Analysis 1, Solutions to Übungsblatt Nr. 10

Problem 1 (Continuity)

(a) Let us start by verifying that the function f is continuous at $x = 0$. We make use of Definition 5.27, by which it is enough to show that

$$\forall \epsilon > 0 \exists \delta > 0 \forall y \in [0,1] : |0 - y| < \delta \Rightarrow |f(0) - f(y)| < \epsilon.$$

Let $\epsilon > 0$ be given, and choose $\delta = \epsilon$. Let y be any number in $[0,1]$ such that

$$|0 - y| = |y| < \epsilon.$$

We split the analysis into two cases, depending on whether y is a dyadic number or not. In the first case, $y \in \mathbb{Y}$, we have that

$$|f(0) - f(y)| = |0 - f(y)| = |f(y)| = |y| < \epsilon.$$

In the second case, $y \in [0,1] \setminus \mathbb{Y}$, we have that

$$|f(0) - f(y)| = |0 - 0| = 0 < \epsilon.$$

In both cases we conclude that

$$|f(0) - f(y)| < \epsilon.$$

It follows that the function f is continuous at 0, as desired.

Let $x_0 \in [0,1] \setminus \{0\} =: (0,1]$ be given. Our next goal is to show that the function f is discontinuous at $x = x_0$. Again by Definition 5.27, this amounts to showing

$$\exists \epsilon > 0 \forall \delta > 0 \exists x \in [0,1] : |x_0 - x| < \delta \wedge |f(x_0) - f(x)| \geq \epsilon.$$

We claim that taking $\epsilon = x_0/2$ will do the job. (Notice that this is a positive number *since* $x_0 \neq 0$.) Again we split the analysis into two cases, according to whether x_0 is a dyadic number or not.

In the first case, $x_0 \in \mathbb{Y}$, let $\epsilon := x_0/2$ and let $\delta > 0$ be given. Pick $x \in [0,1] \setminus \mathbb{Y}$ such that

$$|x_0 - x| < \delta.$$

That such x exists can be seen in a number of ways (e.g. by using the fact that $\{\sqrt{2}/n\}_{n \geq 2}$ is a sequence of elements in $[0,1] \setminus \mathbb{Y}$ converging to 0). Since $x_0 \in \mathbb{Y}$, we have that $f(x_0) = x_0$. Since $x \in [0,1] \setminus \mathbb{Y}$, we have that $f(x) = 0$. But then

$$|f(x_0) - f(x)| = |x_0 - 0| = x_0 \geq \frac{x_0}{2} = \epsilon.$$

In the second case, $x_0 \in [0,1] \setminus \mathbb{Y}$, let again $\epsilon := x_0/2$ and let $\delta > 0$ be given. We lose no generality in assuming that $\delta < x_0/2$, for if $|x - x_0| < \delta$ holds for some $\delta = \delta_0$, it will also hold for any $\delta > \delta_0$. Pick $x \in \mathbb{Y} \cap [0,1]$ such that

$$|x - x_0| < \delta.$$

That such x exists is an immediate consequence of Theorem 2.18 (Archimedes). Reasoning similarly as before, we conclude that

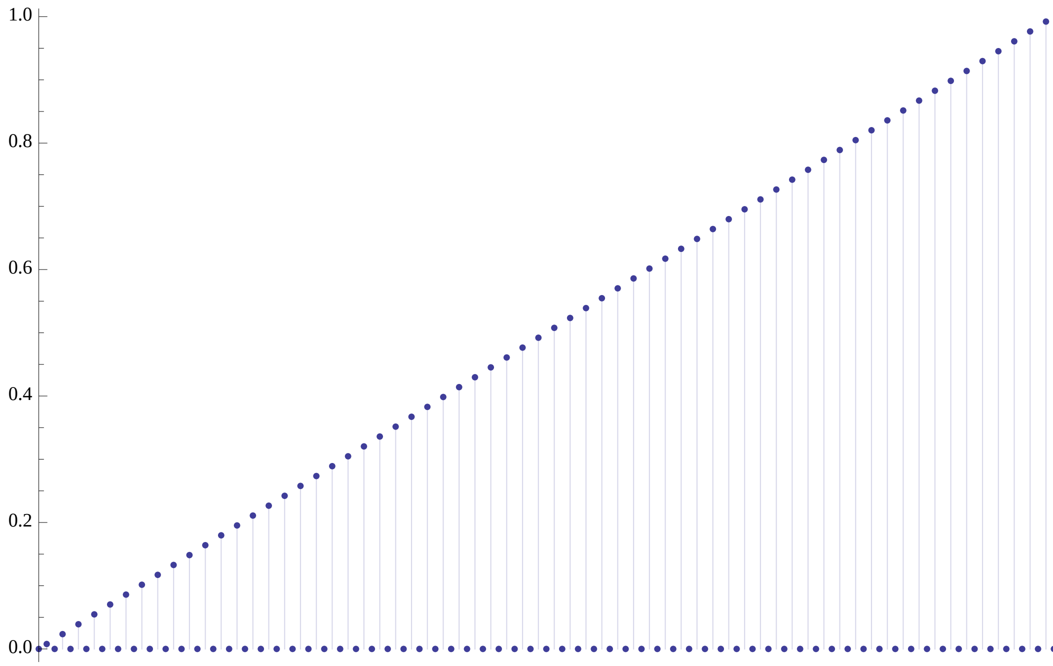
$$|f(x_0) - f(x)| = |0 - x| = x > x_0 - \delta \geq \frac{x_0}{2} = \epsilon.$$

In both cases, we were able to produce an absolute $\epsilon > 0$ and a number $x \in [0,1]$ (depending on δ but on nothing else) such that

$$|x_0 - x| < \delta \text{ but } |f(x_0) - f(x)| \geq \epsilon.$$

This means that the function f is discontinuous at $x = x_0$, as claimed.

An approximate graphical representation of the function f is as follows (in this particular representation, points in $[0,1] \cap \bigcup_{k=0}^{6} \mathbb{Y}_k$ appear with the correct f-value):



(b) Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f([a, b]) \subset [a, b]$. We want to show that $f(x_0) = x_0$ for some $x_0 \in [a, b]$.

We lose no generality in assuming that $f(a) \neq a$ and $f(b) \neq b$, for otherwise we would be done. If $f(a) \neq a$ and $f([a, b]) \subset [a, b]$, then $f(a) > a$, or $a - f(a) < 0$. Similarly, $f(b) < b$, or $b - f(b) > 0$. Consider the auxiliary function

$$\begin{aligned} g : [a, b] &\to \mathbb{R} \\ x &\mapsto g(x) := x - f(x). \end{aligned}$$

Then g is a continuous function (since it is the sum of two continuous functions) which satisfies

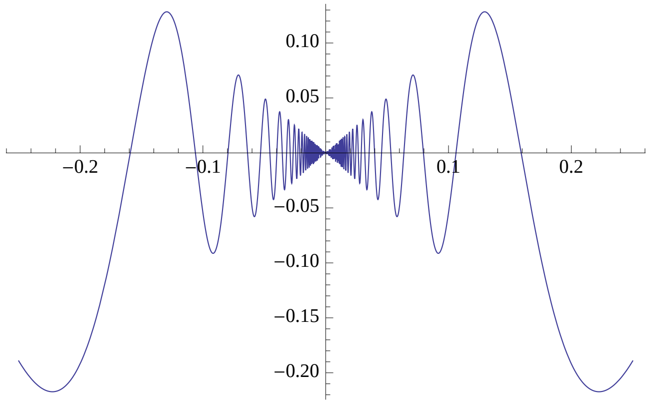
$$\begin{aligned} g(a) &= a - f(a) < 0, \text{ and} \\ g(b) &= b - f(b) > 0. \end{aligned}$$

A straightforward modification of the Intermediate Value Theorem (Theorem[1] 5.15 from the Skript) implies that the function g must have a zero in the interval (a, b). In other words, there exists $x_0 \in (a, b)$ such that $g(x_0) = 0$. This happens if and only if $f(x_0) = x_0$ i.e. if and only if x_0 is a fixed point for f, and we are done.

[1]Note carefully that the monotonicity assumption can be dropped.

Problem 2 (Differentiability)

(a) Here's a sketch of the function f_1 on the interval $[-1/4, 1/4]$:



We want to show that the function f_1 is *not* differentiable at $x = 0$. This amounts to showing that the limit

$$\lim_{h\to 0} \frac{f_1(0+h) - f_1(0)}{h} = \lim_{h\to 0} \frac{f_1(h) - 0}{h} = \lim_{h\to 0} \frac{1}{\cancel{h}} \cancel{h} \sin\left(\frac{1}{h}\right) = \lim_{h\to 0} \sin\left(\frac{1}{h}\right)$$

does not exist. This might already have been discussed in Problem 4(a) from PB10, but we recall it here. Define the sequences

$$a_n := \frac{1}{\pi n} \text{ and } b_n := \frac{1}{\frac{\pi}{2} + \pi n}.$$

Then

$$\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n,$$

but

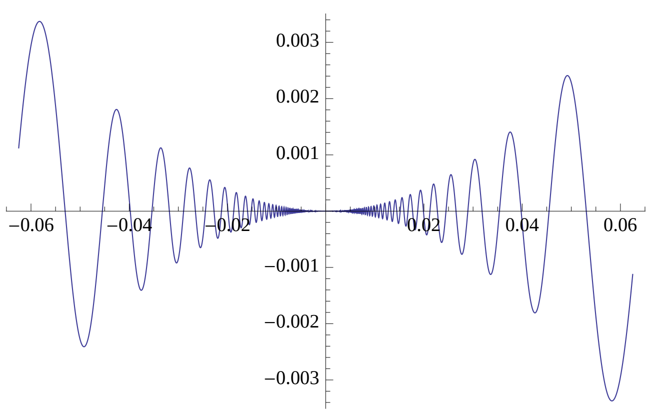
$$\lim_{n\to\infty} \sin\left(\frac{1}{a_n}\right) = \lim_{n\to\infty} \sin(\pi n) = 0 \neq 1 = \limsup_{n\to\infty} \left\{ \sin\left(\frac{\pi}{2} + \pi n\right) \right\} = \limsup_{n\to\infty} \left\{ \sin\left(\frac{1}{b_n}\right) \right\}.$$

This shows that the limit

$$\lim_{h\to 0} \sin\left(\frac{1}{h}\right)$$

does not exist, and therefore the function f_1 is not differentiable at 0, as desired.

(b) Here's a sketch of the function f_2 on the interval $[-1/16, 1/16]$:



This time we need to show that the limit

$$\lim_{h\to 0} \frac{f_2(0+h) - f_2(0)}{h} = \lim_{h\to 0} \frac{f_2(h) - 0}{h} = \lim_{h\to 0} \frac{1}{h} h^2 \sin\left(\frac{1}{h}\right) = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right)$$

exists and is a finite real number. Indeed, we claim that

$$\lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0, \tag{1}$$

and this is the content of Problem 4(b) from PB10. We recall it here: first of all,

$$\lim_{h\to 0^+} h\sin\left(\frac{1}{h}\right) = \lim_{h\to 0^+} \frac{\sin\left(\frac{1}{h}\right)}{\frac{1}{h}} = \lim_{t\to+\infty} \frac{\sin t}{t}.$$

In the solution of Problem 1(a) from ÜB9, we have already argued that $\lim_{t\to+\infty} \frac{\sin t}{t} = 0$, and so

$$\lim_{h\to 0^+} h\sin\left(\frac{1}{h}\right) = 0.$$

Similarly,

$$\lim_{h\to 0^-} h\sin\left(\frac{1}{h}\right) = 0.$$

Since the two side limits equal zero, (1) follows. Moreover,

$$f_2'(0) = \lim_{h\to 0} \frac{f_2(0+h) - f_2(0)}{h} = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0.$$

(c) From part (b) we already know that $f_2'(0) = 0$. If $x \neq 0$, then the product rule implies that

$$f_2'(x) = \left(x^2\sin\left(\frac{1}{x}\right)\right)' = (x^2)'\sin\left(\frac{1}{x}\right) + x^2\left(\sin\left(\frac{1}{x}\right)\right)'.$$

We have seen in class that $(x^2)' = 2x$. On the other hand, it is easy to check that $(x^{-1})' = -x^{-2}$, and then the chain rule implies that

$$\left(\sin\left(\frac{1}{x}\right)\right)' = \left(\frac{1}{x}\right)'\sin'\left(\frac{1}{x}\right) = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$$

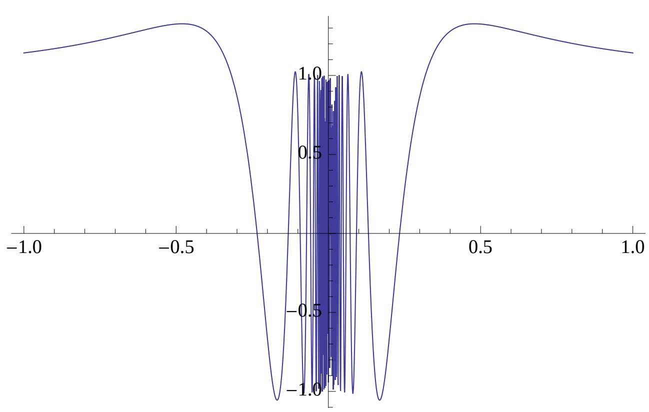
since $\sin' = \cos$. It follows that

$$\begin{aligned} f_2'(x) &= (x^2)'\sin\left(\frac{1}{x}\right) + x^2\left(\sin\left(\frac{1}{x}\right)\right)' \\ &= 2x\sin\left(\frac{1}{x}\right) + x^2\left(-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)\right) \\ &= 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right). \end{aligned}$$

We conclude that

$$f_2'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \in (-1,1)\setminus\{0\}, \\ 0 & \text{if } x = 0. \end{cases}$$

This is a rather pathological function (for instance, it is not continuous at $x = 0$) which is nevertheless bounded on the interval $(-1, 1)$ and looks approximately as follows:



Problem 3 (A differential inequality)

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying

$$f'(x) > f(x),\ \forall x \in \mathbb{R}. \tag{2}$$

(a) Let $x_0 \in \mathbb{R}$ be such that $f(x_0) = 0$. By definition,

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Since $f(x_0) = 0$, assumption (2) at $x = x_0$ translates into $f'(x_0) > 0$, or

$$\lim_{h\to 0} \frac{f(x_0 + h) - 0}{h} > 0.$$

In particular,

$$\lim_{h\to 0^+} \frac{f(x_0 + h)}{h} > 0.$$

We can therefore guarantee the existence of $h_0 > 0$ such that

$$\frac{f(x_0 + h)}{h} > 0$$

provided $0 < h < h_0$. Pick such h. Since $h > 0$, this implies that

$$f(x_0 + h) > 0.$$

Set $x := x_0 + h$. Then $x > x_0$ because $h > 0$, and $f(x) = f(x_0 + h) > 0$, and we are done.

(b) Since the exponential function is (strictly) positive on the whole real line, assumption (2) is equivalent to the following statement:

$$e^{-x} f'(x) > e^{-x} f(x), \forall x \in \mathbb{R}.$$

This can be rewritten as

$$e^{-x} f'(x) - e^{-x} f(x) > 0, \forall x \in \mathbb{R},$$

or even

$$(e^{-x} f(x))' > 0, \forall x \in \mathbb{R}. \tag{3}$$

This last reformulation is a consequence of the product rule, which in particular states that

$$(e^{-x} f(x))' = (e^{-x})' f(x) + e^{-x} f'(x) = -e^{-x} f(x) + e^{-x} f'(x).$$

By a slight refinement of Theorem 6.3.2, from (3) we have that

$$\int_{x_0}^{x} (e^{-t} f(t))' dt > 0, \quad \forall x > x_0.$$

On the other hand, from the second part of the fundamental theorem of calculus (Theorem 7.4), we have that

$$\int_{x_0}^{x} (e^{-t} f(t))' dt = e^{-x} f(x) - e^{-x_0} f(x_0) = e^{-x} f(x)$$

since $f(x_0) = 0$. It follows that

$$e^{-x} f(x) > 0, \forall x > x_0.$$

Again because of strict positivity of the exponential function, this is equivalent to the desired conclusion,

$$f(x) > 0, \forall x > x_0.$$

Problem 4 (Series)

(a) Let us start by assuming that the "condensed" series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} < \infty$$

converges. Then

$$\begin{aligned} 0 \leq \sum_{k=1}^{\infty} a_k &= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + (a_8 + \ldots + a_{15}) + \ldots \\ &\leq a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + (a_8 + \ldots + a_8) + \ldots \\ &= \sum_{k=0}^{\infty} 2^k a_{2^k}, \end{aligned}$$

and so the series $\sum_{k=1}^{\infty} a_k$ converges as well. Here we just used the fact that the sequence $\{a_k\}$ is nonincreasing. Conversely, assume that the series $\sum_{k=1}^{\infty} a_k$ converges. Then

$$\begin{aligned} 0 \leq \sum_{k=0}^{\infty} 2^k a_{2^k} &= a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots \\ &= a_1 + 2\Big(a_2 + 2a_4 + 4a_8 + \ldots\Big) \\ &\leq a_1 + 2\Big(a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \ldots\Big) \\ &= -a_1 + 2\sum_{k=1}^{\infty} a_k, \end{aligned}$$

and so the series $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges as well.

Remarks. Strictly speaking, the various instances of '...' make the former proof non-rigorous. One should work with partial sums instead, truncated at N say, and prove each statement by induction on N.

The result we just proved goes in the literature under the name of "Cauchy condensation criterion", and it provides a useful tool to verify the convergence of series which would otherwise be hard to study. As a further application of this criterion, the reader is encouraged to revisit the α-series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}, \quad \text{for } 0 < \alpha < \infty.$$

(b) Yes. The proof outlined in part (a) goes essentially through with only minor modifications (omitted). Under the assumptions of the problem, one can prove more generally that, given a natural number $m \in \mathbb{N} : m \geq 2$, the series $\sum_{k=1}^{\infty} a_k$ converges if and only the series $\sum_{k=0}^{\infty} m^k a_{m^k}$ converges.

(c) Let us use the criterion of part (b) with $m = 10$. For this purpose, define

$$b_n := \frac{1}{nd(n)}$$

and note that

$$d(10^n) = d(1\underbrace{0\ldots0}_{n\ 0's}) = n + 1. \tag{4}$$

It follows that

$$10^n b_{10^n} = \cancel{10^n}\frac{1}{\cancel{10^n}d(10^n)} = \frac{1}{d(10^n)} = \frac{1}{n+1},$$

and so the series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{nd(n)} \tag{5}$$

converges if and only if the series

$$\sum_{n=0}^{\infty} 10^n b_{10^n} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{k=1}^{\infty} \frac{1}{k}$$

converges. But this is the harmonic series, which is by now well-known to be divergent. It follows that the series (5) diverges as well.

Further define

$$c_n := \frac{1}{nd(n)d(d(n))} \text{ and } d_n := \frac{1}{nd(n)d(d(n))d(d(d(n)))}.$$

Using (4), we have that

$$10^n c_{10^n} = \cancel{10^n} \frac{1}{\cancel{10^n} d(10^n)d(d(10^n)))} = \frac{1}{(n+1)d(n+1)} = b_{n+1}$$

and that

$$10^n d_{10^n} = \cancel{10^n} \frac{1}{\cancel{10^n} d(10^n)d(d(10^n))d(d(d(10^n))))} = \frac{1}{(n+1)d(n+1)d(d(n+1))} = c_{n+1}.$$

From part (b), it follows that the series

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{nd(n)d(d(n))}$$

diverges if and only if the series

$$\sum_{n=0}^{\infty} 10^n c_{10^n} = \sum_{n=0}^{\infty} b_{n+1} = \sum_{n=1}^{\infty} \frac{1}{nd(n)}$$

diverges, which we just verified to be the case. Similarly, the series

$$\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty} \frac{1}{nd(n)d(d(n))d(d(d(n)))}$$

diverges if and only if the series

$$\sum_{n=0}^{\infty} 10^n d_{10^n} = \sum_{n=0}^{\infty} c_{n+1} = \sum_{n=1}^{\infty} \frac{1}{nd(n)d(d(n))}$$

diverges, which we just verified to be the case. The conclusion is that all three series

$$\sum_{n=1}^{\infty} \frac{1}{nd(n)}, \sum_{n=1}^{\infty} \frac{1}{nd(n)d(d(n))}, \text{ and } \sum_{n=1}^{\infty} \frac{1}{nd(n)d(d(n))d(d(d(n)))}$$

diverge, as desired.