dhombres came three

6th Luso-Brazilian Meeting on the History of Mathematics,
S. João del Rei, Minas Gerais, August 2011

distinguished historians of Mathematics: Professor Enrico Giusti, who had previously been in Lisbon at the Colloquium on Anastácio da Cunha; Professor Giorgio Israel (1945–2015) of the University of Rome La Sapienza; and Professor Ahmed Djebbar (1941–) of the University of Lille. They brought not only abundant written material for distribution among the participants, but also some of their doctoral students, who gave presentations alongside Portuguese participants. The Summer School took place from 16 to 21 July at the *Colégio da Herdade da Mitra* in Évora.

This meeting was also significant for another reason. One of the participants was Sergio Nobre, at the time a Brazilian doctoral student working in Leipzig, in what was then East Germany, under the supervision of the distinguished historian of Mathematics Hans Wussing (1927–2011). He came to Évora on the recommendation of Professor Ubiratan D'Ambrosio, who had kept abreast of developments in Portugal and, on seeing who was organising the Summer School, immediately contacted Sergio Nobre. He told him that the *Summer School* would likely be of a very high standard and that he should attend, as it would be an excellent opportunity to enrich his training. This was the beginning of a personal and professional relationship with the historians of Portuguese Mathematics and with the SNHM, which proved fruitful, as we shall see, and continues to the present day.

3 The SNHM from 1993 to the Present

3.1 Introduction

In 1991 and 1992 there were no SNHM activities. This was very likely due primarily to the fact that some of its most active members were abroad working at universities, and there was insufficient motivation among those who remained in the country to resume the

4th Iberian Meeting on the History of Mathematics
Leiria Museum, June 2023

38th SNHM Meeting
Coimbra, June 2025

Seminar's activities. From 1993, however, the work of the Seminar resumed, and since then there have been no further years without activity.

Until the mid-1990s the Seminar had no formal regulations, operating instead by consensus among its members. During the presidency of Professor Graciano Neves de Oliveira at the SPM (1996–2000), he made us understand that if the SNHM wished to expand, it was necessary to have a set of regulations. We accordingly drafted these and submitted them to the members of the Seminar, who approved them. They broadly followed the recommendations of Grattan-Guinness, to which we added elements drawn from our own experience within the SNHM. In this document, the Seminar is defined as an autonomous section of the SPM, whose central aim is to develop research in the history of Mathematics, and especially in the History of Portuguese Mathematics, serving as a point of connection and support for historians of Portuguese mathematics.

The SNHM has a General Council, which currently has 27 members, of whom 7 are founders, and which includes researchers from the Universities of Aveiro, Coimbra, Évora, Lisbon, Minho, Nova de Lisboa, Porto and Trás-os-Montes e Alto Douro, as well as the Naval School, the Polytechnic of Leiria and the Polytechnic of Viseu. The General Council is responsible for electing the Secretariat, which consists of three members, one of whom serves as General Coordinator of the SNHM, for periods of three years. Elections are currently held by electronic vote. Lists standing for the Secretariat must present an action programme for their term, which is sent to all members of the General Council. The Secretariat directs the activities of the SNHM during its term of office. At the end of the term, the Secretariat must send a report on its activities over the three-year period to members of the General Council. Every six years, the General Council should review its composition and may adjust if it is determined that any member has, unambiguously and consistently, ceased to contribute to the activities of the SNHM. Membership of the General Council is achieved by a substantiated proposal from one of its members, accompanied by the candidate's CV, followed by a vote.

Since its integration as an autonomous section of the SPM, even before regulations were formally established, the SPM has always provided informal support to the Seminar's activities. From the period of Nuno Crato's presidency of the SPM (2004–2010) onwards, an annual allocation for SNHM expenses has been established, essentially to cover the travel and accommodation costs in Portugal of a foreign speaker for one of its meetings. Due to budgetary constraints, only speakers based in Europe can be invited.

Location	Frequency	Institution
Lisbon	9	Museum of Science of UL – 4 / Interdisciplinary Institute of UL (former IFM) – 3 / Faculty of Sciences, University of Lisbon – 1 / Academy of Sciences of Lisbon/National Museum of Natural History and Science (MUNHAC) – 1
Coimbra	8	Department of Mathematics, UC
Aveiro	4	Department of Mathematics, UA
Porto	2	Faculty of Sciences, University of Porto
Évora	2	Department of Mathematics, UE
Monte da Caparica	2	Faculty of Sciences and Technology, UNL
Alfeite	2	Naval School
Amadora	1	Military Academy
Braga	1	University of Minho
Castelo Branco	1	Higher School of Education
Gouveia	1	Municipal Library Auditorium
Leiria	1	Online meeting (due to Covid-19 pandemic)
Marco de Canavezes	1	Municipal Auditorium
Mértola	1	Auditorium of the Natural Park of the Guadiana Valley
Santiago do Cacém	1	Municipal Auditorium
Viseu	1	Higher School of Education

Table 1. Locations of SNHM meetings – 1988–2025

Name	Country	Institution	No. of Meetings
Ubiratan D'Ambrosio	Brazil	UNICAMP / University of São Paulo	4
Eberhard Knobloch	Germany	Technische Universität Berlin	3
Ivor Grattan-Guinness	England	London Royal Society	3
José Chabas	Spain	University Pompeu Fabra, Barcelona	3
Jean Dhombres	France	University of Nantes / CNRS	2
Jens Høyrup	Denmark	Roskilde University	2
Luis Español	Spain	University of La Rioja, Logroño	2
Maria Isabel Maroto	Spain	University of Valladolid	2
Mariano Esteban Piñeiro	Spain	University of Valladolid	2
Sergio Nobre	Brazil	UNESP, Rio Claro	2
Ugo Baldini	Italy	University of Padua	2
Vitor Navarro Brotons	Spain	University of Valencia	2
Wagner Valente	Brazil	Federal University of São Paulo	2

Table 2. Foreign guests with more than one participation

3.2 The National Meetings

Since 1993, SNHM meetings have been held regularly — 34 in 33 years. In all of them, as stipulated in the Regulations, there has always been at least one foreign guest. Over the 38 meetings held to date, 59 non-Portuguese researchers from 14 countries have presented 77 communications. Brazil has the highest number of participants (10), followed by France (9), Germany, Spain and England (all with 8) and Italy (5). Outside Europe, the United States (3), Canada (2), Argentina (1), Algeria (1) and the People's Republic of China (1) are represented. The number of participants changes from meeting to meeting, but most frequently there are between 30 and 50 people attending each event.

Whenever possible, we have sought to hold Seminar meetings outside the major cities, in locations where historians of Mathematics were active, so that these events might contribute to disseminating the history of Mathematics in those places. Between 2013 and 2023, we consecutively held 11 meetings outside the three largest cities: Coimbra, Lisbon and Porto. On two occasions, we held meetings in the birthplace of prominent mathematicians: José Sebastião e Silva (1914–1972) in Mértola in 2014, and José Monteiro da Rocha in Marco de Canavezes in 2019. Both intended to celebrate the two mathematicians, the former on the 100th anniversary of his birth, the later on the 200th anniversary of his passing.

The next meeting, the 39th, is scheduled at the Naval School, on 8 and 9 May 2026.

3.3 The Luso-Brazilian Meetings and the Iberian Meetings on the History of Mathematics

Sergio Nobre and Ubiratan D'Ambrosio were part of the main driving force in the development of research and dissemination of the history of Mathematics in Brazil. They were instrumental in creating the *Brazilian Seminars on the History of Mathematics*, which began in 1995, and later led to the founding of the *Brazilian Society for the History of Mathematics* (SBHMat). Sergio Nobre was also one of the founders of the *Brazilian Journal of the History of Mathematics* in 2001.

The collaboration between Brazilian and Portuguese historians led to the creation of the *Luso-Brazilian Meetings on the History of Mathematics*, with the first being organised in Coimbra in 1993. Since then, they have been held periodically and alternately in Brazil and Portugal. Following Coimbra, in August/September 1993, eight further meetings have taken place: Águas de São Pedro, São Paulo, March 1997; Coimbra, February 2000; Natal, Rio Grande do Norte, October 2004; Castelo Branco, October 2007; São João del Rei, Minas Gerais, August 2011; Óbidos, October 2014; Foz do Iguaçu, Paraná, August 2018; Setúbal, October 2022. All meetings were very well attended; those held in Brazil always had more than 100 participants, those in Portugal slightly less. All meetings, with the exception of the third (Coimbra, 2000), have their Proceedings published. Regarding the missing Proceedings, the local organiser has promised to publish them in a near future, as he holds most of the texts of the communications delivered. From the 5th Meeting (Castelo Branco, 2007) onwards, a peer review system was introduced to ensure the quality of articles included in the Proceedings. Those of the 5th Meeting were generously funded by the Municipality of Castelo Branco. Those subsequently published in Portugal (Óbidos 2014, and Setúbal 2022) were funded from registration fee revenue, so their publication entailed no cost to the SPM. Print runs were limited, primarily for the editors, for libraries in Portugal and Brazil, for plenary speakers, and for the SPM, which retained the surplus copies of each publication. To give an idea of the scope of these Proceedings, we note the page counts for the last three meetings held in Portugal: 5th Meeting, Castelo Branco, 570 pages; 7th Meeting, Óbidos 2014, two volumes, 598+574 pages; 9th Meeting, Setúbal 2022, 688 pages. The next meeting, the 10th, will take place in Belém do Pará, from 11 to 14 August 2026.

Following the centenary celebrations of the *Royal Spanish Mathematical Society* in Ávila in 2011, Spanish and Portuguese researchers in attendance decided to launch the *Iberian Meetings on the History of Mathematics*, to be held periodically every three years, alternating between Spain and Portugal. Four meetings have been held to date: Santiago de Compostela, January 2013; Coimbra, July 2016; Seville, July 2019; Leiria, June 2023 (the latter was originally planned for 2022 but had to be postponed by one year due to the Covid-19 pandemic). Attendance at these meetings has ranged between 25 and 40 participants. Unlike the *Luso-Brazilian Meetings*, these gatherings have not produced published Proceedings. Only for the 4th Meeting extended abstracts were published as a special issue of the *Bulletin* of the SPM. We hope to continue with this format for future meetings. The venue and date for the 5th Meeting have not yet been determined, but the event is expected to be held in 2027, as we have the Luso-Brazilian meeting this year and our Spanish colleagues also have a major event in 2026.

4 Concluding Remarks

A few final notes to complement what has been said above regarding our meetings. All participants at SNHM events receive a booklet whose cover is the meeting's poster and which contains the programme, a map of the city where it is held, information about the invited speaker(s), a note on the history of the SNHM with a list of all meetings held, including those of the Luso-Brazilian and Iberian series, and the abstracts of the communications. In most cases, this booklet has between 32 and 36 pages.

Since the 23rd Meeting, held in Évora in 2010, a *Supplement* to the SPM *Bulletin* has been produced with extended abstracts of the communications from each meeting. Each volume is between 60 and 80 pages. The SPM decided to discontinue the print edition of the *Bulletin*, and consequently of its *Supplement* as well. Issue No. 80 of December 2022 was the last to appear in print. The SNHM decided to continue producing limited print runs of the *Supplement*, funded voluntarily by meeting participants. We thus produced a print run of *Supplement* No. 82 containing the extended abstracts from the 35th and 36th Meetings (164 pages), and we are currently preparing a limited print run with the extended abstracts from the 37th and 38th Meetings. As said above, we followed the same approach for the *4th Iberian Meeting*, whose extended abstracts constitute issue No. 81 of the SPM *Bulletin* (112 pages), with printing funded by the Spanish and Portuguese participants who wished to acquire a copy. We always print a surplus of copies to supply our partner institutions, including, of course, the SPM. We understand the budgetary constraints that led to the SPM Board's decision, but we consider essential that printed copies of the *Supplement* continue to be produced.

There is much more that could be said about the activities of Portuguese historians of Mathematics and Mathematics Education during the period 1988–2025, the centres they established, the work they produced, and the meetings they organised. In the article to be published in the *Memórias da Academia das Ciências*, I develop this topic, which I have not addressed here. We maintain the same commitment to working for the research and dissemination of the history of Mathematics, including the history of Mathematics Education, with which we began the SNHM in 1988–38 years ago now. The Seminar is open to all who wish to learn and collaborate in these endeavours.

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Second Atlantic Conference in Nonlinear PDEs

3–7 November 2025

by **Simão Correia***, **Diogo Oliveira e Silva***, **João Pedro Ramos**** and **Hugo Tavares***

The second edition of the *Atlantic Conference in Nonlinear Partial Differential Equations* took place at Instituto Superior Técnico (IST), Lisbon, from November 3 to 7, 2025. The event brought together nearly 100 researchers from around the world to discuss recent advances in harmonic analysis, dispersive PDEs, and elliptic PDEs.

The program featured 17 plenary speakers from eight countries and 38 contributed talks in three thematic sessions, with speakers representing 17 countries across 5 continents. A lively poster session showcased the work of graduate students and postdoctoral researchers, sparking animated discussions that continued well beyond the coffee breaks.

A conference dinner at Zambeze Restaurant, overlooking the Tejo, crowned the social program. The atmosphere remained sunny throughout—at least among the participants, if not always in the Lisbon skies.

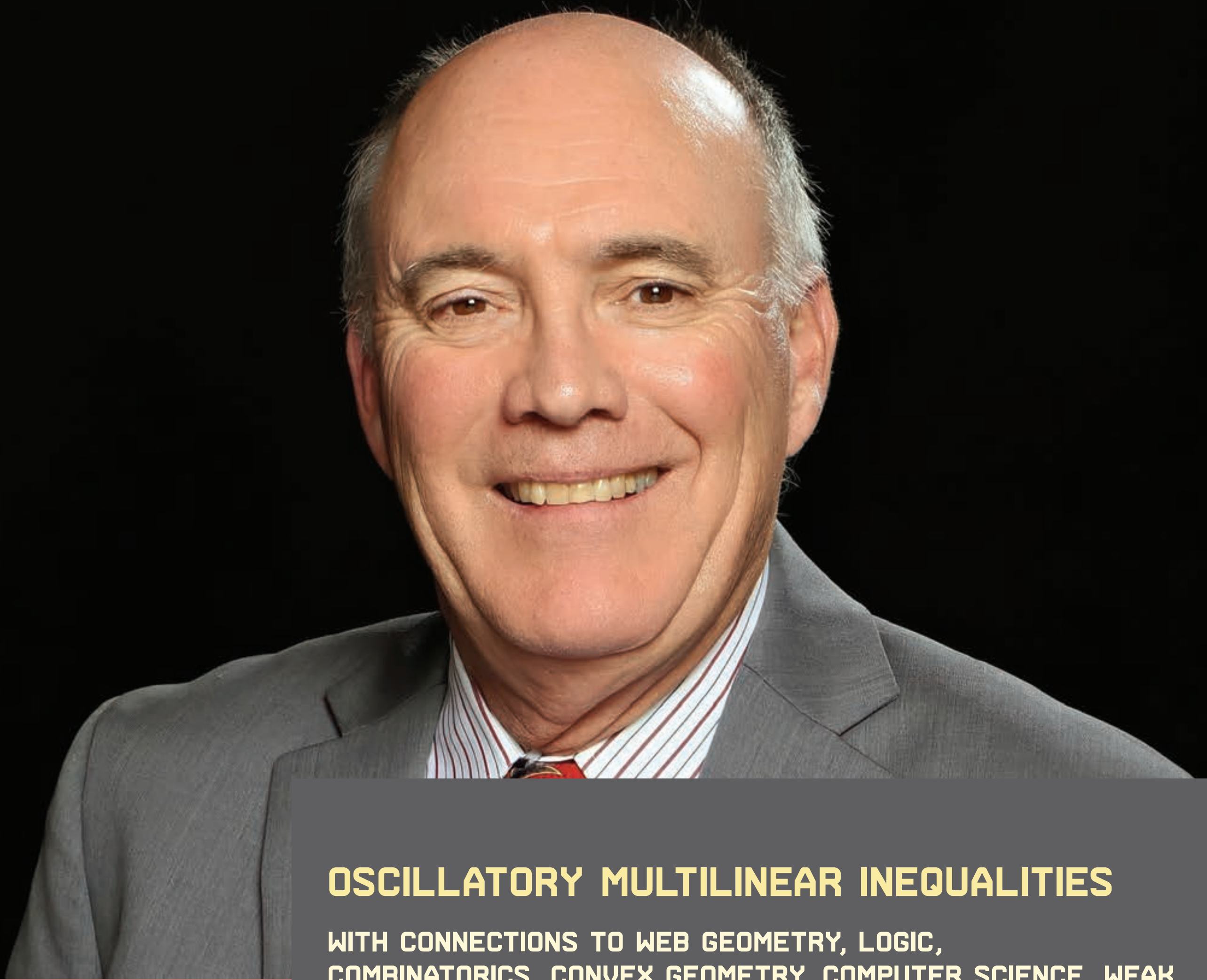
The event was funded by the *Basque Center for Applied Mathematics*, CAMGSD, CEMS.UL, CIM, DM-IST, FCT, FLAD, *Institut Universitaire de France*, *Severo Ochoa/María de Maeztu Excellence* and *Université de Versailles Saint-Quentin-en-Yvelines/Université Paris-Saclay*.

The third edition of the conference is tentatively scheduled for Fall 2027, once again at IST. See you there!

* CAMGSD & Instituto Superior Técnico
** IMPA

PEDRO NUNES LECTURES

MICHAEL CHRIST
UC BERKELEY



OSCILLATORY MULTILINEAR INEQUALITIES

WITH CONNECTIONS TO WEB GEOMETRY, LOGIC, COMBINATORICS, CONVEX GEOMETRY, COMPUTER SCIENCE, WEAK CONTINUITY, ERGODIC THEORY, AND MORE

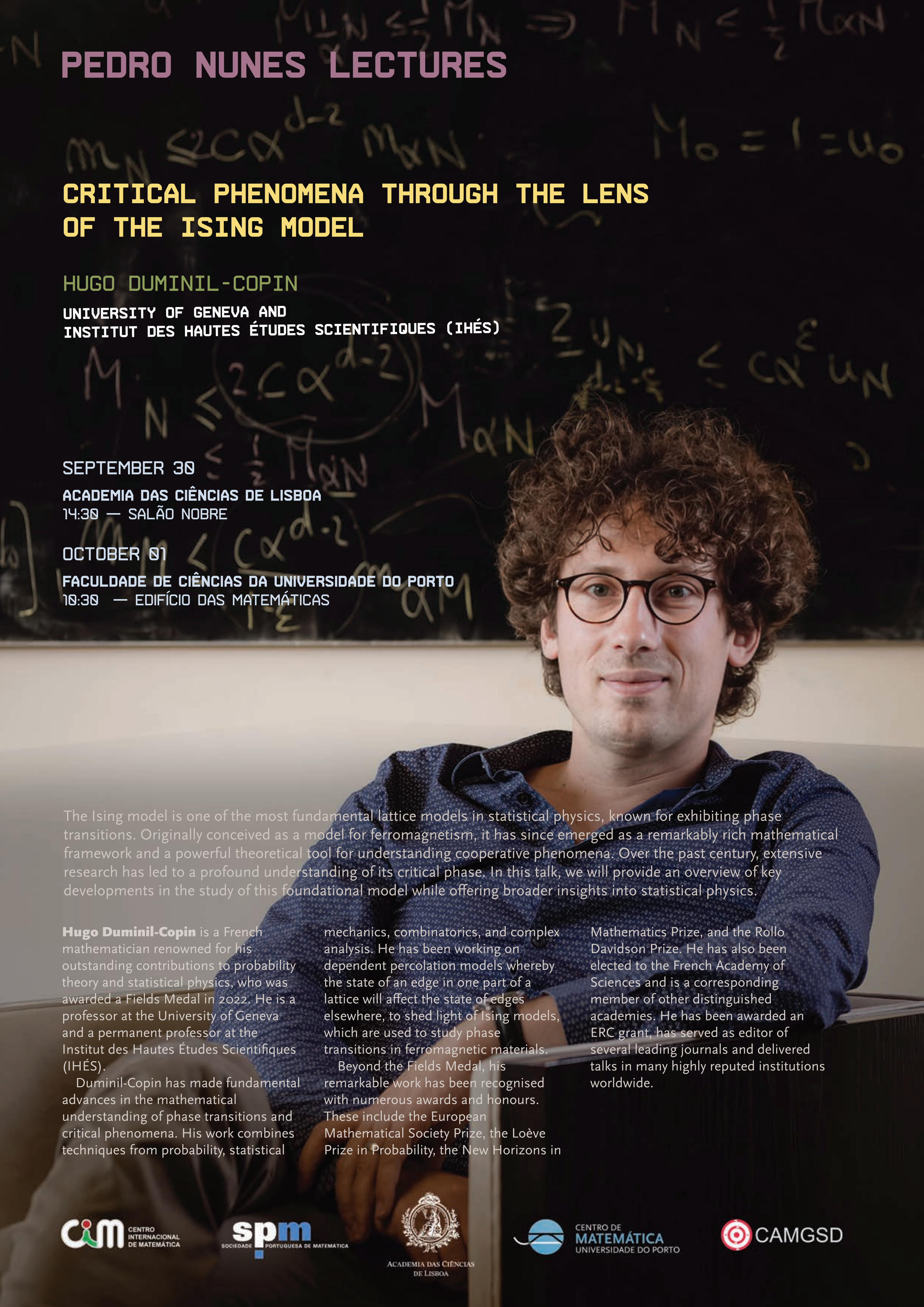
MAY 7 — 2025
INSTITUTO SUPERIOR TÉCNICO
14H — ANFITEATRO ABREU FARO

MAY 9 — 2025
DEPARTAMENTO DE MATEMÁTICA DA FCUP
14.30 — SALA 0.07

Michael Christ (born 7 June 1955) is an American mathematician and Emeritus Professor at the University of California, Berkeley.

His research has Fourier analysis at its core and encompasses partial differential equations, complex analysis in several variables, and topics in mathematical physics. Among his many outstanding research contributions, we highlight the definitive analysis of global regularity of solutions of ∂-bar problems on pseudoconvex domains, the proof (with Colliander and Tao) of the ill-posedness of low-regularity solutions of the nonlinear Schrödinger equations, and the characterization (with Kiselev) of the absolutely continuous spectrum and generalized eigenfunctions for second-order ODE with potentials on the real line.

Michael has been an invited lecturer twice at the International Congress of Mathematicians, first in Kyoto in 1990 and then in Berlin in 1998. He has received numerous honors and awards, including an NSF Presidential Young Investigator Award and a Sloan Fellowship, the Bergman Prize from the AMS, and a Miller Research Professorship. He was elected to the American Academy of Arts and Sciences in 2007.



PEDRO NUNES LECTURES

CRITICAL PHENOMENA THROUGH THE LENS OF THE ISING MODEL

HUGO DUMINIL-COPIN

UNIVERSITY OF GENEVA AND
INSTITUT DES HAUTES ÉTUDES SCIENTIFIQUES (IHÉS)

SEPTEMBER 30
ACADEMIA DAS CIÊNCIAS DE LISBOA
14:30 — SALÃO NOBRE

OCTOBER 01
FACULDADE DE CIÊNCIAS DA UNIVERSIDADE DO PORTO
10:30 — EDIFÍCIO DAS MATEMÁTICAS

The Ising model is one of the most fundamental lattice models in statistical physics, known for exhibiting phase transitions. Originally conceived as a model for ferromagnetism, it has since emerged as a remarkably rich mathematical framework and a powerful theoretical tool for understanding cooperative phenomena. Over the past century, extensive research has led to a profound understanding of its critical phase. In this talk, we will provide an overview of key developments in the study of this foundational model while offering broader insights into statistical physics.

Hugo Duminil-Copin is a French mathematician renowned for his outstanding contributions to probability theory and statistical physics, who was awarded a Fields Medal in 2022. He is a professor at the University of Geneva and a permanent professor at the Institut des Hautes Études Scientifiques (IHÉS).

Duminil-Copin has made fundamental advances in the mathematical understanding of phase transitions and critical phenomena. His work combines techniques from probability, statistical mechanics, combinatorics, and complex analysis. He has been working on dependent percolation models whereby the state of an edge in one part of a lattice will affect the state of edges elsewhere, to shed light of Ising models, which are used to study phase transitions in ferromagnetic materials.

Beyond the Fields Medal, his remarkable work has been recognised with numerous awards and honours. These include the European Mathematical Society Prize, the Loève Prize in Probability, the New Horizons in Mathematics Prize, and the Rollo Davidson Prize. He has also been elected to the French Academy of Sciences and is a corresponding member of other distinguished academies. He has been awarded an ERC grant, has served as editor of several leading journals and delivered talks in many highly reputed institutions worldwide.

CENTRO INTERNACIONAL DE MATEMÁTICA · SOCIEDADE PORTUGUESA DE MATEMÁTICA · ACADEMIA DAS CIÊNCIAS DE LISBOA · CENTRO DE MATEMÁTICA UNIVERSIDADE DO PORTO · CAMGSD