Instituto Superior Técnico - Álgebra Linear - 2^o Semestre 2017/2018

Determinantes (resolução)

1. **(i)** (312645) é par pois tem 4 inversões.
(ii) (234516) é par pois tem 4 inversões.
(iii) (654321) é ímpar pois tem 15 inversões.
(iv) (123456) é par pois tem 0 inversões.
(v) (546321) é ímpar pois tem 13 inversões.
(vi) (453261) é par pois tem 10 inversões.
(vii) (634125) é ímpar pois tem 9 inversões.
(viii) (123465) é ímpar pois tem 1 inversão.

2. **(i)** (234516) é par pois tem 4 inversões e (312645) é par pois tem 4 inversões. Logo, tem-se

$$+a_{23}a_{31}a_{42}a_{56}a_{14}a_{65}$$

uma vez que (234516) e (312645) têm a mesma paridade.

(ii) (123456) é par pois tem 0 inversões e (654321) é ímpar pois tem 15 inversões. Logo, tem-se

$$-a_{16}a_{25}a_{34}a_{43}a_{52}a_{61}$$

uma vez que (123456) e (654321) têm paridades diferentes.

(iii) (546321) é ímpar pois tem 13 inversões e (453261) é par pois tem 10 inversões. Logo, tem-se

$$-a_{54}a_{45}a_{63}a_{32}a_{26}a_{11}$$

uma vez que (546321) e (453261) têm paridades diferentes.

(iv) (123465) é ímpar pois tem 1 inversão e (634125) é ímpar pois tem 9 inversões. Logo, tem-se

$$+a_{16}a_{23}a_{34}a_{41}a_{62}a_{55}$$

uma vez que (123465) e (634125) têm a mesma paridade.

3. **(i)** (123) é par pois tem 0 inversões e (321) é ímpar pois tem 3 inversões. Atendendo à definição de determinante, tem-se

$$\begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\, a_{13}a_{22}a_{31}$$

uma vez que (123) e (321) têm paridades diferentes.

(ii) (1234) é par pois tem 0 inversões e (4321) é par pois tem 6 inversões. Atendendo à definição de determinante, tem-se

$$\begin{vmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{14}a_{23}a_{32}a_{41}$$

uma vez que (1234) e (4321) têm a mesma paridade.

(iii) Ao efectuar $n-1+n-2+...+2+1$ trocas de linhas: linha 1 com cada uma das $n-1$ linhas que lhe estão abaixo, nova linha 1 com cada uma das $n-2$ linhas que lhe estão abaixo (excluindo a nova linha n) e assim sucessivamente, tem-se

$$\det \begin{bmatrix} 0 & \cdots & 0 & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a_{n-1\,2} & \cdots & a_{n-1\,n} \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} = (-1)^{n\frac{n-1}{2}} \det \begin{bmatrix} a_{n1} & \cdots & \cdots & a_{nn} \\ 0 & a_{n-1\,2} & \cdots & a_{n-1\,n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & a_{1n} \end{bmatrix} =$$

$$= (-1)^{n\frac{n-1}{2}} a_{1n}...a_{n-1\,2}a_{n1}$$

atendendo a que

$$n-1+n-2+...+2+1 = (n-1)\frac{n-1+1}{2} = n\frac{n-1}{2}.$$

4.

(i) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2 \neq 0$, logo a matriz é invertível.

(ii) $\begin{vmatrix} 18563 & 18573 \\ 21472 & 21482 \end{vmatrix} = \begin{vmatrix} 18563 & 10 \\ 21472 & 10 \end{vmatrix} = \begin{vmatrix} 18563 & 10 \\ 2909 & 0 \end{vmatrix} = -29090 \neq 0$, logo a matriz é invertível.

(iii) $\begin{vmatrix} 1+\sqrt{2} & 2-\sqrt{3} \\ 2+\sqrt{3} & 1-\sqrt{2} \end{vmatrix} = 1-2-(4-3) = -2 \neq 0$, logo a matriz é invertível.

(iv) $\begin{vmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{vmatrix} = \cos^2\theta - (-\operatorname{sen}^2\theta) = 1 \neq 0$, logo a matriz é invertível.

(v) $\begin{vmatrix} -2 & 0 & 1 \\ 5 & 3 & 0 \\ -5 & 1 & 2 \end{vmatrix} = -12+5-(-15) = 8 \neq 0$, logo a matriz é invertível.

(vi) $\begin{vmatrix} 2 & 3 & 2 \\ 5 & -1 & -3 \\ -2 & 1 & 1 \end{vmatrix} = -2+18+10-4-15-(-6) = 13 \neq 0$, logo a matriz é invertível.

(vii) $\begin{vmatrix} -2 & 1 & 1 \\ 5 & -1 & -3 \\ 2 & 3 & 2 \end{vmatrix} = -\begin{vmatrix} 2 & 3 & 2 \\ 5 & -1 & -3 \\ -2 & 1 & 1 \end{vmatrix} \underset{\text{por (vi)}}{=} -13 \neq 0$, logo a matriz é invertível.

(viii) $\begin{vmatrix} 8 & 12 & 8 \\ 5 & -1 & -3 \\ -2 & 1 & 1 \end{vmatrix} = 4\begin{vmatrix} 2 & 3 & 2 \\ 5 & -1 & -3 \\ -2 & 1 & 1 \end{vmatrix} \underset{\text{por (vi)}}{=} 52 \neq 0$, logo a matriz é invertível.

(ix) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 4 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$, logo a matriz não é invertível.

(x) $\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$, logo a matriz é invertível.

(xi) $\begin{vmatrix} -2 & -2 & 8 & 6 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3 & -23 \\ 0 & 0 & 0 & 5 \end{vmatrix} = 30 \neq 0$, logo a matriz é invertível.

(xii)

$$\begin{vmatrix} 1 & 3 & -1 & 1 \\ -1 & -2 & -1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 0 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -2 & 2 \\ 2 & 1 & 1 & 1 \\ 2 & 0 & -2 & 0 \end{vmatrix} = 2(-1)^{4+1} \begin{vmatrix} 3 & -1 & 1 \\ 1 & -2 & 2 \\ 1 & 1 & 1 \end{vmatrix} + (-2)(-1)^{4+3} \begin{vmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{vmatrix} =$$

$$= -2\,[-6 + 1 + (-2) - (-1) - (-2) - 6] + 2\,[1 + 12 - 2 - 2] = 20 + 18 = 38 \neq 0,$$

logo a matriz é invertível.

(xiii) $\begin{vmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & -3 & -23 \\ 0 & 1 & 2 & 0 \\ -2 & -2 & 8 & 6 \end{vmatrix} = \begin{vmatrix} -2 & -2 & 8 & 6 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3 & -23 \\ 0 & 0 & 0 & 5 \end{vmatrix} \underset{\text{por (xi)}}{=} 30 \neq 0$, logo a matriz é invertível.

(xiv) $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix} = \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \\ 7 & 9 & 11 & 13 \end{vmatrix} = \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 3 & 5 & 7 & 9 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 3 & 5 & 7 & 9 \\ 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix} =$

$= 0$, logo a matriz não é invertível.

(xv) $\begin{vmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{vmatrix} = 5(-1)^{5+1} \begin{vmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \end{vmatrix} = 5 \left[4(-1)^{1+1} \begin{vmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{vmatrix} \right] = 120 \neq 0$, logo a matriz é invertível.

(xvi) $\begin{vmatrix} 7 & -4 & 0 & 5 \\ 1 & 2 & 0 & -2 \\ 2 & 1 & -3 & 8 \\ 2 & -2 & 0 & 4 \end{vmatrix} = -3\,(-1)^6 \begin{vmatrix} 7 & -4 & 5 \\ 1 & 2 & -2 \\ 2 & -2 & 4 \end{vmatrix} = -3 \begin{vmatrix} 1 & 2 & -2 \\ 2 & -2 & 4 \\ 7 & -4 & 5 \end{vmatrix} = -3 \begin{vmatrix} 1 & 2 & -2 \\ 0 & -6 & 8 \\ 0 & -18 & 19 \end{vmatrix} =$

$= -3\,(6\,(-19 + 3 \times 8)) = -3\,(6\,(-19 + 3 \times 8)) = -3 \times 30 = -90 \neq 0$, logo a matriz é invertível.

(xvii) $\begin{vmatrix} 9 & 0 & -6 & 4 & -2 \\ 5 & -3 & 0 & -1 & 2 \\ 8 & 0 & 4 & 1 & 1 \\ -6 & 0 & 3 & -1 & 3 \\ 7 & 0 & 5 & 2 & 3 \end{vmatrix} = -3\,(-1)^4 \begin{vmatrix} 9 & -6 & 4 & -2 \\ 8 & 4 & 1 & 1 \\ -6 & 3 & -1 & 3 \\ 7 & 5 & 2 & 3 \end{vmatrix} = 3 \begin{vmatrix} -2 & -6 & 4 & 9 \\ 1 & 4 & 1 & 8 \\ 3 & 3 & -1 & -6 \\ 3 & 5 & 2 & 7 \end{vmatrix} =$

$$-3\begin{vmatrix} 1 & 4 & 1 & 8 \\ -2 & -6 & 4 & 9 \\ 3 & 3 & -1 & -6 \\ 3 & 5 & 2 & 7 \end{vmatrix} = -3\begin{vmatrix} 1 & 4 & 1 & 8 \\ 0 & 2 & 6 & 25 \\ 0 & -9 & -4 & -30 \\ 0 & -7 & -1 & -17 \end{vmatrix} = -3\begin{vmatrix} 2 & 6 & 25 \\ -7 & 2 & -5 \\ -7 & -1 & -17 \end{vmatrix} = -3\begin{vmatrix} 2 & 6 & 25 \\ -7 & 2 & -5 \\ 0 & -3 & -12 \end{vmatrix} =$$

$$-3\begin{vmatrix} 2 & 0 & 1 \\ -7 & 2 & -5 \\ 0 & -3 & -12 \end{vmatrix} = -3\begin{vmatrix} 2 & 0 & 1 \\ -7 & 2 & -13 \\ 0 & -3 & 0 \end{vmatrix} = -3\,(21-78) = 171 \neq 0,$$ logo a matriz é invertível.

(xviii) $$\begin{vmatrix} 2 & 1 & 0 & 0 \\ -7 & 3 & 0 & -2 \\ \pi & 3 & 1 & 1 \\ 3 & 9 & 0 & 0 \end{vmatrix} = (-1)^6\begin{vmatrix} 2 & 1 & 0 \\ -7 & 3 & -2 \\ 3 & 9 & 0 \end{vmatrix} = -2\,(-1)^5\begin{vmatrix} 2 & 1 \\ 3 & 9 \end{vmatrix} = 30 \neq 0,$$ logo a matriz é invertível.

(xix) $$\begin{vmatrix} 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{vmatrix} = (-1)\,(-1)^{4+2}\begin{vmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{vmatrix} \underset{\substack{(2314) \\ \text{é par}}}{=} -4.$$

(xx) $$\begin{vmatrix} n & n-1 & \vdots & 2 & 1 \\ n-1 & n-1 & \vdots & 2 & 1 \\ \cdots & \cdots & \ddots & \vdots & \vdots \\ 2 & 2 & \cdots & 2 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & \cdots & 2 & 2 \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ 1 & 2 & \cdots & n-1 & n-1 \\ 1 & 2 & \cdots & n-1 & n \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ 0 & 1 & \cdots & n-2 & n-2 \\ 0 & 1 & \cdots & n-2 & n-1 \end{vmatrix} = \cdots = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 \end{vmatrix} = 1 \neq 0,$$ logo a matriz é invertível.

5. (i) Seja $A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ e & 0 & b & 0 & 0 \\ 0 & f & 0 & c & 0 \\ 0 & 0 & g & 0 & d \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$, com $a, b, c, d, e, f, g, h \in \mathbb{R}$. Se $a = 0$ ou $h = 0$ então $\det A = 0$, isto é, A não é invertível. Se $a \neq 0$ e $h \neq 0$ então

$$\det A = \begin{vmatrix} 0 & a & 0 & 0 & 0 \\ e & 0 & b & 0 & 0 \\ 0 & f & 0 & c & 0 \\ 0 & 0 & g & 0 & d \\ 0 & 0 & 0 & h & 0 \end{vmatrix} = \begin{vmatrix} 0 & a & 0 & 0 & 0 \\ e & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g & 0 & d \\ 0 & 0 & 0 & h & 0 \end{vmatrix} = 0,$$

isto é, A não é invertível. Logo, A não é invertível para quaisquer $a, b, c, d, e, f, g, h \in \mathbb{R}$.
Em alternativa, pelo teorema de Laplace

$$\det A = (-1)^5 f \begin{vmatrix} 0 & 0 & 0 & 0 \\ e & b & 0 & 0 \\ 0 & g & 0 & d \\ 0 & 0 & h & 0 \end{vmatrix} + (-1)^7 c \begin{vmatrix} 0 & a & 0 & 0 \\ e & 0 & b & 0 \\ 0 & 0 & g & d \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0.$$

(ii) Diga, para que valores de $a, b, c, d, e, f, g, h, i, j \in \mathbb{R}$, é invertível a seguinte matriz

$$\det \begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 \\ f & 0 & b & 0 & 0 & 0 \\ 0 & g & 0 & c & 0 & 0 \\ 0 & 0 & h & 0 & d & 0 \\ 0 & 0 & 0 & i & 0 & e \\ 0 & 0 & 0 & 0 & j & 0 \end{bmatrix} = a(-1)^3 \det \begin{bmatrix} f & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & h & 0 & d & 0 \\ 0 & 0 & i & 0 & e \\ 0 & 0 & 0 & j & 0 \end{bmatrix} =$$

$$= -ac(-1)^5 \det \begin{bmatrix} f & b & 0 & 0 \\ 0 & h & d & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & j & 0 \end{bmatrix} = ace(-1)^7 \det \begin{bmatrix} f & b & 0 \\ 0 & h & d \\ 0 & 0 & j \end{bmatrix} = -acefhj,$$

pelo que a matriz será invertível se e só se $a, c, e, f, h, j \in \mathbb{R} \setminus \{0\}$.

6. Determinemos todos os valores do escalar λ para os quais a matriz $A - \lambda I$ não é invertível, isto é, todos os valores próprios de A.

(i) $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 3 \\ 2 & -1-\lambda \end{vmatrix} = \lambda(1+\lambda) - 6 = \lambda^2 + \lambda - 6.$
Logo, $\det(A - \lambda I) = 0 \Leftrightarrow (\lambda = 2 \text{ ou } \lambda = -3)$.

(ii) $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & -2 \\ 2 & -2 & -\lambda \end{vmatrix} = \lambda\left(1-\lambda^2\right) + 4(1+\lambda) - 4(1-\lambda) =$
$= \lambda\left[\left(1-\lambda^2\right) + 8\right]$. Logo, $\det(A - \lambda I) = 0 \Leftrightarrow (\lambda = 0 \text{ ou } \lambda = -3 \text{ ou } \lambda = 3)$.

(iii) $\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} = \det \begin{bmatrix} 1-\lambda & 1 & 1 \\ \lambda & -\lambda & 0 \\ \lambda & 0 & -\lambda \end{bmatrix} =$

$$= \det \begin{bmatrix} 2-\lambda & 1 & 1 \\ \lambda & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = \det \begin{bmatrix} 3-\lambda & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = (-1)^3 (\lambda - 3)\lambda^2.$$

Logo, $\det(A - \lambda I) = 0 \Leftrightarrow (\lambda = 0 \text{ ou } \lambda = 3)$.

(iv)

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 1 \\ 1 & \cdots & 1 & 1-\lambda \end{bmatrix}_{n\times n} \underset{\substack{-L_1+L_i \to L_i \\ i=2,\ldots,n}}{=} \det \begin{bmatrix} 1-\lambda & 1 & 1 & \cdots & 1 \\ \lambda & -\lambda & 0 & \cdots & 0 \\ \vdots & 0 & -\lambda & \ddots & \vdots \\ & \vdots & \ddots & \ddots & 0 \\ \lambda & 0 & \cdots & 0 & -\lambda \end{bmatrix} =$$

$$\underset{\substack{C_i+C_1\to C_1\\ i=2,\ldots,n}}{=} \det\begin{bmatrix} n-\lambda & 1 & 1 & \cdots & 1\\ 0 & -\lambda & 0 & \cdots & 0\\ \vdots & \ddots & -\lambda & \ddots & \vdots\\ & & & \ddots & 0\\ 0 & \cdots & & 0 & -\lambda \end{bmatrix} = (-1)^n(\lambda-n)\lambda^{n-1}.$$

Logo, $\det(A-\lambda I)=0 \Leftrightarrow (\lambda=0 \text{ ou } \lambda=n)$.

7. $\begin{bmatrix} 0 & \pi\\ -1 & \pi \end{bmatrix}$, $\begin{bmatrix} \frac{\pi}{2} & \pi^2\\ \frac{1}{4}-\frac{1}{\pi} & \frac{\pi}{2} \end{bmatrix}$, $\begin{bmatrix} \pi & 1\\ -\pi & 0 \end{bmatrix}$.

8. (i) $A^{-1}=\dfrac{1}{\det A}(\operatorname{cof} A)^T=\dfrac{1}{-2}\begin{bmatrix} 4 & -3\\ -2 & 1 \end{bmatrix}^T=\begin{bmatrix} -2 & 1\\ 3/2 & -1/2 \end{bmatrix}$

(ii) $A^{-1}=\dfrac{1}{\det A}(\operatorname{cof} A)^T=\dfrac{1}{1}\begin{bmatrix} 1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1 \end{bmatrix}^T=\begin{bmatrix} 1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1 \end{bmatrix}$

(iii) $A^{-1}=\dfrac{1}{\det A}(\operatorname{cof} A)^T=\dfrac{1}{6}\begin{bmatrix} 18 & 0 & 6\\ 24 & 0 & 6\\ 4 & 1 & 1 \end{bmatrix}^T=\begin{bmatrix} 3 & 4 & 2/3\\ 0 & 0 & 1/6\\ 1 & 1 & 1/6 \end{bmatrix}$

9.

$$\det A = 2(-1)^{3+4}\det\begin{bmatrix} 1 & -3 & 0 & 1\\ -2 & 0 & -1 & 1\\ 0 & 0 & -2 & 1\\ 2 & 0 & -1 & 1 \end{bmatrix} = -2(-3)(-1)^{1+2}\det\begin{bmatrix} -2 & -1 & 1\\ 0 & -2 & 1\\ 2 & -1 & 1 \end{bmatrix} = -24 \neq 0$$

logo A é invertível e

$$(A^{-1})_{(2,3)} = \left(\tfrac{1}{\det A}(\operatorname{cof} A)^T\right)_{(2,3)} = \tfrac{1}{\det A}(\operatorname{cof} A)_{(3,2)} = -\tfrac{1}{24}(-1)^{3+2}\det\begin{bmatrix} 1 & 0 & 0 & 1\\ -2 & -1 & 0 & 1\\ 0 & -2 & 0 & 1\\ 2 & -1 & 0 & 1 \end{bmatrix} = 0.$$

10. $I-\begin{bmatrix} 0 & 1\\ 2 & 2 \end{bmatrix}A = A\det\begin{bmatrix} 0 & 1\\ 2 & 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -2 & 1\\ 2 & 0 \end{bmatrix}A=I \Leftrightarrow A=\begin{bmatrix} -2 & 1\\ 2 & 0 \end{bmatrix}^{-1}=\begin{bmatrix} 0 & \frac{1}{2}\\ 1 & 1 \end{bmatrix}$.

11.

$$\begin{aligned}\det\left(A^TA+A\det A-5A\right) &= \det\left(\left(A^T+I\det A-5I\right)A\right)=\det\left(A^T+I\det A-5I\right)\det A \underset{\det A=4}{=}\\ &= 4\det\left(\begin{bmatrix} -1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1 \end{bmatrix}-I\right)=4\det\begin{bmatrix} -2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2 \end{bmatrix}=0,\end{aligned}$$

logo a matriz $A^TA+A\det A-5A$ é não invertível.

12.

$$\det\left[-2A^{-1}A^T\det\left(A^{-2}\right)\right]=\left[-2\det(A^{-2})\right]^3\frac{1}{\det A}\det\left(A^T\right)=(-2)^3\frac{1}{(\det A)^6}=\frac{(-2)^3}{(-2)^6}=-\frac{1}{8}.$$

13.

$$\begin{aligned}\det\begin{bmatrix}0&a&0&1\\1&0&2&0\\0&-2&0&1\\1&0&a&0\end{bmatrix} &= 1\Leftrightarrow a\,(-1)^{1+2}\det\begin{bmatrix}1&2&0\\0&0&1\\1&a&0\end{bmatrix}+(-1)^{1+4}\det\begin{bmatrix}1&0&2\\0&-2&0\\1&0&a\end{bmatrix}=1\Leftrightarrow\\
&\Leftrightarrow -a\,(-1)^{2+3}\det\begin{bmatrix}1&2\\1&a\end{bmatrix}+(-1)\,(-2)\,(-1)^{2+2}\det\begin{bmatrix}1&2\\1&a\end{bmatrix}=1\Leftrightarrow\\
&\Leftrightarrow a\,(a-2)+2\,(a-2)=1\Leftrightarrow a^2-4=1\Leftrightarrow(a=\sqrt{5}\ \text{ ou }\ a=-\sqrt{5}).\end{aligned}$$

14.

$$\begin{aligned}\det\begin{bmatrix}1&-1&0&0\\0&-1&0&2\\1&1&2&-2\\-1&0&-2&1\end{bmatrix} &= \det\begin{bmatrix}1&-1&0&0\\0&-1&0&2\\1&0&2&0\\-1&0&-2&1\end{bmatrix}=\det\begin{bmatrix}1&-1&0&0\\0&-1&0&2\\1&0&2&0\\0&0&0&1\end{bmatrix}=\\
&= \det\begin{bmatrix}1&-1&0\\0&-1&0\\1&0&2\end{bmatrix}=2\det\begin{bmatrix}1&-1\\0&-1\end{bmatrix}=-2.\end{aligned}$$

Logo

$$\begin{aligned}\det\left(A^3+A^4\right) &= (\det A)^3\det(I+A)=-8\det\begin{bmatrix}2&-1&0&0\\0&0&0&2\\1&1&3&-2\\-1&0&-2&2\end{bmatrix}=\\
&= -8\det\begin{bmatrix}2&-1&0&0\\0&0&0&2\\1&1&3&-2\\-1&0&-2&2\end{bmatrix}=-8\times2\det\begin{bmatrix}2&-1&0\\1&1&3\\-1&0&-2\end{bmatrix}=48.\end{aligned}$$

15.

$$\det\left(A^6-A^5\right)=(\det A)^5\det(A-I)=\left(\det\begin{bmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}\right)^5\det\begin{bmatrix}-1&0&0&1\\0&-1&0&-2\\0&0&-1&0\\0&0&0&3\end{bmatrix}=3.$$

16. $AXB - B = AX\det\left(A^TA\right) \Leftrightarrow AX\left(B - I\underbrace{\det\left(A^TA\right)}_{=1}\right) = B \Leftrightarrow$

$$\Leftrightarrow X = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

17. (i) Tem-se $\det A = \begin{vmatrix} -3 & 2 & 0 & 2 \\ -1 & 0 & 0 & 3 \\ 0 & 9 & -2 & 0 \\ 3 & -1 & 0 & 2 \end{vmatrix} = (-2)(-1)^{3+3}\begin{vmatrix} -3 & 2 & 2 \\ -1 & 0 & 3 \\ 3 & -1 & 2 \end{vmatrix} =$
$= (-2)(2 + 18 - (-4) - 9) = -30 \neq 0$ pelo que A é invertível. Logo,

$$\left(A^{-1}\right)_{(2,2)} = \frac{1}{\det A}\left((\operatorname{cof} A)^T\right)_{(2,2)} = \frac{1}{\det A}(\operatorname{cof} A)_{(2,2)} = \frac{1}{-30}(-1)^{2+2}\begin{vmatrix} -3 & 0 & 2 \\ 0 & -2 & 0 \\ 3 & 0 & 2 \end{vmatrix} = -\frac{4}{5}.$$

$$\det B = \begin{vmatrix} -1 & 0 & 0 & 2 \\ 4 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & -1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & -1 \end{vmatrix} = (-1)(-1)^{1+1}\begin{vmatrix} 0 & 1 & 8 \\ 1 & 0 & -3 \\ 0 & 2 & -1 \end{vmatrix} = -17 \neq 0$$

pelo que B é invertível. Logo,

$$\left(B^{-1}\right)_{(2,3)} = \frac{1}{\det B}\left((\operatorname{cof} B)^T\right)_{(2,3)} = \frac{1}{\det B}(\operatorname{cof} B)_{(3,2)} = \frac{1}{-17}(-1)^{3+2}\begin{vmatrix} -1 & 0 & 2 \\ 4 & 1 & 0 \\ 0 & 2 & 2 \end{vmatrix} = \frac{14}{17}.$$

(ii) $\det(A+B) = \det\begin{bmatrix} -4 & 2 & 0 & 4 \\ 3 & 0 & 1 & 3 \\ 0 & 10 & -2 & -3 \\ 3 & -2 & 2 & 4 \end{bmatrix} = -134 \neq -47 = -30 - 17 = \det A + \det B$

$\det(A-B) = \det\begin{bmatrix} -2 & 2 & 0 & 0 \\ -5 & 0 & -1 & 3 \\ 0 & 8 & -2 & 3 \\ 3 & 0 & -2 & 0 \end{bmatrix} = 138 \neq -13 = -30 + 17 = \det A - \det B$

18. (i) $x = \dfrac{\begin{vmatrix} 1 & 3 \\ 3 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix}} = \dfrac{-2}{-1} = 2$ e $y = \dfrac{\begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix}} = \dfrac{1}{-1} = -1$

(ii)

$$x=\frac{\begin{vmatrix}1&1&0\\1&0&1\\-1&2&2\end{vmatrix}}{\begin{vmatrix}1&1&0\\2&0&1\\1&2&2\end{vmatrix}}=\frac{-5}{-5}=1,\quad y=\frac{\begin{vmatrix}1&1&0\\2&1&1\\1&-1&2\end{vmatrix}}{\begin{vmatrix}1&1&0\\2&0&1\\1&2&2\end{vmatrix}}=0\ \text{ e }\ z=\frac{\begin{vmatrix}1&1&1\\2&0&1\\1&2&-1\end{vmatrix}}{\begin{vmatrix}1&1&0\\2&0&1\\1&2&2\end{vmatrix}}=\frac{5}{-5}=-1$$

19. $\det C=\begin{vmatrix}1&0&1\\2&3&2\\0&1&-2\end{vmatrix}=-6\neq 0$, logo C é invertível. $\det D=\begin{vmatrix}9&-8&-1\\-7&-3&0\\-2&0&0\end{vmatrix}=6\neq 0$, logo D é invertível.

(i) $\det\left(2C^{-1}\right)=2^3\,\dfrac{1}{\det C}=-\dfrac{4}{3}$

(ii) $\det\left(C^3\left(2C\right)^{-1}\right)=\left(\det C\right)^3\dfrac{1}{2^3}\dfrac{1}{\det C}=\left(\det C\right)^2\dfrac{1}{8}=\dfrac{9}{2}$

(iii) $\det\left(\left(C^T2C\right)^{-1}\right)=\det\left(C^{-1}\dfrac{1}{2}\left(C^{-1}\right)^T\right)=\dfrac{1}{2^3}\det\left(C^{-1}\right)\det\left(C^{-1}\right)=\dfrac{1}{2^3}\left(\dfrac{1}{\det C}\right)^2=\dfrac{1}{288}$

(iv) $\det\left(C^T\frac{1}{2}C^{-2}\right)=\dfrac{1}{2^3}\det\left(C^T\right)\det\left(C^{-2}\right)=\dfrac{1}{2^3}\det C\dfrac{1}{\left(\det C\right)^2}=\dfrac{1}{2^3}\dfrac{1}{\det C}=-\dfrac{1}{48}$

(v)

$$\det\left(C^2+2D\right)=\det\left(\begin{bmatrix}1&1&-1\\8&11&4\\2&1&6\end{bmatrix}+2\begin{bmatrix}9&-8&-1\\-7&-3&0\\-2&0&0\end{bmatrix}\right)=\det\begin{bmatrix}19&-15&-3\\-6&5&4\\-2&1&6\end{bmatrix}=62$$

20. $\det(2A^TB^{-3})=2^3\det\left(A^T\right)\det(B^{-3})=8\det A\dfrac{1}{\left(\det B\right)^3}=64\sqrt{3}.$

21. **(i)** $\begin{vmatrix}1&1&a\\2&b&1\\c&2&2\end{vmatrix}=-2$ **(ii)** $\begin{vmatrix}a+3&-1&-2\\b+3&b-2&-2\\c+3&2-c&-c\end{vmatrix}=-2$ **(iii)** $\det\left(\frac{1}{2}B^TB\right)=\frac{1}{2}$

22. $\begin{vmatrix}a&b&0&c\\d&e&0&f\\g&h&0&i\\x&y&-1&z\end{vmatrix}=5\Leftrightarrow\begin{vmatrix}a&b&c\\d&e&f\\g&h&i\end{vmatrix}=5$

(i) $\begin{vmatrix}d&e&f\\g&h&i\\a&b&c\end{vmatrix}=5$ **(ii)** $\begin{vmatrix}-a&-b&-c\\2d&2e&2f\\-g&-h&-i\end{vmatrix}=10$ **(iii)** $\begin{vmatrix}a+d&b+e&c+f\\d&e&f\\g&h&i\end{vmatrix}=5$

(iv) $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = -5$

23. **(i)** $\begin{vmatrix} a & b & c \\ 6 & 3 & 0 \\ -\frac{1}{2} & -1 & -\frac{1}{2} \end{vmatrix} = -\dfrac{3}{2}$ **(ii)** $\begin{vmatrix} a & b & c \\ 2a+2 & 2b+1 & 2c \\ a+1 & b+2 & c+1 \end{vmatrix} = 1$

(iii) $\begin{vmatrix} a-1 & b-2 & c-1 \\ 3 & 3 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 1$

(iv) $\begin{vmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 3a+1 & 3b+2 & 3c+1 \end{vmatrix} = -\begin{vmatrix} -1 & 1 & 1 \\ 2 & 1 & 0 \\ 3a+1 & 3b+2 & 3c+1 \end{vmatrix} =$

$= -\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 3a+1 & 3b+2 & 3c+1 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 3a & 3b & 3c \end{vmatrix} = 3\begin{vmatrix} a & b & c \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 3$

24. $\begin{vmatrix} 1 & 2 & \alpha \\ \beta & \beta\alpha+\beta^2 & 2\beta \\ \beta\alpha & \alpha & \alpha \end{vmatrix} = -\alpha\beta.$

25. $\begin{vmatrix} \lambda & 1 & 1 & 1 & 1 & 1 \\ \lambda & \lambda+1 & 2 & 2 & 2 & 2 \\ \lambda & \lambda+1 & \lambda+2 & 3 & 3 & 3 \\ \lambda & \lambda+1 & \lambda+2 & \lambda+3 & 4 & 4 \\ \lambda & \lambda+1 & \lambda+2 & \lambda+3 & \lambda+4 & 5 \\ \lambda & \lambda+1 & \lambda+2 & \lambda+3 & \lambda+4 & \lambda+5 \end{vmatrix} = \begin{vmatrix} \lambda & 1 & 1 & 1 & 1 & 1 \\ 0 & \lambda & 1 & 1 & 1 & 1 \\ 0 & 0 & \lambda & 1 & 1 & 1 \\ 0 & 0 & 0 & \lambda & 1 & 1 \\ 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{vmatrix} = \lambda^6.$

26. $\begin{vmatrix} \lambda & \lambda & \lambda & \dots & & \lambda \\ 1 & \lambda+1 & 1 & \dots & & 1 \\ \vdots & 1 & \lambda+1 & \ddots & & \\ & \vdots & 1 & \ddots & 1 & \vdots \\ & & \vdots & \ddots & \lambda+1 & 1 \\ 1 & 1 & 1 & \dots & 1 & \lambda+1 \end{vmatrix} = \begin{vmatrix} \lambda & 0 & 0 & \dots & & 0 \\ 1 & \lambda & 0 & \dots & & 0 \\ \vdots & 0 & \lambda & \ddots & & \\ & \vdots & 0 & \ddots & 0 & \vdots \\ & & \vdots & \ddots & \lambda & 0 \\ 1 & 0 & 0 & \dots & 0 & \lambda \end{vmatrix} = \lambda^n.$

27. Sejam $\beta \neq 0$ e $A = (a_{ij})_{n\times n}$.

$$\det\left(\beta^{i-j}a_{ij}\right) = \begin{vmatrix} a_{11} & \frac{1}{\beta}a_{12} & \frac{1}{\beta^2}a_{13} & \dots & \frac{1}{\beta^{n-2}}a_{1\ n-1} & \frac{1}{\beta^{n-1}}a_{1n} \\ \beta a_{21} & a_{22} & \frac{1}{\beta}a_{23} & \dots & \frac{1}{\beta^{n-3}}a_{2\ n-1} & \frac{1}{\beta^{n-2}}a_{2n} \\ \vdots & \beta a_{32} & a_{33} & \ddots & & \\ & \vdots & \beta a_{43} & \ddots & \frac{1}{\beta}a_{n-2\ n-1} & \vdots \\ & & \vdots & \ddots & a_{n-1\ n-1} & \frac{1}{\beta}a_{n-1\ n} \\ \beta^{n-1}a_{n1} & \beta^{n-2}a_{n2} & \beta^{n-3}a_{n3} & \dots & \beta a_{n\ n-1} & a_{nn} \end{vmatrix} =$$

$$= \frac{1}{\beta} \cdots \frac{1}{\beta^{n-2}} \frac{1}{\beta^{n-1}} \begin{vmatrix} \beta^{n-1}a_{11} & \beta^{n-2}a_{12} & \beta^{n-3}a_{13} & \cdots & \beta a_{1\ n-1} & a_{1n} \\ \beta^{n-1}a_{21} & \beta^{n-2}a_{22} & \beta^{n-3}a_{23} & \cdots & \beta a_{2\ n-1} & a_{2n} \\ \vdots & \beta^{n-2}a_{32} & \beta^{n-3}a_{33} & \ddots & & \\ & \vdots & \beta^{n-3}a_{43} & \ddots & \beta a_{n-2\ n-1} & \vdots \\ & & \vdots & \ddots & \beta a_{n-1\ n-1} & a_{n-1\ n} \\ \beta^{n-1}a_{n1} & \beta^{n-2}a_{n2} & \beta^{n-3}a_{n3} & \cdots & \beta a_{n\ n-1} & a_{nn} \end{vmatrix} =$$

$$= \frac{1}{\beta} \cdots \frac{1}{\beta^{n-2}} \frac{1}{\beta^{n-1}} \beta^{n-1}\beta^{n-2} \cdots \beta \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1\ n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2\ n-1} & a_{2n} \\ \vdots & a_{32} & a_{33} & \ddots & & \\ & \vdots & a_{43} & \ddots & a_{n-2\ n-1} & \vdots \\ & & \vdots & \ddots & a_{n-1\ n-1} & a_{n-1\ n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n\ n-1} & a_{nn} \end{vmatrix} = \det A.$$

28. Seja

$$A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix},$$

com $a, b, c \in \mathbb{R}$. A matriz A é invertível se e só se $\det A \neq 0$. Tem-se

$$\det A = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = 0$$

se $a = b$ ou $a = c$. Se $a \neq b$ e $a \neq c$ então

$$\det A = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & c^2-a^2-(c-a)(b+a) \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c-a)\left[(c+a)-(b+a)\right] \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c-a)(c-b) \end{vmatrix} = 0$$

se $b = c$.

Logo, a matriz A é invertível se e só se $a \neq b$, $a \neq c$ e $b \neq c$.

29. (i)

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & y_1 & y_1 \\ x_2 & x_2 & y_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x_1 & y_1-x_1 & y_1-x_1 \\ x_2 & 0 & y_2-x_2 \end{vmatrix} = (y_1-x_1)(-1)^{2+2} \det \begin{vmatrix} 1 & 0 \\ x_2 & y_2-x_2 \end{vmatrix} = (y_1-x_1)(y_2-x_2).$$

(ii)

$$\det \begin{bmatrix} (x_1)^3 & (x_1)^2 & x_1 & 1 \\ (x_2)^3 & (x_2)^2 & x_2 & 1 \\ (x_3)^3 & (x_3)^2 & x_3 & 1 \\ (x_4)^3 & (x_4)^2 & x_4 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & x_1 & (x_1)^2 & (x_1)^3 \\ 1 & x_2 & (x_2)^2 & (x_2)^3 \\ 1 & x_3 & (x_3)^2 & (x_3)^3 \\ 1 & x_4 & (x_4)^2 & (x_4)^3 \end{bmatrix} =$$

$$= \det \begin{bmatrix} 1 & x_1 & (x_1)^2 & (x_1)^3-(x_1)^3 \\ 0 & x_2-x_1 & (x_2)^2-(x_1)^2 & (x_2)^3-(x_1)^3 \\ 0 & x_3-x_1 & (x_3)^2-(x_1)^2 & (x_3)^3-(x_1)^3 \\ 0 & x_4-x_1 & (x_4)^2-(x_1)^2 & (x_4)^3-(x_1)^3 \end{bmatrix} =$$

$$= (x_4-x_1)(x_3-x_1)(x_2-x_1)\det \begin{bmatrix} 1 & x_1 & (x_1)^2 & (x_1)^3-(x_1)^3 \\ 0 & 1 & x_2+x_1 & x_1^2+x_1x_2+x_2^2 \\ 0 & 1 & x_3+x_1 & x_1^2+x_1x_3+x_3^2 \\ 0 & 1 & x_4+x_1 & x_1^2+x_1x_4+x_4^2 \end{bmatrix} =$$

$$= (x_4-x_1)(x_3-x_1)(x_2-x_1)\det \begin{bmatrix} 1 & x_1 & (x_1)^2 & (x_1)^3-(x_1)^3 \\ 0 & 1 & x_2+x_1 & x_1^2+x_1x_2+x_2^2 \\ 0 & 0 & x_3-x_2 & x_1(x_3-x_2)+x_3^2-x_2^2 \\ 0 & 0 & x_4-x_2 & x_1(x_4-x_2)+x_4^2-x_2^2 \end{bmatrix} =$$

$$= (x_4-x_1)(x_3-x_1)(x_2-x_1)(x_3-x_2)(x_4-x_2)\det \begin{bmatrix} 1 & x_1 & (x_1)^2 & (x_1)^3-(x_1)^3 \\ 0 & 1 & x_2+x_1 & x_1^2+x_1x_2+x_2^2 \\ 0 & 0 & 1 & x_1+x_3+x_2 \\ 0 & 0 & 1 & x_1+x_4+x_2 \end{bmatrix} =$$

$$= (x_4-x_1)(x_3-x_1)(x_2-x_1)(x_3-x_2)(x_4-x_2)\det \begin{bmatrix} 1 & x_1 & (x_1)^2 & (x_1)^3-(x_1)^3 \\ 0 & 1 & x_2+x_1 & x_1^2+x_1x_2+x_2^2 \\ 0 & 0 & 1 & x_1+x_3+x_2 \\ 0 & 0 & 0 & x_4-x_3 \end{bmatrix} =$$

$$= (x_4-x_1)(x_3-x_1)(x_2-x_1)(x_3-x_2)(x_4-x_2)(x_4-x_3) =$$

$$= (x_1-x_2)(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)(x_3-x_4)$$

30. (i) $$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

(ii) $$\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 2a_1 & a_1-b_1 & c_1 \\ 2a_2 & a_2-b_2 & c_2 \\ 2a_3 & a_3-b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 2a_1 & -b_1 & c_1 \\ 2a_2 & -b_2 & c_2 \\ 2a_3 & -b_3 & c_3 \end{vmatrix} = -2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(iii) $$\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

31. $$\begin{vmatrix} a_1+b_1 & c_1+d_1 \\ a_2+b_2 & c_2+d_2 \end{vmatrix} = (a_1+b_1)(c_2+d_2) - (a_2+b_2)(c_1+d_1) =$$

$$= a_1c_2 - a_2c_1 + a_1d_2 - b_2c_1 + b_1c_2 - a_2d_1 + b_1d_2 - b_2d_1 =$$

$$= \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 \\ b_2 & d_2 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 \\ a_2 & c_2 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}.$$

32. $\begin{vmatrix} 0 & 0 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$ e $\begin{vmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{vmatrix} = 0$

33. **(i)** $\begin{vmatrix} 1 & x & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} 1 & x & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0 \Leftrightarrow -1 + x = 0 \Leftrightarrow x = 1$

(ii) $\begin{vmatrix} x & x & x & x \\ x & 4 & x & x \\ x & x & 4 & x \\ x & x & x & 4 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} x & x & x & x \\ 0 & 4-x & 0 & 0 \\ 0 & 0 & 4-x & 0 \\ 0 & 0 & 0 & 4-x \end{vmatrix} = 0 \Leftrightarrow x\,(4-x)^3 = 0 \Leftrightarrow (x = 0 \text{ ou } x = 4)$

(iii) $\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} x & 1 & 1 & 1 \\ 0 & x-1 & 0 & 1-x \\ 0 & 0 & x-1 & 1-x \\ 1 & 1 & 1 & x \end{vmatrix} = 0 \Leftrightarrow$

$\Leftrightarrow \begin{vmatrix} x & 1-x & 1-x & 1-x \\ 0 & x-1 & 0 & 1-x \\ 0 & 0 & x-1 & 1-x \\ 1 & 0 & 0 & x-1 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} 0 & 1-x & 1-x & 1-x^2 \\ 0 & x-1 & 0 & 1-x \\ 0 & 0 & x-1 & 1-x \\ 1 & 0 & 0 & x-1 \end{vmatrix} = 0 \Leftrightarrow$

$\Leftrightarrow \begin{vmatrix} 0 & 0 & 0 & 3-2x-x^2 \\ 0 & x-1 & 0 & 1-x \\ 0 & 0 & x-1 & 1-x \\ 1 & 0 & 0 & 0 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} 0 & 0 & 0 & 3-2x-x^2 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 0 \Leftrightarrow$

$\Leftrightarrow -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & 3-2x-x^2 \end{vmatrix} = 0 \Leftrightarrow -(x-1)^2\,(3-2x-x^2) = 0 \Leftrightarrow (x = 1 \text{ ou } x = -3)$

34. $\begin{vmatrix} 2 & 1 & 8 \\ -1 & 0 & 10 \\ 3 & -7 & 4 \end{vmatrix} \underset{-3C_2+C_3\to C_3}{=} \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 10 \\ 3 & -7 & 25 \end{vmatrix} = 5 \begin{vmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & -7 & 5 \end{vmatrix}.$

35. Sendo $A \in \mathcal{M}_{5\times 6}(\mathbb{R})$, tem-se $A^T A \in \mathcal{M}_{6\times 6}(\mathbb{R})$. Como $\mathcal{N}(A) \subseteq \mathcal{N}\left(A^T A\right)$ e

$$\dim \mathcal{N}(A) = 6 - \operatorname{car} A \geq 6 - 5 = 1$$

então

$$1 \leq \dim \mathcal{N}(A) \leq \dim \mathcal{N}\left(A^T A\right).$$

Logo $\dim \mathcal{N}\left(A^T A\right) \neq 0$ pelo que $A^T A$ não é invertível, isto é,

$$\det\left(A^T A\right) = 0.$$