

# Analysis 1, Solutions to Übungsblatt Nr. 4

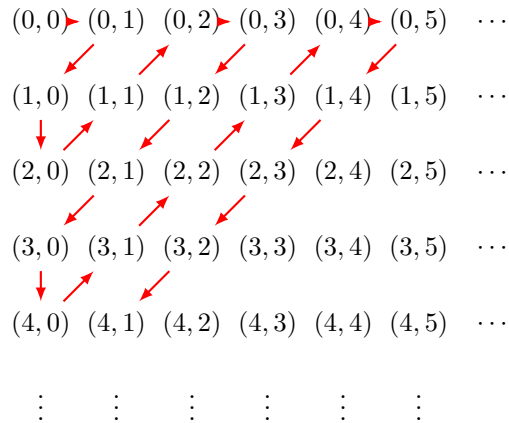
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## Problem 1 (Cardinality)

(a) Let us use the letter  $\mathcal{F}$  to denote the set of all functions  $f : \{0, 1\} \rightarrow \mathbb{N}$ .

An element  $f \in \mathcal{F}$  is completely determined by the values it takes at 0 and at 1, i.e. by looking at the natural number  $f(0)$  and *then* at the natural number  $f(1)$ . This information can be encoded in the “ordered pair”  $(f(0), f(1))$ .

The following picture illustrates a map, denoted  $\Phi$ , from the set natural numbers  $\mathbb{N}$  onto the set  $\mathcal{F}$ :



which should be interpreted as follows:

$$\begin{aligned}
 0 &\xrightarrow{\Phi} (f_0 : \{0, 1\} \rightarrow \mathbb{N}, f_0(0) = 0, f_0(1) = 0) \\
 1 &\xrightarrow{\Phi} (f_1 : \{0, 1\} \rightarrow \mathbb{N}, f_1(0) = 0, f_1(1) = 1) \\
 2 &\xrightarrow{\Phi} (f_2 : \{0, 1\} \rightarrow \mathbb{N}, f_2(0) = 1, f_2(1) = 0) \\
 3 &\xrightarrow{\Phi} (f_3 : \{0, 1\} \rightarrow \mathbb{N}, f_3(0) = 2, f_3(1) = 0) \\
 4 &\xrightarrow{\Phi} (f_4 : \{0, 1\} \rightarrow \mathbb{N}, f_4(0) = 1, f_4(1) = 1) \\
 &\vdots
 \end{aligned}$$

By construction, it is clear that the map  $\Phi : \mathbb{N} \rightarrow \mathcal{F}$  just described is well-defined and bijective. Theorem 1.33 implies in particular that there exists an injective map  $\Psi : \mathcal{F} \rightarrow \mathbb{N}$ . It follows that the set  $\mathcal{F}$  is countable, as desired.

*Remark.* The set  $\mathcal{F}$  is usually denoted in the literature by “ $\mathbb{N} \times \mathbb{N}$ ” to emphasize the fact that it consists of “ordered pairs” of natural numbers. We will obey this tradition in the problems below.

(b) Let  $\mathbb{Q}_+$  denote the set of positive rationals, i.e.

$$\mathbb{Q}_+ := \{q \in \mathbb{Q} : q > 0\}.$$

For every  $q \in \mathbb{Q}_+$ , there exists at least one pair  $(n, m) \in \mathbb{N} \times \mathbb{N}$  such that  $q = \frac{m}{n}$ . Therefore we can find an injection  $\iota : \mathbb{Q}_+ \rightarrow \mathbb{N} \times \mathbb{N}, q \mapsto (n, m)$ . In part (a) we proved that the set  $\mathbb{N} \times \mathbb{N}$  is countable, i.e. there exists an injection  $\phi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ . Since the composition of injective functions is injective, it follows that the function  $\phi \circ \iota$  provides an injection from  $\mathbb{Q}_+$  to  $\mathbb{N}$ . It follows that  $\mathbb{Q}_+$  is countable.

In a similar way, the set  $\mathbb{Q}_-$  of negative rationals is countable. It follows from Problem 0 of this week's Präsenzblatt that the set

$$\mathbb{Q} = \mathbb{Q}_- \cup \{0\} \cup \mathbb{Q}_+$$

is countable, as desired.

- (c) Aiming at a contradiction, let  $\varphi : \mathbb{N} \rightarrow \mathbb{Q}$  be an order-preserving bijection from the set of natural numbers to the set of rationals. Consider the rational numbers  $\varphi(0)$  and  $\varphi(1)$ . Since  $\varphi$  is in particular an injection, it follows that  $\varphi(0) \neq \varphi(1)$ . Since  $0 < 1$ , we actually have that  $\varphi(0) < \varphi(1)$  since  $\varphi$  is order-preserving by assumption. Consider the midpoint  $q$  of the rational numbers  $\varphi(0)$  and  $\varphi(1)$ ,

$$q := \frac{\varphi(0) + \varphi(1)}{2}.$$

This is still a rational number, which satisfies  $\varphi(0) < q < \varphi(1)$ . We claim that  $\varphi(n) \neq q$  for every  $n \in \mathbb{N}$ . We already know that  $\varphi(0) \neq q$  and that  $\varphi(1) \neq q$ . Any other natural number  $n \in \mathbb{N} \setminus \{0, 1\}$  satisfies  $1 < n$ , and so  $\varphi(1) < \varphi(n)$  by the order-preserving property of  $\varphi$ . It follows that

$$q < \varphi(1) < \varphi(n),$$

and so  $\varphi(n) \neq q$ , as claimed. Thus  $\varphi$  does not surject onto  $\mathbb{Q}$ , and as such it cannot be a bijection. The contradiction resulted from assuming the existence of an order-preserving bijection from the set of natural numbers to the set of rationals. Thus no such order-preserving bijection exists, as we wanted to prove.

## Problem 2 (Recursion)

- (a) Let us start by using induction on  $m$  to prove that, for every  $m \in \mathbb{N}$ ,

$$\forall n \in \mathbb{N} : \nu(n) + m = \nu(n + m), \tag{1}$$

which, using the definition of addition  $+$ , can be stated in the following equivalent form:

$$\forall n \in \mathbb{N} : \nu^m(\nu(n)) = \nu(\nu^m(n)).$$

The base case  $m = 0$  follows immediately from the first property of addition established in Theorem 1.24. Indeed, for any  $n \in \mathbb{N}$ ,

$$\nu^0(\nu(n)) = \nu(n) + 0 = \nu(n) = \nu(n + 0) = \nu(\nu^0(n)).$$

Assuming the induction hypothesis

$$\forall n \in \mathbb{N} : \nu^m(\nu(n)) = \nu(\nu^m(n)), \tag{2}$$

let us prove that

$$\forall n \in \mathbb{N} : \nu^{\nu(m)}(\nu(n)) = \nu(\nu^{\nu(m)}(n)).$$

With this purpose in mind, let  $n \in \mathbb{N}$  be arbitrary. Then:

$$\nu^{\nu(m)}(\nu(n)) \stackrel{(i)}{=} \nu(\nu^m(\nu(n))) \stackrel{(ii)}{=} \nu(\nu(\nu^m(n))) \stackrel{(iii)}{=} \nu(\nu^{\nu(m)}(n)),$$

as desired. Here, (i) and (iii) are a consequence of property 2 of Theorem 1.22, whereas (ii) amounts to the induction hypothesis (2). Thus (1) is established for every  $m \in \mathbb{N}$ .

Let us now use induction on  $n$  to show that, for every  $n \in \mathbb{N}$  and every  $p \in \text{Dom}(g)$ ,

$$\forall m \in \mathbb{N} : g^{n+m}(p) = g^n(g^m(p)).$$

The base case  $n = 0$  is again a consequence of the fact that 0 is the neutral element for addition. In fact, for any  $m \in \mathbb{N}$ ,

$$g^{0+m}(p) = g^m(p) = g^0(g^m(p)).$$

Assuming the induction hypothesis

$$\forall m \in \mathbb{N} : g^{n+m}(p) = g^n(g^m(p)), \quad (3)$$

let us prove that

$$\forall m \in \mathbb{N} : g^{\nu(n)+m}(p) = g^{\nu(n)}(g^m(p)).$$

With that purpose in mind, let  $m \in \mathbb{N}$  be arbitrary. Then

$$g^{\nu(n)+m}(p) \stackrel{(i)}{=} g^{\nu(n+m)}(p) \stackrel{(ii)}{=} g(g^{n+m}(p)) \stackrel{(iii)}{=} g(g^n(g^m(p))) \stackrel{(iv)}{=} g^{\nu(n)}(g^m(p)),$$

as desired. Here, (i) is a consequence of property (1) proved above and (iii) follows from the induction hypothesis (3). On the other hand, steps (ii) and (iv) are a consequence of property 2 of Theorem 1.22. More precisely: since  $g$  satisfies the hypothesis of Theorem 1.22 and  $p \in \text{Dom}(g)$ , there exists a sequence  $h : \mathbb{N} \rightarrow \text{Dom}(g)$  such that  $h(0) = p$  and

$$\forall x \in \mathbb{N} : h(\nu(x)) = g(h(x)). \quad (*)$$

In particular, since (by definition)  $h(k) = g^k(p)$  for every  $k \in \mathbb{N}$ ,

$$g^{\nu(n+m)}(p) = h(\nu(n+m)) \stackrel{(*)}{=} g(h(n+m)) = g(g^{n+m}(p)).$$

This establishes (ii), and a similar argument establishes (iv). This concludes the proof.

(b) Part (a) with  $g = \nu$  and  $p = k \in \mathbb{N} = \text{Dom}(\nu)$  tells us that, for every natural numbers  $n, m$ ,

$$\nu^{n+m}(k) = \nu^n(\nu^m(k)). \quad (4)$$

Then

$$\begin{aligned} k + (n+m) &= \nu^{n+m}(k) \quad (\text{by definition of } +) \\ &= \nu^n(\nu^m(k)) \quad (\text{by (4)}) \\ &= \nu^n(k+m) \quad (\text{by definition of } +) \\ &= (k+m) + n. \quad (\text{by definition of } +) \end{aligned} \quad (5)$$

The result follows from this and the commutativity property (C) of addition which was already proved in class. Indeed,

$$(n+m) + k \stackrel{(C)}{=} k + (n+m) \stackrel{(5)}{=} (k+m) + n \stackrel{(C)}{=} n + (k+m) \stackrel{(C)}{=} n + (m+k).$$

### Problem 3 (Surjectivity and injectivity)

(a) From the definition of surjectivity, the hypothesis implies that

$$\forall y \in Y \exists x \in X : f(x) = y.$$

Applying the axiom of choice (Rule 34) to this statement and the identity function on  $Y$ , denoted  $I_Y$ , we conclude the existence of a function  $g$  for which

$$\forall y : (f(g(y)) = y \wedge g(y) \neq g \wedge I_Y(y) \neq I_Y) \vee (g(y) = g \wedge I_Y(y) = I_Y). \quad (6)$$

If  $y \in Y$ , then  $I_Y(y) \neq I_Y$ , and so  $g(y) \neq g$  (and  $f(g(y)) = y$ ). Conversely, if  $g(y) \neq g$ , then  $I_Y(y) \neq I_Y$  and so  $y \in Y$ . This shows that  $\text{Dom}(g) = Y$ . Also, if  $x \in \text{Ran}(g)$ , then

$$\exists y : g(y) = x \wedge x \neq g.$$

In particular,  $f(x) = f(g(y)) = y$  because  $g(y) \neq g$ . Since  $y \neq f$  (because  $y \in \text{Dom}(g) = Y = \text{Ran}(f)$ ), it follows that  $x \in \text{Dom}(f) = X$ . This shows that  $\text{Ran}(g) \subset X$ .

Hence we will be done once we show that  $g$  is injective, i.e.,

$$\forall y_1 \in Y \forall y_2 \in Y : g(y_1) \neq g(y_2) \vee y_1 = y_2.$$

Let  $y_1, y_2 \in Y$  be such that  $g(y_1) = g(y_2)$ . We want to show that  $y_1 = y_2$ . Since  $I_Y(y_1) \neq I_Y$  and  $I_Y(y_2) \neq I_Y$ , it follows from (6) that  $f(g(y_1)) = y_1$  and that  $f(g(y_2)) = y_2$ . But then

$$y_1 = f(g(y_1)) = f(g(y_2)) = y_2,$$

as desired.

(b) By definition of injectivity,

$$\forall x_1 \in X \forall x_2 \in X : f(x_1) \neq f(x_2) \vee x_1 = x_2.$$

In particular, given  $y \in \text{Ran}(f) \subset Y$ , there exists exactly one  $x \in X$  such that  $f(x) = y$ . Define  $g$  on such  $y$  to be equal to that specific  $x$ , i.e.  $g(y) = x$ . On the other hand, if  $y \in Y \setminus \text{Ran}(f)$ , pick a fixed  $x_0 \in X$  and define  $g(y) = x_0$  (this can be done because without loss of generality  $X \neq \emptyset$ ). The function  $g$  thus defined: for  $y \in Y$ ,

$$g(y) = \begin{cases} x & \text{if } y \in \text{Ran}(f) \text{ and } f(x) = y \\ x_0 & \text{if } y \in Y \setminus \text{Ran}(f), \end{cases}$$

and  $g(y) = g$  otherwise, satisfies:

- (i)  $\text{Dom}(g) = Y$ ,
- (ii)  $\text{Ran}(g) \subset X$ ,
- (iii)  $g$  is surjective.

Parts (i) and (ii) follow directly from the way the function  $g$  was constructed. To check (iii), we additionally have to verify that  $X \subset \text{Ran}(g)$ . But this is immediate since  $g(f(x)) = x$  for every  $x \in X$ .

#### Problem 4 (Arithmetic Mean-Geometric Mean inequality)

*Proof 1.* This proof uses only elementary arithmetic rules and mathematical induction.

The base case  $n = 1$  is trivial to verify since  $x_1 = (x_1/1)^1$ . Let us assume that the statement has been verified for all choices of  $n$  nonnegative real numbers. Consider  $n + 1$  nonnegative real numbers  $x_1, x_2, \dots, x_n, x_{n+1}$  with arithmetic mean  $A$  defined via

$$(n + 1)A := x_1 + x_2 + \dots + x_n + x_{n+1}. \quad (7)$$

If all the numbers  $x_i$  equal  $A$ , we are done. Otherwise we can find one number which is strictly larger than  $A$  and one number which is strictly smaller than  $A$ , say  $x_n > A$  and  $x_{n+1} < A$ . In particular, the real numbers  $x_n - A$  and  $A - x_{n+1}$  are (strictly) positive, and so

$$(x_n - A)(A - x_{n+1}) > 0. \quad (8)$$

Let us now consider the  $n$  numbers  $x_1, x_2, \dots, x_{n-1}, x$ , where  $x := x_n + x_{n+1} - A$ . Note that  $x$  is a positive real number. Indeed, since  $x_{n+1} \geq 0$ ,

$$x = x_n + x_{n+1} - A \geq x_n - A > 0.$$

The crucial observation is that  $A$  is still the arithmetic mean of the  $n$  numbers  $x_1, x_2, \dots, x_{n-1}, x$ . Indeed, from (7) it follows at once that

$$nA = x_1 + x_2 + \dots + x_{n-1} + \underbrace{x_n + x_{n+1} - A}_{=x}.$$

Using the induction hypothesis, we thus conclude that  $A^n \geq x_1 x_2 \cdots x_{n-1} x$ , and so

$$A^{n+1} = A^n \cdot A \geq (x_1 x_2 \cdots x_{n-1} x) A. \quad (9)$$

Now,

$$\begin{aligned}
xA - x_n x_{n+1} &= (x_n + x_{n+1} - A)A - x_n x_{n+1} \quad (\text{by definition of } x) \\
&= (x_n - A)(A - x_{n+1}) \quad (\text{expand brackets}) \\
&> 0, \quad (\text{by (8)})
\end{aligned}$$

and so

$$xA > x_n x_{n+1} \geq 0, \tag{10}$$

and thus  $A > 0$ . Thus, if at least one of the numbers  $x_1, x_2, \dots, x_{n-1}$  is zero, then we already have strict inequality in (9). Otherwise the right-hand side of (9) is positive and *strict* inequality can be derived from (9) and (10):

$$\begin{aligned}
A^{n+1} &\geq (x_1 x_2 \cdots x_{n-1} x)A \\
&= x_1 x_2 \cdots x_{n-1} (xA) \\
&> x_1 x_2 \cdots x_{n-1} (x_n x_{n+1}) \\
&= x_1 x_2 \cdots x_{n+1}.
\end{aligned}$$

In particular, the inequality is an equality if and only if all the  $x_i$  are the same.

*Proof 2.* (Sketch) An alternative elegant proof due to Cauchy is available and involves a non-standard kind of induction which sometimes goes by the name of “forward-backward induction”. To briefly describe it, let  $P(n)$  denote the statement of the inequality we want to prove,

$$x_1 \cdot x_2 \cdots x_n \leq \left( \frac{x_1 + x_2 + \dots + x_n}{n} \right)^n.$$

For  $n = 2$ , we have that

$$x_1 \cdot x_2 \leq \left( \frac{x_1 + x_2}{2} \right)^2 \text{ if and only if } (x_1 - x_2)^2 \geq 0,$$

which is true. Then we proceed in the following two steps, which will clearly imply the full result:

- (i)  $P(n) \Rightarrow P(n - 1)$ ;
- (ii)  $P(n)$  and  $P(2) \Rightarrow P(n - 1)$ .

For part (i), let  $A := \frac{1}{n-1} \sum_{k=1}^{n-1} x_k$ , and observe that  $P(n)$  implies

$$(x_1 \cdot x_2 \cdots x_{n-1})A \leq \left( \frac{\sum_{k=1}^{n-1} x_k + A}{n} \right)^n = \left( \frac{(n-1)A + A}{n} \right)^n = A^n,$$

and hence

$$x_1 \cdot x_2 \cdots x_{n-1} \leq A^{n-1} = \left( \frac{x_1 + x_2 + \dots + x_{n-1}}{n-1} \right)^{n-1},$$

which is  $P(n - 1)$ .

Part (ii) is left to the reader.