Characterizing Computable Analysis with Differential Equations

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Outline

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Characterizing Computable Analysis

- Results of the form "$\mathcal{CA} =$ FA".
- Real Recursive Functions introduced by C. Moore 1996.
  Function algebras with operations like this:

  Solve a differential equation and keep the result.

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  Characterize Elementary Computable and Computable.

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Computable Analysis

Definition

- $f(x) \in \mathbf{C}(\mathbb{R})$:

  There is a computable function $F^x(n)$ with an oracle for the real number x such that $|f(x) - F^x(n)| \leq 1/n$.

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Function Algebras

Definition

Suppose $\mathcal{B}$ is a set of functions (i.e. the basic functions) and $\mathcal{O}$ is a set of operations. Then $\text{FA}[\mathcal{B};\mathcal{O}]$ is the smallest set of functions containing $\mathcal{B}$ and closed under $\mathcal{O}$.

Basic Functions:

- Constant functions: $0, 1, -1, \pi$
- Projection functions “P” (example: $U(x, y) = x$)
- $\theta_k(x) = \begin{cases} 0, & x < 0; \\ x^k, & x \geq 0. \end{cases}$

Operation: comp (Composes the given functions).

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The Differential Equation Operation

Definition

ODE is the operation:

- Input: Functions: $\overrightarrow{\mathbf{g}}(\bar{x})$, $\overrightarrow{\mathbf{f}}(y,\bar{u},\bar{x})$.
- Output: $h_1(y,\bar{x})$ where $(h_1,\ldots,h_n)$ is the solution to the IVP:

$$\overrightarrow{\mathbf{h}}(0,\bar{x}) = \overrightarrow{\mathbf{g}}(\bar{x})$$
$$\frac{\partial}{\partial y}\overrightarrow{\mathbf{h}} = \overrightarrow{\mathbf{f}}(y,\overrightarrow{\mathbf{h}},\bar{x})$$

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The Limit Operation

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LIM* is the operation:

- Input: $f(t, \bar{x})$
- Output: $F(\bar{x}) = \lim_{t \to \infty} f(t, \bar{x})$, if 1) the limit exists, 2) $|F(\bar{x}) - f(t, \bar{x})| \leq 1/t$, and 3) F is $\mathcal{C}^2$.

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Elementary Computability.

- Let $\mathcal{L}_k$ abbreviate FA$[0, 1, -1, \pi, \theta_k, P; comp, LI]$.
- Let $\mathcal{L}$ abbreviate FA$[0, 1, -1, P; comp, LI]$.

Originally Bournez and Hainry, extended by us:

Theorem

$\mathbf{E}(\mathbb{R}) = \mathcal{L}_k(LIM) = \mathcal{L}(LIM)$, for $k \geq 3$.

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Computability: Previous Approach

Definition

The operation UMU:
Input: $f(t, \bar{x})$ (with unique root and other conditions)
Output: Function $F(\bar{x})$ = the unique t such that $f(t, \bar{x}) = 0$.

Definition

Let RT_k be $\mathrm{FA}[0, 1, -1, \theta_k, \mathrm{P}; \mathrm{comp}, \mathrm{CLI}, \mathrm{UMU}]$

Theorem (Bournez and Hainry 2005, 2006)

$\mathcal{C}^2 \cap [\mathbf{C}(\mathbb{R})] = [RT_k(LIM^*)]$, for $k \geq 3$

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Let DF_k be $\mathrm{FA}[0, 1, -1, \theta_k, \mathrm{P}; \mathrm{comp}, \mathrm{ODE}]$

Our Result:

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- General Goal: Provide alternative model for Computable Analysis.
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Useful Function Algebras

Definition

The operation Inverse:
Input: $f(t, \bar{x})$ (which bijection in t and other conditions)
Output: The inverse of f

- Let IV_k be $\mathrm{FA}[0, 1, -1, \theta_k, \mathrm{P}; \mathrm{comp}, \mathrm{LI}, \mathrm{Inverse}]$
- Let $\mathrm{IV}_k^{(c)}$ be the functions of IV_k that can be defined using c or less applications of the operation Inverse.
- Let $\mathrm{RT}_k^{(c)}$ be the functions of RT_k that can be defined using c or less applications of the operation UMU.

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- Let $\mathrm{RT}_k^{(c)}$ be the functions of RT_k that can be defined using c or less applications of the operation UMU.

Outline

1. $\mathrm{RT}_k^{(c)} \subseteq \mathrm{IV}_k^{(c)} \subseteq \mathrm{DF}_{k-c} \subseteq \mathbf{C}(\mathbb{R})$, for any $c \geq 0$ and $k \geq c+3$ (now, we would like something like "$\mathbf{C}(\mathbb{R}) \subseteq \mathrm{RT}_k^{(c)}$")
2. For some constant "bh", and for any $k \geq 3$, we have:

$$\mathcal{C}^2 \cap [\mathbf{C}(\mathbb{R})] \subseteq [\mathrm{RT}_k^{(\mathrm{bh})}(\mathrm{LIM}^*)].$$

3. Closing under limits and considering compact restrictions to complete proof.

Obtains our theorem:

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$\mathrm{RT}_k^{(c)} \subseteq \mathrm{IV}_k^{(c)}$

Simulate any application of UMU with a single application of Inverse. Given a function $f(t)$ to find its root:

- Find $f^{-1}(t)$ with Inverse.
- The root if then $f^{-1}(0)$.

$\mathrm{IV}_k^{(c)} \subseteq \mathrm{DF}_{k-c}$

Main point: For $f \in \mathrm{DF}_r$ we can find its inverse because:

$$(f^{-1}(t))' = \frac{1}{f'(f^{-1}(t))}$$

But ... then we also need $f' \in \mathrm{DF}_r$.

In general, if $f \in \mathrm{DF}_r$ then $f' \in \mathrm{DF}_{r-1}$. Thus:

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$\mathrm{DF}_k \subseteq \mathbf{C}(\mathbb{R})$

Induction on DF_k:

- Basic Functions
- Composition
- The ODE Operation (the main step)

Two approaches:

1. Use Collins/Graça 2008 (this conference).
   Straightforward Induction: *Any function of DF_k is computable*
2. Use Graça/Zhong/Buescu 2007.
   Induction: *Any function in DF_k and any partial derivative of it is computable, and furthermore have r.e.-open domains.*

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Finishing the Proof

Thus we have shown:
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Conclusion

We have a new characterization of Computable Analysis. While it seems to be an improvement, we consider ways to further improve it:

- Show it is useful!
- Remove the restriction to $\mathcal{C}^2$ functions (more than 90% sure it can be done).
- Simplify classes to their "analytic versions" (i.e. remove θ_k function ... 75% sure it is true, though it looks difficult).

Thus we conjecture:

$$[\mathbf{C}(\mathbb{R})] = [\mathrm{RT}(\mathrm{LIM})] = [\mathrm{IV}(\mathrm{LIM})] = [\mathrm{DF}(\mathrm{LIM})]$$

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