Analysis 1, Solutions to Übungsblatt Nr. 5

Problem 1 (Fibonacci Numbers)

(a) Let us use induction. The base case $n = 1$ is easy to check:

$$F_0 F_2 - F_1^2 = 0 \cdot 1 - 1^2 = -1 = (-1)^1.$$

Assuming that $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ has been verified, let us prove that

$$F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}.$$

Appealing to the recurrence relation defining the sequence $\{F_n\}$ and using the induction hypothesis in the second-to-last equality below, we indeed have that

$$\begin{aligned} F_n F_{n+2} - F_{n+1}^2 &= F_n(F_{n+1} + F_n) - (F_n + F_{n-1})^2 \\ &= F_n F_{n+1} + \cancel{F_n^2} - \cancel{F_n^2} - 2F_n F_{n-1} - F_{n-1}^2 \\ &= F_n(F_{n+1} - F_{n-1}) - F_{n-1}(F_n + F_{n-1}) \\ &= -(F_{n-1}F_{n+1} - F_n^2) \\ &= -(-1)^n = (-1)^{n+1}. \end{aligned}$$

(b) Start by noting that the real numbers λ and μ are the distinct roots of the quadratic equation

$$x^2 = 1 + x.$$

In particular, $\lambda^2 = 1 + \lambda$ and $\mu^2 = 1 + \mu$.

After these preliminary observations, let us again use induction to prove the claim. The base case $n = 1$ is just as easy: since $\lambda \neq \mu$,

$$F_1 = 1 = \frac{\lambda - \mu}{\lambda - \mu} = \frac{\lambda^1 - \mu^1}{\lambda - \mu}.$$

Incidentally notice that the case $n = 0$ also holds trivially. This is important because of the slightly different kind of inductive argument we will be using (thanks to Felix Beck for pointing this out).

Assuming that $F_k = (\lambda^k - \mu^k)/(\lambda - \mu)$ has already been shown *for every natural* $k \leq n$, let us compute:

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &\overset{(1)}{=} \frac{\lambda^n - \mu^n}{\lambda - \mu} + \frac{\lambda^{n-1} - \mu^{n-1}}{\lambda - \mu} \\ &= \frac{\lambda^{n-1}(1 + \lambda) - \mu^{n-1}(1 + \mu)}{\lambda - \mu} \\ &\overset{(2)}{=} \frac{\lambda^{n-1}\lambda^2 - \mu^{n-1}\mu^2}{\lambda - \mu} \\ &= \frac{\lambda^{n+1} - \mu^{n+1}}{\lambda - \mu}. \end{aligned}$$

Here, in (1) we used the induction hypothesis for $k = n$ *and for* $k = n - 1$, and in (2) we used the identities $1 + \lambda = \lambda^2$ and $1 + \mu = \mu^2$. The proof is therefore complete.

Remark. This method of induction, which consists of assuming not only that the statement holds for n but also that it is true for every k less than or equal to n, sometimes goes in the literature under the name of "complete induction".

(c) We make use of the following binomial expansion, valid for any $n \in \mathbb{N}$ and $t \in \mathbb{R}$:

$$t^n = ((t-1)+1)^n = \sum_{k=0}^{n} \binom{n}{k} (t-1)^k. \tag{1}$$

For $\omega \in \{\lambda, \mu\}$, we know from part (b) that $\omega^2 - 1 = \omega$. Using this together with formula (1) for $t = \omega^2$, one gets that

$$\omega^{2n} = (\omega^2)^n = \sum_{k=0}^{n} \binom{n}{k} (\omega^2 - 1)^k = \sum_{k=0}^{n} \binom{n}{k} \omega^k.$$

It follows that

$$\lambda^{2n} - \mu^{2n} = \sum_{k=0}^{n} \binom{n}{k} (\lambda^k - \mu^k).$$

Dividing both sides of this identity by $\lambda - \mu \neq 0$, we conclude that

$$F_{2n} \overset{(b)}{=} \frac{\lambda^{2n} - \mu^{2n}}{\lambda - \mu} = \sum_{k=0}^{n} \binom{n}{k} \frac{\lambda^k - \mu^k}{\lambda - \mu} \overset{(b)}{=} \sum_{k=0}^{n} \binom{n}{k} F_k,$$

as desired.

(d) Again using induction, observe that the base case $n = 1$ amounts to

$$F_0 + F_1 = 0 + 1 = 2 - 1 = F_3 - 1.$$

Assume that $F_{n+2} = 1 + (F_0 + F_1 + \ldots + F_n)$ has already been proved. Then:

$$\begin{aligned} F_{n+3} =& F_{n+2} + F_{n+1} \\ =& 1 + (F_0 + F_1 + \ldots + F_n) + F_{n+1} \\ =& 1 + (F_0 + F_1 + \ldots + F_n + F_{n+1}), \end{aligned}$$

as desired. So this turned out to be the easiest property to verify!

Remark. Fibonacci numbers can be interpreted combinatorially, and as such it is no surprise that several of these identities can be proved combinatorially too.

We leave it as an amusing exercise to check that F_n counts the number of lists of 1's and 2's with sum $n-1$. Using this fact, we offer an alternative, combinatorial proof of part (d):

There are F_{n+2} lists of 1's and 2's with sum $n+1$. There is one list of all 1's. The remaining lists each contain at least one 2. There are F_{n-j} lists in which the last 2 is followed by j terms that are 1 because such lists are determined by the part of the list preceding the last 2, which has sum $(n+1) - 2 - j = (n-j) - 1$. Here j can be any element of the set $\{0, 1, ..., n-1\}$. Thus

$$F_{n+2} = 1 + \sum_{j=0}^{n-1} F_{n-j} = 1 + \sum_{k=1}^{n} F_k = 1 + \sum_{k=0}^{n} F_k,$$

where the second to last equality follows from changing indices $k = n - j$, and the last equality follows by adding $F_0 = 0$. So the identity is shown.

Problem 2 (Ordering $\mathbb{X}$)

(a) Let $x, y, z \in \mathbb{X}$ be such that $x \leq y$ and $y \leq z$. We want to show that $x \leq z$, i.e.

$$\forall y' \in \mathbb{Y} : y' < z \vee x \leq y'.$$

Rewriting the assumptions,

$$\forall y' \in \mathbb{Y} : y' < y \vee x \leq y' \tag{2}$$
$$\forall y' \in \mathbb{Y} : y' < z \vee y \leq y'. \tag{3}$$

Let $y' \in \mathbb{Y}$. From (2) and (3) we know that

$$y' < y \vee x \leq y'$$
$$y' < z \vee y \leq y',$$

and so

$$(y' < y \vee x \leq y') \wedge (y' < z \vee y \leq y').$$

By distributivity,

$$(y' < y \wedge y' < z) \vee (y' < y \wedge y \leq y') \vee (x \leq y' \wedge y' < z) \vee (x \leq y' \wedge y \leq y'),$$

and so we have to analyze four cases. In the first case $y' < z$ holds, whereas $x \leq y'$ holds in the third and fourth cases. The second case leads to a contradiction since $y' < y$ implies $y' \in \alpha(y)$ and $y \leq y'$ implies $y' \notin \alpha(y)$. As such, it cannot happen, and we are done.

(b) Let $x, y \in \mathbb{X}$ be given, and assume that $x \leq y$ does *not* hold. Our mission is to show that in this case $y \leq x$. Since the statement $x \leq y$ does not hold, we have that there exists $y' \in \mathbb{Y}$ such that $y' < y$ fails *and* $x \leq y'$ fails. Since $y' < y$ fails, we conclude that $y' \notin \alpha(y)$ and so $y \leq y'$. Similarly, since $x \leq y'$ fails, we conclude that $y' \in \alpha(x)$ and so $y' < x$. We claim that $y' < x$ implies that $y' \leq x$. This claim might seem obvious, but one should carefully notice that we are still in the process of building the fundamental properties of the order relations '$<$' and '$\leq$'. Therefore let us prove it:

Proof of the claim: The hypothesis $y' < x$ can be restated as $y' \in \alpha(x)$. We want to show that $y' \leq x$ i.e.

$$\forall y'' \in \mathbb{Y} : y'' < x \vee y' \leq y''. \tag{4}$$

Let $y'' \in \mathbb{Y}$ be such that $y' \leq y''$ fails, i.e. $y'' \notin \alpha(y')$ fails. Then $y'' \in \alpha(y')$. Since $y'' \in \alpha(y')$ ad $y' \in \alpha(x)$, it follows that $y'' \in \alpha(x)$. In other words, $y'' < x$. This verifies (4) and finishes the proof of the claim.

Knowing that $y \leq y'$ and $y' \leq x$, we conclude by the transitivity of $\leq$ proved in part (a) that $y \leq x$, as desired.

(c) *Preliminary remark. In what follows we will use the following facts without further justification: for any $a, b \in \mathbb{X}$*

$$a \leq b \textit{ if and only if } a + a \leq b + b, \textit{ and} \tag{5}$$
$$a < b \textit{ if and only if } a + a < b + b. \tag{6}$$

As an example, let us prove the forward implication of (5)*: if $a \leq b$, then $a + a \leq b + a$ by Theorem 2.19. Similarly $b + a \leq b + b$. By the transitivity of $\leq$ proved in part (a), it follows that $a + a \leq b + b$, as desired. The reverse implication can be proved by contradiction, and statement* (6) *is similar. We leave the details to the reader.*

Let $x, y \in \mathbb{X}$ be given. Let us start by proving that the set

$$D := \{z \in \mathbb{Y} : z + z < x + y\}$$

is a Dedekind cut. We need to show two things:

(i) $\forall a \in D \forall b \in \mathbb{Y} : a < b \vee b \in D$.
Let $a \in D$ and $b \in \mathbb{Y}$. Let us assume that $a < b$ does not hold and prove that then b must necessarily belong to the set D. Since a and b are both dyadic numbers, and $a < b$ does not hold, we have that $b \leq a$. Since $a \in D$, we have that in particular $a + a < x + y$. From $b \leq a$ it follows that $b + b \leq a + a$. By definition of '$<$', it follows that $b + b < x + y$. We conclude that $b \in D$, as desired.

(ii) $\forall a \in D \exists b \in \mathbb{Y} : a < b \wedge b \in D$.

Since $a \in D$, we have that $a \in \mathbb{Y}$ and $a + a < x + y$. Since $a + a \in \mathbb{Y}$, the statement $a + a < x + y$ can be rewritten as

$$\exists x' \in \mathbb{Y} \exists y' \in \mathbb{Y} : x' \leq x \wedge y' \leq y \wedge a + a < x' + y',$$

in light of *the definition of addition on* $\mathbb{X}$ (cf. Skript, top of p. 45). Let $x' \in \mathbb{Y}$ and $y' \in \mathbb{Y}$ be such that

$$x' \leq x \wedge y' \leq y \wedge a + a < x' + y',$$

and let $b' \in \mathbb{Y}$ be the middle point between the (dyadic) points $a + a$ and $x' + y'$, as promised by Theorem 2.8. This means that

$$b' + b' = (a + a) + (x' + y').$$

Applying Theorem 2.8 again we conclude the existence of $b \in \mathbb{Y}$ such that $b + b = b'$. We will be done once we prove that $a < b$ and that $b + b < x + y$. The former follows from the chain of inequalities

$$(a + a) + (a + a) < (a + a) + (x' + y') = (b + b) + (b + b),$$

whereas the latter follows from

$$(b + b) + (b + b) = (a + a) + (x' + y') < (x' + y') + (x' + y') \leq (x + y) + (x + y).$$

Having shown that D is a Dedekind cut, let $w \in \mathbb{X}$ be such that $\alpha(w) = D$. Such w exists and is unique by construction of the function $\alpha : \mathbb{X} \to A$. Our next goal is to show that $w + w = x + y$, which is equivalent to $\alpha(w + w) = \alpha(x + y)$. We accomplish this in two steps:

Step 1. $\alpha(w + w) \subset \alpha(x + y)$.

Let $z \in \alpha(w + w)$. Then $z \in \mathbb{Y}$ is such that $z < w + w$. By Theorem 2.8 there exists $z' \in \mathbb{Y}$ such that $z' + z' = z$, and so $z' + z' < w + w$. This implies that $z' < w$, and so $z' \in \alpha(w)$. But $\alpha(w) = D$, and so $z' + z' < x + y$. Since $z' + z' = z$, it follows that $z < x + y$, and so $z \in \alpha(x + y)$.

Step 2. $\alpha(x + y) \subset \alpha(w + w)$.

Let $z \in \alpha(x + y)$. Then $z \in \mathbb{Y}$ is such that $z < x + y$. By Theorem 2.8 there exists $z' \in \mathbb{Y}$ such that $z' + z' = z$. Then $z' + z' < x + y$, and so $z' \in D$. But $D = \alpha(w)$, and so $z' < w$. This implies that $z' + z' < w + w$. Since $z = z' + z'$, it follows that $z \in \alpha(w + w)$, as desired.

Problem 3 (Dyadic Intervals)

(a) Let $x \in \mathbb{X}$ be such that $x \neq \infty$. We are going to use induction on k to show that, for every $k \in \mathbb{N}$,

$$\exists n \in \mathbb{N} : x \in I_{k,n}. \tag{7}$$

We will then show that such n is unique, thereby finishing the proof. We call these two steps *existence* and *uniqueness*.

Existence. Let us start by checking the base case $k = 0$. We have to show that there exists $n \in \mathbb{N}$ such that $x \in I_{0,n}$, i.e. $n \leq x < n + 1$. Because of Theorem 2.18 (Archimedes 2), there exists $y \in \mathbb{Y}$ such that

$$y \leq x < y + \frac{1}{2}.$$

Theorem 2.9 (Archimedes 1) then implies that there exists $m_\star \in \mathbb{N}$ such that $y \leq \frac{m_\star}{2}$. Consider the set

$$S := \{m \in \mathbb{N} : y \leq \frac{m}{2}\}.$$

The set S is nonempty because $m_\star \in S$. Then Theorem 2.11 implies that S has a minimal element, say m_0, which satisfies $\frac{m_0 - 1}{2} < y \leq \frac{m_0}{2}$. Observe that we lose no generality in assuming that $m_0 \neq 0$ (so that $m_0 - 1 \in \mathbb{N}$ is defined as the unique natural number whose successor equals m_0), for otherwise $y = 0$ and $0 \leq x < 1$.

If m_0 is odd, then taking $n = \frac{m_0-1}{2} \in \mathbb{N}$ will do the job. Indeed,

$$n = \frac{m_0 - 1}{2} < y \leq x < y + \frac{1}{2} \leq \frac{m_0 + 1}{2} = n + 1,$$

and so $n \leq x < n + 1$. If m_0 is even, then $\frac{m_0}{2}$ is a natural number. By the previous remark, it is w.l.o.g. nonzero. By the totality of order proved in part (b) of Problem 2, we have that either $x \leq \frac{m_0}{2}$ or $\frac{m_0}{2} \leq x$. In the first case we can take $n = \frac{m_0}{2} - 1$ if $x < \frac{m_0}{2}$, and in the second case we can take $n = \frac{m_0}{2}$. This concludes the verification of the base case.

Assuming that (7) holds for k, let us prove it for $k+1$. By induction hypothesis, there exists $n \in \mathbb{N}$ such that $x \in I_{k,n}$, i.e.

$$\frac{n}{2^k} \leq x < \frac{n+1}{2^k}.$$

This can be rewritten as

$$\frac{n+n}{2^{k+1}} \leq x < \frac{(n+1)+(n+1)}{2^{k+1}}.$$

Consider the midpoint $\frac{n+n+1}{2^{k+1}}$ of the interval $I_{k,n}$. Again because of totality of order, either $x \leq \frac{n+n+1}{2^{k+1}}$ or $\frac{n+n+1}{2^{k+1}} \leq x$. This can be strengthened to either $x < \frac{n+n+1}{2^{k+1}}$ or $\frac{n+n+1}{2^{k+1}} \leq x$. In the first case,

$$\frac{n+n}{2^{k+1}} \leq x < \frac{n+n+1}{2^{k+1}},$$

i.e. $x \in I_{k+1,n+n}$, whereas in the second case

$$\frac{n+n+1}{2^{k+1}} \leq x < \frac{n+n+2}{2^{k+1}},$$

i.e. $x \in I_{k+1,n+n+1}$. This proves the induction step and finishes the argument.

Uniqueness. Assume that $x \in I_{k,n} \cap I_{k,m}$ for some $k, n, m \in \mathbb{N}$. We want to show that $n = m$. The hypothesis implies that

$$\frac{n}{2^k} \leq x < \frac{n+1}{2^k} \text{ and } \frac{m}{2^k} \leq x < \frac{m+1}{2^k}.$$

By transitivity, this implies that

$$\frac{n}{2^k} \leq x < \frac{m+1}{2^k} \text{ and } \frac{m}{2^k} \leq x < \frac{n+1}{2^k},$$

and so in particular

$$\frac{n}{2^k} < \frac{m+1}{2^k} \text{ and } \frac{m}{2^k} < \frac{n+1}{2^k}.$$

It follows that $n < m+1$ and $m < n+1$. This implies that $n = m$. (Why? Well, assume without loss of generality that $n \leq m$. Since $m < n+1$ and $m \in \mathbb{N}$, it follows that $m = n$, as desired.)

The proof is complete.

(b) Let $I_1 := I_{j,n}$ and $I_2 := I_{k,m}$ be two dyadic intervals. If $j = k$, then $I_1 = I_2$ or $I_1 \cap I_2 = \emptyset$ by virtue of part (a). Therefore we can assume that $j \neq k$. Suppose without loss of generality that $j < k$. By a straightforward modification of Theorem 1.26, this implies that there exists $\ell \in \mathbb{N} \setminus \{0\}$ such that $j + \ell = k$. Further assume that $I_1 \cap I_2 \neq \emptyset$. We will be done once we show that $I_1 \supset I_2$. We shall proceed by induction on ℓ, the "degree of separation" between the scales at which the intervals I_1 and I_2 live.

If $\ell = 1$, then let $I_1 = I_{j,n}$ and $I_2 = I_{j+1,m}$, and let $x \in I_1 \cap I_2$. Note that

$$\frac{n}{2^j} \leq x < \frac{n+1}{2^j}$$

because $x \in I_1$, and

$$\frac{m}{2^{j+1}} \leq x < \frac{m+1}{2^{j+1}}$$

because $x \in I_2$. Since

$$\frac{n}{2^j} = \frac{n+n}{2^{j+1}} \text{ and } \frac{n+1}{2^j} = \frac{(n+1)+(n+1)}{2^{j+1}},$$

it follows by transitivity of $\leq$ that $n+n < m+1$ and that $m < (n+1)+(n+1)$. Since only natural numbers are involved in the preceding two inequalities, it follows that $n+n \leq m$ and that $m+1 \leq (n+1)+(n+1)$. This implies that

$$\frac{n}{2^j} = \frac{n+n}{2^{j+1}} \leq \frac{m}{2^{j+1}} < \frac{m+1}{2^{j+1}} \leq \frac{(n+1)+(n+1)}{2^{j+1}} = \frac{n+1}{2^j},$$

from which we can finally conclude that $I_1 \supset I_2$. This establishes the base case.

Let us assume that the statement has been verified for ℓ and let us prove it for $\ell+1$. This time set $I_1 = I_{j,n}$ and $I_2 = I_{j+\ell+1,m}$, and let $x \in I_1 \cap I_2$. Since $x \in I_1$,

$$\frac{n}{2^j} \leq x < \frac{n+1}{2^j}.$$

Since $x \in I_2$,

$$\frac{m}{2^{j+\ell+1}} \leq x < \frac{m+1}{2^{j+\ell+1}}.$$

Assume that m is an even integer, and let $m_0 \in \mathbb{N}$ be such that $m = m_0 + m_0$. In particular, since

$$\frac{m}{2^{j+\ell+1}} \leq x < \frac{m+2}{2^{j+\ell+1}},$$

we have that

$$\frac{m_0}{2^{j+\ell}} \leq x < \frac{m_0+1}{2^{j+\ell}},$$

and so $x \in I_3 := I_{j+\ell,m_0}$. It follows that $I_1 \cap I_3 \neq \emptyset$. Since I_3 is an interval of generation $j+\ell$, the induction hypothesis applies and implies that $I_1 \supset I_3$. Since $I_3 \supset I_2$, we then conclude that $I_1 \supset I_2$. The case when m is an odd integer is analogous, one just observes that $m+1$ is even and applies a similar reasoning to the right endpoint of the interval I_2. The details are left to the reader.

Problem 4 (Isoperimetric inequality for the 3-cube)

Let us consider a rectangular box (sometimes also called a "cuboid") in 3-space of dimensions $a, b, c > 0$. As pointed out in the statement of the problem, the surface area of such a box is given by

$$A = 2(ab + bc + ca). \tag{8}$$

Its volume equals the product of the three side lengths, i.e. abc. On the other hand, consider a cube of surface area A. Its edges have side length given by $(A/6)^{1/2}$, and so its volume equals $(A/6)^{3/2}$. To show that among all rectangular boxes of a given surface area, the cube is the one that encapsulates the largest volume, thus amounts to showing that the volume of any such box is less than or equal to the volume of the corresponding cube, and that equality holds if and only if the box is a cube. Algebraically, we are after the following inequality:

$$abc \leq \Big(\frac{A}{6}\Big)^{3/2},$$

which in view of (8) can be rewritten as

$$abc \leq \Big(\frac{ab+bc+ca}{3}\Big)^{3/2}. \tag{9}$$

Applying the AMGM inequality (proved in last week's problem set) to the three positive numbers $ab, bc, ca > 0$, we see that

$$(abc)^2 = (ab)(bc)(ca) \leq \Big(\frac{ab+bc+ca}{3}\Big)^3,$$

which implies our desired inequality (9) after taking square roots on both sides. Moreover, equality is known to hold if and only if $ab = bc = ca$. Since all of the three numbers a, b, c are nonzero, this is turn implies that $a = b = c$, as desired.

Thus the cube is the unique solution to this easy isoperimetric problem. A more challenging problem of a similar flavor asks the following: *among all curves of a given length, which one encapsulates the largest area?* The dedicated reader should start thinking about providing a proof for the obvious conjecture immediately!