Analysis 1, Solutions to Übungsblatt Nr. 11

Problem 1 (Complex numbers)

(a_1) Start by noting that

$$\frac{1+2i}{3-4i} = \frac{(1+2i)(3+4i)}{(3-4i)(3+4i)} = \frac{(1\cdot 3 - 2\cdot 4) + (1\cdot 4 + 2\cdot 3)i}{3^2+4^2} = \frac{-5+10i}{25} = -\frac{1}{5} + \frac{2}{5}i.$$

It follows that

$$\Re\Big(\frac{1+2i}{3-4i}\Big) = \Re\Big(-\frac{1}{5}+\frac{2}{5}i\Big) = -\frac{1}{5}$$
$$\Im\Big(\frac{1+2i}{3-4i}\Big) = \Im\Big(-\frac{1}{5}+\frac{2}{5}i\Big) = \frac{2}{5},$$

and

$$\Big|\frac{1+2i}{3-4i}\Big| = \Big|-\frac{1}{5}+\frac{2}{5}i\Big| = \sqrt{\Big(\frac{1}{5}\Big)^2 + \Big(\frac{2}{5}\Big)^2} = \sqrt{\frac{1^2+2^2}{5^2}} = \frac{1}{\sqrt{5}}.$$

(a_2) Start by noting that

$$\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{1^2 - 2(1)(i) + (-i)^2}{1^2+1^2} = \frac{-2i}{2} = -i.$$

On the other hand, the quantity $(-i)^n$ equals

$$1, -i, -1, \text{ or } i$$

depending on whether n is congruent to $0, 1, 2$ or $3 \pmod 4$, respectively. Since

$$2015 = 4\cdot 503 + 3,$$

we have that

$$\Big(\frac{1-i}{1+i}\Big)^{2015} = (-i)^{2015} = (-i)^{4\cdot 503+3} = (-i)^3 = i,$$

a purely imaginary number. It follows that

$$\Re\Big\{\Big(\frac{1-i}{1+i}\Big)^{2015}\Big\} = 0, \ \Im\Big\{\Big(\frac{1-i}{1+i}\Big)^{2015}\Big\} = 1, \text{ and } \Big|\Big(\frac{1-i}{1+i}\Big)^{2015}\Big| = 1.$$

(a_3) This one is easy. By Euler's formula, we have that

$$\sqrt{i} = i^{\frac{1}{2}} = (e^{i\frac{\pi}{2}})^{\frac{1}{2}} = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}},$$

and so

$$\Re(\sqrt{i}) = \frac{1}{\sqrt{2}}, \ \Im(\sqrt{i}) = \frac{1}{\sqrt{2}}, \text{ and } |\sqrt{i}| = 1.$$

(a_4) Recall that, for every $z \in \mathbb{C}$,

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}.$$

Plugging in $z = i$ and recalling that $i^2 = -1$, this formula yields

$$\cos i = \frac{e^{i^2}+e^{-i^2}}{2} = \frac{e^{-1}+e}{2},$$

a real number. It follows that

$$\Re(\cos(i)) = \frac{e^{-1}+e}{2}, \Im(\cos(i)) = 0, \text{ and } |\cos(i)| = \frac{e^{-1}+e}{2}.$$

(b) Let $a, b, c, d \in \mathbb{R}$ be real numbers such that $\zeta = a + bi$ and $\eta = c + di$. Then $\overline{\zeta} = a - bi$ and

$$\begin{aligned}
|1 - \overline{\zeta}\eta|^2 - |\zeta - \eta|^2 &= |1 - (a - bi)(c + di)|^2 - |(a + bi) - (c + di)|^2 \\
&= |(1 - ac - bd) + (bc - ad)i|^2 - |(a - c) + (b - d)i|^2 \\
&= (1 - ac - bd)^2 + (bc - ad)^2 - (a - c)^2 - (b - d)^2 \\
&= 1 - a^2 - b^2 - c^2 - d^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \\
&= (1 - a^2 - b^2)(1 - c^2 - d^2) \\
&= (1 - |a + bi|^2)(1 - |c + di|^2) \\
&= (1 - |\zeta|^2)(1 - |\eta|^2).
\end{aligned}$$

(c_1) Substituting $w = z^2$, we can start by solving the quadratic equation

$$w^2 - 2w + 4 = 0.$$

This has the solutions

$$w = \frac{2 \pm \sqrt{4 - 4 \cdot 4}}{2} = \frac{2 \pm 2\sqrt{1 - 4}}{2} = 1 \pm \sqrt{3}i.$$

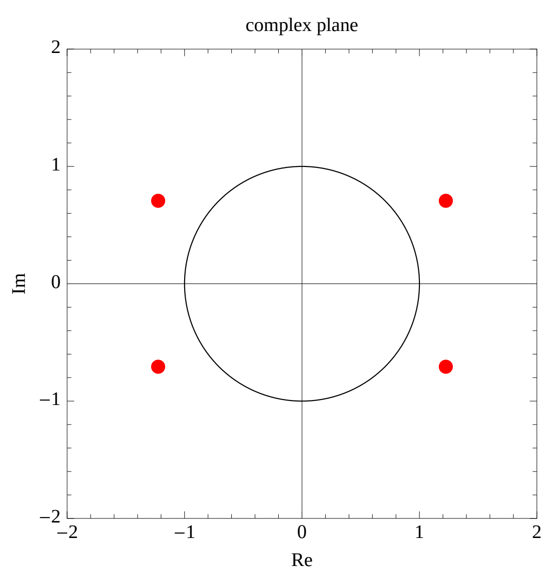
It follows that the solutions z to the original equation satisfy $z^2 = 1 \pm \sqrt{3}i$. From this and from the fundamental theorem of algebra one concludes that the complete list of solutions to the original equation is

$$z_1 = \sqrt{1 + \sqrt{3}i},\ z_2 = \sqrt{1 - \sqrt{3}i},\ z_3 = -\sqrt{1 + \sqrt{3}i},\ \text{and } z_4 = -\sqrt{1 - \sqrt{3}i}.$$

These can be put in normal form

$$z_1 = \sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}}i,\ z_2 = \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}}i,\ z_3 = -\sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}}i,\ \text{and } z_4 = -\sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}}i.$$

and represented (together with the unit circle) in the complex plane as follows:



(c_2) Write $z = x + iy$ for some $x, y \in \mathbb{R}$. Then the equation $e^z = i$ can be rewritten as $e^x e^{iy} = i$, which in view of Euler's formula is equivalent to

$$(e^x \cos y) + (e^x \sin y)i = i.$$

In particular,

$$e^x \cos y = 0 \text{ and } e^x \sin y = 1.$$

Since the exponential function is strictly positive on the whole real line, the first equation means that necessarily we must have $\cos y = 0$, which happens for real y if and only if

$$y = \frac{\pi}{2} + n\pi, \text{ for some } n \in \mathbb{Z}.$$

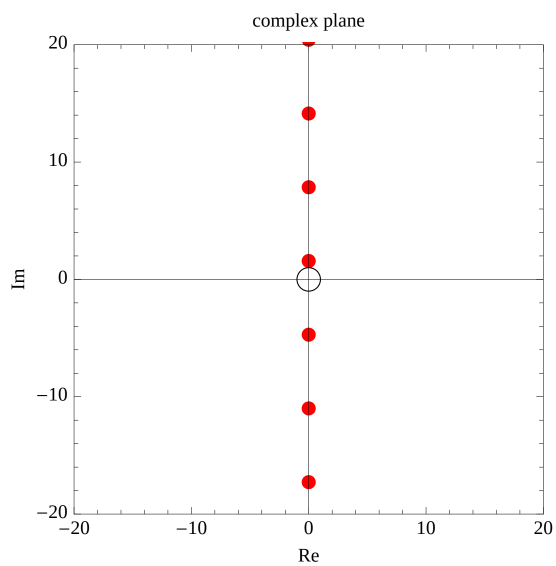
For such y,

$$\sin y = \sin\left(\frac{\pi}{2} + n\pi\right) = (-1)^n,$$

and so for the second equation $e^x \sin y = 1$ to be fulfilled, one needs n to be an even integer, and $x = 0$. In other words, the candidates z to being a solution of the original equation $e^z = i$ are the numbers of the form

$$z_n = \left(\frac{\pi}{2} + 2n\pi\right)i, \quad (n \in \mathbb{Z}). \tag{1}$$

The first six elements of this sequence are depicted here (together with the unit circle):



It is straightforward to check that every element of the sequence $\{z_n\}$ defined in (1) is indeed a solution to the original equation, and we are done.

Problem 2 (Power series)

(a) We start by recognizing the first power series as a geometric series:

$$\sum_{n=1}^{\infty} 3^{\frac{n}{2}} e^{-n} x^n = \sum_{n=1}^{\infty} \left(\frac{\sqrt{3}x}{e}\right)^n.$$

This converges if

$$\left|\frac{\sqrt{3}x}{e}\right| < 1 \Leftrightarrow |x| < \frac{e}{\sqrt{3}}$$

and diverges if

$$\left|\frac{\sqrt{3}x}{e}\right| > 1 \Leftrightarrow |x| > \frac{e}{\sqrt{3}},$$

and so its radius of convergence equals $r_1 = e/\sqrt{3}$.

For the second example, we make use of the so-called ratio test, which states that the radius of convergence r of a power series $\sum_{n=1}^{\infty} a_n x^n$ equals

$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

provided this limit exists. We leave the proof of the ratio test as an exercise to the reader. In this case, we have that

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{a_n}{a_{n+1}} = \frac{n!}{n^n} \frac{(n+1)^{n+1}}{(n+1)!} = \left(1 + \frac{1}{n}\right)^n,$$

which is a sequence that converges to e. Thus $r_2 = e$.

For the third and fourth examples, we make use of the Cauchy-Hadamard theorem, which states that the radius of convergence r of a power series $\sum_{n=1}^{\infty} a_n x^n$ is given by

$$r = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

Sketch of the proof of the Cauchy-Hadamard theorem. Let $R := (\limsup_{n\to\infty} |a_n|^{1/n})^{-1}$. We will show that the power series $\sum_{n=1}^{\infty} a_n x^n$ converges for $|x| < R$, and then that it diverges for $|x| > R$.

For the first part, let $\epsilon > 0$ be given. By definition of lim sup, there are only finitely many a_n for which

$$|a_n|^{1/n} \geq R^{-1} + \epsilon.$$

This means that

$$|a_n| < (R^{-1} + \epsilon)^n$$

for all but a finite number of a_n. In particular, by comparison with a geometric series, the power series $\sum_{n=1}^{\infty} a_n x^n$ is seen to be convergent for every $x \in \mathbb{R}$ satisfying $|x| < \frac{1}{R^{-1}+\epsilon}$. Since $\epsilon > 0$ can be chosen arbitrarily small, this means that the power series in question converges for $|x| < R$, as desired.

For the second part, let $\epsilon > 0$ again be given. By definition of lim sup,

$$|a_n| \geq (R^{-1} - \epsilon)^n$$

for infinitely many a_n. It follows that, if $|x| = \frac{1}{R^{-1}-\epsilon} > R$, then the power series $\sum_{n=1}^{\infty} a_n x^n$ cannot converge because its nth term does not tend to 0. □

For the third example,

$$|a_n|^{1/n} = \left(\frac{n^3 |\sin n|}{3^n}\right)^{1/n} = \frac{n^{3/n} |\sin n|^{1/n}}{3}.$$

We already know that $n^{1/n} \to 1$ as $n \to \infty$, and so

$$\lim_{n\to\infty} n^{3/n} = (\lim_{n\to\infty} n^{1/n})^3 = 1^3 = 1.$$

On the other hand, $0 < |\sin n| \leq 1$ for every $n \in \mathbb{N}_{\geq 1}$. Moreover we have that

$$\sin x > \frac{1}{2} \text{ provided } \frac{\pi}{6} + 2\pi n < x < \frac{5\pi}{6} + 2\pi n, \text{ for some } n \in \mathbb{Z}.$$

This, together with the observation that, for each $n \in \mathbb{Z}$, the interval

$$\left[\frac{\pi}{6} + 2\pi n, \frac{5\pi}{6} + 2\pi n\right]$$

has length $2\pi/3 > 1$ and therefore contains an integer, allows us to show that

$$1 = \lim_{n\to\infty} \left(\frac{1}{2}\right)^{1/n} \leq \limsup_{n\to\infty} |\sin n|^{1/n} \leq 1$$

and therefore conclude that

$$\limsup_{n\to\infty} |\sin n|^{1/n} = 1.$$

It follows that

$$\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1\cdot 1}{3} = \frac{1}{3},$$

and, by Cauchy-Hadamard, $r_3 = 3$.

Remark. The sequence $\{|\sin n|^{1/n}\}$ *actually converges (and its limit must therefore equal 1), but the proof of this fact is a bit trickier. The interested reader is urged to think about it.*

For the fourth example,

$$|a_n|^{1/n} = \left[n\left(1+\frac{3}{n}\right)^{n^2}\right]^{1/n} = n^{1/n}\left(1+\frac{3}{n}\right)^n.$$

As before, we have that $\lim_{n\to\infty} n^{1/n} = 1$. Moreover,

$$\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n = e^3.$$

It follows that

$$\lim_{n\to\infty} |a_n|^{1/n} = 1\cdot e^3 = e^3,$$

and, by Cauchy-Hadamard, $r_4 = e^{-3}$.

(b) Again by Cauchy-Hadamard we have that

$$r_a := (\limsup_{n\to\infty} |a_n|^{1/n})^{-1} \text{ and } r_b := (\limsup_{n\to\infty} |b_n|^{1/n})^{-1} \text{ and } r := (\limsup_{n\to\infty} |a_n b_n|^{1/n})^{-1}.$$

The desired inequality $r \geq r_a \cdot r_b$ is therefore equivalent to

$$(\limsup_{n\to\infty} |a_n b_n|^{1/n})^{-1} \geq (\limsup_{n\to\infty} |a_n|^{1/n})^{-1} \cdot (\limsup_{n\to\infty} |b_n|^{1/n})^{-1},$$

which can be rewritten as

$$(\limsup_{n\to\infty} |a_n|^{1/n}) \cdot (\limsup_{n\to\infty} |b_n|^{1/n}) \geq (\limsup_{n\to\infty} |a_n b_n|^{1/n}).$$

This follows from an elementary property about lim sup which we state but do not prove: if $\{\alpha_n\}$ and $\{\beta_n\}$ are arbitrary sequences of *nonnegative* real numbers, then

$$\limsup_{n\to\infty}(\alpha_n\beta_n) \leq \Big(\limsup_{n\to\infty} \alpha_n\Big)\cdot\Big(\limsup_{n\to\infty} \beta_n\Big).$$

Note that strict inequality might occur in this last inequality: as a simple but instructive example, consider

$$\alpha_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \quad \text{and } \beta_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Problem 3 (Integration by parts)

(a) We have already proved that the product rule

$$(fg)' = f'g + fg' \tag{2}$$

holds in the case when both f and g are monotonically increasing, convex functions: this is the content of Theorem 8.2 from the Skript. In particular, (2) holds for *absolutely monotonic* functions, for these are in particular increasing (since the first derivative is nonnegative) and convex (since the second derivative is nonnegative).

We claim that (2) continues to hold if f, g are merely assumed to be real analytic as in the assumptions of the problem. This can be checked as follows: by definition, there exist absolutely monotonic functions f_1, f_2, g_1, g_2 such that

$$f = f_1 - f_2 \text{ and } g = g_1 - g_2.$$

As such,

$$fg = (f_1 - f_2)(g_1 - g_2) = f_1g_1 - f_1g_2 - f_2g_1 + f_2g_2,$$

and so

$$\begin{aligned}
(fg)' &= (f_1g_1 - f_1g_2 - f_2g_1 + f_2g_2)' = (f_1g_1)' - (f_1g_2)' - (f_2g_1)' + (f_2g_2)' \\
&= (f_1'g_1 + f_1g_1') - (f_1'g_2 + f_1g_2') - (f_2'g_1 + f_2g_1') + (f_2'g_2 + f_2g_2') \\
&= (f_1'g_1 - f_1'g_2 - f_2'g_1 + f_2'g_2) + (f_1g_1' - f_1g_2' - f_2g_1' + f_2g_2') \\
&= (f_1' - f_2')(g_1 - g_2) + (f_1 - f_2)(g_1' - g_2') \\
&= (f_1 - f_2)'(g_1 - g_2) + (f_1 - f_2)(g_1 - g_2)' \\
&= f'g + fg',
\end{aligned}$$

as desired.

Next, we can integrate both sides of identity (2) from a to b and obtain

$$\int_a^b (fg)'(x)dx = \int_a^b \Big(f'(x)g(x) + f(x)g'(x)\Big)dx. \tag{3}$$

Using the fundamental theorem of calculus,[1] we obtain for the left-hand side:

$$\int_a^b (fg)'(x)dx = f(b)g(b) - f(a)g(a). \tag{4}$$

The desired integration-by-parts formula follows from linearity of the integral together with identities (3) and (4).

(b_1) We integrate by parts once:

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx &= \int_{-\pi}^{\pi} \Big(\frac{\sin(nx)}{n}\Big)' \cos(mx)dx \\
&= \frac{\sin(n\pi)}{n}\cos(m\pi) - \frac{\sin(-n\pi)}{n}\cos(-m\pi) - \int_{-\pi}^{\pi} \frac{\sin(nx)}{n}(\cos(mx))'dx \\
&= \frac{0}{n}(-1)^m - \frac{0}{n}(-1)^{-m} - \int_{-\pi}^{\pi} \frac{\sin(nx)}{n}(-m\sin(mx))dx \\
&= \frac{m}{n}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx,
\end{aligned}$$

and then once again:

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx &= \int_{-\pi}^{\pi} \Big(-\frac{\cos(nx)}{n}\Big)' \sin(mx)dx \\
&= 0 - 0 + \int_{-\pi}^{\pi} \frac{\cos(nx)}{n}(\sin(mx))'dx \\
&= \frac{m}{n}\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx
\end{aligned}$$

[1] Caveat: the fundamental theorem of calculus as presented in class (Theorems 7.2 and 7.4 from the Skript) holds for monotonically increasing, right-continuous functions only. The adjustments to make it work for general real analytic functions are similar to what we did before, and are therefore omitted.

to conclude that

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \left(\frac{m}{n}\right)^2 \int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx.$$

From this identity it immediate that

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = 0 \text{ unless } m = n.$$

If $m = n = 0$, then

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \int_{\pi}^{\pi} \underbrace{\cos 0}_{=1}\underbrace{\cos 0}_{=1} dx = \int_{-\pi}^{\pi} dx = 2\pi.$$

Finally, if $m = n \neq 0$, then

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \int_{-\pi}^{\pi} \cos^2(nx)dx = \int_{-\pi}^{\pi} \frac{1+\cos(2nx)}{2}dx = \pi.$$

Here we used the trigonometric identity, valid for any real t,

$$\cos^2 t = \frac{1+\cos 2t}{2},$$

and the fact that, for any $k \in \mathbb{N}_{\geq 1}$,

$$\int_{-\pi}^{\pi} \cos(kt)dt = \int_{-\pi}^{\pi} \left(\frac{\sin(kt)}{k}\right)' dt = \frac{\sin(k\pi)}{k} - \frac{\sin(-k\pi)}{k} = 0 - 0 = 0.$$

All in all we have that

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \begin{cases} 2\pi & \text{if } n = m = 0, \\ \pi & \text{if } n = m \neq 0, \\ 0 & \text{if } n \neq m. \end{cases}$$

This innocent-looking formula is a possible starting point for the study of Fourier analysis.

(b_2) Our goal is to compute

$$I_n := \int_0^{\frac{\pi}{2}} \sin^{2n}(x)dx$$

for $n \in \mathbb{N}$. Start by computing

$$I_0 = \int_0^{\frac{\pi}{2}} \sin^0(x)dx = \int_0^{\frac{\pi}{2}} 1dx = \frac{\pi}{2}$$

and

$$I_1 = \int_0^{\frac{\pi}{2}} \sin^2(x)dx = \int_0^{\frac{\pi}{2}} \frac{1-\cos(2x)}{2}dx = \frac{1}{2}\cdot\frac{\pi}{2}$$

and

$$\begin{aligned} I_2 &= \int_0^{\frac{\pi}{2}} \sin^4(x)dx = \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos(2x)}{2}\right)^2 dx = \int_0^{\frac{\pi}{2}} \frac{1-2\cos(2x)+\cos^2(2x)}{4}dx \\ &= \int_0^{\frac{\pi}{2}} \frac{1-2\cos(2x)+\frac{1+\cos(4x)}{2}}{4}dx = \int_0^{\frac{\pi}{2}} \left(\frac{3}{8}-\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)\right)dx = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}. \end{aligned}$$

Hopefully we start to see a pattern emerging. We might be tempted to conjecture that, for higher values of n,

$$I_n = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\ldots\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}. \tag{5}$$

To verify our conjecture, let us integrate I_n by parts once:

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \sin^{2n}(x)dx = \int_0^{\frac{\pi}{2}} \sin(x)\sin^{2n-1}(x)dx = \int_0^{\frac{\pi}{2}} (-\cos(x))'\sin^{2n-1}(x)dx \\ &= \int_0^{\frac{\pi}{2}} \cos(x)(\sin^{2n-1}(x))'dx = (2n-1)\int_0^{\frac{\pi}{2}} \cos^2(x)\sin^{2n-2}(x)dx. \end{aligned} \tag{6}$$

This seems not to be quite enough, and so one might be tempted to integrate by parts again. However, if one just recalls the mother of all trigonometric identities,

$$\sin^2(x) + \cos^2(x) = 1,$$

valid for every $x \in \mathbb{R}$, one sees that (6) can be rewritten as

$$I_n = (2n-1)(I_{n-1} - I_n),$$

or equivalently

$$I_n = \frac{2n-1}{2n} I_{n-1}.$$

The result (5) now follows by a straightforward induction on n, and can be equivalently rewritten as

$$I_n = \frac{(2n)!}{2^{2n}(n!)^2}\frac{\pi}{2}. \quad (n \in \mathbb{N})$$

This is known as (a special case of) Wallis' formula, and has a number of applications in classical analysis. Can you think of any?

Problem 4 (Mean value theorem and Cauchy's generalization)

(a) We provide a proof which does not minimize length but introduces and builds upon several useful results in analysis. Let us proceed in five steps.

Step 1. *Every bounded sequence $\{x_n\}$ in $\mathbb{R}$ has a convergent subsequence.*

Proof. By a slight variation of the Heine-Borel theorem seen in class (Theorem 3.11 from the Skript), the sequence $\{x_n\}$ has a monotonic subsequence. This subsequence is necessarily bounded as well. A bounded, monotonic sequence converges to a finite limit, and we are done. □

Step 2. *Let $g : [a, b] \to \mathbb{R}$ be a continuous function. Then g is bounded.*

Proof. Aiming at a contradiction, suppose that the function g is not bounded from above on the interval $[a, b]$. This implies that, for every natural number $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $g(x_n) > n$. This defines a sequence $\{x_n\}$ taking values in $[a, b]$. Since $[a, b]$ is bounded, Step 1 implies that there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let x be its limit (which necessarily belongs to $[a, b]$). Since the function g is continuous at x, the sequence $\{g(x_{n_k})\}$ converges to the real number $g(x)$. But $g(x_{n_k}) > n_k \geq k$ for every $k \in \mathbb{N}$, which implies that the sequence $\{g(x_{n_k})\}$ diverges to $+\infty$, a contradiction. The contradiction resulted from assuming that g is not bounded from above. A similar reasoning shows that g is bounded from below, and therefore g is bounded, as desired. □

Step 3. *If $h : [a, b] \to \mathbb{R}$ is a continuous function, then h must attain a maximum and a minimum (each at least once).*

Proof. By Step 2, the function h is bounded from above. Let

$$M := \sup_{x\in[a,b]} h(x).$$

Our goal is to find $y \in [a,b]$ such that $h(y) = M$. Let $n \in \mathbb{N}$. Since M is the least upper bound for h on the interval $[a,b]$, the number $M - 1/n$ is not an upper bound for h on $[a,b]$. Therefore, there exists $y_n \in [a,b]$ such that

$$M - \frac{1}{n} < h(y_n).$$

This defines a sequence $\{y_n\}$ taking values in $[a,b]$. Since M is an upper bound for h on $[a,b]$, we additionally have that, for every $n \in \mathbb{N}$,

$$M - \frac{1}{n} < h(y_n) \leq M.$$

It follows that the sequence $\{h(y_n)\}$ converges to M. By Step 1, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ which converges to some $y \in [a,b]$. Since h is continuous at y, the sequence $\{h(y_{n_k})\}$ converges to $h(y)$. But $\{h(y_{n_k})\}$ is a subsequence of $\{h(y_n)\}$, and the latter is already known to converge to M; therefore so does the former. It follows that $M = h(y)$, which means that the function h attains its supremum. Proceed similarly for the infimum. □

Step 4. *If $r : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) and such that $r(a) = r(b)$, then there exists $c \in (a,b)$ such that $r'(c) = 0$.*

Proof. By Step 3, the function r attains both its maximum and its minimum in $[a,b]$. If these are both attained at the endpoints of $[a,b]$, then r is constant on $[a,b]$ and so the derivative of r is identically zero, and we are done. Suppose now that the maximum is attained at an interior point $c \in (a,b)$. (The argument for the minimum is entirely analogous: one just considers the function $-r$ instead of r.) For a real δ such that $c + \delta \in [a,b]$, the value of $r(c+\delta)$ is smaller than or equal to $r(c)$ because r attains its maximum at c. Therefore, for every $\delta > 0$,

$$\frac{r(c+\delta) - r(c)}{\delta} \leq 0,$$

and so

$$r'(c_+) := \lim_{\delta \to 0^+} \frac{r(c+\delta) - r(c)}{\delta} \leq 0.$$

(Note that this limit exists by our assumptions on the function r.) Similarly, for every $\delta < 0$, the inequality turns around because the denominator is now negative, and we get

$$r'(c_-) := \lim_{\delta \to 0^-} \frac{r(c+\delta) - r(c)}{\delta} \geq 0.$$

Since r is differentiable at c, we conclude that

$$r'(c) = r'(c_-) = r'(c_+) = 0.$$

□

Step 5. *Conclude.*

Let $f : [a,b] \to \mathbb{R}$ be a continuous function which is continuously differentiable in the open interval (a,b). Consider the auxiliary function

$$s(x) := f(x) - cx, \tag{7}$$

where c is a constant to be chosen below. Since f is continuous on $[a,b]$ and continuously differentiable on (a,b), the same is true for s. We now want to choose c in such a way that the function s satisfies the extra condition of Step 4 above. Namely,

$$s(a) = s(b) \Leftrightarrow f(a) - ca = f(b) - cb \Leftrightarrow c = \frac{f(b) - f(a)}{b - a}.$$

Let us take this choice of c. Since the function s satisfies all the assumptions of Step 4, we conclude the existence of a point $x_1 \in (a,b)$ for which $s'(x_1) = 0$. It follows from (7) that

$$f'(x_1) = s'(x_1) + c = 0 + c = \frac{f(b) - f(a)}{b - a},$$

and we are done.

Remark. Steps 1, 2, 3 and 4 go sometimes in the literature under the names of *Bolzano-Weierstrass theorem*, *boundedness theorem*, *extreme value theorem* and *Rolle's theorem*, respectively.

(b) This turns out to be an important generalization of part (a) - originally due to none other than our old friend August-Louis Cauchy - which can be proved in *almost the same way.*

Given functions f, g satisfying the conditions of the problem, define the auxiliary function

$$s(x) := f(x) - cg(x).$$

The constant c is chosen, as before, in such a way that $s(a) = s(b)$. This happens if and only if

$$c = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

With this choice of c, the function s satisfies the hypotheses of Step 4 of part (a). In particular, there exists $x_2 \in (a, b)$ such that $s'(x_2) = 0$. This means that

$$0 = s'(x_2) = f'(x_2) - cg'(x_2) = f'(x_2) - \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right) g'(x_2),$$

and so

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_2)}{g'(x_2)},$$

as desired.