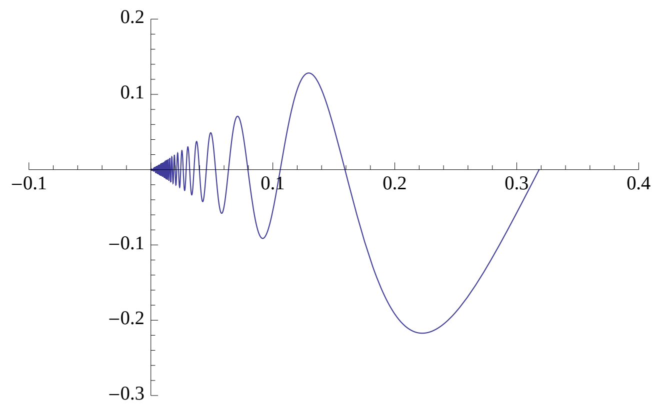
Analysis 1, Solutions to Übungsblatt Nr. 9

Problem 1 (Continuity)

(a) Here's what a graphical representation of the function f looks like:



As expected, the only problematic points are $x = 0$ and $x = \frac{1}{\pi}$, since the function f at all other points is either (i) constant and therefore continuous (such is the case if $x < 0$ or $x > \frac{1}{\pi}$), or (ii) given by the product of continuous functions and therefore continuous (such is the case if $0 < x < \frac{1}{\pi}$).

At $x = 0$ we have that

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x \cdot \sin\left(\frac{1}{x}\right) = \lim_{x \to 0^+} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{t \to +\infty} \frac{\sin t}{t} = 0.$$

This last limit follows since the function $x \mapsto \sin x$ is bounded from below and from above (by -1 and 1, respectively) on the whole real line, and $\lim_{t \to +\infty}(1/t) = 0$. As a consequence,

$$\lim_{x \to 0^+} f(x) = 0 = \lim_{x \to 0^-} f(x),$$

and the function f is continuous at 0.

If $x = \frac{1}{\pi}$, then

$$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^-} x \cdot \sin\left(\frac{1}{x}\right) = \frac{1}{\pi}\sin(\pi) = \frac{0}{\pi} = 0 = \lim_{x \to \pi^+} f(x),$$

and the function f is continuous at $\frac{1}{\pi}$.

We conclude that the function f is continuous at every point of its domain, and is therefore a continuous function.

(b) We want to show that f is not of bounded variation. Since the function f can only be nonzero on the bounded interval $(0, \frac{1}{\pi})$, this amounts to showing that f is not locally of bounded variation. Aiming at a contradiction, suppose it is. By definition, this means that

$$f = f_1 - f_2,$$

for some monotonically increasing functions f_1 and f_2. Since the functions f_1 and f_2 are monotonically increasing, the quantities

$$V(f_1) := \sup_P \sum_{i=1}^n |f_1(x_i) - f_1(x_{i-1})| \text{ and } V(f_2) := \sup_P \sum_{i=1}^n |f_2(x_i) - f_2(x_{i-1})|$$

are finite. (In both cases, the supremum is taken over all finite partitions $P = \{0 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = \frac{1}{\pi}\}$ of the interval $[0, \frac{1}{\pi}]$.) To see why this is the case, note that $x_j < x_{j+1}$ implies $f_1(x_j) \leq f_1(x_{j+1})$ because of monotonicity, and similarly for f_2. It follows that, for any partition $0 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = \frac{1}{\pi}$ of the interval $[0, \frac{1}{\pi}]$,

$$\sum_{i=1}^{n} |f_1(x_i) - f_1(x_{i-1})| = (\cancel{f_1(x_1)} - f_1(x_0)) + (\cancel{f_1(x_2)} - \cancel{f_1(x_1)}) + \ldots + (f_1(x_n) - \cancel{f_1(x_{n-1})})$$
$$= f_1(x_n) - f_1(x_0) = f_1(1/\pi) - f_1(0)$$

since the sum telescopes. Since this value is independent of the chosen partition, and since $f_1(1/\pi) - f_1(0) < \infty$, we conclude that

$$V(f_1) = f_1(1/\pi) - f_1(0) < \infty.$$

Similarly for $V(f_2) < \infty$. Using the triangle inequality, it can be shown that

$$V(f) = V(f_1 - f_2) = V(f_1 + (-f_2)) \leq V(f_1) + V(-f_2) = V(f_1) + V(f_2) = (f_1(1/\pi) - f_1(0)) + (f_2(1/\pi) - f_2(0)) < \infty.$$

On the other hand, we claim that $V(f) = \infty$. This is in contradiction to the previous line. The absurd resulted from assuming that f is of (local) bounded variation. So we will be done once we verify the claim.

With that purpose in mind, define the sequence

$$\alpha_k = \frac{1}{\frac{\pi}{2} + \pi k},$$

choose a number $n \in \mathbb{N}$ (say, even) and consider the following partition of the interval $[0, \frac{1}{\pi}]$:

$$P_n := \Big\{0 = x_0 < \alpha_n < \alpha_{n-1} < \ldots < \alpha_3 < \alpha_2 < x_n = \frac{1}{\pi}\Big\}.$$

In other words, $x_0 = 0$, $x_k = \alpha_{n-(k-1)}$ if $k \in \{1, 2, \ldots, n-1\}$ and $x_n = \frac{1}{\pi}$. For the partition P_n, we have that

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + |f(x_3) - f(x_2)| + \ldots + |f(x_{n-1}) - f(x_{n-2})| + |f(x_n) - f(x_{n-1})|$$
$$= \left|\frac{1}{\frac{\pi}{2} + \pi n} \cdot 1 - 0\right| + \left|\frac{1}{\frac{\pi}{2} + \pi(n-1)} \cdot (-1) - \frac{1}{\frac{\pi}{2} + \pi n} \cdot 1\right| + \left|\frac{1}{\frac{\pi}{2} + \pi(n-2)} \cdot 1 - \frac{1}{\frac{\pi}{2} + \pi(n-1)} \cdot (-1)\right| + \ldots$$
$$\ldots + \left|\frac{1}{\frac{\pi}{2} + 2\pi} \cdot 1 - \frac{1}{\frac{\pi}{2} + 3\pi} \cdot (-1)\right| + \left|0 - \frac{1}{\frac{\pi}{2} + 2\pi} \cdot 1\right|$$
$$\geq \frac{1}{\frac{\pi}{2} + \pi n} + \frac{2}{\frac{\pi}{2} + \pi n} + \frac{2}{\frac{\pi}{2} + \pi(n-1)} + \ldots + \frac{2}{\frac{\pi}{2} + 3\pi} + \frac{1}{\frac{\pi}{2} + 2\pi}.$$

Thus we have the lower bound

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \geq \sum_{k=2}^{n} \frac{1}{\frac{\pi}{2} + k\pi} \geq \frac{1}{\pi} \sum_{k=2}^{n} \frac{1}{k+1}.$$

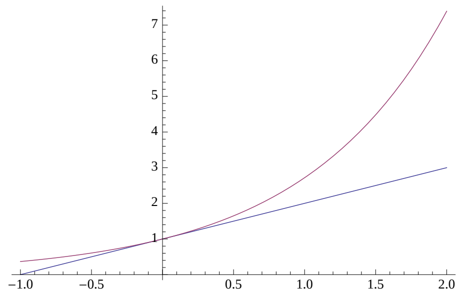
By choosing n arbitrarily large, we see that the sum on the (LHS) of this chain of inequalities, corresponding to the partition P_n, can become arbitrarily large (this uses a fact that should be well-known by now: *the harmonic series diverges*). It follows that $V(f) = \infty$, as claimed.

Problem 2 (Convexity and Hölder's inequality)

(a) For the first time in this course we utilize the exponential function, $x \mapsto \exp(x) := e^x$. More precisely, we make use of the following bound

$$1 + x \leq e^x, \tag{1}$$

which is valid for any $x \in \mathbb{R}$. Estimate (1) is graphically clear:



and it can be proved with the help of Bernoulli's inequality (Problem 4(a), PB 5). The change of variables $x \to x - 1$ translates bound (1) into

$$x \leq e^{x-1}, \ \forall x \in \mathbb{R}. \tag{2}$$

Applying this bound to each of the multiplicands x_j for $j = 1, 2, \ldots, n$, we find

$$x_j \leq e^{x_j - 1} \text{ and so } x_j^{\lambda_j} \leq e^{\lambda_j x_j - \lambda_j}.$$

Multiplying these bounds together (for $1 \leq j \leq n$) yields

$$x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n} \leq e^{\lambda_1 x_1 - \lambda_1} e^{\lambda_2 x_2 - \lambda_2} \ldots e^{\lambda_n x_n - \lambda_n} = \exp\Big(\sum_{j=1}^n \lambda_j x_j - \sum_{j=1}^n \lambda_j\Big) = \exp\Big(\sum_{j=1}^n \lambda_j x_j - 1\Big),$$

and so

$$x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n} \leq \exp\Big(\sum_{j=1}^n \lambda_j x_j - 1\Big). \tag{3}$$

This means that the quantity $R^{(\lambda)}(x_1, x_2, \ldots, x_n) := \exp\Big(\sum_{j=1}^n \lambda_j x_j - 1\Big)$ is an upper bound for the geometric mean $G := x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n}$. Bound (2) with $x := \sum_{j=1}^n \lambda_j x_j$ implies

$$\sum_{j=1}^n \lambda_j x_j \leq \exp\Big(\sum_{j=1}^n \lambda_j x_j - 1\Big) = R^{(\lambda)}(x_1, x_2, \ldots, x_n),$$

and so $R^{(\lambda)}$ is *also* a bound for the arithmetic mean $A := \sum_{j=1}^n \lambda_j x_j$. Combined, these two inequalities yield

$$\max\{x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n}, \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n\} \leq \exp\Big(\sum_{j=1}^n \lambda_j x_j - 1\Big). \tag{4}$$

Our goal is to show that $G \leq A$, and it may seem a bit surprising to know that we can do it from a bound for the *maximum* between A and G (which is all that (4) gives). Once again (recall the proof of Cauchy-Schwarz from ÜB 6), normalization comes to the rescue! Consider new variables

$$a_k := \frac{x_k}{A} \text{ where } A = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n,$$

and apply bound (3) to these new variables. One gets that

$$\left(\frac{x_1}{A}\right)^{\lambda_1}\left(\frac{x_2}{A}\right)^{\lambda_2}\cdots\left(\frac{x_n}{A}\right)^{\lambda_n} \le \exp\Big(\Big\{\sum_{j=1}^{n}\lambda_j\frac{x_j}{A}\Big\}-1\Big). \tag{5}$$

The right-hand side of this last inequality equals

$$\exp\Big(\Big\{\sum_{j=1}^{n}\lambda_j\frac{x_j}{A}\Big\}-1\Big) = \exp\Big(\Big\{\frac{1}{A}\sum_{j=1}^{n}\lambda_j x_j\Big\}-1\Big) = \exp\Big(\frac{A}{A}-1\Big) = \exp(0) = 1,$$

and so (5) translates into

$$\left(\frac{x_1}{A}\right)^{\lambda_1}\left(\frac{x_2}{A}\right)^{\lambda_2}\cdots\left(\frac{x_n}{A}\right)^{\lambda_n} \le 1,$$

which is equivalent to

$${x_1}^{\lambda_1}{x_2}^{\lambda_2}\cdots{x_n}^{\lambda_n} \le A^{\lambda_1+\lambda_2+\ldots+\lambda_n}.$$

Since $\sum_{j=1}^{n}\lambda_j = 1$, this means that

$$G \le A.$$

The proof is complete.

(b) Let $p, q > 1$ be such that $\frac{1}{p}+\frac{1}{q} = 1$. Note in particular that $pq = p+q$. From part (a) we know that, for any positive real numbers $a, b > 0$,

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q.$$

In particular, for any $\alpha > 0$, this implies that

$$a_k b_k = (\alpha a_k)\Big(\frac{b_k}{\alpha}\Big) \le \frac{1}{p}\alpha^p a_k^p + \frac{1}{q}\frac{b_k^q}{\alpha^q}.$$

Summing this over $k \in \{1, 2, \ldots, n\}$ yields

$$\sum_{k=1}^{n} a_k b_k \le \frac{\alpha^p}{p}\sum_{k=1}^{n} a_k^p + \frac{1}{q\alpha^q}\sum_{k=1}^{n} b_k^q. \tag{6}$$

We still have the freedom to choose $\alpha > 0$, and we will do it in such a way as to equalize[1] both summands of the convex combination which is the right-hand side of inequality (6). In other words, choose α such that

$$\alpha^p \sum_{k=1}^{n} a_k^p = \frac{1}{\alpha^q}\sum_{k=1}^{n} b_k^q.$$

This means that

$$\alpha^{p+q} = \frac{\sum_{k=1}^{n} b_k^q}{\sum_{k=1}^{n} a_k^p},$$

and so

$$\alpha^p = \Big(\frac{\sum_{k=1}^{n} b_k^q}{\sum_{k=1}^{n} a_k^p}\Big)^{\frac{p}{p+q}}.$$

With this choice of α, estimate (6) translates into

$$\sum_{k=1}^{n} a_k b_k \le \Big(\frac{1}{p}+\frac{1}{q}\Big)\Big[\alpha^p\sum_{k=1}^{n} a_k^p\Big] = 1\cdot\Big[\Big(\frac{\sum_{k=1}^{n} b_k^q}{\sum_{k=1}^{n} a_k^p}\Big)^{\frac{p}{p+q}}\sum_{k=1}^{n} a_k^p\Big]$$
$$= \Big(\sum_{k=1}^{n} a_k^p\Big)^{1-\frac{p}{p+q}}\Big(\sum_{k=1}^{n} b_k^q\Big)^{\frac{p}{p+q}} = \Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{q}{p+q}}\Big(\sum_{k=1}^{n} b_k^q\Big)^{\frac{p}{p+q}} = \Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}}\Big(\sum_{k=1}^{n} b_k^q\Big)^{\frac{1}{q}}.$$

This completes the proof of the discrete version of Hölder's inequality.

[1] In the literature this is often referred to as "optimizing in α".

Problem 3 (Riemann Sums) Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined via $f(x) = x$ and $g(x) = x^2$. Let $a, b \in \mathbb{Y}_{k_0}$ be such that $a < b$, say

$$a = \frac{n'}{2^{k_0}} \text{ and } b = \frac{n' + m'}{2^{k_0}}.$$

If $k \geq k_0$, then $a, b \in \mathbb{Y}_k$ as well. To compare the dyadic representations of a and b in $\mathbb{Y}_{k_0}$ and $\mathbb{Y}_k$, respectively, let $j \in \mathbb{N}$ be such that $k_0 + j = k$. Then:

$$a = \frac{n'}{2^{k_0}} = \frac{n' 2^j}{2^{k_0+j}} =: \frac{n}{2^k} \text{ and } b = \frac{n' + m'}{2^{k_0}} = \frac{(n' + m')2^j}{2^{k_0+j}} =: \frac{n+m}{2^k}. \tag{7}$$

(This serves as the definition for n and m.)

(a) Having k, n, m at our disposal, we can follow Definition 6.1 and compute

$$\begin{aligned} L_k(f, a, b) &= \sum_{\ell=0}^{m-1} \frac{1}{2^k} f\Big(\frac{n+\ell}{2^k}\Big) = \sum_{\ell=0}^{m-1} \frac{1}{2^k} \Big(\frac{n+\ell}{2^k}\Big) \\ &= \frac{1}{2^k \cdot 2^k} \Big\{ n \sum_{\ell=0}^{m-1} 1 + \sum_{\ell=0}^{m-1} \ell \Big\} = \frac{1}{2^k \cdot 2^k} \Big\{ nm + \frac{(m-1)m}{2} \Big\} \end{aligned}$$

and

$$\begin{aligned} L_k(g, a, b) &= \sum_{\ell=0}^{m-1} \frac{1}{2^k} g\Big(\frac{n+\ell}{2^k}\Big) = \sum_{\ell=0}^{m-1} \frac{1}{2^k} \Big(\frac{n+\ell}{2^k}\Big)^2 \\ &= \frac{1}{2^k \cdot 2^k \cdot 2^k} \Big\{ n^2 \sum_{\ell=0}^{m-1} 1 + 2n \sum_{\ell=0}^{m-1} \ell + \sum_{\ell=0}^{m-1} \ell^2 \Big\} \\ &= \frac{1}{2^k \cdot 2^k \cdot 2^k} \Big\{ n^2 m + n(m-1)m + \frac{(m-1)m(2m-1)}{6} \Big\}. \end{aligned}$$

Here we used the formulas

$$\sum_{\ell=0}^{m-1} \ell = \frac{(m-1)m}{2} \text{ and } \sum_{\ell=0}^{m-1} \ell^2 = \frac{(m-1)m(2m-1)}{6},$$

which were already proved by induction in ÜB2 and PB2, respectively.

(b) Still following Definition 6.1,

$$\begin{aligned} U_k(f, a, b) &= \sum_{\ell=0}^{m-1} \frac{1}{2^k} f\Big(\frac{n+\ell+1}{2^k}\Big) = \sum_{\ell=0}^{m-1} \frac{1}{2^k} \Big(\frac{n+\ell+1}{2^k}\Big) \\ &= \frac{1}{2^k \cdot 2^k} \Big\{ (n+1) \sum_{\ell=0}^{m-1} 1 + \sum_{\ell=0}^{m-1} \ell \Big\} = \frac{1}{2^k \cdot 2^k} \Big\{ (n+1)m + \frac{(m-1)m}{2} \Big\} \end{aligned}$$

and

$$\begin{aligned} U_k(g, a, b) &= \sum_{\ell=0}^{m-1} \frac{1}{2^k} g\Big(\frac{n+\ell+1}{2^k}\Big) = \sum_{\ell=0}^{m-1} \frac{1}{2^k} \Big(\frac{n+\ell+1}{2^k}\Big)^2 \\ &= \frac{1}{2^k \cdot 2^k \cdot 2^k} \Big\{ (n+1)^2 \sum_{\ell=0}^{m-1} 1 + 2(n+1) \sum_{\ell=0}^{m-1} \ell + \sum_{\ell=0}^{m-1} \ell^2 \Big\} \\ &= \frac{1}{2^k \cdot 2^k \cdot 2^k} \Big\{ (n+1)^2 m + (n+1)(m-1)m + \frac{(m-1)m(2m-1)}{6} \Big\}. \end{aligned}$$

(c) By Definition 6.2,

$$\int_a^b x dx = \sup_{k>k_0} L_k(f,a,b).$$

Note that the sequence $(L_k)_{k>k_0}$ is increasing in light of Observation 2 from p. 99 in the Skript, and so

$$\sup_{k>k_0} L_k(f,a,b) = \lim_{k\to\infty} L_k(f,a,b).$$

From part (a) we know that

$$\begin{aligned} L_k(f,a,b) &= \frac{1}{2^k\cdot 2^k}\Big\{nm + \frac{(m-1)m}{2}\Big\} = \frac{1}{2^k\cdot 2^k}\Big\{\frac{2nm+m^2-m}{2}\Big\} = \frac{1}{2^k\cdot 2^k}\Big\{\frac{n^2+2nm+m^2-n^2-m}{2}\Big\} \\ &= \frac{1}{2^k\cdot 2^k}\Big\{\frac{(n+m)^2-n^2-m}{2}\Big\} = \frac{1}{2}\Big\{\frac{(n+m)^2-n^2-m}{2^k\cdot 2^k}\Big\} = \frac{1}{2}\Big\{\Big(\frac{n+m}{2^k}\Big)^2 - \Big(\frac{n}{2^k}\Big)^2 - \frac{1}{2^k}\frac{m}{2^k}\Big\}. \end{aligned}$$

Recalling the dyadic representations (7) for a and b, we finally have that

$$L_k(f,a,b) = \frac{1}{2}\Big\{b^2 - a^2 - \frac{b-a}{2^k}\Big\}.$$

Since $b-a<\infty$,

$$\lim_{k\to\infty} \frac{b-a}{2^k} = 0,$$

and so

$$\int_a^b x dx = \lim_{k\to\infty} L_k(f,a,b) = \frac{b^2-a^2}{2}.$$

To compute the other integral, note that

$$\begin{aligned} L_k(g,a,b) &= \frac{1}{2^k\cdot 2^k\cdot 2^k}\Big\{n^2m + n(m-1)m + \frac{(m-1)m(2m-1)}{6}\Big\} \\ &= \frac{1}{3\cdot 2^k\cdot 2^k\cdot 2^k}\Big(3n^2m + 3nm^2 + m^3 - 3nm - \frac{3}{2}m^2 + \frac{m}{2}\Big) \\ &= \frac{1}{3\cdot 2^k\cdot 2^k\cdot 2^k}\Big((n+m)^3 - n^3 - 3nm - \frac{3}{2}m^2 + \frac{m}{2}\Big) \\ &= \frac{1}{3}\Big\{\Big(\frac{n+m}{2^k}\Big)^3 - \Big(\frac{n}{2^k}\Big)^3 - \frac{3}{2^k}\frac{n}{2^k}\frac{m}{2^k} - \frac{3}{2}\frac{1}{2^k}\Big(\frac{m}{2^k}\Big)^2 + \frac{1}{2}\frac{1}{2^k\cdot 2^k}\frac{m}{2^k}\Big\} \\ &= \frac{1}{3}\Big\{b^3 - a^3 - \frac{3}{2^k}a(b-a) - \frac{3}{2}\frac{1}{2^k}(b-a)^2 + \frac{1}{2}\frac{1}{2^k\cdot 2^k}(b-a)\Big\}. \end{aligned}$$

Again, since a and $b-a$ are finite numbers, we have that

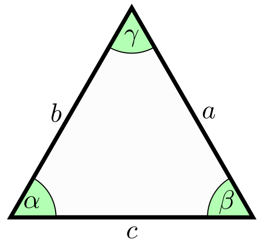
$$\lim_{k\to\infty}\Big(-\frac{3}{2^k}a(b-a) - \frac{3}{2}\frac{1}{2^k}(b-a)^2 + \frac{1}{2}\frac{1}{2^k\cdot 2^k}(b-a)\Big) = 0.$$

We finally conclude that

$$\int_a^b x^2 dx = \lim_{k\to\infty} L_k(g,a,b) = \frac{b^3-a^3}{3}.$$

Problem 4 (A geometric inequality)

Consider a generic nondegenerate triangle in the plane whose side lengths are a, b, c. Label with α, β, γ the angles opposite to the sides with lengths a, b, c, respectively, as illustrated in the following picture:



Our starting point is the following general observation: all the trigonometric functions are convex (or concave) if their arguments are restricted to an appropriate domain, and as a consequence there are many interesting geometric consequences of Jensen's inequality (Problem 4, PB9).

There are at least three different ways to express the area A in terms of the side lengths and trigonometric functions of the interior angles, and we make use of them all:

$$A = \frac{ab}{2}\sin\gamma = \frac{bc}{2}\sin\alpha = \frac{ca}{2}\sin\beta.$$

The different pairwise products of side lengths can therefore be expressed as

$$ab = \frac{2A}{\sin\gamma},\ bc = \frac{2A}{\sin\alpha},\ ca = \frac{2A}{\sin\beta}.$$

Taking the average of these representations, we find that

$$\max\{ab, bc, ca\} \geq \frac{ab+bc+ca}{3} = \frac{\frac{2A}{\sin\gamma} + \frac{2A}{\sin\alpha} + \frac{2A}{\sin\beta}}{3} = 2A\Big(\frac{1}{3}\frac{1}{\sin\alpha} + \frac{1}{3}\frac{1}{\sin\beta} + \frac{1}{3}\frac{1}{\sin\gamma}\Big). \tag{8}$$

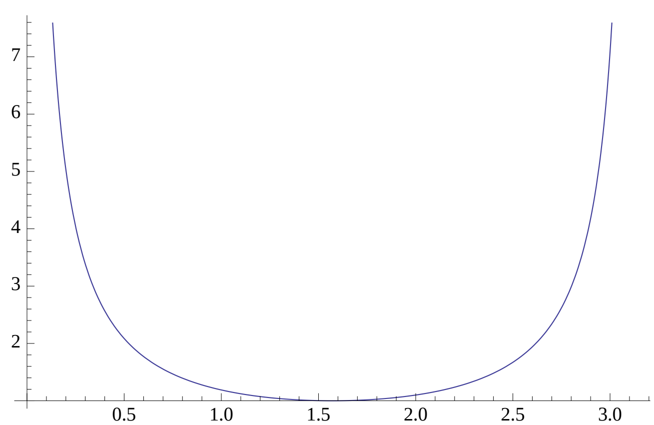
Now, the interior angles of any triangle in the plane add up to 180°, and so

$$\alpha + \beta + \gamma = \pi,$$

or equivalently

$$\frac{\alpha}{3} + \frac{\beta}{3} + \frac{\gamma}{3} = \frac{\pi}{3}.$$

The last observation is that the function $x \mapsto f(x) := \frac{1}{\sin x}$ is convex on the interval $(0, \pi)$. The following picture illustrates this:



Jensen's inequality implies in particular that

$$f\Big(\frac{\pi}{3}\Big) = f\Big(\frac{\alpha}{3} + \frac{\beta}{3} + \frac{\gamma}{3}\Big) \leq \frac{1}{3}f(\alpha) + \frac{1}{3}f(\beta) + \frac{1}{3}f(\gamma) = \frac{1}{3}\frac{1}{\sin\alpha} + \frac{1}{3}\frac{1}{\sin\beta} + \frac{1}{3}\frac{1}{\sin\gamma}.$$

Going back to (8), we finally conclude that

$$\begin{aligned}\max\{ab, bc, ca\} &\geq 2A\Big(\frac{1}{3}\frac{1}{\sin\alpha} + \frac{1}{3}\frac{1}{\sin\beta} + \frac{1}{3}\frac{1}{\sin\gamma}\Big) \\ &= 2A\Big(\frac{1}{3}f(\alpha) + \frac{1}{3}f(\beta) + \frac{1}{3}f(\gamma)\Big) \\ &\geq 2Af\Big(\frac{\alpha}{3} + \frac{\beta}{3} + \frac{\gamma}{3}\Big) \\ &= 2Af\Big(\frac{\pi}{3}\Big) = \frac{2A}{\sin(\pi/3)} = \frac{2A}{\sqrt{3}/2} = \frac{4A}{\sqrt{3}}.\end{aligned}$$

In other words, each planar triangle of area A has at least two sides for which the product of the side lengths is at least $4A/\sqrt{3} \sim 2.31A$. This is what we wanted to show.

As a final remark, notice that the inequality is sharp: equilateral triangles turn it into an equality.