IST - Álgebra Linear - 2^o semestre 2018/2019

Determinantes

1. Classifique quanto à paridade as seguintes permutações de números de 1 a 6:

   (i) (312645) **(ii)** (234516) **(iii)** (654321) **(iv)** (123456)

   (v) (546321) **(vi)** (453261) **(vii)** (634125) **(viii)** (123465)

2. Na expressão do determinante de uma matriz do tipo 6×6 diga qual o sinal que afecta cada uma das seguintes parcelas:

   (i) $a_{23}a_{31}a_{42}a_{56}a_{14}a_{65}$ **(ii)** $a_{16}a_{25}a_{34}a_{43}a_{52}a_{61}$

   (iii) $a_{54}a_{45}a_{63}a_{32}a_{26}a_{11}$ **(iv)** $a_{16}a_{23}a_{34}a_{41}a_{62}a_{55}$

3. Verifique que

   (i) $\begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\, a_{13}a_{22}a_{31}$ **(ii)** $\begin{vmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{14}a_{23}a_{32}a_{41}$

   (iii)
$$\det \begin{bmatrix} 0 & \cdots & 0 & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a_{n-1\ 2} & \cdots & a_{n-1\ n} \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} = (-1)^{n\frac{n-1}{2}}\, a_{1n}...a_{n-1\ 2}a_{n1}$$

4. Calcule os seguintes determinantes e diga quais são as matrizes invertíveis:

   (i) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ **(ii)** $\begin{vmatrix} 18563 & 18573 \\ 21472 & 21482 \end{vmatrix}$ **(iii)** $\begin{vmatrix} 1+\sqrt{2} & 2-\sqrt{3} \\ 2+\sqrt{3} & 1-\sqrt{2} \end{vmatrix}$

   (iv) $\begin{vmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{vmatrix}$ **(v)** $\begin{vmatrix} -2 & 0 & 1 \\ 5 & 3 & 0 \\ -5 & 1 & 2 \end{vmatrix}$ **(vi)** $\begin{vmatrix} 2 & 3 & 2 \\ 5 & -1 & -3 \\ -2 & 1 & 1 \end{vmatrix}$

   (vii) $\begin{vmatrix} -2 & 1 & 1 \\ 5 & -1 & -3 \\ 2 & 3 & 2 \end{vmatrix}$ **(viii)** $\begin{vmatrix} 8 & 12 & 8 \\ 5 & -1 & -3 \\ -2 & 1 & 1 \end{vmatrix}$ **(ix)** $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ **(x)** $\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

   (xi) $\begin{vmatrix} -2 & -2 & 8 & 6 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3 & -23 \\ 0 & 0 & 0 & 5 \end{vmatrix}$ **(xii)** $\begin{vmatrix} 1 & 3 & -1 & 1 \\ -1 & -2 & -1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 0 & -2 & 0 \end{vmatrix}$ **(xiii)** $\begin{vmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & -3 & -23 \\ 0 & 1 & 2 & 0 \\ -2 & -2 & 8 & 6 \end{vmatrix}$

   (xiv) $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$ **(xv)** $\begin{vmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{vmatrix}$ **(xvi)** $\begin{vmatrix} 7 & -4 & 0 & 5 \\ 1 & 2 & 0 & -2 \\ 2 & 1 & -3 & 8 \\ 2 & -2 & 0 & 4 \end{vmatrix}$

(xvii) $\begin{vmatrix} 9 & 0 & -6 & 4 & -2 \\ 5 & -3 & 0 & -1 & 2 \\ 8 & 0 & 4 & 1 & 1 \\ -6 & 0 & 3 & -1 & 3 \\ 7 & 0 & 5 & 2 & 3 \end{vmatrix}$ **(xviii)** $\begin{vmatrix} 2 & 1 & 0 & 0 \\ -7 & 3 & 0 & -2 \\ \pi & 3 & 1 & 1 \\ 3 & 9 & 0 & 0 \end{vmatrix}$

(xix) $\begin{vmatrix} 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{vmatrix}$ **(xx)** $\begin{vmatrix} n & n-1 & \vdots & 2 & 1 \\ n-1 & n-1 & \vdots & 2 & 1 \\ \cdots & \cdots & \ddots & \vdots & \vdots \\ 2 & 2 & \cdots & 2 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix}$

5. **(i)** Verifique que a matriz

$$\begin{bmatrix} 0 & a & 0 & 0 & 0 \\ e & 0 & b & 0 & 0 \\ 0 & f & 0 & c & 0 \\ 0 & 0 & g & 0 & d \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

não é invertível para quaisquer $a, b, c, d, e, f, g, h \in \mathbb{R}$.

(ii) Diga, para que valores de $a, b, c, d, e, f, g, h, i, j \in \mathbb{R}$, é invertível a seguinte matriz

$$\begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 \\ f & 0 & b & 0 & 0 & 0 \\ 0 & g & 0 & c & 0 & 0 \\ 0 & 0 & h & 0 & d & 0 \\ 0 & 0 & 0 & i & 0 & e \\ 0 & 0 & 0 & 0 & j & 0 \end{bmatrix}$$

6. Determine todos os valores do escalar λ para os quais a matriz $A - \lambda I$ é não invertível, onde A é dada por:

(i) $\begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **(iv)** $\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{n \times n}$

7. Indique três matrizes A do tipo 2×2 tais que $\operatorname{tr} A = \pi = \det A$.

8. Use a fórmula de inversão de matrizes para inverter:

(i) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **(iii)** $\begin{bmatrix} -1 & 0 & 4 \\ 1 & -1 & -3 \\ 0 & 6 & 0 \end{bmatrix}$

9. Seja

$$A = \begin{bmatrix} 1 & -3 & 0 & 0 & 1 \\ -2 & 0 & -1 & 0 & 1 \\ -3 & -1 & 2 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \\ 2 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

Justifique que A é invertível e calcule a entrada $(2,3)$ de A^{-1}.

10. Determine a matriz A do tipo 2×2 tal que

$$I - \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} A = A \det \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}.$$

11.

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Diga, justificando, se a matriz:

$$A^T A + A \det A - 5A$$

é invertível.

12. Seja $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. Calcule $\det\left[-2A^{-1}A^T \det\left(A^{-2}\right)\right]$.

13. Calcule os valores de a para os quais $\det \begin{bmatrix} 0 & a & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & a & 0 \end{bmatrix} = 1.$

14. Seja $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 2 \\ 1 & 1 & 2 & -2 \\ -1 & 0 & -2 & 1 \end{bmatrix}$. Calcule $\det\left(A^3 + A^4\right)$.

15. Seja $A = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 3 & -3 \\ -2 & 1 & -2 & 2 \\ 0 & -2 & 1 & 0 \end{bmatrix}$. Calcule $\det\left(A^6 - A^5\right)$.

16. Sejam $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ e $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. Determine a matriz X tal que

$$AXB - B = AX \det\left(A^T A\right).$$

17. Sejam

$$A = \begin{bmatrix} -3 & 2 & 0 & 2 \\ -1 & 0 & 0 & 3 \\ 0 & 9 & -2 & 0 \\ 3 & -1 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 & 0 & 2 \\ 4 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & -1 & 2 & 2 \end{bmatrix}.$$

(i) Justifique que A e B são invertíveis e determine as entradas $(2,2)$ de A^{-1} e $(2,3)$ de B^{-1}.

(ii) Verifique que $\det(A+B) \neq \det A + \det B$ e $\det(A-B) \neq \det A - \det B$.

18. Use a regra de Cramer para calcular as soluções dos sistemas:

(i) $\begin{cases} 2x+3y=1 \\ 5x+7y=3 \end{cases}$ **(ii)** $\begin{cases} x+y=1 \\ 2x+z\ =1 \\ x+2y+2z=-1 \end{cases}$

19. Sejam $C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ 0 & 1 & -2 \end{bmatrix}$ e $D = \begin{bmatrix} 9 & -8 & -1 \\ -7 & -3 & 0 \\ -2 & 0 & 0 \end{bmatrix}$.

Verifique que C e D são invertíveis e calcule:

(i) $\det(2C^{-1})$

(ii) $\det\left(C^3(2C)^{-1}\right)$

(iii) $\det\left(\left(C^T 2C\right)^{-1}\right)$

(iv) $\det\left(C^T \frac{1}{2} C^{-2}\right)$

(v) $\det(C^2+2D)$

20. Sejam A e B matrizes 3×3 tais que $\det A = \sqrt{3}$ e $\det B = \frac{1}{2}$. Calcule $\det(2A^T B^{-3})$.

21. Sejam $a,b,c \in \mathbb{R}$. Seja $B = \begin{bmatrix} a & 1 & 2 \\ 1 & b & 2 \\ 1 & 2 & c \end{bmatrix}$. Sabendo que $\det B = 2$, calcule:

(i) $\begin{vmatrix} 1 & 1 & a \\ 2 & b & 1 \\ c & 2 & 2 \end{vmatrix}$ **(ii)** $\begin{vmatrix} a+3 & -1 & -2 \\ b+3 & b-2 & -2 \\ c+3 & 2-c & -c \end{vmatrix}$ **(iii)** $\det\left(\frac{1}{2}B^T B\right)$

22. Sejam $a,b,c,d,e,f \in \mathbb{R}$. Sabendo que

$$\begin{vmatrix} a & b & 0 & c \\ d & e & 0 & f \\ g & h & 0 & i \\ x & y & -1 & z \end{vmatrix} = 5,$$

calcule:

(i) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$ **(ii)** $\begin{vmatrix} -a & -b & -c \\ 2d & 2e & 2f \\ -g & -h & -i \end{vmatrix}$ **(iii)** $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix}$

(iv) $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$

23. Sejam $a, b, c \in \mathbb{R}$. Sabendo que

$$\begin{vmatrix} a & b & c \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 1,$$

calcule:

(i) $\begin{vmatrix} a & b & c \\ 6 & 3 & 0 \\ -\frac{1}{2} & -1 & -\frac{1}{2} \end{vmatrix}$ **(ii)** $\begin{vmatrix} a & b & c \\ 2a+2 & 2b+1 & 2c \\ a+1 & b+2 & c+1 \end{vmatrix}$

(iii) $\begin{vmatrix} a-1 & b-2 & c-1 \\ 3 & 3 & 1 \\ 1 & 2 & 1 \end{vmatrix}$ **(iv)** $\begin{vmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 3a+1 & 3b+2 & 3c+1 \end{vmatrix}$

24. Sejam $\alpha, \beta \in \mathbb{R}$. Sabendo que $\begin{vmatrix} 1 & 2 & \alpha \\ \beta & 1 & 1 \\ 1 & \alpha+\beta & 2 \end{vmatrix} = 1$, calcule $\begin{vmatrix} 1 & 2 & \alpha \\ \beta & \beta\alpha+\beta^2 & 2\beta \\ \beta\alpha & \alpha & \alpha \end{vmatrix}$.

25. Seja $\lambda \in \mathbb{R}$. Verifique que

$$\begin{vmatrix} \lambda & 1 & 1 & 1 & 1 & 1 \\ \lambda & \lambda+1 & 2 & 2 & 2 & 2 \\ \lambda & \lambda+1 & \lambda+2 & 3 & 3 & 3 \\ \lambda & \lambda+1 & \lambda+2 & \lambda+3 & 4 & 4 \\ \lambda & \lambda+1 & \lambda+2 & \lambda+3 & \lambda+4 & 5 \\ \lambda & \lambda+1 & \lambda+2 & \lambda+3 & \lambda+4 & \lambda+5 \end{vmatrix} = \lambda^6.$$

26. Seja $\lambda \in \mathbb{R}$. Calcule o determinante da seguinte matriz do tipo $n \times n$.

$$\begin{bmatrix} \lambda & \lambda & \lambda & \cdots & & \lambda \\ 1 & \lambda+1 & 1 & \cdots & & 1 \\ \vdots & 1 & \lambda+1 & \ddots & & \\ & \vdots & 1 & \ddots & 1 & \vdots \\ & & \vdots & \ddots & \lambda+1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & \lambda+1 \end{bmatrix}$$

27. Sejam $\beta \neq 0$ e $A = (a_{ij})_{n \times n}$. Mostre que $\det A = \det\left(\beta^{i-j} a_{ij}\right)$.

28. Que condições devem os parâmetros reais a, b e c verificar para que a matriz $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$ seja invertível?

29. Verifique que

(i) $\det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & y_1 & y_1 \\ x_2 & x_2 & y_2 \end{bmatrix} = (y_1 - x_1)(y_2 - x_2)$

(ii) $\det\begin{bmatrix} (x_1)^3 & (x_1)^2 & x_1 & 1 \\ (x_2)^3 & (x_2)^2 & x_2 & 1 \\ (x_3)^3 & (x_3)^2 & x_3 & 1 \\ (x_4)^3 & (x_4)^2 & x_4 & 1 \end{bmatrix} = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)$

30. Mostre que:

(i) $\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$

(ii) $\begin{vmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ c_1+a_1 & c_2+a_2 & c_3+a_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{vmatrix} = 2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(iii) $\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix} = -2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(iv) $\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

31. Verifique que

$$\begin{vmatrix} a_1+b_1 & c_1+d_1 \\ a_2+b_2 & c_2+d_2 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 \\ b_2 & d_2 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 \\ a_2 & c_2 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}.$$

32. Sem calcular explicitamente o determinante, mostre que para $x = 0$ e $x = 2$ se tem

$$\begin{vmatrix} x^2 & x & 2 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{vmatrix} = 0.$$

33. Resolva as seguintes equações.

(i) $\begin{vmatrix} 1 & x & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 0$ **(ii)** $\begin{vmatrix} x & x & x & x \\ x & 4 & x & x \\ x & x & 4 & x \\ x & x & x & 4 \end{vmatrix} = 0$ **(iii)** $\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = 0$

34. Sem calcular o determinante, verifique que

$$\begin{vmatrix} 2 & 1 & 8 \\ -1 & 0 & 10 \\ 3 & -7 & 4 \end{vmatrix}$$

é múltiplo de 5.

35. Seja A uma matriz real do tipo 5×6. Calcule, justificando, $\det\left(A^T A\right)$.