

# Analysis 1, Solutions to Übungsblatt Nr. 6

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## Problem 1 (Cauchy criterion for converging sequences)

Let us start by proving that a sequence that converges to a finite limit satisfies Cauchy's criterion. With that purpose in mind, let  $f : \mathbb{N} \rightarrow \mathbb{X}$  be a sequence for which

$$\liminf_{n \rightarrow \infty} f(n) = \limsup_{n \rightarrow \infty} f(n) < \infty. \quad (1)$$

As usual, consider the auxiliary sequences  $g, h : \mathbb{N} \rightarrow \mathbb{X}$  defined by

$$\begin{aligned} g(n) &:= \inf\{f(m) : m \geq n\} \\ h(n) &:= \sup\{f(m) : m \geq n\}. \end{aligned}$$

The sequence  $g$  is monotonically increasing and the sequence  $h$  is monotonically decreasing. Let

$$A := \sup_{n \in \mathbb{N}} g(n) \text{ and } B := \inf_{n \in \mathbb{N}} h(n).$$

By hypothesis,  $A = B < \infty$ . By Theorem 2.36 and a straightforward modification thereof (for monotonically decreasing sequences), we know that

$$\forall \epsilon > 0, \exists n_1 \in \mathbb{N} : \forall m > n_1, \quad g(m) \leq A < g(m) + \epsilon, \quad (2)$$

$$\forall \epsilon > 0, \exists n_2 \in \mathbb{N} : \forall m' > n_2, \quad h(m') < A + \epsilon \leq h(m') + \epsilon. \quad (3)$$

Moreover we have that, for every  $n \in \mathbb{N}$ ,

$$g(n) \leq f(n) \leq h(n).$$

Now, let  $\epsilon > 0$  be given. Since  $\epsilon/2 > 0$  as well, we know from (2) and (3) that there exist natural numbers  $n_1, n_2 \in \mathbb{N}$  with the following properties:

$$\begin{aligned} g(m) &\leq A < g(m) + \epsilon/2 \text{ if } m > n_1, \text{ and} \\ h(m') &< A + \epsilon/2 \leq h(m') + \epsilon/2 \text{ if } m' > n_2. \end{aligned}$$

Let  $n := \max\{n_1, n_2\}$ . Then, for  $m, m' > n$ , we have that

$$\begin{aligned} A &< g(m) + \epsilon/2 \leq f(m) + \epsilon/2 \leq h(m) + \epsilon/2 < A + \epsilon, \\ A &< g(m') + \epsilon/2 \leq f(m') + \epsilon/2 \leq h(m') + \epsilon/2 < A + \epsilon, \end{aligned}$$

and so in particular

$$\begin{aligned} f(m) &\leq A + \epsilon/2, \text{ and} \\ A &< f(m') + \epsilon/2. \end{aligned}$$

Since  $A < \infty$ , we conclude from adding these two inequalities that

$$f(m) < f(m') + \epsilon.$$

The proof that  $f(m') < f(m) + \epsilon$  is analogous and left to the reader.

Now we prove that a sequence that satisfies Cauchy's criterion converges to a finite limit. With that purpose in mind, let  $f : \mathbb{N} \rightarrow \mathbb{X}$  be a sequence satisfying

$$\forall \epsilon > 0 \exists n \in \mathbb{N} \forall m \geq n, m' \geq n : f(m) < f(m') + \epsilon \wedge f(m') < f(m) + \epsilon, \quad (4)$$

and consider the auxiliary sequences  $g$  and  $h$  as defined above. Let  $\epsilon > 0$  be given. Applying (4), we conclude the existence of a natural number  $n_3 \in \mathbb{N}$  such that

$$f(m) < f(m') + \epsilon/4 \wedge f(m') < f(m) + \epsilon/4 \text{ if } m, m' > n_3.$$

In what follows, we will summarize this by writing more succinctly

$$|f(m) - f(m')| < \epsilon/4 \text{ if } m, m' > n_3. \quad (5)$$

We claim that  $|g(n_3) - h(n_3)| < \epsilon/2$ . This implies that  $|g(n) - h(n)| < \epsilon/2$  for every  $n \geq n_3$ . In fact, since  $g$  is monotonically increasing and  $h$  is monotonically decreasing, if  $n \geq n_3$ , then

$$g(n_3) \leq g(n) \leq f(n) \leq h(n) \leq h(n_3),$$

and so

$$|h(n) - g(n)| \leq |h(n_3) - g(n_3)| < \epsilon/2.$$

It remains to verify the claim. Suppose not, i.e. suppose that  $|h(n_3) - g(n_3)| \geq \epsilon/2$ . Since  $h(n_3) \geq g(n_3)$ , this can be rewritten as

$$h(n_3) \geq g(n_3) + \epsilon/2.$$

By definition,  $g(n_3) = \inf\{f(m) : m \geq n_3\}$  and  $h(n_3) = \sup\{f(m) : m \geq n_3\}$ . Recalling that the infimum of a set is its largest lower bound, and the supremum of a set is its smallest upper bound, we see that, for every  $\delta > 0$ , and so in particular for  $\delta := \epsilon/100$ , there exist natural numbers  $m, m' > n_3$  such that

$$g(n_3) + \delta > f(m) \text{ and } f(m') + \delta > h(n_3).$$

It follows that

$$f(m) + \epsilon/2 < g(n_3) + \delta + \epsilon/2 \leq h(n_3) + \delta < f(m') + 2\delta.$$

Recalling that  $\delta = \epsilon/100$ , it follows that

$$f(m') + \epsilon/50 = f(m') + 2\delta > f(m) + \epsilon/2,$$

and so

$$f(m') > f(m) + 12\epsilon/25 > f(m) + \epsilon/4,$$

which is absurd in light of (5). This establishes the claim.

So now we know that

$$|g(n) - h(n)| < \epsilon/2 \text{ for every } n \geq n_3. \quad (6)$$

Denoting as before  $A := \sup_n g(n)$  and  $B := \inf_n h(n)$ , we will be done once we show that  $A = B$ . It will suffice to show that  $A \geq B$ , for the reverse inequality holds for a general sequence  $f$  in light of Theorem 3.4. We claim that  $A + \epsilon \geq B$ , and establish this by contradiction. Suppose not, i.e. suppose

$$A + \epsilon < B.$$

Since  $g(n_3) \leq A$  and  $B \leq h(n_3)$ , we would then have that

$$g(n_3) + \epsilon \leq A + \epsilon < B < h(n_3),$$

and so

$$g(n_3) + \epsilon < h(n_3),$$

in contradiction to (6). The contradiction shows that  $A + \epsilon \geq B$ . Since  $\epsilon > 0$  could have been chosen arbitrarily small, this implies that  $A \geq B$ , and so  $A = B$ , as desired. The proof is now complete.

## Problem 2 (Series, quotients and inequalities)

- (a) From Problem 2(c) in Übungsblatt 2 we have an expression for the partial sums of the series we are trying to compute, namely

$$\sum_{k=0}^n x^k = \frac{1 - x^{n+1}}{1 - x}.$$

Note carefully that up to now only the condition  $x \neq 1$  was used. Now, since  $0 < x < 1$ , we have that  $\lim_{n \rightarrow \infty} x^{n+1} = 0$ , and so

$$\sum_{n=0}^{\infty} x^n = \lim_{n \rightarrow \infty} \sum_{k=0}^n x^k = \lim_{n \rightarrow \infty} \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x}.$$

- (b) Let  $N \in \mathbb{N}$  be as promised by the hypothesis of the problem. We claim that  $a_{N+k} \leq y^k a_N$  for every  $k \in \mathbb{N}$ . This can be verified by induction, the base case  $k = 0$  amounting to the identity  $a_N = a_N$ . Assuming the statement has been proved for  $k$ , compute:

$$a_{N+(k+1)} = a_{(N+k)+1} \leq y a_{N+k} \leq y(y^k a_N) = y^{k+1} a_N,$$

as desired. Here we used the assumption of the problem and the induction hypothesis, respectively.

It follows that:

$$\begin{aligned} \sum_{n=0}^{\infty} a_n &= \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} a_n = \sum_{n=0}^{N-1} a_n + \sum_{k=0}^{\infty} a_{N+k} \\ &\leq \sum_{n=0}^{N-1} a_n + a_N \sum_{k=0}^{\infty} y^k = \sum_{n=0}^{N-1} a_n + \frac{a_N}{1 - y}. \end{aligned} \quad (7)$$

Here, we just changed indices  $n = N+k$ , applied the claim, and appealed to part (a) (which is legitimate since  $0 < y < 1$ ). The right-hand side of (7) is clearly a finite number, and the proof is complete.

- (c) By the elementary inequality proved in Problem 3(d) from Übungsblatt 1 we have that, for each  $n \in \mathbb{N}$ ,

$$a_n b_n \leq \frac{1}{2} a_n^2 + \frac{1}{2} b_n^2.$$

It follows that

$$\sum_{n=0}^{\infty} a_n b_n \leq \frac{1}{2} \sum_{n=0}^{\infty} a_n^2 + \frac{1}{2} \sum_{n=0}^{\infty} b_n^2, \quad (8)$$

and so the series on the left-hand side of this inequality is finite provided both of the series on the right-hand side are finite, as desired.

For the second part, let us appeal to a common “normalization” trick. If neither of the sequences  $a, b$  is made up of all zeros (in which case the inequality we are after holds trivially), then we can introduce the modified sequences

$$\tilde{a}_n = a_n / \left( \sum_{k=0}^{\infty} a_k^2 \right)^{1/2} \quad \text{and} \quad \tilde{b}_n = b_n / \left( \sum_{k=0}^{\infty} b_k^2 \right)^{1/2}.$$

These are “normalized” in the sense that

$$\sum_{n=0}^{\infty} \tilde{a}_n^2 = \sum_{n=0}^{\infty} \left\{ a_n^2 / \left( \sum_{k=0}^{\infty} a_k^2 \right) \right\} = \frac{\sum_{n=0}^{\infty} a_n^2}{\sum_{k=0}^{\infty} a_k^2} = 1,$$

and similarly

$$\sum_{n=0}^{\infty} \tilde{b}_n^2 = 1.$$

Applying (8) to the sequences  $\tilde{a}$  and  $\tilde{b}$  yields

$$\sum_{n=0}^{\infty} \tilde{a}_n \tilde{b}_n \leq \frac{1}{2} \sum_{n=0}^{\infty} \tilde{a}_n^2 + \frac{1}{2} \sum_{n=0}^{\infty} \tilde{b}_n^2,$$

which can be rewritten in terms of the original sequences  $a$  and  $b$  as

$$\sum_{n=0}^{\infty} \left\{ a_n / \left( \sum_{k=0}^{\infty} a_k^2 \right)^{1/2} \right\} \left\{ b_n / \left( \sum_{k=0}^{\infty} b_k^2 \right)^{1/2} \right\} \leq \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1.$$

Clearing denominators, we find our old friend Cauchy-Schwarz, this time covering the case of possibly infinite sequences:

$$\sum_{n=0}^{\infty} a_n b_n \leq \left( \sum_{n=0}^{\infty} a_n^2 \right)^{1/2} \left( \sum_{n=0}^{\infty} b_n^2 \right)^{1/2}.$$

(d) Using the Cauchy-Schwarz inequality proved in part (c), we have that

$$\sum_{n=1}^{\infty} \frac{a_n}{n} \leq \left( \sum_{n=1}^{\infty} a_n^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2}.$$

Since the series  $\sum_{n=0}^{\infty} a_n^2 < \infty$  by hypothesis, we have that  $\sum_{n=1}^{\infty} a_n^2 < \infty$ , and so will be done once we check that  $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ . For that purpose, let us follow the hint. Start by noting that, if  $n \geq 2$ ,

$$\frac{1}{n^2} \leq \frac{1}{n(n-1)}.$$

On the other hand, for the same range of  $n$ ,

$$\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

It follows that the series with general term  $(n(n-1))^{-1}$  telescopes, i.e. its partial sums obey

$$\begin{aligned} \sum_{n=2}^N \frac{1}{n(n-1)} &= \sum_{n=2}^N \left( \frac{1}{n-1} - \frac{1}{n} \right) = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \dots + \frac{1}{N-2} - \frac{1}{N-1} + \left( \frac{1}{N-1} - \frac{1}{N} \right) \\ &= 1 + \underbrace{\left( \frac{1}{2} - \frac{1}{2} \right)}_{=0} + \underbrace{\left( \frac{1}{3} - \frac{1}{3} \right)}_{=0} + \dots + \underbrace{\left( \frac{1}{N-1} - \frac{1}{N-1} \right)}_{=0} - \frac{1}{N} \\ &= 1 - \frac{1}{N}. \end{aligned}$$

It follows that

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \lim_{N \rightarrow \infty} \sum_{n=2}^N \frac{1}{n(n-1)} = \lim_{N \rightarrow \infty} \left( 1 - \frac{1}{N} \right) = 1,$$

and so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \leq 1 + \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1 + 1 = 2 < \infty.$$

**Problem 3 ( $\alpha$ -test)**

- (a) Let  $n$  be a natural number greater than or equal to 1. By definition of maximum, we have that  $L(n) \in \{k \in \mathbb{N} : 2^k \leq n\}$ , and so

$$2^{L(n)} \leq n.$$

On the other hand,

$$n < 2^{L(n)+1},$$

for if  $2^{L(n)+1} \leq n$ , then  $L(n) + 1$  would be an element of the set  $\{k \in \mathbb{N} : 2^k \leq n\}$  which is strictly larger than  $L(n)$ , thereby contradicting the assumption that  $L(n)$  is the maximum of that set. Therefore we have that

$$2^{L(n)} \leq n < 2^{L(n)+1},$$

as desired.

- (b) The crucial combinatorial observation is that, for  $j \in \{0, 1, \dots, k-1\}$ , there are  $2^j$  numbers for which  $L(n) = j$ . In fact, these are the elements of the set  $\{2^j, 2^j + 1, 2^j + 2, \dots, 2^j + (2^j - 1)\}$ . Additionally note that

$$\sum_{j=0}^{k-1} 2^j = \frac{1-2^k}{1-2} = 2^k - 1,$$

and so the original sum can be decomposed as

$$\sum_{n=1}^{2^k-1} \left(\frac{1}{2^{L(n)}}\right)^\alpha = \sum_{j=0}^{k-1} \left( \sum_{\substack{n \in \mathbb{N} \setminus \{0\}: \\ L(n)=j}} \left(\frac{1}{2^{L(n)}}\right)^\alpha \right).$$

In the right-hand side of this last identity, for each  $j \in \{0, 1, \dots, k-1\}$ , the inner summation runs over all natural numbers  $n \geq 1$  such that  $L(n) = j$ . It follows that

$$\sum_{n=1}^{2^k-1} \left(\frac{1}{2^{L(n)}}\right)^\alpha = \sum_{j=0}^{k-1} \left( \sum_{\substack{n \in \mathbb{N} \setminus \{0\}: \\ L(n)=j}} \left(\frac{1}{2^{L(n)}}\right)^\alpha \right) = \sum_{j=0}^{k-1} \frac{1}{2^{j\alpha}} 2^j = \sum_{j=0}^{k-1} 2^{j(1-\alpha)} = \sum_{j=0}^{k-1} (2^{1-\alpha})^j.$$

If  $\alpha \neq 1$ , then  $2^{1-\alpha} \neq 1$ , and appealing again to the formula for the sum of the first  $k$  terms of a geometric series one gets that

$$\sum_{n=1}^{2^k-1} \left(\frac{1}{2^{L(n)}}\right)^\alpha = \sum_{j=0}^{k-1} (2^{1-\alpha})^j = \frac{1 - (2^{1-\alpha})^k}{1 - 2^{1-\alpha}}.$$

If  $0 \leq \alpha < 1$ , then  $2^{1-\alpha} > 1$ ,

$$\lim_{k \rightarrow \infty} (2^{1-\alpha})^k = \infty$$

and

$$\sum_{n=1}^{\infty} \left(\frac{1}{2^{L(n)}}\right)^\alpha = \lim_{k \rightarrow \infty} \sum_{n=1}^{2^k-1} \left(\frac{1}{2^{L(n)}}\right)^\alpha = \lim_{k \rightarrow \infty} \frac{1 - (2^{1-\alpha})^k}{1 - 2^{1-\alpha}} = \infty.$$

If, on the other hand,  $1 < \alpha < \infty$ , then  $2^{1-\alpha} < 1$  and

$$\lim_{k \rightarrow \infty} (2^{1-\alpha})^k = 0.$$

It follows that

$$\sum_{n=1}^{\infty} \left(\frac{1}{2^{L(n)}}\right)^{\alpha} = \lim_{k \rightarrow \infty} \sum_{n=1}^{2^k-1} \left(\frac{1}{2^{L(n)}}\right)^{\alpha} = \lim_{k \rightarrow \infty} \frac{1 - (2^{1-\alpha})^k}{1 - 2^{1-\alpha}} = \frac{1}{1 - 2^{1-\alpha}} < \infty.$$

Finally, if  $\alpha = 1$ , then

$$\sum_{n=1}^{2^k-1} \frac{1}{2^{L(n)}} = \sum_{j=0}^{k-1} (2^{1-1})^j = \sum_{j=0}^{k-1} 1 = k,$$

and so the series

$$\sum_{n=1}^{\infty} \frac{1}{2^{L(n)}} = \lim_{k \rightarrow \infty} \sum_{n=1}^{2^k-1} \frac{1}{2^{L(n)}} = \lim_{k \rightarrow \infty} k = \infty$$

again diverges.

(c) This time we start with the case  $\alpha = 1$ . It follows from part (a) that

$$\sum_{n=1}^{\infty} \frac{1}{n} > \sum_{n=1}^{\infty} \frac{1}{2^{L(n)+1}} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^{L(n)}}.$$

From the last paragraph of part (b) we know that this last series diverges, and so *the harmonic series diverges* as well:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \infty.$$

If  $0 \leq \alpha < 1$ , then parts (a) and (b) imply that

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} > \sum_{n=1}^{\infty} \left(\frac{1}{2^{L(n)+1}}\right)^{\alpha} = \frac{1}{2^{\alpha}} \sum_{n=1}^{\infty} \left(\frac{1}{2^{L(n)}}\right)^{\alpha} = \infty,$$

and so the series with general term  $n^{-\alpha}$  ( $0 \leq \alpha < 1$ ) diverges as well.

Finally, if  $1 < \alpha < \infty$ , then appealing again to parts (a) and (b), one has that

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \leq \sum_{n=1}^{\infty} \left(\frac{1}{2^{L(n)}}\right)^{\alpha} = \frac{1}{1 - 2^{1-\alpha}} < \infty.$$

All in all, we conclude that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$$

converges if and only if  $1 < \alpha < \infty$ .

#### Problem 4 (Quasi Stirling formula)

(a) Let us start by proving that the sequence  $\{\alpha_n\}$  is monotonically increasing. In light of Theorem 2.29, it will be enough to show that  $\alpha_n \leq \alpha_{n+1}$  for every  $n \geq 1$ , i.e.

$$\left(1 + \frac{1}{n}\right)^n \leq \left(1 + \frac{1}{n+1}\right)^{n+1}.$$

This follows from an elementary application of the AM-GM inequality (Problem 4, Übungsblatt 4) to the  $n+1$  positive real numbers

$$x_1 = 1, \text{ and } x_2 = x_3 = \dots = x_{n+1} = 1 + \frac{1}{n},$$

for then

$$\left(1 + \frac{1}{n}\right)^n = x_1 x_2 x_3 \dots x_{n+1} \leq \left(\frac{x_1 + x_2 + x_3 + \dots + x_{n+1}}{n+1}\right)^{n+1} = \left(\frac{1 + n\left(1 + \frac{1}{n}\right)}{n+1}\right)^{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1},$$

as desired.

Let us now focus on the sequence  $\{\beta_n\}$ , for which we want to show

$$\left(1 + \frac{1}{n}\right)^{n+1} \geq \left(1 + \frac{1}{n+1}\right)^{n+2}$$

for every  $n \geq 1$ . We shall use Bernoulli's inequality which was already discussed in Präsenzübung 5. A variant thereof (whose easy proof runs by induction on  $m$  and will therefore be omitted) states that if  $m \in \mathbb{N}$  and  $x \in \mathbb{X}$ ,

$$(1+x)^m \geq 1+mx.$$

Using this with  $x = \frac{1}{n^2+2n}$  and  $m = n+1$ , one gets that

$$\frac{\left(1 + \frac{1}{n}\right)^{n+1}}{\left(1 + \frac{1}{n+1}\right)^{n+1}} = \left(1 + \frac{1}{n^2+2n}\right)^{n+1} \geq 1 + \frac{n+1}{n^2+2n} \geq 1 + \frac{1}{n+1}.$$

It follows that

$$\left(1 + \frac{1}{n}\right)^{n+1} \geq \left(1 + \frac{1}{n+1}\right) \left(1 + \frac{1}{n+1}\right)^{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+2},$$

as desired.

- (b) Start by noting that the sequence  $\alpha$  converges since it is monotonically increasing. This is the partial content of Theorem 3.5. It is also a bounded sequence since, for every natural number  $n \geq 2$ ,

$$\left(1 + \frac{1}{n}\right) \left(1 - \frac{1}{n}\right) = 1 - \frac{1}{n^2} \leq 1,$$

and so such numbers satisfy

$$\alpha_n = \left(1 + \frac{1}{n}\right)^n \leq \left(1 - \frac{1}{n}\right)^{-n} \leq \left(1 - \frac{1}{2}\right)^{-2} = 4. \quad (9)$$

It follows that  $\lim_{n \rightarrow \infty} \alpha_n$  is a (finite) nonnegative real number, which we will denote by  $e$ .

Now we relate the sequences  $\alpha$  and  $\beta$ . Note that, for  $n \geq 2$ , the previous estimate (9) implies

$$\beta_n - \alpha_n = \left(1 + \frac{1}{n}\right)^n \left(\left(1 + \frac{1}{n}\right) - 1\right) \leq \frac{4}{n}.$$

Since  $1 + 1/n \geq 1$ , we further have that  $\beta_n \geq \alpha_n$ , and so

$$\alpha_n - \frac{4}{n} \leq \beta_n \leq \alpha_n + \frac{4}{n}$$

for  $n \geq 2$ . It follows that

$$\liminf_{n \rightarrow \infty} \alpha_n \leq \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \alpha_n.$$

Since we already know that  $\liminf_{n \rightarrow \infty} \alpha_n = \limsup_{n \rightarrow \infty} \alpha_n = e$ , it follows that

$$\liminf_{n \rightarrow \infty} \beta_n = \limsup_{n \rightarrow \infty} \beta_n,$$

and that the value of this common limit equals  $e$ .

*Remark.* It can be shown that  $e = \sum_{n=0}^{\infty} \frac{1}{n!} \simeq 2.7182818284590452354\dots$

(c) Let us start by proving that, for every natural number  $n \geq 2$ ,

$$\left(1 + \frac{1}{1}\right)^1 \left(1 + \frac{1}{2}\right)^2 \left(1 + \frac{1}{3}\right)^3 \dots \left(1 + \frac{1}{n-1}\right)^{n-1} = \frac{n^n}{n!}.$$

We use induction on  $n$ , the base case  $n = 2$  being trivial:

$$\left(1 + \frac{1}{1}\right)^1 = 2 = \frac{2^2}{2!}.$$

Assuming the result holds for  $n$ , let us verify it for  $n + 1$ :

$$\begin{aligned} \left(1 + \frac{1}{1}\right)^1 \left(1 + \frac{1}{2}\right)^2 \left(1 + \frac{1}{3}\right)^3 \dots \left(1 + \frac{1}{n-1}\right)^{n-1} \left(1 + \frac{1}{n}\right)^n &= \frac{n^n}{n!} \left(1 + \frac{1}{n}\right)^n = \frac{\cancel{n^n} (n+1)^n}{n! \cancel{n^n}} \\ &= \frac{(n+1)^n}{n!} = \frac{(n+1)^n}{n!} \frac{n+1}{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}. \end{aligned}$$

This proves the first identity.

The second one states that, for every natural number  $n \geq 2$ ,

$$\left(1 + \frac{1}{1}\right)^2 \left(1 + \frac{1}{2}\right)^3 \left(1 + \frac{1}{3}\right)^4 \dots \left(1 + \frac{1}{n-1}\right)^n = \frac{n^n}{(n-1)!},$$

and we follow a similar route. The base case  $n = 2$  amounts to checking that

$$\left(1 + \frac{1}{1}\right)^2 = 4 = \frac{2^2}{1!}.$$

Assuming that the statement has been verified for  $n$ , let us check it for  $n + 1$ :

$$\begin{aligned} \left(1 + \frac{1}{1}\right)^2 \left(1 + \frac{1}{2}\right)^3 \left(1 + \frac{1}{3}\right)^4 \dots \left(1 + \frac{1}{n-1}\right)^n \left(1 + \frac{1}{n}\right)^{n+1} &= \frac{n^n}{(n-1)!} \left(1 + \frac{1}{n}\right)^{n+1} = \frac{n^n}{(n-1)!} \frac{(n+1)^{n+1}}{n^{n+1}} \\ &= \frac{\cancel{n^n} (n+1)^{n+1}}{(n-1)! \cancel{n \cdot n^n}} = \frac{(n+1)^{n+1}}{n!}. \end{aligned}$$

This proves the second identity.

(d) The first identity proved in part (c) translates into

$$\alpha_1 \alpha_2 \alpha_3 \dots \alpha_{n-1} = \frac{n^n}{n!}, \tag{10}$$

whereas the second one translates into

$$\beta_1 \beta_2 \beta_3 \dots \beta_{n-1} = \frac{n^n}{(n-1)!}. \tag{11}$$

Since the sequence  $\alpha$  is monotonically increasing, we have that

$$\alpha_n \leq \lim_{n \rightarrow \infty} \alpha_n = e, \quad \text{for every } n \in \mathbb{N} \setminus \{0\}.$$

Similarly, since the sequence  $\beta$  is monotonically decreasing, we have that

$$\lim_{n \rightarrow \infty} \beta_n = e \leq \beta_n, \quad \text{for every } n \in \mathbb{N} \setminus \{0\}.$$

It follows that



$$\alpha_1\alpha_2\alpha_3\cdots\alpha_{n-1} \leq e^{n-1} \leq \beta_1\beta_2\beta_3\cdots\beta_{n-1},$$

and so (10) and (11) imply that

$$\frac{n^n}{n!} \leq e^{n-1} \leq \frac{n^n}{(n-1)!}.$$

These inequalities can be rewritten as

$$e\left(\frac{n}{e}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n,$$

and we are done.