Research Summary

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I work in logic and combinatorics. Within logic, I have focused on Computable Analysis and Bounded Arithmetic. Computable Analysis and Real Recursive Functions are two models of computability that deal with real numbers (as opposed to the classical situation which is restricted to natural numbers). The former model is defined using Turing Machines, while the latter is defined in a very different manner, using differential equations. We (joint with Manuel Campagnolo) have demonstrated that despite the difference in presentation, the two models can be specified in a manner that makes them equivalent. I have also begun to (unofficially) advise a Ph.D. student at the Graduate Center (CUNY) in the area of Computable Analysis. Bounded Arithmetic is a weak logical theory contained in Peano Arithmetic; my research in this area has focused on the issue of provability of standard theorems of finite combinatorics, especially Ramsey theory. I have shown that the axioms of Bounded Arithmetic are strong enough to prove certain combinatorial theorems, but not strong enough to prove other theorems. More recently, I have begun work on a graph-theoretic game called Cops and Robber. The key parameter that researchers study is the cop number, that is, the fewest number of cops needed to "catch" the robber. We (joint with David Offner) have focused on the hypercube graph, determining exactly how a number of variations in the rules of the game effect the cop number, thus extending and generalizing the work of others. I now discuss these three areas (for a more detailed *research statement* and preprints of papers, see my web site: http://www.math.ist.utl.pt/∼ojakian/).

Combinatorics

I have mostly completed a paper on the combinatorial game Cops and Robber (joint work with David Offner of Westmister College, New Wilmington, PA); the in-progress paper is available upon request. The game of Cops and Robber is a two player game played on a graph. One player controls some number of cops; the other player controls a single robber. In the standard game, first the cops choose some vertices, then the robber chooses a vertex; the players then alternate turns. On a turn, the robber may stay still or move to an adjacent vertex, likewise for the cops. The cop number of a graph is the least number of cops needed to guarantee the robber will eventually be caught (i.e. the robber is caught if the robber and any cop ever occupy the same vertex). The n-cube (or hypercube) is the graph whose vertices are the length n binary strings with an edge between strings that differ at exactly one position. Various authors have investigated the cop number of the hypercube for a couple of variations in the rules of the game. We have developed a systematic way to describe a whole spectrum of variations in the rules of the game, and have determined the cop number for many of these variations. We generalize earlier work, obtaining earlier results as special cases of our work. Our research has the advantage of being accessible to undergraduates and of interest to serious mathematical researchers. In fact I recently gave well received talks at both an undergraduate math seminar at York College, CUNY, and at the Graduate Center combinatorics seminar.

Computable Analysis

My work in Computable Analysis (joint with Manuel Campagnolo, from Instituto Superior de Agronomia, Lisbon, Portugal) has involved showing that various classes of functions defined using the Computable Analysis model can be exactly characterized by the Real Recursive Functions model. A function is in the Computable Analysis model if a Turing machine can convert rational approximations of the real number input into rational approximations of the output of the function. Since computation involves the discrete steps of a Turing machine, Computable Analysis is called a discrete-time model. The functions of the Real Recursive Functions model are defined via function algebras, i.e. some basic real functions are closed under various operations (a key operation allows us to solve differential equations). Since computation proceeds continuously rather than in a step by step manner, this kind of model is called an analog-time model. We have demonstrated that despite the difference in presentation, the two models (one being discrete-time and the other being analog-time) can be specified in a manner that makes them equivalent.

We have built upon the work of Bournez and Hainry to obtain characterizations of Computable Analysis. We have devised a natural function algebra (very similar to the primitive recursive real functions, a widely studied function algebra in this area) which yields exactly the same functions as those given by Computable Analysis. It is also typical to consider restricting the functions of Computable Analysis to those which can be computed within some time constraint. We have also devised a natural function algebra to exactly characterize the functions of Computable Analysis which are computable in elementary time. A natural direction for future work is to characterize other typical computational classes over the reals. For example, we have partial results on the counting hierarchy, while other researchers have results on the polynomial time computable functions.

Besides improving upon earlier results, our work introduces a new technique we call the *method of approximation*. Our approach appears to be more general and has allowed us to prove stronger results. A direction for future work would be to develop our apparently more general methods into a full blown general theory.

I have begun (unofficially) advising a Ph.D. student at the CUNY Graduate Center on a different topic within Computable Analysis. We have been looking at computablility aspects of topology, focusing on fixed point theorems.

Bounded Arithmetic

My work in Bounded Arithmetic has shown how to prove standard theorems of finite combinatorics from a weak set of axioms; in some cases I have shown that the axioms are insufficient to prove a theorem. Bounded Arithmetic refers to a hierarchy of theories which is of interest because of its connection to the polynomial-time hierarchy of computational complexity. The Bounded Arithmetic hierarchy begins with a theory that has been referred to as “polynomial-time reasoning” or “feasible reasoning” since objects shown to exist in this theory can be constructed in an efficient and feasible manner (i.e. with a polynomial-time function). The theories higher up in the hierarchy formalize (what is believed to be) increasingly less feasible reasoning. The goal of my research in this area has been to position theorems of finite combinatorics within this hierarchy. I have focused on theorems that are proved using probabilistic methods or linear algebra methods, with an emphasis on their application to Ramsey theory.

The Ramsey number $R_r(k)$ is the smallest number N such that if each of the edges of the complete graph on N vertices is assigned one of r colors, then there must be a size k subset X of the vertices, such that all of the edges with vertices in X are assigned the same color (X is called a monochromatic set). The ultimate goal of Ramsey theory is to evaluate $R_r(k)$ exactly for various choices of r and k; since this is often extremely difficult, researchers in combinatorics work on upper and lower bounds. Building on the work of others, I have shown how to prove both upper and lower bounds in Bounded Arithmetic, and have shown that a certain known upper bound can not be proven near the bottom of the Bounded Arithmetic hierarchy.

In the case of the lower bounds, combinatorialists make an interesting distinction between constructive and non-constructive lower bound proofs. A lower bound on a Ramsey number consists of showing that there exists a coloring of a graph with no monochromatic set of the specified size. Combinatorialists refer to a proof as constructive if it exhibits this coloring explicitly and non-constructive if it demonstrates the existence of such a coloring without describing the coloring. The non-constructive lower bounds are typically proven using the probabilistic method. I show how a number of proofs that use the probabilistic method can be carried out in Bounded Arithmetic, including a Ramsey lower bound proof. The constructive lower bounds are proved in a variety of ways, though a popular approach uses bounds on set systems of particular kinds. I show that a number of bounds on set systems can be proven in Bounded Arithmetic, and show how these bounds can be used to prove constructive Ramsey lower bounds in Bounded Arithmetic.

By formalizing the combinatorial facts within an axiomatic framework some interesting issues arise which would not otherwise be apparent. One expected, but interesting phenomenon that occurs in both the case of Ramsey lower bounds and set system bounds is the following: Theorems with better bounds use theories higher up in the hierarchy of Bounded Arithmetic. Another point relates to the informal distinction combinatorialists make between constructive and non-constructive Ramsey lower bounds. From the point of view of Bounded Arithmetic, this distinction becomes precise: The constructive lower bounds can be proven using the feasible reasoning at the bottom of the Bounded Arithmetic hierarchy, while the non-constructive lower bounds using probabilistic arguments, formalize higher up in the hierarchy.

I intend to develop this program more fully, positioning other theorems of finite combinatorics within the hierarchy of Bounded Arithmetic. To facilitate this program, I have work in progress on a higher-order type theory for Bounded Arithmetic. When working in the standard first-order theory of Bounded Arithmetic, the relevant objects are natural numbers, thus forcing us to "code" all our mathematical objects as numbers. Once completed, my type theory should allow us to avoid such cumbersome codings of combinatorial objects, letting us work more directly with the objects in question.