IST - Álgebra Linear - 2^o semestre 2018/2019

Sistemas lineares e matrizes (resolução)

1. (i) $\begin{bmatrix} 4 & -2 & | & 5 \\ -6 & 3 & | & 1 \end{bmatrix} \underset{\frac{3}{2}L_1+L_2\to L_2}{\longrightarrow} \begin{bmatrix} 4 & -2 & | & 5 \\ 0 & 0 & | & \frac{17}{2} \end{bmatrix}$. Logo, o sistema não tem solução (é impossível). $CS = \varnothing$.

(ii) $\begin{bmatrix} 2 & 3 & | & 1 \\ 5 & 7 & | & 3 \end{bmatrix} \underset{-\frac{5}{2}L_1+L_2\to L_2}{\longrightarrow} \begin{bmatrix} 2 & 3 & | & 1 \\ 0 & -\frac{1}{2} & | & \frac{1}{2} \end{bmatrix}$. Logo, $\begin{cases} 2x+3y=1 \\ -\frac{1}{2}y = \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} x = 2 \\ y = -1. \end{cases}$

A solução geral do sistema é $CS = \{(2,-1)\}$.

(iii) $\begin{bmatrix} 2 & 4 & | & 10 \\ 3 & 6 & | & 15 \end{bmatrix} \underset{-\frac{3}{2}L_1+L_2\to L_2}{\longrightarrow} \begin{bmatrix} 2 & 4 & | & 10 \\ 0 & 0 & | & 0 \end{bmatrix}$. Logo, $2x+4y=10 \Leftrightarrow x = 5-2y$.

A solução geral do sistema é $CS = \{(5-2y, y) : y \in \mathbb{R}\}$.

(iv) $\begin{bmatrix} 2 & 1 & -3 & | & 5 \\ 3 & -2 & 2 & | & 5 \\ 5 & -3 & -1 & | & 16 \end{bmatrix} \underset{\substack{-\frac{3}{2}L_1+L_2\to L_2 \\ -\frac{5}{2}L_1+L_3\to L_3}}{\longrightarrow} \begin{bmatrix} 2 & 1 & -3 & | & 5 \\ 0 & -7/2 & 13/2 & | & -5/2 \\ 0 & -11/2 & 13/2 & | & 7/2 \end{bmatrix} \underset{-\frac{11}{7}L_2+L_3\to L_3}{\longrightarrow}$

$\underset{-\frac{11}{7}L_2+L_3\to L_3}{\longrightarrow} \begin{bmatrix} 2 & 1 & -3 & | & 5 \\ 0 & -7/2 & 13/2 & | & -5/2 \\ 0 & 0 & -26/7 & | & 52/7 \end{bmatrix}$.

Logo, $\begin{cases} 2x+y-3z=5 \\ -\frac{7}{2}y+\frac{13}{2}z = -\frac{5}{2} \\ -\frac{26}{7}z = \frac{52}{7} \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = -3 \\ z = -2. \end{cases}$ A solução geral do sistema é $CS = \{(1,-3,-2)\}$.

(v) $\begin{bmatrix} 1 & 2 & 3 & | & 3 \\ 2 & 3 & 8 & | & 4 \\ 3 & 2 & 17 & | & 1 \end{bmatrix} \underset{\substack{-2L_1+L_2\to L_2 \\ -3L_1+L_3\to L_3}}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 & | & 3 \\ 0 & -1 & 2 & | & -2 \\ 0 & -4 & 8 & | & -8 \end{bmatrix} \underset{-4L_2+L_3\to L_3}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 & | & 3 \\ 0 & -1 & 2 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$.

Logo, $\begin{cases} x+2y+3z=3 \\ -y+2z=-2 \end{cases} \Leftrightarrow \begin{cases} x = -7z-1 \\ y = 2z+2. \end{cases}$

A solução geral do sistema é $CS = \{(-7z-1, 2z+2, z) : z \in \mathbb{R}\}$.

(vi) $\begin{bmatrix} 2 & 3 & -2 & | & 5 \\ 1 & -2 & 3 & | & 2 \\ 4 & -1 & 4 & | & 1 \end{bmatrix} \underset{\substack{-\frac{1}{2}L_1+L_2\to L_2 \\ -2L_1+L_3\to L_3}}{\longrightarrow} \begin{bmatrix} 2 & 3 & -2 & | & 5 \\ 0 & -7/2 & 4 & | & -1/2 \\ 0 & -7 & 8 & | & -9 \end{bmatrix} \underset{-2L_2+L_3\to L_3}{\longrightarrow}$

$$\xrightarrow[-2L_2+L_3\to L_3]{} \left[\begin{array}{ccc|c} 2 & 3 & -2 & 5 \\ 0 & -7/2 & 4 & -1/2 \\ 0 & 0 & 0 & -8 \end{array}\right].$$

Logo, o sistema não tem solução (é impossível). $CS = \varnothing$.

(vii) $$\left[\begin{array}{cccc|c} 1 & 5 & 4 & -13 & 3 \\ 3 & -1 & 2 & 5 & 2 \\ 2 & 2 & 3 & -4 & 1 \end{array}\right] \xrightarrow[\substack{-3L_1+L_2\to L_2 \\ -2L_1+L_3\to L_3}]{} \left[\begin{array}{cccc|c} 1 & 5 & 4 & -13 & 3 \\ 0 & -16 & -10 & 44 & -7 \\ 0 & -8 & -5 & 22 & -5 \end{array}\right] \xrightarrow[-\frac{1}{2}L_2+L_3\to L_3]{}$$

$$\xrightarrow[-\frac{1}{2}L_2+L_3\to L_3]{} \left[\begin{array}{cccc|c} 1 & 5 & 4 & -13 & 3 \\ 0 & -16 & -10 & 44 & -7 \\ 0 & 0 & 0 & 0 & -\frac{3}{2} \end{array}\right].$$

Logo, o sistema não tem solução (é impossível). $CS = \varnothing$.

(viii) $$\left[\begin{array}{cccc|c} 0 & 0 & 2 & 3 & 4 \\ 2 & 0 & -6 & 9 & 7 \\ 2 & 2 & -5 & 2 & 4 \\ 0 & 100 & 150 & -200 & 50 \end{array}\right] \xrightarrow[\substack{L_1\leftrightarrow L_3 \\ \frac{1}{50}L_4\to L_4}]{} \left[\begin{array}{cccc|c} 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \end{array}\right] \xrightarrow[-L_1+L_2\to L_2]{}$$

$$\xrightarrow[-L_1+L_2\to L_2]{} \left[\begin{array}{cccc|c} 2 & 2 & -5 & 2 & 4 \\ 0 & -2 & -1 & 7 & 3 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \end{array}\right] \xrightarrow[L_2+L_4\to L_4]{} \left[\begin{array}{cccc|c} 2 & 2 & -5 & 2 & 4 \\ 0 & -2 & -1 & 7 & 3 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{array}\right] \xrightarrow[-L_3+L_4\to L_4]{}$$

$$\xrightarrow[-L_3+L_4\to L_4]{} \left[\begin{array}{cccc|c} 2 & 2 & -5 & 2 & 4 \\ 0 & -2 & -1 & 7 & 3 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Logo, $\begin{cases} 2x_1 + 2x_2 - 5x_3 + 2x_4 = 4 \\ -2x_2 - x_3 + 7x_4 = 3 \\ 2x_3 + 3x_4 = 4 \end{cases} \Leftrightarrow \begin{cases} x_1 = \frac{19}{2} - 9x_4 \\ x_2 = \frac{17}{4}x_4 - \frac{5}{2} \\ x_3 = -\frac{3}{2}x_4 + 2 \end{cases}$

A solução geral do sistema é dada por

$$CS = \left\{ \left(\frac{19}{2} - 9s, \frac{17}{4}s - \frac{5}{2}, -\frac{3}{2}s + 2, s \right) : s \in \mathbb{R} \right\}.$$

(ix) $$\left[\begin{array}{cc|c} 2 & 3 & 3 \\ 1 & -2 & 5 \\ 3 & 2 & 7 \end{array}\right] \xrightarrow[L_1\leftrightarrow L_2]{} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 2 & 3 & 3 \\ 3 & 2 & 7 \end{array}\right] \xrightarrow[\substack{-2L_1+L_2\to L_2 \\ -3L_1+L_3\to L_3}]{}$$

$$\xrightarrow[\substack{-2L_1+L_2\to L_2\\ -3L_1+L_3\to L_3}]{} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 7 & -7 \\ 0 & 8 & -8 \end{array}\right] \xrightarrow[-\frac{8}{7}L_2+L_3\to L_3]{} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{array}\right].$$

Logo, $\begin{cases} x-2y=5 \\ 7y=-7 \end{cases} \Leftrightarrow \begin{cases} x=3 \\ y=-1. \end{cases}$ A solução geral do sistema é $CS=\{(3,-1)\}$.

(x) $\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 3 \\ 2 & 4 & 4 & 3 & 9 \\ 3 & 6 & -1 & 8 & 10 \end{array}\right] \xrightarrow[\substack{-2L_1+L_2\to L_2\\ -3L_1+L_3\to L_3}]{} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 3 \\ 0 & 0 & 6 & -3 & 3 \\ 0 & 0 & 2 & -1 & 1 \end{array}\right] \xrightarrow[-\frac{1}{3}L_2+L_3\to L_3]{}$

$$\xrightarrow[-\frac{1}{3}L_2+L_3\to L_3]{} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 3 \\ 0 & 0 & 6 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Logo, $\begin{cases} x+2y-z+3w=3 \\ 6z-3w=3 \end{cases} \Leftrightarrow \begin{cases} x=-2y-\frac{5}{2}w+\frac{7}{2} \\ z=\frac{1}{2}w+\frac{1}{2}. \end{cases}$

A solução geral do sistema é $CS=\left\{\left(-2y-\frac{5}{2}w+\frac{7}{2}, y, \frac{1}{2}w+\frac{1}{2}, w\right) : y,w\in\mathbb{R}\right\}$.

2. a) Sejam $A=\begin{bmatrix} 1 & 2 & -3 \\ 3 & -1 & 2 \\ 1 & -5 & 8 \end{bmatrix}$ e $B_{a,b,c}=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

$[A \mid B_{a,b,c}] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 3 & -1 & 2 & b \\ 1 & -5 & 8 & c \end{array}\right] \xrightarrow[\substack{-3L_1+L_2\to L_2\\ -L_1+L_3\to L_3}]{}$

$$\xrightarrow[\substack{-3L_1+L_2\to L_2\\ -L_1+L_3\to L_3}]{} \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -7 & 11 & b-3a \\ 0 & -7 & 11 & c-a \end{array}\right] \xrightarrow[-L_2+L_3\to L_3]{} \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -7 & 11 & b-3a \\ 0 & 0 & 0 & c-b+2a \end{array}\right].$$

Para que haja solução é necessário que car A = car $[A \mid B_{a,b,c}]$, isto é, é necessário que

$$c-b+2a=0.$$

(b) Sejam $A=\begin{bmatrix} 1 & -2 & 4 \\ 2 & 3 & -1 \\ 3 & 1 & 2 \end{bmatrix}$ e $B_{a,b,c}=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

$[A \mid B_{a,b,c}] = \left[\begin{array}{ccc|c} 1 & -2 & 4 & a \\ 2 & 3 & -1 & b \\ 3 & 1 & 2 & c \end{array}\right] \xrightarrow[\substack{-2L_1+L_2\to L_2\\ -3L_1+L_3\to L_3}]{}$

$$\xrightarrow[\substack{-2L_1+L_2\to L_2\\ -3L_1+L_3\to L_3}]{} \left[\begin{array}{ccc|c} 1 & -2 & 4 & a \\ 0 & 7 & -9 & b-2a \\ 0 & 7 & -10 & c-3a \end{array}\right] \xrightarrow[-L_2+L_3\to L_3]{} \left[\begin{array}{ccc|c} 1 & -2 & 4 & a \\ 0 & 7 & -9 & b-2a \\ 0 & 0 & -1 & c-b-a \end{array}\right].$$

Como $\text{car}\, A = \text{car}\, [A \mid B_{a,b,c}]$, este sistema tem solução para quaisquer valores de a, b, c.

3. Sendo $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ \alpha & 0 & 1 \end{bmatrix}$, tem-se $[A \mid B] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ \alpha & 0 & 1 & 1 \end{array}\right] \underset{\cdots}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1-\alpha & 1-\alpha \end{array}\right]$, usando o método de eliminação de Gauss. O sistema tem solução única se e só se $\text{car}\, A = 3 = \text{car}\, [A \mid B]$, isto é, se e só se $\alpha \in \mathbb{R} \backslash \{1\}$.

4. Sendo $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & \alpha \\ \alpha & 0 & \alpha \\ -1 & 0 & -1 \end{bmatrix}$, tem-se

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ -1 & 0 & \alpha & -1 \\ \alpha & 0 & \alpha & 1 \\ -1 & 0 & -1 & -1 \end{array}\right] \underset{\cdots}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 0 & \alpha+1 & 0 \\ 0 & 0 & 0 & 1-\alpha \\ 0 & 0 & 0 & 0 \end{array}\right],$$

usando o método de eliminação de Gauss.

Se $\alpha \in \mathbb{R} \backslash \{1\}$ então $\text{car}\, A = 2 < 3 = \text{car}\, [A \mid B]$ e assim o sistema é impossível.

Se $\alpha = 1$ então $\text{car}\, A = 2 = \text{car}\, [A \mid B]$ e assim o sistema é possível e indeterminado tendo por solução geral:

$$\{(1, y, 0) : y \in \mathbb{R}\}.$$

5. a) Usando o método de eliminação de Gauss, tem-se

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 10 \\ 2 & 7 & 2 & 20 \\ 1 & 5 & \alpha & 10 \end{array}\right] \underset{\substack{-2L_1+L_2 \to L_2 \\ -L_1+L_3 \to L_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 10 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & \alpha-2 & 0 \end{array}\right] \underset{L_2+L_3 \to L_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 10 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & \alpha-4 & 0 \end{array}\right].$$

Se $\alpha \neq 4$ então o sistema é possível e determinado, existindo uma única solução. Se $\alpha = 4$ então o sistema é possível e indeterminado, existindo um n^o infinito de soluções.

b) Para $\alpha = 4$, tem-se o sistema de equações lineares

$$\begin{cases} x + 4y + 2z = 10 \\ -y - 2z = 0. \end{cases} \Leftrightarrow \begin{cases} x = 10 + 6z \\ y = -2z. \end{cases}$$

A solução geral do sistema é dada por: $CS = \{(10 + 6z, -2z, z) : z \in \mathbb{R}\}$.

6. $[A \mid B] = \left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ \alpha & 0 & -1 & -2 \\ 0 & \alpha & -1 & -1 \end{array}\right] \underset{-\alpha L_1+L_2 \to L_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & \alpha & -1 & \alpha-2 \\ 0 & \alpha & -1 & -1 \end{array}\right] \underset{-L_2+L_3 \to L_3}{\longrightarrow}$
$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & \alpha & -1 & \alpha-2 \\ 0 & 0 & 0 & 1-\alpha \end{array}\right].$

O sistema é possível e indeterminado se e só se car $A = \text{car}\,[A \mid B] < 3$ ($= n^o$ de colunas de A) se e só se $\alpha = 1$. Se $\alpha = 1$ então a solução geral é: $\{(z-2, z-1, z) : z \in \mathbb{R}\}$.

7. (i) Sejam $A_\alpha = \begin{bmatrix} 1 & 2 & \alpha \\ 2 & \alpha & 8 \end{bmatrix}$ e $B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$$[A_\alpha \mid B] = \left[\begin{array}{ccc|c} 1 & 2 & \alpha & 1 \\ 2 & \alpha & 8 & 3 \end{array}\right] \underset{-2L_1+L_2 \to L_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & \alpha & 1 \\ 0 & \alpha-4 & 8-2\alpha & 1 \end{array}\right].$$

Se $\alpha \neq 4$ então car $A_\alpha = \text{car}\,[A_\alpha \mid B] = 2 < 3 = \text{n}^o$ de incógnitas do sistema. Logo o sistema é possível e indeterminado, tendo-se

$$\begin{cases} x + 2y + \alpha z = 1 \\ (\alpha - 4)\, y + (8 - 2\alpha)\, z = 1 \end{cases} \Leftrightarrow \begin{cases} x = 1 - \dfrac{2}{\alpha - 4} - (\alpha + 4)\, z \\ \\ y = \dfrac{1}{\alpha - 4} + 2z. \end{cases}$$

A solução geral deste sistema é então dada por

$$CS_\alpha = \left\{ \left(1 - \frac{2}{\alpha - 4} - (\alpha + 4)\, z, \frac{1}{\alpha - 4} + 2z, z \right) : z \in \mathbb{R} \right\}.$$

Se $\alpha = 4$ então $\underbrace{\text{car}\, A_\alpha}_{=1} < \underbrace{\text{car}\,[A_\alpha \mid B]}_{=2}$. Logo, o sistema não tem solução (é impossível). $CS_4 = \varnothing$.

(ii) Sejam $A_\alpha = \begin{bmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{bmatrix}$ e $B_\alpha = \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix}$. $[A_\alpha \mid B_\alpha] = \left[\begin{array}{ccc|c} \alpha & 1 & 1 & 1 \\ 1 & \alpha & 1 & \alpha \\ 1 & 1 & \alpha & \alpha^2 \end{array}\right] \underset{L_1 \leftrightarrow L_3}{\longrightarrow}$

$$\underset{L_1 \leftrightarrow L_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & \alpha & \alpha^2 \\ 1 & \alpha & 1 & \alpha \\ \alpha & 1 & 1 & 1 \end{array}\right] \underset{\substack{-L_1+L_2 \to L_2 \\ -\alpha L_1 + L_3 \to L_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & \alpha & \alpha^2 \\ 0 & \alpha-1 & 1-\alpha & \alpha - \alpha^2 \\ 0 & 1-\alpha & 1-\alpha^2 & 1-\alpha^3 \end{array}\right] \underset{L_2+L_3 \to L_3}{\longrightarrow}$$

$$\underset{L_2+L_3 \to L_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & \alpha & \alpha^2 \\ 0 & \alpha-1 & 1-\alpha & \alpha\,(1-\alpha) \\ 0 & 0 & (1-\alpha)\,(\alpha+2) & (1+\alpha)\,(1-\alpha^2) \end{array}\right].$$

Se $\alpha = 1$ então car $A_\alpha = \text{car}\,[A_\alpha \mid B_\alpha] = 1 < 3 = \text{n}^o$ de incógnitas do sistema. Logo o sistema é possível e indeterminado, tendo-se $x + y + z = 1$. A solução geral deste sistema é então dada por $CS_1 = \{(1 - y - z, y, z) : y, z \in \mathbb{R}\}$.

Se $\alpha = -2$ então $\underbrace{\text{car}\, A_\alpha}_{=2} < \underbrace{\text{car}\,[A_\alpha \mid B_\alpha]}_{=3}$. O sistema não tem solução (é impossível). $CS_{-2} = \varnothing$.

Se $\alpha \neq 1$ e $\alpha \neq -2$ então car $A_\alpha = \text{car}\,[A_\alpha \mid B_\alpha] = 3 = \text{n}^o$ de incógnitas do sistema. Logo o sistema é possível e determinado, tendo-se

$$\begin{cases} x + y + \alpha z = \alpha^2 \\ (\alpha - 1)\, y + (1 - \alpha)\, z = \alpha\,(1 - \alpha) \\ (1 - \alpha)\,(\alpha + 2)\, z = (1 + \alpha)\,(1 - \alpha^2) \end{cases} \Leftrightarrow \begin{cases} x = -\,(\alpha + 1)\,/\,(\alpha + 2) \\ y = 1/\,(\alpha + 2) \\ z = (1 + \alpha)^2\,/\,(\alpha + 2)\,. \end{cases}$$

A solução geral do sistema é então dada por

$$CS_\alpha = \left\{ \left(-\frac{\alpha + 1}{\alpha + 2}, \frac{1}{\alpha + 2}, \frac{(1 + \alpha)^2}{\alpha + 2} \right) \right\}.$$

(iii) Sejam $A_\alpha = \begin{bmatrix} 1 & 1 & \alpha \\ 3 & 4 & 2 \\ 2 & 3 & -1 \end{bmatrix}$ e $B_\alpha = \begin{bmatrix} 2 \\ \alpha \\ 1 \end{bmatrix}$. $[A_\alpha \mid B_\alpha] = \begin{bmatrix} 1 & 1 & \alpha & | & 2 \\ 3 & 4 & 2 & | & \alpha \\ 2 & 3 & -1 & | & 1 \end{bmatrix} \underset{\substack{-3L_1+L_2 \to L_2 \\ -2L_1+L_3 \to L_3}}{\longrightarrow}$

$$\underset{\substack{-3L_1+L_2 \to L_2 \\ -2L_1+L_3 \to L_3}}{\longrightarrow} \begin{bmatrix} 1 & 1 & \alpha & | & 2 \\ 0 & 1 & 2 - 3\alpha & | & \alpha - 6 \\ 0 & 1 & -1 - 2\alpha & | & -3 \end{bmatrix} \underset{-L_2+L_3 \to L_3}{\longrightarrow} \begin{bmatrix} 1 & 1 & \alpha & | & 2 \\ 0 & 1 & 2 - 3\alpha & | & \alpha - 6 \\ 0 & 0 & -3 + \alpha & | & 3 - \alpha \end{bmatrix}.$$

Se $\alpha = 3$ então car $A_\alpha = \text{car}\,[A_\alpha \mid B_\alpha] = 2 < 3 = \text{n}^o$ de incógnitas do sistema. Logo o sistema é possível e indeterminado, tendo-se

$$\begin{cases} x + y + 3z = 2 \\ y - 7z = -3 \end{cases} \Leftrightarrow \begin{cases} x = 5 - 10z \\ \\ y = -3 + 7z. \end{cases}$$

A solução geral deste sistema é então dada por

$$CS_3 = \{(5 - 10z, -3 + 7z, z) : z \in \mathbb{R}\}\,.$$

Se $\alpha \neq 3$ então car $A_\alpha = \text{car}\,[A_\alpha \mid B_\alpha] = 3 = \text{n}^o$ de incógnitas do sistema. Logo o sistema é possível e determinado, tendo-se

$$\begin{cases} x + y + \alpha z = 2 \\ y + (2 - 3\alpha)\, z = \alpha - 6 \\ (-3 + \alpha)\, z = 3 - \alpha \end{cases} \Leftrightarrow \begin{cases} x = 6 + 3\alpha \\ y = -4 - 2\alpha \\ z = -1. \end{cases}$$

A solução geral do sistema é então dada por

$$CS_\alpha = \{(6 + 3\alpha, -4 - 2\alpha, -1)\}\,.$$

8. (i) Sejam

$$A_\alpha = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 7 & -2 \\ 1 & 5 & \alpha \end{bmatrix} \quad \text{e} \quad B_\beta = \begin{bmatrix} 10 \\ 10 \\ \beta \end{bmatrix}.$$

$$[A_\alpha \mid B_\beta] = \begin{bmatrix} 1 & 4 & 3 & | & 10 \\ 2 & 7 & -2 & | & 10 \\ 1 & 5 & \alpha & | & \beta \end{bmatrix} \underset{\substack{-2L_1+L_2\to L_2 \\ -L_1+L_3\to L_3}}{\longrightarrow}$$

$$\underset{\substack{-2L_1+L_2\to L_2 \\ -L_1+L_3\to L_3}}{\longrightarrow} \begin{bmatrix} 1 & 4 & 3 & | & 10 \\ 0 & -1 & -8 & | & -10 \\ 0 & 1 & \alpha-3 & | & \beta-10 \end{bmatrix} \underset{L_2+L_3\to L_3}{\longrightarrow} \begin{bmatrix} 1 & 4 & 3 & | & 10 \\ 0 & -1 & -8 & | & -10 \\ 0 & 0 & \alpha-11 & | & \beta-20 \end{bmatrix}.$$

Se $\alpha = 11$ e $\beta = 20$ então $\operatorname{car} A_\alpha = \operatorname{car}[A_\alpha \mid B_\beta] = 2 < 3 =$ n^o de incógnitas do sistema. Logo o sistema é possível e indeterminado, tendo-se

$$\begin{cases} x + 4y + 3z = 10 \\ -y - 8z = -10 \end{cases} \Leftrightarrow \begin{cases} x = -30 + 29z \\ y = 10 - 8z. \end{cases}$$

Se $\alpha = 11$ e $\beta \neq -20$ então $\underbrace{\operatorname{car} A_\alpha}_{=2} < \underbrace{\operatorname{car}[A_\alpha \mid B_\beta]}_{=3}$. Logo, o sistema não tem solução (é impossível). $CS_{\alpha,\beta} = \varnothing$.

Se $\alpha \neq 11$ então $\operatorname{car} A_\alpha = \operatorname{car}[A_\alpha \mid B_\beta] = 3 =$ n^o de incógnitas do sistema. Logo o sistema é possível e determinado, tendo-se

$$\begin{cases} x + 4y + 3z = 10 \\ -y - 8z = -10 \\ (\alpha - 11)\, z = \beta - 20 \end{cases} \Leftrightarrow \begin{cases} x = -\left(30\alpha - 29\beta + 250\right) / \left(\alpha - 11\right) \\ y = \left(10\alpha - 8\beta + 50\right) / \left(\alpha - 11\right) \\ z = \left(\beta - 20\right) / \left(\alpha - 11\right). \end{cases}$$

A solução geral do sistema é então dada por

$$CS_{\alpha,\beta} = \left\{ \left(-\frac{30\alpha - 29\beta + 250}{\alpha - 11}, \frac{10\alpha - 8\beta + 50}{\alpha - 11}, \frac{\beta - 20}{\alpha - 11} \right) \right\}.$$

(ii) Sejam $A_\alpha = \begin{bmatrix} \alpha & 1 & -1 & \alpha \\ 1 & -2 & 2 & 1 \\ 1 & -1 & 1 & \alpha+1 \end{bmatrix}$ e $B_\beta = \begin{bmatrix} 0 \\ 1 \\ \beta \end{bmatrix}$.

$$[A_\alpha \mid B_\beta] = \begin{bmatrix} \alpha & 1 & -1 & \alpha & | & 0 \\ 1 & -2 & 2 & 1 & | & 1 \\ 1 & -1 & 1 & \alpha+1 & | & \beta \end{bmatrix} \underset{L_1 \leftrightarrow L_3}{\longrightarrow}$$

$$\underset{L_1 \leftrightarrow L_3}{\longrightarrow} \begin{bmatrix} 1 & -1 & 1 & \alpha+1 & | & \beta \\ 1 & -2 & 2 & 1 & | & 1 \\ \alpha & 1 & -1 & \alpha & | & 0 \end{bmatrix} \underset{\substack{-L_1+L_2\to L_2 \\ -\alpha L_1+L_3\to L_3}}{\longrightarrow}$$

$$\underset{\substack{-L_1+L_2\to L_2\\ -\alpha L_1+L_3\to L_3}}{\longrightarrow}\left[\begin{array}{cccccc}1 & -1 & 1 & \alpha+1 & | & \beta\\ 0 & -1 & 1 & -\alpha & | & 1-\beta\\ 0 & \alpha+1 & -\alpha-1 & -\alpha^2 & | & -\alpha\beta\end{array}\right]\underset{(\alpha+1)L_2+L_3\to L_3}{\longrightarrow}$$

$$\underset{(\alpha+1)L_2+L_3\to L_3}{\longrightarrow}\left[\begin{array}{cccccc}1 & -1 & 1 & \alpha+1 & | & \beta\\ 0 & -1 & 1 & -\alpha & | & 1-\beta\\ 0 & 0 & 0 & (-2\alpha-1)\,\alpha & | & \alpha-2\alpha\beta+1-\beta\end{array}\right].$$

Se $\alpha \neq 0$ e $\alpha \neq -\dfrac{1}{2}$ então car $A_\alpha =$ car $[A_\alpha \mid B_\beta] = 3 < 4 =$ n^o de incógnitas do sistema. Logo o sistema é possível e indeterminado, tendo-se

$$\begin{cases} x-y+z+(\alpha+1)\,w=\beta\\ -y+z-\alpha w=1-\beta\\ (-2\alpha-1)\,\alpha w=\alpha+1+(-2\alpha-1)\,\beta\end{cases}\Leftrightarrow\begin{cases} x=-\frac{\alpha+1}{2\alpha+1}-1-\frac{(\alpha+1)^2}{(-2\alpha-1)\alpha}-\frac{\beta}{\alpha}\\ y=z-\frac{\alpha+1}{2\alpha+1}-1\\ w=\frac{\alpha+1}{(-2\alpha-1)\alpha}+\frac{\beta}{\alpha}.\end{cases}$$

A solução geral do sistema é então dada por

$$CS_{\alpha,\beta}=\left\{\left(-\frac{\alpha+1}{2\alpha+1}-1-\frac{(\alpha+1)^2}{(-2\alpha-1)\,\alpha}-\frac{\beta}{\alpha},z-\frac{\alpha+1}{2\alpha+1}-1,z,\frac{\alpha+1}{(-2\alpha-1)\,\alpha}+\frac{\beta}{\alpha}\right)\right\}.$$

Se $\alpha = 0$ e $\beta = 1$ então car $A_\alpha =$ car $[A_\alpha \mid B_\beta] = 2 < 4 =$ n^o de incógnitas do sistema. Logo o sistema é possível e indeterminado, tendo-se

$$\begin{cases} x-y+z+w=1\\ -y+z=0\end{cases}\Leftrightarrow\begin{cases} x=1-w\\ y=z.\end{cases}$$

A solução geral deste sistema é então dada por $CS_{\alpha,\beta} = \{(1-w, z, z, w) : z, w \in \mathbb{R}\}$.

Se ($\alpha = 0$ e $\beta \neq 1$) ou $\alpha = -\dfrac{1}{2}$ então $\underbrace{\text{car}\,A_\alpha}_{=2} < \underbrace{\text{car}\,[A_\alpha \mid B_\beta]}_{=3}$. Logo, o sistema não tem solução (é impossível). $CS_{\alpha,\beta} = \varnothing$.

9. $\begin{cases} 2z+\alpha w=\beta\\ x+y+z+3w=1\\ 2x+2y\ +z+w=2\\ x+y\ +3z+14w=4\end{cases}$

Sejam $A_\alpha = \begin{bmatrix}0&0&2&\alpha\\1&1&1&3\\2&2&1&1\\1&1&3&14\end{bmatrix}$ e $B_\beta = \begin{bmatrix}\beta\\1\\2\\4\end{bmatrix}$.

$$[A_\alpha \mid B_\beta]=\left[\begin{array}{cccccc}0&0&2&\alpha&|&\beta\\1&1&1&3&|&1\\2&2&1&1&|&2\\1&1&3&14&|&4\end{array}\right]\underset{L_1\leftrightarrow L_3}{\longrightarrow}$$

$$\xrightarrow[L_1\leftrightarrow L_3]{} \left[\begin{array}{cccc|c} 2 & 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 & 1 \\ 0 & 0 & 2 & \alpha & \beta \\ 1 & 1 & 3 & 14 & 4 \end{array}\right] \xrightarrow[L_1\leftrightarrow L_2]{} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 1 \\ 2 & 2 & 1 & 1 & 2 \\ 0 & 0 & 2 & \alpha & \beta \\ 1 & 1 & 3 & 14 & 4 \end{array}\right] \underset{-L_1+L_4\to L_4}{\xrightarrow[-2L_1+L_2\to L_2]{}}$$

$$\underset{-L_1+L_4\to L_4}{\xrightarrow[-2L_1+L_2\to L_2]{}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 1 \\ 0 & 0 & -1 & -5 & 0 \\ 0 & 0 & 2 & \alpha & \beta \\ 0 & 0 & 2 & 11 & 3 \end{array}\right] \underset{2L_2+L_4\to L_4}{\xrightarrow[2L_2+L_3\to L_3]{}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 1 \\ 0 & 0 & -1 & -5 & 0 \\ 0 & 0 & 0 & \alpha-10 & \beta \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[L_1\leftrightarrow L_2]{}$$

$$\xrightarrow[L_1\leftrightarrow L_2]{} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 1 \\ 0 & 0 & -1 & -5 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & \alpha-10 & \beta \end{array}\right] \xrightarrow[-(\alpha-10)L_3+L_4\to L_4]{} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 1 \\ 0 & 0 & -1 & -5 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & -3(\alpha-10)+\beta \end{array}\right].$$

Se $\beta = 3(\alpha-10)$ então $\operatorname{car} A_\alpha = \operatorname{car}[A_\alpha \mid B_\beta] = 3 < 4 = \text{n}^o$ de incógnitas do sistema. Logo o sistema é possível e indeterminado, tendo-se

$$\begin{cases} x+y+z+3w=1 \\ -z-5w=0 \\ w=3 \end{cases} \Leftrightarrow \begin{cases} x=7-y \\ z=-15 \\ w=3. \end{cases}.$$

A solução geral deste sistema é então dada por $CS_{\alpha,\beta} = \{(7-y, y, -15, 3) : y \in \mathbb{R}\}$.

Se $\beta \neq 3(\alpha-10)$ então $\underbrace{\operatorname{car} A_\alpha}_{=3} < \underbrace{\operatorname{car}[A_\alpha \mid B_\beta]}_{=4}$. Logo, o sistema não tem solução (é impossível). $CS_{\alpha,\beta} = \varnothing$.

10. Usando o método de eliminação de Gauss, tem-se

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 2 & 1 & 4 \\ -1 & 2 & 5 & -4 \\ \alpha & 2\alpha & 3 & 0 \end{array}\right] \xrightarrow[\substack{-3L_1+L_2\to L_2 \\ L_1+L_3\to L_3 \\ -\alpha L_1+L_4\to L_4}]{} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 4 \\ 0 & 4 & 8 & -4 \\ 0 & 0 & 3(1-\alpha) & 0 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 4 \\ 0 & 0 & 3(1-\alpha) & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

(i) O sistema é possível e determinado se e só se $\alpha \in \mathbb{R}\backslash\{1\}$.

(ii) Para $\alpha = 1$, tem-se o sistema de equações lineares $\begin{cases} x + 2y + 3z = 0 \\ -4y - 8z = 4 \end{cases} \Leftrightarrow \begin{cases} x = z + 2 \\ y = -2z - 1. \end{cases}$

A solução geral do sistema é dada por: $CS = \{(z+2, -2z-1, z) : z \in \mathbb{R}\}$.

11. (i)

$$\left[\begin{array}{ccc|c} -1 & 0 & \alpha & 0 \\ 2 & 1 & \alpha & -1 \\ 1 & -\alpha & 0 & \alpha \\ 0 & 1 & 3\alpha & -1 \end{array}\right] \underset{\substack{2L_1+L_2 \to L_2 \\ L_1+L_3 \to L_3}}{\longrightarrow} \left[\begin{array}{ccc|c} -1 & 0 & \alpha & 0 \\ 0 & 1 & 3\alpha & -1 \\ 0 & -\alpha & \alpha & \alpha \\ 0 & 1 & 3\alpha & -1 \end{array}\right] \underset{\substack{\alpha L_2+L_3 \to L_3 \\ -L_2+L_4 \to L_4}}{\longrightarrow}$$

$$\underset{\substack{\alpha L_2+L_3 \to L_3 \\ -L_2+L_4 \to L_4}}{\longrightarrow} \left[\begin{array}{ccc|c} -1 & 0 & \alpha & 0 \\ 0 & 1 & 3\alpha & -1 \\ 0 & 0 & \alpha(1+3\alpha) & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

O sistema é possível e determinado se e só se car $A =$ car $[A \mid B]) = 3$ $(= n^o$ de colunas de $A)$ se e só se $\alpha \in \mathbb{R}\backslash \left\{-\frac{1}{3}, 0\right\}$. Se $\alpha = -\frac{1}{3}$ então a solução geral é: $\left\{\left(-\frac{1}{3}z, z-1, z\right) : z \in \mathbb{R}\right\}$.

12. (i) Por exemplo: $\begin{cases} x = 1 \\ y = 1 \\ z = 1. \end{cases}$

(ii) Sejam $x = t$, $y = 1 - 2t$ e $z = 1$. Tem-se então o seguinte sistema:

$$\begin{cases} 2x + y = 1 \\ \\ z = 1. \end{cases}$$

(iii) Sejam $x = 1 + t$ e $y = 1 - t$. Logo $x + y = 2$.

(iv) Sejam $x = 3t$, $y = 2t$ e $z = t$. Tem-se então o seguinte sistema:

$$\begin{cases} x - 3z = 0 \\ \\ y - 2z = 0. \end{cases}$$

(v) Sejam $x = 3t - s$, $y = t + 2s - 1$ e $z = s - 2t + 1$. Logo $s = 3t - x$ e assim

$$y = t + 2(3t - x) - 1 = 7t - 2x - 1 \Leftrightarrow t = \frac{y + 2x + 1}{7}.$$

Deste modo:

$$s = 3\frac{y + 2x + 1}{7} - x = \frac{3y - x + 3}{7}$$

Com
$$s=\frac{3y-x+3}{7} \quad \text{e} \quad t=\frac{y+2x+1}{7}$$
Tem-se então a seguinte equação linear:
$$z=s-2t+1=\frac{3y-x+3}{7}-2\frac{y+2x+1}{7}+1.$$
Isto é:
$$5x-y+7z=8.$$

(vi) Seja $CS=\{(1-s,s-t,2s,t-1):s,t\in\mathbb{R}\}$.

Sejam $x=1-s,\quad y=s-t,\ \ z=2s,\quad w=t-1$. Uma vez que $s=1-x \quad \text{e} \quad t=w+1$, tem-se então o seguinte sistema linear não homogéneo
$$\begin{cases} y=1-x-(w+1) \\ z=2(1-x) \end{cases} \Leftrightarrow \begin{cases} x+y+w=0 \\ 2x+z=2 \end{cases}$$

(vii) Por exemplo: $\begin{cases} x+y=1 \\ \\ x+y=0. \end{cases}$

13. Sendo $A=\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, tem-se $(A-5I)\,u=\mathbf{0} \Leftrightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} u=\mathbf{0} \Leftrightarrow u\in\{(s,2s):s\in\mathbb{R}\}$.
Logo
$$\{u\neq\mathbf{0}:Au=5u\}=\{(s,2s):s\in\mathbb{R}\backslash\{0\}\}.$$

14. Pretende-se determinar $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ tal que $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
Tem-se então
$$\begin{cases} a+2c=a+3b \\ b+2d=2a+4b \\ 3a+4c=c+3d \\ 3b+4d=2c+4d \end{cases} \Leftrightarrow \begin{cases} c=\frac{3}{2}b \\ d=a+\frac{3}{2}b \end{cases}$$
As matrizes reais que comutam com $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ são da forma: $\begin{bmatrix} a & b \\ \frac{3}{2}b & a+\frac{3}{2}b \end{bmatrix}$, com $a,b\in\mathbb{R}$.

15. Sendo x o n^o de livros e y o n^o de caixas, tem-se $\begin{cases} x-7y=1 \\ x-8y=-7 \end{cases} \Leftrightarrow \begin{cases} x=57 \\ y=8. \end{cases}$

A solução geral do sistema é $\{(57,8)\}$.

16. **(i)** Para que o gráfico da função polinomial $p(x) = ax^3 + bx^2 + cx + d$ passe pelos pontos $P_1 = (0, 10), P_2 = (1, 7), P_3 = (3, -11)$ e $P_4 = (4, -14)$, é necessário que

$$\begin{cases} p(0) = 10 \\ p(1) = 7 \\ p(3) = -11 \\ p(4) = -14. \end{cases}$$

O que é equivalente a existir solução para o seguinte sistema de equações lineares nas variáveis a, b, c e d:

$$\begin{cases} d = 10 \\ a + b + c + d = 7 \\ 27a + 9b + 3c + d = -11 \\ 64a + 16b + 4c + d = -14. \end{cases}$$

Ou seja:

$$\begin{cases} d = 10 \\ a + b + c = -3 \\ 27a + 9b + 3c = -21 \\ 16a + 4b + c = -6. \end{cases}$$

Atendendo a que:

$$\begin{bmatrix} 1 & 1 & 1 & | & -3 \\ 27 & 9 & 3 & | & -21 \\ 16 & 4 & 1 & | & -6 \end{bmatrix} \underset{\substack{-27L_1+L_2 \to L_2 \\ -16L_1+L_3 \to L_3}}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & | & -3 \\ 0 & -18 & -24 & | & 60 \\ 0 & -12 & -15 & | & 42 \end{bmatrix} \underset{\frac{1}{6}L_2 \to L_2}{\longrightarrow}$$

$$\underset{\frac{1}{6}L_2 \to L_2}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & | & -3 \\ 0 & -3 & -4 & | & 10 \\ 0 & -12 & -15 & | & 42 \end{bmatrix} \underset{-4L_2+L_3 \to L_3}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & | & -3 \\ 0 & -3 & -4 & | & 10 \\ 0 & 0 & 1 & | & 2 \end{bmatrix},$$

tem-se

$$\begin{cases} a = 1 \\ b = -6 \\ c = 2 \\ d = 10. \end{cases}$$

(ii) Para que os pontos $P_1 = (-2, 7), P_2 = (-4, 5)$ e $P_3 = (4, -3)$ pertençam à circunferência de equação $x^2 + y^2 + ax + by + c = 0$, é necessário que

$$\begin{cases} (-2)^2 + 7^2 + a(-2) + 7b + c = 0 \\ (-4)^2 + 5^2 + a(-4) + 5b + c = 0 \\ 4^2 + (-3)^2 + 4a + b(-3) + c = 0. \end{cases}$$

O que é equivalente a existir solução para o seguinte sistema de equações lineares nas variáveis a, b e c:

$$\begin{cases} -2a + 7b + c = -53 \\ -4a + 5b + c = -41 \\ 4a - 3b + c = -25. \end{cases}$$

Atendendo a que:

$$\left[\begin{array}{ccc|c} -2 & 7 & 1 & -53 \\ -4 & 5 & 1 & -41 \\ 4 & -3 & 1 & -25 \end{array}\right] \underset{\substack{-2L_1+L_2\to L_2 \\ 2L_1+L_3\to L_3}}{\longrightarrow} \left[\begin{array}{ccc|c} -2 & 7 & 1 & -53 \\ 0 & -9 & -1 & 65 \\ 0 & 11 & 3 & -131 \end{array}\right] \underset{\frac{11}{9}L_2+L_3\to L_3}{\longrightarrow}$$

$$\underset{\frac{11}{9}L_2+L_3\to L_3}{\longrightarrow} \left[\begin{array}{ccc|c} -2 & 7 & 1 & -53 \\ 0 & -9 & -1 & 65 \\ 0 & 0 & 16/9 & -464/9 \end{array}\right],$$

tem-se

$$\begin{cases} a = -2 \\ b = -4 \\ c = -29. \end{cases}$$

17. **(i)** $\begin{bmatrix} -\frac{1}{3} & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -\pi \end{bmatrix} = [-2\pi - 1]$ **(ii)** Não é possível.

(iii) $\begin{bmatrix} \frac{3}{2} & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & 2 & 4 \\ -3 & \sqrt{5} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{11}{2} & 3-2\sqrt{5} & 5 \\ \frac{8}{3} & 2-\sqrt{5} & \frac{7}{2} \end{bmatrix}$

(iv) $2\begin{bmatrix} 1 & 0 \\ 3 & -\frac{1}{2} \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 0 & -6 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ \frac{20}{3} & -2 \end{bmatrix}$ **(v)** Não é possível. **(vi)** Não é possível.

(vii) $\begin{bmatrix} \sqrt{2} \\ -3 \end{bmatrix} \begin{bmatrix} 4 & -\frac{1}{2} & 2 \end{bmatrix} = \begin{bmatrix} 4\sqrt{2} & -\frac{1}{2}\sqrt{2} & 2\sqrt{2} \\ -12 & \frac{3}{2} & -6 \end{bmatrix}$

(viii) $\left(\begin{bmatrix} 2 \\ -\frac{1}{4} \\ 1 \end{bmatrix} 2 \begin{bmatrix} \sqrt{2} & -3 \end{bmatrix}\right)^T = \begin{bmatrix} 4\sqrt{2} & -\frac{1}{2}\sqrt{2} & 2\sqrt{2} \\ -12 & \frac{3}{2} & -6 \end{bmatrix}$

(ix) $\left(2\begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{2} & -1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{2} & -1 \end{bmatrix} - \begin{bmatrix} \frac{8}{9} & \frac{1}{3} & 1 \\ \frac{1}{3} & \frac{1}{2} & 1 \\ \frac{5}{3} & 1 & \frac{5}{2} \end{bmatrix}\right)^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(x) $\begin{bmatrix} 1 & -\frac{1}{2} & 0 & -2 \\ 0 & -1 & -\frac{1}{4} & 0 \\ 6 & 2 & 5 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ -2 & 4 \\ -\frac{1}{3} & -3 \end{bmatrix} = \begin{bmatrix} -1 & -18 \\ \frac{5}{6} & -10 \\ -\frac{7}{6} & -16 \\ -\frac{7}{3} & -3 \end{bmatrix}$

(xi) $\begin{bmatrix} 1 & 0 \\ -2 & 4 \\ -\frac{1}{3} & -3 \end{bmatrix}^T \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -2 \\ 0 & -1 & -\frac{1}{4} & 0 \\ 6 & 2 & 5 & 1 \end{bmatrix} = \begin{bmatrix} -1 & \frac{5}{6} & -\frac{7}{6} & -\frac{7}{3} \\ -18 & -10 & -16 & -3 \end{bmatrix}$

18. **(i)** $\begin{bmatrix} 1 & -4 & 9 & -16 \\ -1 & 4 & -9 & 16 \\ 1 & -4 & 9 & -16 \\ -1 & 4 & -9 & 16 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 0 & 2 & 3 & 4 \\ -2 & 0 & 3 & 4 \\ -3 & -3 & 0 & 4 \\ -4 & -4 & -4 & 0 \end{bmatrix}$ $(a_{ii} = -a_{ii} \Leftrightarrow a_{ii} = 0)$

(iii) $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ 0 & 0 & \frac{1}{5} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{7} \end{bmatrix}$ **(iv)** $\begin{bmatrix} 1 & -2 & -2 & -3 \\ 1 & 2 & -3 & -2 \\ 2 & 1 & 3 & -4 \\ 3 & 2 & 1 & 4 \end{bmatrix}$

19. **(i)** $\begin{bmatrix} 1 & \alpha & \alpha \\ -2 & 1 & 2 \\ 3 & -2-\alpha & -1 \end{bmatrix} \underset{\substack{2L_1+L_2\to L_2 \\ -3L_1+L_3\to L_3}}{\longrightarrow} \begin{bmatrix} 1 & \alpha & \alpha \\ 0 & 1+2\alpha & 2+2\alpha \\ 0 & -2-4\alpha & -1-3\alpha \end{bmatrix} \underset{2L_2+L_3\to L_3}{\longrightarrow}$

$\underset{2L_2+L_3\to L_3}{\longrightarrow} \begin{bmatrix} 1 & \alpha & \alpha \\ 0 & 1+2\alpha & 2(1+\alpha) \\ 0 & 0 & 3+\alpha \end{bmatrix}$

Seja $A_\alpha = \begin{bmatrix} 1 & \alpha & \alpha \\ -2 & 1 & 2 \\ 3 & -2-\alpha & -1 \end{bmatrix}$. Se $\alpha \neq -3$ e $\alpha \neq -\dfrac{1}{2}$ então $\operatorname{car} A_\alpha = 3$ e $\operatorname{nul} A_\alpha = 0$.

Se $\alpha = -3$ ou $\alpha = -\dfrac{1}{2}$ então $\operatorname{car} A_\alpha = 2$ e $\operatorname{nul} A_\alpha = 1$.

Assim, A_α é invertível se e só se $\alpha \neq -3$ e $\alpha \neq -\dfrac{1}{2}$, uma vez que é só neste caso que $\operatorname{car} A_\alpha =$ n^o de colunas de A_α.

(ii) $\begin{bmatrix} 2 & \alpha^2 & -\alpha \\ 2 & 1 & 1 \\ 0 & \alpha^2-1 & \alpha+1 \end{bmatrix} \underset{-L_1+L_2\to L_2}{\longrightarrow} \begin{bmatrix} 2 & \alpha^2 & -\alpha \\ 0 & 1-\alpha^2 & 1+\alpha \\ 0 & \alpha^2-1 & \alpha+1 \end{bmatrix} \underset{L_2+L_3\to L_3}{\longrightarrow}$

$\underset{L_2+L_3\to L_3}{\longrightarrow} \begin{bmatrix} 2 & \alpha^2 & -\alpha \\ 0 & (1-\alpha)(1+\alpha) & 1+\alpha \\ 0 & 0 & 2(\alpha+1) \end{bmatrix}.$

Seja $A_\alpha = \begin{bmatrix} 2 & \alpha^2 & -\alpha \\ 2 & 1 & 1 \\ 0 & \alpha^2-1 & \alpha+1 \end{bmatrix}$. Se $\alpha = -1$ então $\operatorname{car} A_\alpha = 1$ e $\operatorname{nul} A_\alpha = 2$.

Se $\alpha = 1$ então $\operatorname{car} A_\alpha = 2$ e $\operatorname{nul} A_\alpha = 1$.

Se $\alpha \neq -1$ e $\alpha \neq 1$ então $\operatorname{car} A_\alpha = 3$ e $\operatorname{nul} A_\alpha = 0$.

Assim, A_α é invertível se e só se $\alpha \neq -1$ e $\alpha \neq 1$, uma vez que é só neste caso que $\operatorname{car} A_\alpha = \text{n}^o$ de colunas de A_α.

(iii) $$\begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -\alpha^2 & -1 \\ 2 & 0 & -2 & -2 \end{bmatrix} \underset{\substack{L_1+L_3\to L_3 \\ 2L_1+L_4\to L_4}}{\longrightarrow} \begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & -1 & 0 \\ 0 & 0 & (1-\alpha)(1+\alpha) & \alpha-1 \\ 0 & 0 & 0 & 2(\alpha-1) \end{bmatrix}.$$

Seja $A_\alpha = \begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -\alpha^2 & -1 \\ 2 & 0 & -2 & -2 \end{bmatrix}$. Se $\alpha = 1$ então $\operatorname{car} A_\alpha = 2$ e $\operatorname{nul} A_\alpha = 2$.

Se $\alpha = -1$ então $\operatorname{car} A_\alpha = 3$ e $\operatorname{nul} A_\alpha = 1$.

Se $\alpha \neq 1$ e $\alpha \neq -1$ então $\operatorname{car} A_\alpha = 4$ e $\operatorname{nul} A_\alpha = 0$.

Assim, A_α é invertível se e só se $\alpha \neq 1$ e $\alpha \neq -1$, uma vez que é só neste caso que $\operatorname{car} A_\alpha = \text{n}^o$ de colunas de A_α.

20. Existem 16 matrizes 2×2 só com 0 e 1 nas respectivas entradas. 6 são invertíveis:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

21. Seja $(a_{ij}) \in \mathcal{M}_{2\times 2}(\mathbb{R})$ tal que

$$a_{ij} = 3i + 2j$$

Como

$$a_{12} = 3 \times 1 + 2 \times 2 = 7 \neq 8 = 3 \times 2 + 2 \times 1 = a_{21}$$

então A não é simétrica.

22. **(i)** $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \underset{-L_1+L_3\to L_3}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Assim, sendo $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, tem-se $\operatorname{car} A = 2$ e $\operatorname{nul} A = 1$. Pivots: 1 e 1.

(ii) $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \underset{\substack{-5L_1+L_2\to L_2 \\ -9L_1+L_3\to L_3}}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix} \underset{-2L_2+L_3\to L_3}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Assim, sendo $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\operatorname{car} A = 2$ e $\operatorname{nul} A = 2$. Pivots: 1 e -4.

(iii) $\begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 2 & -1 \end{bmatrix} \underset{L_1 \leftrightarrow L_2}{\longrightarrow} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & 2 & -1 \end{bmatrix} \underset{-L_1+L_3 \to L_3}{\longrightarrow}$

$\underset{-L_1+L_3 \to L_3}{\longrightarrow} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & 1 & -1 \end{bmatrix} \underset{-2L_2+L_3 \to L_3}{\longrightarrow} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -3 \end{bmatrix}$.

Assim, sendo $A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -3 \end{bmatrix}$, tem-se $\operatorname{car} A = 3$ e $\operatorname{nul} A = 1$. Pivots: $1, 1$ e 3.

(iv) $\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix} \underset{\substack{-2L_1+L_3 \to L_3 \\ -4L_1+L_4 \to L_4}}{\longrightarrow} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & -11 & 5 & -3 \\ 0 & -11 & 5 & -3 \end{bmatrix} \underset{\substack{L_2+L_3 \to L_3 \\ L_2+L_4 \to L_4}}{\longrightarrow} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Assim, sendo $A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$, tem-se $\operatorname{car} A = 2$ e $\operatorname{nul} A = 2$. Pivots: 1 e 11.

(v) $\begin{bmatrix} 2 & 10 & -6 & 8 & -4 \\ -1 & -5 & 3 & -4 & 2 \\ -2 & -10 & 6 & -8 & 4 \end{bmatrix} \underset{\substack{\frac{1}{2}L_1+L_2 \to L_2 \\ L_1+L_3 \to L_3}}{\longrightarrow} \begin{bmatrix} 2 & 10 & -6 & 8 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Assim, sendo $A = \begin{bmatrix} 2 & 10 & -6 & 8 & -4 \\ -1 & -5 & 3 & -4 & 2 \\ -2 & -10 & 6 & -8 & 4 \end{bmatrix}$, $\operatorname{car} A = 1$ e $\operatorname{nul} A = 4$. Pivot: 2.

(vi) Seja $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. $\operatorname{car} A = 0$, $\operatorname{nul} A = 2$. Não existem pivots.

(vii) Sendo $A = \begin{bmatrix} 5 & -1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$, tem-se $\operatorname{car} A = 2$ e $\operatorname{nul} A = 1$. Pivots: 5 e 2.

(viii) $\begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} \underset{L_1 \leftrightarrow L_3}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \underset{\substack{-2L_1+L_2 \to L_2 \\ -3L_1+L_3 \to L_3}}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Assim, sendo $A = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$, tem-se $\operatorname{car} A = 1$ e $\operatorname{nul} A = 2$. Pivot: 1.

(ix) $\begin{bmatrix} 2 & 1 \\ 2 & 4 \\ -1 & -2 \end{bmatrix} \underset{L_1 \leftrightarrow L_3}{\longrightarrow} \begin{bmatrix} -1 & -2 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \underset{\substack{2L_1+L_2 \to L_2 \\ 2L_1+L_3 \to L_3}}{\longrightarrow} \begin{bmatrix} -1 & -2 \\ 0 & 0 \\ 0 & -3 \end{bmatrix} \underset{L_1 \leftrightarrow L_3}{\longrightarrow} \begin{bmatrix} -1 & -2 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}$.

Assim, sendo $A = \begin{bmatrix} 2 & 1 \\ 2 & 4 \\ -1 & -2 \end{bmatrix}$, $\operatorname{car} A = 2$ e $\operatorname{nul} A = 0$. Pivots: -1 e -3.

(x) $\begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ -1 & 1 & 3 & -2 & -1 \\ 2 & 7 & -1 & 9 & 8 \\ 3 & 3 & -2 & 4 & -6 \end{bmatrix} \underset{\substack{L_1+L_2 \to L_2 \\ -2L_1+L_3 \to L_3 \\ -3L_1+L_4 \to L_4}}{\longrightarrow} \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & 3 & 2 & 1 & 1 \\ 0 & 3 & 1 & 3 & 4 \\ 0 & -3 & 1 & -5 & -12 \end{bmatrix} \underset{\substack{-L_2+L_3 \to L_3 \\ L_2+L_4 \to L_4}}{\longrightarrow}$

$\underset{\substack{-L_2+L_3 \to L_3 \\ L_2+L_4 \to L_4}}{\longrightarrow} \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & 3 & 2 & 1 & 1 \\ 0 & 0 & -1 & 2 & 3 \\ 0 & 0 & 3 & -4 & -11 \end{bmatrix} \underset{3L_3+L_4 \to L_4}{\longrightarrow} \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & 3 & 2 & 1 & 1 \\ 0 & 0 & -1 & 2 & 3 \\ 0 & 0 & 0 & 2 & -2 \end{bmatrix}$.

Assim, sendo $A = \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ -1 & 1 & 3 & -2 & -1 \\ 2 & 7 & -1 & 9 & 8 \\ 3 & 3 & -2 & 4 & -6 \end{bmatrix}$, tem-se $\operatorname{car} A = 4$ e $\operatorname{nul} A = 1$. Pivots: $1, 3, -1$ e 2.

23. **(i)** $\begin{bmatrix} 1 & 0 & 1 \\ -1 & \alpha & \alpha \\ 0 & \alpha & 1 \end{bmatrix} \underset{L_1+L_2 \to L_2}{\longrightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \alpha & \alpha+1 \\ 0 & \alpha & 1 \end{bmatrix} \underset{-L_2+L_3 \to L_3}{\longrightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \alpha & \alpha+1 \\ 0 & 0 & \alpha \end{bmatrix}$.

Seja $A_\alpha = \begin{bmatrix} 1 & 0 & 1 \\ -1 & \alpha & \alpha \\ 0 & \alpha & 1 \end{bmatrix}$. Se $\alpha \neq 0$ então $\operatorname{car} A_\alpha = 3$ e $\operatorname{nul} A_\alpha = 0$.

Se $\alpha = 0$ então $\operatorname{car} A_\alpha = 2$ e $\operatorname{nul} A_\alpha = 1$.

Assim, A_α é invertível se e só se $\alpha \neq 0$, uma vez que é só neste caso que $\operatorname{car} A_\alpha = \text{n}^o$ de colunas de A_α.

(ii) $\begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & 0 & 0 \\ 3 & 0 & \alpha & 0 \\ -1 & -1 & 1 & 2 \end{bmatrix} \underset{\substack{3L_1+L_3 \to L_3 \\ -L_1+L_4 \to L_4}}{\longrightarrow} \begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha+3 & 3\alpha \\ 0 & -1 & 0 & 2-\alpha \end{bmatrix} \underset{L_2+L_4 \to L_4}{\longrightarrow}$

$$\underset{L_2+L_4\to L_4}{\longrightarrow}\begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha+3 & 3\alpha \\ 0 & 0 & 0 & 2-\alpha \end{bmatrix}.$$

Seja $A_\alpha = \begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & 0 & 0 \\ 3 & 0 & \alpha & 0 \\ -1 & -1 & 1 & 2 \end{bmatrix}$. Se $\alpha = 2$ ou $\alpha = -3$ então $\operatorname{car} A_\alpha = 3$ e $\operatorname{nul} A_\alpha = 1$.

Se $\alpha \neq 2$ e $\alpha \neq -3$ então $\operatorname{car} A_\alpha = 4$ e $\operatorname{nul} A_\alpha = 0$.

Assim, A_α é invertível se e só se $\alpha \neq 2$ e $\alpha \neq -3$, uma vez que é só neste caso que $\operatorname{car} A_\alpha = \text{n}^o$ de colunas de A_α.

(iii) $\begin{bmatrix} 1 & -1 & \alpha & 0 \\ 1 & \alpha & -1 & 0 \\ 1 & -1 & \alpha^3 & 0 \\ -1 & 1 & -\alpha & \alpha^2-1 \end{bmatrix} \underset{\substack{-L_1+L_2\to L_2 \\ -L_1+L_3\to L_3 \\ L_1+L_4\to L_4}}{\longrightarrow}$

$$\underset{\substack{-L_1+L_2\to L_2 \\ -L_1+L_3\to L_3 \\ L_1+L_4\to L_4}}{\longrightarrow}\begin{bmatrix} 1 & -1 & \alpha & 0 \\ 0 & \alpha+1 & -1-\alpha & 0 \\ 0 & 0 & \alpha(\alpha-1)(\alpha+1) & 0 \\ 0 & 0 & 0 & (\alpha-1)(\alpha+1) \end{bmatrix}.$$

Seja $A_\alpha = \begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -\alpha^2 & -1 \\ 2 & 0 & -2 & -2 \end{bmatrix}$. Se $\alpha = 1$ então $\operatorname{car} A_\alpha = 2$ e $\operatorname{nul} A_\alpha = 2$.

Se $\alpha = 0$ então $\operatorname{car} A_\alpha = 3$ e $\operatorname{nul} A_\alpha = 1$.

Se $\alpha = -1$ então $\operatorname{car} A_\alpha = 1$ e $\operatorname{nul} A_\alpha = 3$.

Se $\alpha = 1$ então $\operatorname{car} A_\alpha = 2$ e $\operatorname{nul} A_\alpha = 2$.

Se $\alpha \neq 0$ e $\alpha \neq 1$ e $\alpha \neq -1$ então $\operatorname{car} A_\alpha = 4$ e $\operatorname{nul} A_\alpha = 0$.

Assim, A_α é invertível se e só se $\alpha \in \mathbb{R}\backslash\{-1,0,1\}$, uma vez que é só neste caso que $\operatorname{car} A_\alpha = \text{n}^o$ de colunas de A_α.

24. (i) $\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 4 & 0 & 6 & 0 & 1 & 0 \\ 1 & 8 & 1 & 0 & 0 & 1 \end{array}\right] \underset{\substack{-4L_1+L_2\to L_2 \\ -L_1+L_3\to L_3}}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -8 & 2 & -4 & 1 & 0 \\ 0 & 6 & 0 & -1 & 0 & 1 \end{array}\right] \underset{\frac{3}{4}L_2+L_3\to L_3}{\longrightarrow}$

$$\underset{\frac{3}{4}L_2+L_3\to L_3}{\longrightarrow}\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -8 & 2 & -4 & 1 & 0 \\ 0 & 0 & \frac{3}{2} & -4 & \frac{3}{4} & 1 \end{array}\right] \underset{\frac{2}{3}L_3\to L_3}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -8 & 2 & -4 & 1 & 0 \\ 0 & 0 & 1 & -\frac{8}{3} & \frac{1}{2} & \frac{2}{3} \end{array}\right] \underset{\substack{-2L_3+L_2\to L_2 \\ -L_3+L_1\to L_1}}{\longrightarrow}$$

$$\underset{\substack{-2L_3+L_2\to L_2\\ -L_3+L_1\to L_1}}{\longrightarrow}\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{11}{3} & -\frac{1}{2} & -\frac{2}{3}\\ 0 & -8 & 0 & \frac{4}{3} & 0 & -\frac{4}{3}\\ 0 & 0 & 1 & -\frac{8}{3} & \frac{1}{2} & \frac{2}{3}\end{array}\right]\underset{-\frac{1}{8}L_2\to L_2}{\longrightarrow}\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{11}{3} & -\frac{1}{2} & -\frac{2}{3}\\ 0 & 1 & 0 & -\frac{1}{6} & 0 & \frac{1}{6}\\ 0 & 0 & 1 & -\frac{8}{3} & \frac{1}{2} & \frac{2}{3}\end{array}\right]\underset{-2L_2+L_1\to L_1}{\longrightarrow}$$

$$\underset{-2L_2+L_1\to L_1}{\longrightarrow}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -\frac{1}{2} & -1\\ 0 & 1 & 0 & -\frac{1}{6} & 0 & \frac{1}{6}\\ 0 & 0 & 1 & -\frac{8}{3} & \frac{1}{2} & \frac{2}{3}\end{array}\right]. \text{ Logo } \left[\begin{array}{ccc} 1 & 2 & 1\\ 4 & 0 & 6\\ 1 & 8 & 1\end{array}\right]^{-1}=\left[\begin{array}{ccc} 4 & -\frac{1}{2} & -1\\ -\frac{1}{6} & 0 & \frac{1}{6}\\ -\frac{8}{3} & \frac{1}{2} & \frac{2}{3}\end{array}\right].$$

(ii) Para $\alpha \neq k\dfrac{\pi}{2}$, $(k \in \mathbb{Z})$

$$\left[\begin{array}{cc|cc} \cos\alpha & -\operatorname{sen}\alpha & 1 & 0\\ \operatorname{sen}\alpha & \cos\alpha & 0 & 1\end{array}\right]\underset{\substack{(\cos\alpha)L_1\to L_1\\ (\operatorname{sen}\alpha)L_2\to L_2}}{\longrightarrow}\left[\begin{array}{cc|cc} \cos^2\alpha & -\cos\alpha\operatorname{sen}\alpha & \cos\alpha & 0\\ \operatorname{sen}^2\alpha & \operatorname{sen}\alpha\cos\alpha & 0 & \operatorname{sen}\alpha\end{array}\right]\underset{L_2+L_1\to L_1}{\longrightarrow}$$

$$\underset{L_2+L_1\to L_1}{\longrightarrow}\left[\begin{array}{cc|cc} 1 & 0 & \cos\alpha & \operatorname{sen}\alpha\\ \operatorname{sen}^2\alpha & \operatorname{sen}\alpha\cos\alpha & 0 & \operatorname{sen}\alpha\end{array}\right]\underset{(-\operatorname{sen}^2\alpha)L_1+L_2\to L_2}{\longrightarrow}$$

$$\underset{(-\operatorname{sen}^2\alpha)L_1+L_2\to L_2}{\longrightarrow}\left[\begin{array}{cc|cc} 1 & 0 & \cos\alpha & \operatorname{sen}\alpha\\ 0 & \operatorname{sen}\alpha\cos\alpha & -\operatorname{sen}^2\alpha\cos\alpha & \operatorname{sen}\alpha\,(1-\operatorname{sen}^2\alpha)\end{array}\right]\underset{\frac{1}{\operatorname{sen}\alpha\cos\alpha}L_2\to L_2}{\longrightarrow}$$

$\underset{\frac{1}{\operatorname{sen}\alpha\cos\alpha}L_2\to L_2}{\longrightarrow}\left[\begin{array}{cc|cc} 1 & 0 & \cos\alpha & \operatorname{sen}\alpha\\ 0 & 1 & -\operatorname{sen}\alpha & \cos\alpha\end{array}\right]$. Note que $\operatorname{sen}\alpha\cos\alpha \neq 0$ para todo o $\alpha \neq k\dfrac{\pi}{2}$, $(k \in \mathbb{Z})$.

Logo $\left[\begin{array}{cc} \cos\alpha & -\operatorname{sen}\alpha\\ \operatorname{sen}\alpha & \cos\alpha\end{array}\right]^{-1}=\left[\begin{array}{cc} \cos\alpha & \operatorname{sen}\alpha\\ -\operatorname{sen}\alpha & \cos\alpha\end{array}\right]$, para todo o $\alpha \neq k\dfrac{\pi}{2}$, $(k \in \mathbb{Z})$

Se $\alpha = \dfrac{\pi}{2} + 2k\pi$, $(k \in \mathbb{Z})$,

$$\left[\begin{array}{cc} \cos\alpha & -\operatorname{sen}\alpha\\ \operatorname{sen}\alpha & \cos\alpha\end{array}\right]^{-1}=\left[\begin{array}{cc} 0 & -1\\ 1 & 0\end{array}\right]^{-1}=\left[\begin{array}{cc} 0 & 1\\ -1 & 0\end{array}\right]=\left[\begin{array}{cc} \cos\alpha & \operatorname{sen}\alpha\\ -\operatorname{sen}\alpha & \cos\alpha\end{array}\right].$$

Se $\alpha = 2k\pi$, $(k \in \mathbb{Z})$,

$$\left[\begin{array}{cc} \cos\alpha & -\operatorname{sen}\alpha\\ \operatorname{sen}\alpha & \cos\alpha\end{array}\right]^{-1}=\left[\begin{array}{cc} 1 & 0\\ 0 & 1\end{array}\right]^{-1}=\left[\begin{array}{cc} 1 & 0\\ 0 & 1\end{array}\right]=\left[\begin{array}{cc} \cos\alpha & \operatorname{sen}\alpha\\ -\operatorname{sen}\alpha & \cos\alpha\end{array}\right].$$

Se $\alpha = \pi + 2k\pi$, $(k \in \mathbb{Z})$,

$$\left[\begin{array}{cc} \cos\alpha & -\operatorname{sen}\alpha\\ \operatorname{sen}\alpha & \cos\alpha\end{array}\right]^{-1}=\left[\begin{array}{cc} -1 & 0\\ 0 & -1\end{array}\right]^{-1}=\left[\begin{array}{cc} -1 & 0\\ 0 & -1\end{array}\right]=\left[\begin{array}{cc} \cos\alpha & \operatorname{sen}\alpha\\ -\operatorname{sen}\alpha & \cos\alpha\end{array}\right].$$

Se $\alpha = \dfrac{3\pi}{2} + 2k\pi$, $(k \in \mathbb{Z})$,

$$\begin{bmatrix} \cos\alpha & -\operatorname{sen}\alpha \\ \operatorname{sen}\alpha & \cos\alpha \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \operatorname{sen}\alpha \\ -\operatorname{sen}\alpha & \cos\alpha \end{bmatrix}.$$

Logo, para todo o $\alpha \in \mathbb{R}$

$$\begin{bmatrix} \cos\alpha & -\operatorname{sen}\alpha \\ \operatorname{sen}\alpha & \cos\alpha \end{bmatrix}^{-1} = \begin{bmatrix} \cos\alpha & \operatorname{sen}\alpha \\ -\operatorname{sen}\alpha & \cos\alpha \end{bmatrix}.$$

(iii) Seja $k \neq 0$.

$$\left[\begin{array}{cccc|cccc} k & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & k & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & k & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & k & 0 & 0 & 0 & 1 \end{array}\right] \underset{\substack{-\frac{1}{k}L_1+L_2\to L_2 \\ \frac{1}{k}L_3\to L_3 \\ \frac{1}{k}L_4\to L_4}}{\longrightarrow} \left[\begin{array}{cccc|cccc} k & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & -\frac{1}{k} & 1 & 0 & 0 \\ 0 & \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} & 0 \\ 0 & 0 & \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} \end{array}\right] \underset{\substack{-\frac{1}{k^2}L_2+L_3\to L_3 \\ \frac{1}{k}L_1\to L_1}}{\longrightarrow}$$

$$\underset{\substack{-\frac{1}{k^2}L_2+L_3\to L_3 \\ \frac{1}{k}L_1\to L_1}}{\longrightarrow} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ 0 & k & 0 & 0 & -\frac{1}{k} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} & 0 \\ 0 & 0 & \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} \end{array}\right] \underset{\substack{-\frac{1}{k}L_3+L_4\to L_4 \\ \frac{1}{k}L_2\to L_2}}{\longrightarrow}$$

$$\underset{\substack{-\frac{1}{k}L_3+L_4\to L_4 \\ \frac{1}{k}L_2\to L_2}}{\longrightarrow} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{k^2} & \frac{1}{k} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{k^4} & \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} \end{array}\right].$$

$$\text{Logo} \begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{k} & 0 & 0 & 0 \\ -\frac{1}{k^2} & \frac{1}{k} & 0 & 0 \\ \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} & 0 \\ -\frac{1}{k^4} & \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} \end{bmatrix}.$$

(iv) $$\begin{bmatrix} \frac{5}{13} & \frac{2}{13} & \frac{2}{13} & -\frac{8}{13} \\ \frac{2}{13} & -\frac{7}{13} & \frac{6}{13} & \frac{2}{13} \\ \frac{2}{13} & \frac{6}{13} & -\frac{7}{13} & \frac{2}{13} \\ -\frac{8}{13} & \frac{2}{13} & \frac{2}{13} & \frac{5}{13} \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 5 & 2 & 2 & -8 \\ 2 & -7 & 6 & 2 \\ 2 & 6 & -7 & 2 \\ -8 & 2 & 2 & 5 \end{bmatrix}.$$

$$\left[\begin{array}{cccc|cccc} 5 & 2 & 2 & -8 & 1 & 0 & 0 & 0 \\ 2 & -7 & 6 & 2 & 0 & 1 & 0 & 0 \\ 2 & 6 & -7 & 2 & 0 & 0 & 1 & 0 \\ -8 & 2 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \underset{L_1\leftrightarrow L_3}{\longrightarrow} \left[\begin{array}{cccc|cccc} 2 & 6 & -7 & 2 & 0 & 0 & 1 & 0 \\ 2 & -7 & 6 & 2 & 0 & 1 & 0 & 0 \\ 5 & 2 & 2 & -8 & 1 & 0 & 0 & 0 \\ -8 & 2 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \underset{\substack{-L_1+L_2\to L_2 \\ -\frac{5}{2}L_1+L_3\to L_3 \\ 4L_1+L_4\to L_4}}{\longrightarrow}$$

$$
\underset{\substack{-L_1+L_2\to L_2\\ -\frac{5}{2}L_1+L_3\to L_3\\ 4L_1+L_4\to L_4}}{\longrightarrow}
\left[\begin{array}{cccc|cccc}
2 & 6 & -7 & 2 & 0 & 0 & 1 & 0\\
0 & -13 & 13 & 0 & 0 & 1 & -1 & 0\\
0 & -13 & \frac{39}{2} & -13 & 1 & 0 & -\frac{5}{2} & 0\\
0 & 26 & -26 & 13 & 0 & 0 & 4 & 1
\end{array}\right]
\underset{\substack{-L_2+L_3\to L_3\\ 2L_2+L_4\to L_4}}{\longrightarrow}
$$

$$
\underset{\substack{-L_2+L_3\to L_3\\ 2L_2+L_4\to L_4}}{\longrightarrow}
\left[\begin{array}{cccc|cccc}
2 & 6 & -7 & 2 & 0 & 0 & 1 & 0\\
0 & -13 & 13 & 0 & 0 & 1 & -1 & 0\\
0 & 0 & \frac{13}{2} & -13 & 1 & -1 & -\frac{3}{2} & 0\\
0 & 0 & 0 & 13 & 0 & 2 & 2 & 1
\end{array}\right]
\underset{\substack{\frac{1}{2}L_1\to L_1\\ -\frac{1}{13}L_2\to L_2\\ \frac{2}{13}L_3\to L_3\\ \frac{1}{13}L_4\to L_4}}{\longrightarrow}
$$

$$
\underset{\substack{\frac{1}{2}L_1\to L_1\\ -\frac{1}{13}L_2\to L_2\\ \frac{2}{13}L_3\to L_3\\ \frac{1}{13}L_4\to L_4}}{\longrightarrow}
\left[\begin{array}{cccc|cccc}
1 & 3 & -\frac{7}{2} & 1 & 0 & 0 & \frac{1}{2} & 0\\
0 & 1 & -1 & 0 & 0 & -\frac{1}{13} & \frac{1}{13} & 0\\
0 & 0 & 1 & -2 & \frac{2}{13} & -\frac{2}{13} & -\frac{3}{13} & 0\\
0 & 0 & 0 & 1 & 0 & \frac{2}{13} & \frac{2}{13} & \frac{1}{13}
\end{array}\right]
\underset{\substack{-L_4+L_1\to L_1\\ 2L_4+L_3\to L_3}}{\longrightarrow}
$$

$$
\underset{\substack{-L_4+L_1\to L_1\\ 2L_4+L_3\to L_3}}{\longrightarrow}
\left[\begin{array}{cccc|cccc}
1 & 3 & -\frac{7}{2} & 0 & 0 & -\frac{2}{13} & \frac{9}{26} & -\frac{1}{13}\\
0 & 1 & -1 & 0 & 0 & -\frac{1}{13} & \frac{1}{13} & 0\\
0 & 0 & 1 & 0 & \frac{2}{13} & \frac{2}{13} & \frac{1}{13} & \frac{2}{13}\\
0 & 0 & 0 & 1 & 0 & \frac{2}{13} & \frac{2}{13} & \frac{1}{13}
\end{array}\right]
\underset{\substack{L_3+L_2\to L_2\\ \frac{7}{2}L_3+L_1\to L_1}}{\longrightarrow}
$$

$$
\underset{\substack{L_3+L_2\to L_2\\ \frac{7}{2}L_3+L_1\to L_1}}{\longrightarrow}
\left[\begin{array}{cccc|cccc}
1 & 3 & 0 & 0 & \frac{7}{13} & \frac{5}{13} & \frac{8}{13} & \frac{6}{13}\\
0 & 1 & 0 & 0 & \frac{2}{13} & \frac{1}{13} & \frac{2}{13} & \frac{2}{13}\\
0 & 0 & 1 & 0 & \frac{2}{13} & \frac{2}{13} & \frac{1}{13} & \frac{2}{13}\\
0 & 0 & 0 & 1 & 0 & \frac{2}{13} & \frac{2}{13} & \frac{1}{13}
\end{array}\right]
\underset{-3L_2+L_1\to L_1}{\longrightarrow}
\left[\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & \frac{1}{13} & \frac{2}{13} & \frac{2}{13} & 0\\
0 & 1 & 0 & 0 & \frac{2}{13} & \frac{1}{13} & \frac{2}{13} & \frac{2}{13}\\
0 & 0 & 1 & 0 & \frac{2}{13} & \frac{2}{13} & \frac{1}{13} & \frac{2}{13}\\
0 & 0 & 0 & 1 & 0 & \frac{2}{13} & \frac{2}{13} & \frac{1}{13}
\end{array}\right].
$$

Logo
$$
\begin{bmatrix}
\frac{5}{13} & \frac{2}{13} & \frac{2}{13} & -\frac{8}{13}\\
\frac{2}{13} & -\frac{7}{13} & \frac{6}{13} & \frac{2}{13}\\
\frac{2}{13} & \frac{6}{13} & -\frac{7}{13} & \frac{2}{13}\\
-\frac{8}{13} & \frac{2}{13} & \frac{2}{13} & \frac{5}{13}
\end{bmatrix}^{-1}
=
\left(\frac{1}{13}\begin{bmatrix}
5 & 2 & 2 & -8\\
2 & -7 & 6 & 2\\
2 & 6 & -7 & 2\\
-8 & 2 & 2 & 5
\end{bmatrix}\right)^{-1}
=
$$

$$= 13 \begin{bmatrix} \frac{1}{13} & \frac{2}{13} & \frac{2}{13} & 0 \\ \frac{2}{13} & \frac{1}{13} & \frac{2}{13} & \frac{2}{13} \\ \frac{2}{13} & \frac{2}{13} & \frac{1}{13} & \frac{2}{13} \\ 0 & \frac{2}{13} & \frac{2}{13} & \frac{1}{13} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 \\ 0 & 2 & 2 & 1 \end{bmatrix}.$$

(v) $$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & -1 & -1 & 1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 1 & -1 & -1 & 2 \end{bmatrix}.$$

$$\left[\begin{array}{cccc|cccc} 2 & -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[L_1 \leftrightarrow L_4]{} \left[\begin{array}{cccc|cccc} 1 & -1 & -1 & 2 & 0 & 0 & 0 & 1 \\ -1 & 2 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 1 & 0 \\ 2 & -1 & -1 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\substack{L_1+L_2\to L_2 \\ L_1+L_3\to L_3 \\ -2L_1+L_4\to L_4}]{}$$

$$\xrightarrow[\substack{L_1+L_2\to L_2 \\ L_1+L_3\to L_3 \\ -2L_1+L_4\to L_4}]{} \left[\begin{array}{cccc|cccc} 1 & -1 & -1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & -3 & 1 & 0 & 0 & -2 \end{array}\right] \xrightarrow[\substack{L_2+L_3\to L_3 \\ -L_2+L_4\to L_4}]{}$$

$$\xrightarrow[\substack{L_2+L_3\to L_3 \\ -L_2+L_4\to L_4}]{} \left[\begin{array}{cccc|cccc} 1 & -1 & -1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & -4 & 1 & -1 & 0 & -3 \end{array}\right] \xrightarrow[L_3 \leftrightarrow L_4]{}$$

$$\xrightarrow[L_3 \leftrightarrow L_4]{} \left[\begin{array}{cccc|cccc} 1 & -1 & -1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -4 & 1 & -1 & 0 & -3 \\ 0 & 0 & 0 & 2 & 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow[\substack{2L_4+L_3\to L_3 \\ -\frac{1}{2}L_4+L_2\to L_2 \\ -L_4+L_1\to L_1}]{}$$

$$\xrightarrow[\substack{2L_4+L_3\to L_3\\ -\frac{1}{2}L_4+L_2\to L_2\\ -L_4+L_1\to L_1}]{} \left[\begin{array}{cccc|cccc} 1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow[\substack{\frac{1}{2}L_3\to L_3\\ \frac{1}{2}L_4\to L_4}]{}$$

$$\xrightarrow[\substack{\frac{1}{2}L_3\to L_3\\ \frac{1}{2}L_4\to L_4}]{} \left[\begin{array}{cccc|cccc} 1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] \xrightarrow[\substack{L_3+L_2\to L_2\\ L_3+L_1\to L_1}]{}$$

$$\xrightarrow[\substack{L_3+L_2\to L_2\\ L_3+L_1\to L_1}]{} \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] \xrightarrow[L_2+L_1\to L_1]{}$$

$$\xrightarrow[L_2+L_1\to L_1]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{array}\right].$$

$$\text{Logo} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}^{-1} = \left(\frac{1}{2}\begin{bmatrix} 2 & -1 & -1 & 1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 1 & -1 & -1 & 2 \end{bmatrix}\right)^{-1} =$$

$$= 2\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

25.

$$I - A^T B = B^2 \Leftrightarrow A = \left[\left(I - B^2\right) B^{-1}\right]^T =$$

$$= \left[\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^2\right) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}\right]^T = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}.$$

26.

$$A^{-1} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} - I = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}^T \Leftrightarrow A^{-1} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

27. Sejam $A, B, X \in \mathcal{M}_{n \times n}(\mathbb{R})$ matrizes invertíveis tais que

$$(AB)^2 = \begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}.$$

(i)

$$AXB + AB = \mathbf{0} \Leftrightarrow AXB = -AB \Leftrightarrow A^{-1}(AXB)B^{-1} = A^{-1}(-AB)B^{-1} \Leftrightarrow$$

$$\Leftrightarrow \left(A^{-1}A\right) X \left(BB^{-1}\right) = -\left(A^{-1}A\right)\left(BB^{-1}\right) \Leftrightarrow$$

$$\Leftrightarrow IXI = -II \Leftrightarrow X = -I.$$

(ii)

$$BXA - A^{-1}B^{-1} = \mathbf{0} \Leftrightarrow B^{-1}(BXA)A^{-1} = B^{-1}\left(A^{-1}B^{-1}\right)A^{-1} \Leftrightarrow$$

$$\Leftrightarrow \left(B^{-1}B\right) X \left(AA^{-1}\right) = \left(B^{-1}A^{-1}\right)\left(B^{-1}A^{-1}\right) \Leftrightarrow$$

$$\Leftrightarrow IXI = \left[(AB)^2\right]^{-1} \Leftrightarrow$$

$$\Leftrightarrow X = \begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} -9 & 4 \\ 7 & -3 \end{bmatrix}.$$

28.

$$\begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} A^T + (\operatorname{tr} I)\, I = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}^2 - \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow A = \left(\begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}^{-1} \left(\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\right)^T = \begin{bmatrix} 0 & 0 \\ -21 & 12 \end{bmatrix}.$$

29. $\left(A \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}\right)^T = 3 \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}^2 \Leftrightarrow A = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 \\ 3 & -3 \end{bmatrix}.$

30. $A \in \mathcal{M}_{2\times 2}(\mathbb{R})$,

$$\left(2I - \left(3A^{-1}\right)^T\right)^{-1} = \begin{bmatrix} 4 & 3 \\ 7 & 5 \end{bmatrix} \Leftrightarrow 2I - \left(3A^{-1}\right)^T = \begin{bmatrix} 4 & 3 \\ 7 & 5 \end{bmatrix}^{-1} \Leftrightarrow$$

$$\Leftrightarrow 2I - \left(3A^{-1}\right)^T = \begin{bmatrix} -5 & 3 \\ 7 & -4 \end{bmatrix} \Leftrightarrow \left(3A^{-1}\right)^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} -5 & 3 \\ 7 & -4 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow A = \left(\frac{1}{3}\begin{bmatrix} 7 & -3 \\ -7 & 6 \end{bmatrix}^T\right)^{-1} = \begin{bmatrix} \frac{7}{3} & -\frac{7}{3} \\ -1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{6}{7} & 1 \\ \frac{3}{7} & 1 \end{bmatrix}.$$

31.

$$\left(A^T + 4I\right)^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \Leftrightarrow A^T + 4I = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}^{-1} \Leftrightarrow$$

$$\Leftrightarrow A = \left(\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} -9 & 2 \\ 3 & -5 \end{bmatrix}.$$

32. $\begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}^T A \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix} = I \Leftrightarrow A = \left(\begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}\right)^{-1} = I.$

33.

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}^{-1} (2I - A) = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow A = 2I - \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

34. $A^T - I = \left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} - I\right)^{-1} \Leftrightarrow A^T - I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Leftrightarrow A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

35.

$$\left(\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} A\right)^T + 2I = \left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^2 - 2\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\right)^{-1} \Leftrightarrow$$

$$\Leftrightarrow A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}^{-1}\left(\begin{bmatrix} 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 \end{bmatrix} - 2I\right)^T = \begin{bmatrix} -1 & -\frac{1}{8} \\ \frac{7}{8} & -\frac{7}{8} \end{bmatrix}.$$

36. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^T A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + I = 2I \underset{B \text{ é invertível}}{\Leftrightarrow} A = \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1}\right)^T \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} =$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

37. (i) Seja $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ tal que $A^k = \mathbf{0}$ para algum $k \in \mathbb{N}\setminus\{1\}$.

$(I - A)\left(I + A + ... + A^{k-1}\right) = I + A + ... + A^{k-1} - A - A^2 - ... - A^{k-1} - A^k = I - A^k = I$

ou seja, $I - A$ é invertível e

$$(I - A)^{-1} = I + A + ... + A^{k-1}.$$

(ii)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \left(I - \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\right)^{-1} \underset{\text{(i)}}{=}$$

$$= I + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

uma vez que $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

38. $A = \begin{bmatrix} 2 & 2 & -2 \\ 5 & 1 & -3 \\ 1 & 5 & -3 \end{bmatrix}.$

(i) $A^3 = \begin{bmatrix} 2 & 2 & -2 \\ 5 & 1 & -3 \\ 1 & 5 & -3 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

(ii) Por (i): $(I - A)(I + A + A^2) = I$.

39. a)

$$A_\alpha = \begin{bmatrix} 1 & -\alpha & \alpha \\ 2 & 1 & -2 \\ 3 & 2+\alpha & -2 \end{bmatrix} \underset{\substack{-2L_1+L_2\to L_2 \\ -3L_1+L_3\to L_3}}{\longrightarrow} \begin{bmatrix} 1 & -\alpha & \alpha \\ 0 & 1+2\alpha & -2-2\alpha \\ 0 & 2+4\alpha & -2-3\alpha \end{bmatrix} \underset{-2L_2+L_3\to L_3}{\longrightarrow}$$

$$\longrightarrow \begin{bmatrix} 1 & -\alpha & \alpha \\ 0 & 1+2\alpha & -2-2\alpha \\ 0 & 0 & 2+\alpha \end{bmatrix}.$$

Logo, car $A_\alpha = 3$ se e só se $\alpha \neq -\frac{1}{2}$ e $\alpha \neq -2$.

A matriz $A_\alpha = 3$ é invertível se e só se $\alpha \neq -\frac{1}{2}$ e $\alpha \neq -2$.

car $A_\alpha = 2$ se e só se $\alpha = -\frac{1}{2}$ ou $\alpha = -2$.

b) Para $\alpha = 0$,

$$[A_0 \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 3 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \underset{\substack{-2L_1+L_2 \longrightarrow L_2 \\ -3L_1+L_3 \longrightarrow L_3}}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 2 & -2 & -3 & 0 & 1 \end{array}\right] \underset{-2L_2+L_3 \longrightarrow L_3}{\longrightarrow}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right] \underset{L_3+L_2 \longrightarrow L_2}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right] \underset{\frac{1}{2}L_3 \longrightarrow L_3}{\longrightarrow}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right].$$

Logo,

$$A_0^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ 1/2 & -1 & 1/2 \end{bmatrix}.$$

c) $A_0X = B \Longleftrightarrow X = A_0^{-1}B \Longleftrightarrow X = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ 1/2 & -1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$

40. **(i)** $B_{a,b} = \begin{bmatrix} 0 & 0 & a & 1 \\ 2 & 2 & 0 & a \\ 0 & 0 & a & b \\ 3 & 0 & 6 & 0 \end{bmatrix} \underset{L_1 \leftrightarrow L_2}{\longrightarrow} \begin{bmatrix} 2 & 2 & 0 & a \\ 0 & 0 & a & 1 \\ 0 & 0 & a & b \\ 3 & 0 & 6 & 0 \end{bmatrix} \underset{L_2 \leftrightarrow L_4}{\longrightarrow}$

$$\underset{L_2 \leftrightarrow L_4}{\longrightarrow} \begin{bmatrix} 2 & 2 & 0 & a \\ 3 & 0 & 6 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & a & 1 \end{bmatrix} \underset{\substack{-\frac{3}{2}L_1+L_2 \to L_2 \\ -L_3+L_4 \to L_4}}{\longrightarrow} \begin{bmatrix} 2 & 2 & 0 & a \\ 0 & -3 & 6 & -\frac{3}{2}a \\ 0 & 0 & a & b \\ 0 & 0 & 0 & 1-b \end{bmatrix}.$$

Se $a = 0$ ou ($a \neq 0$ e $b = 1$) então car $B_{a,b} = 3$ e nul $B_{a,b} = 1$.

Se $a \neq 0$ e $b \neq 1$ então car $B_{a,b} = 4$ e nul $B_{a,b} = 0$.

(ii) $[B_{1,0} \mid I] = \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 6 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \underset{L_1 \leftrightarrow L_4}{\longrightarrow} \left[\begin{array}{cccc|cccc} 3 & 0 & 6 & 0 & 0 & 0 & 0 & 1 \\ 2 & 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \underset{\substack{-\frac{2}{3}L_1+L_2 \to L_2 \\ -L_3+L_4 \to L_4}}{\longrightarrow}$

$$\underset{\substack{-\frac{2}{3}L_1+L_2 \to L_2 \\ -L_3+L_4 \to L_4}}{\longrightarrow} \left[\begin{array}{cccc|cccc} 3 & 0 & 6 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & -4 & 1 & 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right] \underset{\substack{-L_4+L_2 \to L_2 \\ -6L_3+L_1 \to L_1}}{\longrightarrow}$$

$$\xrightarrow[-6L_3+L_1\to L_1]{-L_4+L_2\to L_2}\left[\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 0 & 0 & -6 & 1 \\ 0 & 2 & -4 & 0 & -1 & 1 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right]\xrightarrow[\frac{1}{3}L_1\to L_1]{\frac{1}{2}L_2\to L_2}$$

$$\xrightarrow[\frac{1}{3}L_1\to L_1]{\frac{1}{2}L_2\to L_2}\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & -2 & \frac{1}{3} \\ 0 & 1 & -2 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right]\xrightarrow[2L_3+L_2\to L_2]{}\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & -2 & \frac{1}{3} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right].$$

Logo $(B_{1,0})^{-1} = \begin{bmatrix} 0 & 0 & -2 & \frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$.

(iii) Como $B_{1,0}$ é invertível,

$$B_{1,0}X = C \Leftrightarrow X = (B_{1,0})^{-1}\,C \Leftrightarrow X = \begin{bmatrix} 0 & 0 & -2 & \frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{19}{3} \\ \frac{19}{3} \\ 3 \\ -2 \end{bmatrix}.$$

(iv) Seja $X = (x_1, x_2, x_3, x_4)$.

$$B_{a,1}X = D \Leftrightarrow \begin{bmatrix} 0 & 0 & a & 1 \\ 2 & 2 & 0 & a \\ 0 & 0 & a & 1 \\ 3 & 0 & 6 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -a \\ 0 \\ -a \\ -6 \end{bmatrix}.$$

A solução geral de $B_{a,1}X = D$ é dada por:

(Solução particular de $B_{a,1}X = D$) + (Solução geral de $B_{a,1}X = \mathbf{0}$).

O vector $(0, 0, -1, 0)$ é uma solução particular de $B_{a,1}X = D$. Determinemos a solução geral de $B_{a,1}X = \mathbf{0}$.

Tem-se $\begin{bmatrix} 0 & 0 & a & 1 \\ 2 & 2 & 0 & a \\ 0 & 0 & a & 1 \\ 3 & 0 & 6 & 0 \end{bmatrix}\xrightarrow[-\frac{3}{2}L_2+L_4\to L_4]{-L_1+L_3\to L_3}\begin{bmatrix} 0 & 0 & a & 1 \\ 2 & 2 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 6 & -\frac{3}{2}a \end{bmatrix}\xrightarrow[L_3\leftrightarrow L_4]{L_1\leftrightarrow L_2}$

$$\xrightarrow[L_3\leftrightarrow L_4]{L_1\leftrightarrow L_2}\begin{bmatrix} 2 & 2 & 0 & a \\ 0 & 0 & a & 1 \\ 0 & -3 & 6 & -\frac{3}{2}a \\ 0 & 0 & 0 & 0 \end{bmatrix}\xrightarrow[L_2\leftrightarrow L_3]{}\begin{bmatrix} 2 & 2 & 0 & a \\ 0 & -3 & 6 & -\frac{3}{2}a \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Logo, $\begin{cases} 2x_1+2x_2+ax_4=0 \\ -3x_2+6x_3-\frac{3}{2}ax_4=0 \\ ax_3+x_4=0 \end{cases} \Leftrightarrow \begin{cases} x=-2x_3 \\ x_2=\left(2+\dfrac{a^2}{2}\right)x_3 \\ x_4=-ax_3. \end{cases}$

Assim, a solução geral de $B_{a,1}X=\mathbf{0}$ é dada por:

$$\left\{(-2s,\left(2+\frac{a^2}{2}\right)s,s,-as):s\in\mathbb{R}\right\}.$$

Logo, a solução geral do sistema linear $B_{a,1}X=D$ é dada por:

$$\{(0,0,-1,0)\}+\left\{\left(-2s,\left(2+\frac{a^2}{2}\right)s,s,-as\right):s\in\mathbb{R}\right\}=\left\{\left(-2s,\left(2+\frac{a^2}{2}\right)s,s-1,-as\right):s\in\mathbb{R}\right\}.$$

Resolução Alternativa.

$$[B_{a,1}\mid D]=\left[\begin{array}{cccc|c} 0&0&a&1&-a\\ 2&2&0&a&0\\ 0&0&a&1&-a\\ 3&0&6&0&-6 \end{array}\right]\underset{\substack{-L_1+L_3\to L_3\\ -\frac{3}{2}L_2+L_4\to L_4}}{\longrightarrow}\left[\begin{array}{cccc|c} 0&0&a&1&-a\\ 2&2&0&a&0\\ 0&0&0&0&0\\ 0&-3&6&-\frac{3}{2}a&-6 \end{array}\right]\underset{\substack{L_1\leftrightarrow L_2\\ L_3\leftrightarrow L_4}}{\longrightarrow}$$

$$\underset{\substack{L_1\leftrightarrow L_2\\ L_3\leftrightarrow L_4}}{\longrightarrow}\left[\begin{array}{cccc|c} 2&2&0&a&0\\ 0&0&a&1&-a\\ 0&-3&6&-\frac{3}{2}a&-6\\ 0&0&0&0&0 \end{array}\right]\underset{L_2\leftrightarrow L_3}{\longrightarrow}\left[\begin{array}{cccc|c} 2&2&0&a&0\\ 0&-3&6&-\frac{3}{2}a&-6\\ 0&0&a&1&-a\\ 0&0&0&0&0 \end{array}\right].$$

Tem-se então $\begin{cases} 2x+2y\ +aw=0 \\ -3y+6z-\dfrac{3}{2}aw=-6 \\ az+w=-a \end{cases} \Leftrightarrow \begin{cases} x=-2z-2 \\ y=\left(\frac{a^2}{2}+2\right)(z+1) \\ w=-a-az. \end{cases}$

Logo, a solução geral do sistema linear $B_{a,1}X=D$ é dada por:

$$\left\{\left(-2s-2,\left(\frac{a^2}{2}+2\right)(s+1),s,-a-as\right):s\in\mathbb{R}\right\}=\left\{\left(-2s,\left(2+\frac{a^2}{2}\right)s,s-1,-as\right):s\in\mathbb{R}\right\}.$$

41. **(i)** e **(ii)** $A_{\lambda,\mu}=\begin{bmatrix} 1&0&-1&0\\ 1&\lambda&\lambda^2+\mu&\lambda\mu\\ 0&1&\lambda&\mu\\ 1&\lambda&\lambda^2+\mu&\lambda+\lambda\mu \end{bmatrix}\underset{L_2\leftrightarrow L_3}{\longrightarrow}\begin{bmatrix} 1&0&-1&0\\ 0&1&\lambda&\mu\\ 1&\lambda&\lambda^2+\mu&\lambda\mu\\ 1&\lambda&\lambda^2+\mu&\lambda+\lambda\mu \end{bmatrix}\underset{\substack{-L_1+L_3\to L_3\\ -L_1+L_4\to L_4}}{\longrightarrow}$

$$\underset{\substack{-L_1+L_3\to L_3\\ -L_1+L_4\to L_4}}{\longrightarrow}\begin{bmatrix} 1&0&-1&0\\ 0&1&\lambda&\mu\\ 0&\lambda&\lambda^2+\mu+1&\lambda\mu\\ 0&\lambda&\lambda^2+\mu+1&\lambda+\lambda\mu \end{bmatrix}\underset{\substack{-\lambda L_2+L_3\to L_3\\ -\lambda L_2+L_4\to L_4}}{\longrightarrow}\begin{bmatrix} 1&0&-1&0\\ 0&1&\lambda&\mu\\ 0&0&\mu+1&0\\ 0&0&\mu+1&\lambda \end{bmatrix}\underset{-L_3+L_4\to L_4}{\longrightarrow}$$

$$\xrightarrow[-L_3+L_4\to L_4]{} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & \lambda & \mu \\ 0 & 0 & \mu+1 & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

Se $\mu = -1$ e $\lambda = 0$ então $\operatorname{car} A = 2$ e $\operatorname{nul} A = 2$.

Se ($\mu = -1$ e $\lambda \neq 0$) ou ($\mu \neq -1$ e $\lambda = 0$) então $\operatorname{car} A = 3$ e $\operatorname{nul} A = 1$.

Se $\mu \neq -1$ e $\lambda \neq 0$ então $\operatorname{car} A = 4$ e $\operatorname{nul} A = 0$.

Assim, $A_{\lambda,\mu}$ é invertível se e só se $\mu \neq -1$ e $\lambda \neq 0$, uma vez que é só neste caso que $\operatorname{car} A_{\lambda,\mu} = \text{n}^o$ de colunas de $A_{\lambda,\mu}$.

42. **(i)** Tem-se

$$A_\beta = \begin{bmatrix} 1 & 0 & \beta & 2 \\ 2 & \beta & \beta^2 & 4 \\ -4 & 0 & -\beta^3 & -8 \\ \beta & 0 & \beta^2 & \beta^2 \end{bmatrix} \xrightarrow[\substack{-2L_1+L_2\to L_2 \\ 4L_1+L_3\to L_3 \\ -\beta L_1+L_4\to L_4}]{} \begin{bmatrix} 1 & 0 & \beta & 2 \\ 0 & \beta & \beta(\beta-2) & 0 \\ 0 & 0 & (2-\beta)(2+\beta)\beta & 0 \\ 0 & 0 & 0 & \beta(\beta-2) \end{bmatrix}.$$

Logo, como $\operatorname{car} A_\beta + \operatorname{nul} A_\beta = 4$,
se $\beta = 0$ então $\operatorname{car} A_\beta = 1$ e $\operatorname{nul} A_\beta = 3$;
se $\beta = 2$ então $\operatorname{car} A_\beta = 2$ e $\operatorname{nul} A_\beta = 2$;
se $\beta = -2$ então $\operatorname{car} A_\beta = 3$ e $\operatorname{nul} A_\beta = 1$;
se $\beta \neq 0$ e $\beta \neq 2$ e $\beta \neq -2$ então $\operatorname{car} A_\alpha = 4$ e $\operatorname{nul} A_\alpha = 0$.
Assim, A_β é invertível se e só se $\beta \in \mathbb{R}\backslash\{-2,0,2\}$, uma vez que é só nestes casos que $\operatorname{car} A_\beta = \text{n}^o$ de colunas de A_β.

(ii) $\left[\begin{array}{c|c} A_1 & I \end{array}\right] =$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 4 & 0 & 1 & 0 & 0 \\ -4 & 0 & -1 & -8 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{-2L_1+L_2\to L_2 \\ 4L_1+L_3\to L_3 \\ -L_1+L_4\to L_4}]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{2L_4+L_1\to L_1 \\ -\frac{1}{3}L_3+L_1\to L_1 \\ \frac{1}{3}L_3+L_2\to L_2}]{}$$

$$\xrightarrow[\substack{2L_4+L_1\to L_1 \\ -\frac{1}{3}L_3+L_1\to L_1 \\ \frac{1}{3}L_3+L_2\to L_2}]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{7}{3} & 0 & -\frac{1}{3} & 2 \\ 0 & 1 & 0 & 0 & -\frac{2}{3} & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 3 & 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{-L_4\to L_4 \\ \frac{1}{3}L_3\to L_3}]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{7}{3} & 0 & -\frac{1}{3} & 2 \\ 0 & 1 & 0 & 0 & -\frac{2}{3} & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & \frac{4}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \end{array}\right]$$

Logo

$$(A_1)^{-1} = \begin{bmatrix} -\frac{7}{3} & 0 & -\frac{1}{3} & 2 \\ -\frac{2}{3} & 1 & \frac{1}{3} & 0 \\ \frac{4}{3} & 0 & \frac{1}{3} & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

43.

$$A_\alpha = \begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -\alpha^2 & -1 \\ 2 & 0 & -2 & -2 \end{bmatrix} \underset{\substack{L_1+L_3\to L_3 \\ 2L_1+L_4\to L_4}}{\longrightarrow} \begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1-\alpha^2 & \alpha-1 \\ 0 & 0 & 0 & 2\alpha-2 \end{bmatrix}.$$

a) Se $\alpha = 1$ então car $A_\alpha = 2$. Se $\alpha = -1$ então car $A_\alpha = 3$. Se $\alpha \neq 1$ e $\alpha \neq -1$ então car $A_\alpha = 4$. Por outro lado, tem-se:

$$\text{car } A_\alpha + \text{nul } A_\alpha = 4 \quad (= \text{n}^o \text{ de colunas de } A_\alpha).$$

Logo, se $\alpha = 1$ então nul $A_\alpha = 2$; se $\alpha = -1$ então nul $A_\alpha = 1$; se $\alpha \neq 1$ e $\alpha \neq -1$ então nul $A_\alpha = 0$.

b) Como A_α é quadrada, A_α é invertível se e só se car $A_\alpha = 4$ ($=$ n^o de colunas de A_α). Assim, A_α é invertível se e só se $\alpha \in \mathbb{R}\backslash \{-1, 1\}$.

c) Para $\alpha = 1$, tem-se:

$$A_1 = \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & -1 \\ 2 & 0 & -2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U'.$$

Seja $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Tem-se

$$A_1X = \mathbf{0} \Leftrightarrow \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(Método de eliminação de Gauss).}$$

$$\Leftrightarrow \begin{cases} -x_1 + x_3 + x_4 = 0 \\ \\ x_2 - x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = x_3 + x_4 \\ x_2 = x_3. \end{cases}$$

Logo, fazendo $x_3 = s$ e $x_4 = t$, a solução geral do sistema é dada por

$$CS = \{(s+t, s, s, t) : s, t \in \mathbb{R}\}.$$

d) Seja $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Tem-se

$$A_1X = B \Leftrightarrow \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & -1 \\ 2 & 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}.$$

A solução geral de $A_1X = B$ é dada por:

$$(\text{Solução particular de } A_1X = B) + (\text{Solução geral de } A_1X = \mathbf{0}).$$

O elemento $(0,0,0,1)$ é uma solução particular de $A_1X = B$. Logo, atendendo à alínea **c)**, a solução geral de $A_1X = B$ é dada por:

$$\{(0,0,0,1)\} + \{(s+t,s,s,t) : s,t \in \mathbb{R}\} = \{(0,0,0,1) + s(1,1,1,0) + t(1,0,0,1) : s,t \in \mathbb{R}\}.$$

Obs. Esta alínea podia ter sido resolvida directamente sem usar **c)**.

44. Seja $A = (a_{ij})_{n\times n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ tal que $Au = \mathbf{0}$ para qualquer $u \in \mathcal{M}_{n\times 1}(\mathbb{R})$.

Para cada $j \in \{1,...,n\}$ fixo, seja $e_j = (\delta_{ij})_{n\times 1} \in \mathcal{M}_{n\times 1}(\mathbb{R})$ em que $\delta_{ij} = \begin{cases} 1 & \text{se} \quad i = j \\ 0 & \text{se} \quad i \neq j. \end{cases}$ Como

$$Ae_j = \mathbf{0}$$

para todo o $j \in \{1,...,n\}$ e por outro lado

$$Ae_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

para todo o $j \in \{1,...,n\}$, então $\begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ para todo o $j \in \{1,...,n\}$ pelo que $A = \mathbf{0}$.

45. Sendo $A, B \in \mathcal{M}_{n\times 1}(\mathbb{R})$

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

e assim

$$AB^T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{bmatrix}.$$

Como A e B são matrizes não nulas, existe $i \in \{1,...,n\}$ tal que $a_i \neq 0$ e existe $j \in \{1,...,n\}$ tal que $b_j \neq 0$, tendo-se

$$AB^T = \begin{bmatrix} a_1b_1 & \cdots & a_1b_j & \cdots & a_1b_n \\ \vdots & \ddots & \vdots & & \vdots \\ a_ib_1 & \cdots & a_ib_j & \cdots & a_ib_n \\ \vdots & & \vdots & \ddots & \vdots \\ a_nb_1 & \cdots & a_nb_j & \cdots & a_nb_n \end{bmatrix}.$$

Aplicando sucessivamente a operação elementar

$$-\frac{a_k}{a_i}L_i + L_k \to L_k$$

para todo o $k = 1, ..., n$ com $k \neq i$, tem-se

$$\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ a_ib_1 & \cdots & a_ib_j & \cdots & a_ib_n \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \underset{L_i \leftrightarrow L_1}{\longrightarrow} \begin{bmatrix} a_ib_1 & \cdots & a_ib_j & \cdots & a_ib_n \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

com $a_ib_j \neq 0$, isto é,

$$\text{car}(AB^T) = 1.$$

46.

$$AB = A + B \Leftrightarrow (A - I)(B - I) = I.$$

Logo $A - I$ é a inversa de $B - I$ tendo-se

$$(A - I)(B - I) = I = (B - I)(A - I),$$

isto é,

$$AB = BA.$$

47. $A^2 = A(BAB) = (ABA)B = B^2$.

48.

$$(A + B)^2 = A^2 + B^2 + AB + BA = A + B + AB + BA.$$

Logo

$$(A + B)^2 = A + B \Leftrightarrow AB = -BA.$$

Mas

$$AB = A^2B = A(AB) = A(-BA) = -(AB)A = -(-BA)A = BA^2 = BA.$$

Logo

$$AB = -BA = -AB = BA \Leftrightarrow AB = BA = \mathbf{0}.$$

49. Seja $A = (a_{ij}) \in \mathcal{M}_{m\times n}(\mathbb{R})$. Tem-se

$$\text{tr}(A^TA) = \text{tr}\left(\sum_{k=1}^{n} a_{ki}a_{kj}\right) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ki}a_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ki}^2.$$

Logo

$$\text{tr}(A^TA) = 0 \Leftrightarrow A = \mathbf{0}.$$