

Determinantes (resolução)

1. (i) (312645) é par pois tem 4 inversões. (ii) (234516) é par pois tem 4 inversões.
 (iii) (654321) é ímpar pois tem 15 inversões. (iv) (123456) é par pois tem 0 inversões.
 (v) (546321) é ímpar pois tem 13 inversões. (vi) (453261) é par pois tem 10 inversões.
 (vii) (634125) é ímpar pois tem 9 inversões. (viii) (123465) é ímpar pois tem 1 inversão.

2. (i) (234516) é par pois tem 4 inversões e (312645) é par pois tem 4 inversões. Logo, tem-se

$$+a_{23}a_{31}a_{42}a_{56}a_{14}a_{65}$$

uma vez que (234516) e (312645) têm a mesma paridade.

- (ii) (123456) é par pois tem 0 inversões e (654321) é ímpar pois tem 15 inversões. Logo, tem-se

$$-a_{16}a_{25}a_{34}a_{43}a_{52}a_{61}$$

uma vez que (123456) e (654321) têm paridades diferentes.

- (iii) (546321) é ímpar pois tem 13 inversões e (453261) é par pois tem 10 inversões. Logo, tem-se

$$-a_{54}a_{45}a_{63}a_{32}a_{26}a_{11}$$

uma vez que (546321) e (453261) têm paridades diferentes.

- (iv) (123465) é ímpar pois tem 1 inversão e (634125) é ímpar pois tem 9 inversões. Logo, tem-se

$$+a_{16}a_{23}a_{34}a_{41}a_{62}a_{55}$$

uma vez que (123465) e (634125) têm a mesma paridade.

3. (i) (123) é par pois tem 0 inversões e (321) é ímpar pois tem 3 inversões. Atendendo à definição de determinante, tem-se

$$\begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{13}a_{22}a_{31}$$

uma vez que (123) e (321) têm paridades diferentes.

- (ii) (1234) é par pois tem 0 inversões e (4321) é par pois tem 6 inversões. Atendendo à definição de determinante, tem-se

$$\begin{vmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{14}a_{23}a_{32}a_{41}$$

uma vez que (1234) e (4321) têm a mesma paridade.

(iii) Ao efectuar $n-1+n-2+\dots+2+1$ trocas de linhas: linha 1 com cada uma das $n-1$ linhas que lhe estão abaixo, nova linha 1 com cada uma das $n-2$ linhas que lhe estão abaixo (excluindo a nova linha n) e assim sucessivamente, tem-se

$$\det \begin{bmatrix} 0 & \cdots & 0 & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a_{n-1\ 2} & \cdots & a_{n-1\ n} \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} = (-1)^{n\frac{n-1}{2}} \det \begin{bmatrix} a_{n1} & \cdots & \cdots & a_{nn} \\ 0 & a_{n-1\ 2} & \cdots & a_{n-1\ n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & a_{1n} \end{bmatrix} =$$

$$= (-1)^{n\frac{n-1}{2}} a_{1n} \dots a_{n-1\ 2} a_{n1}$$

atendendo a que

$$n-1+n-2+\dots+2+1 = (n-1) \frac{n-1+1}{2} = n \frac{n-1}{2}.$$

4.

(i) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$, logo a matriz é invertível.

(ii) $\begin{vmatrix} 18563 & 18573 \\ 21472 & 21482 \end{vmatrix} = \begin{vmatrix} 18563 & 10 \\ 21472 & 10 \end{vmatrix} = \begin{vmatrix} 18563 & 10 \\ 2909 & 0 \end{vmatrix} = -29090 \neq 0$, logo a matriz é invertível.

(iii) $\begin{vmatrix} 1+\sqrt{2} & 2-\sqrt{3} \\ 2+\sqrt{3} & 1-\sqrt{2} \end{vmatrix} = 1-2-(4-3) = -2 \neq 0$, logo a matriz é invertível.

(iv) $\begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} = \cos^2 \theta - (-\sin^2 \theta) = 1 \neq 0$, logo a matriz é invertível.

(v) $\begin{vmatrix} -2 & 0 & 1 \\ 5 & 3 & 0 \\ -5 & 1 & 2 \end{vmatrix} = -12 + 5 - (-15) = 8 \neq 0$, logo a matriz é invertível.

(vi) $\begin{vmatrix} 2 & 3 & 2 \\ 5 & -1 & -3 \\ -2 & 1 & 1 \end{vmatrix} = -2 + 18 + 10 - 4 - 15 - (-6) = 13 \neq 0$, logo a matriz é invertível.

(vii) $\begin{vmatrix} -2 & 1 & 1 \\ 5 & -1 & -3 \\ 2 & 3 & 2 \end{vmatrix} = - \begin{vmatrix} 2 & 3 & 2 \\ 5 & -1 & -3 \\ -2 & 1 & 1 \end{vmatrix} \underset{\text{por (vi)}}{=} -13 \neq 0$, logo a matriz é invertível.

(viii) $\begin{vmatrix} 8 & 12 & 8 \\ 5 & -1 & -3 \\ -2 & 1 & 1 \end{vmatrix} = 4 \begin{vmatrix} 2 & 3 & 2 \\ 5 & -1 & -3 \\ -2 & 1 & 1 \end{vmatrix} \underset{\text{por (vi)}}{=} 52 \neq 0$, logo a matriz é invertível.

(ix) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 4 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$, logo a matriz não é invertível.

$$(x) \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0, \text{ logo a matriz é invertível.}$$

$$(xi) \begin{vmatrix} -2 & -2 & 8 & 6 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3 & -23 \\ 0 & 0 & 0 & 5 \end{vmatrix} = 30 \neq 0, \text{ logo a matriz é invertível.}$$

(xii)

$$\begin{vmatrix} 1 & 3 & -1 & 1 \\ -1 & -2 & -1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 0 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -2 & 2 \\ 2 & 1 & 1 & 1 \\ 2 & 0 & -2 & 0 \end{vmatrix} = 2(-1)^{4+1} \begin{vmatrix} 3 & -1 & 1 \\ 1 & -2 & 2 \\ 1 & 1 & 1 \end{vmatrix} + (-2)(-1)^{4+3} \begin{vmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{vmatrix} =$$

$$= -2[-6 + 1 + (-2) - (-1) - (-2) - 6] + 2[1 + 12 - 2 - 2] = 20 + 18 = 38 \neq 0,$$

logo a matriz é invertível.

$$(xiii) \begin{vmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & -3 & -23 \\ 0 & 1 & 2 & 0 \\ -2 & -2 & 8 & 6 \end{vmatrix} = \begin{vmatrix} -2 & -2 & 8 & 6 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3 & -23 \\ 0 & 0 & 0 & 5 \end{vmatrix} \underset{\text{por (xi)}}{=} 30 \neq 0, \text{ logo a matriz é invertível.}$$

$$(xiv) \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix} = \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \\ 7 & 9 & 11 & 13 \end{vmatrix} = \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 3 & 5 & 7 & 9 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 3 & 5 & 7 & 9 \\ 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix} =$$

$= 0$, logo a matriz não é invertível.

$$(xv) \begin{vmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{vmatrix} = 5(-1)^{5+1} \begin{vmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \end{vmatrix} = 5 \left[4(-1)^{1+1} \begin{vmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{vmatrix} \right] = 120 \neq 0, \text{ logo a}$$

matriz é invertível.

$$(xvi) \begin{vmatrix} 7 & -4 & 0 & 5 \\ 1 & 2 & 0 & -2 \\ 2 & 1 & -3 & 8 \\ 2 & -2 & 0 & 4 \end{vmatrix} = -3(-1)^6 \begin{vmatrix} 7 & -4 & 5 \\ 1 & 2 & -2 \\ 2 & -2 & 4 \end{vmatrix} = -3 \begin{vmatrix} 1 & 2 & -2 \\ 2 & -2 & 4 \\ 7 & -4 & 5 \end{vmatrix} = -3 \begin{vmatrix} 1 & 2 & -2 \\ 0 & -6 & 8 \\ 0 & -18 & 19 \end{vmatrix} =$$

$= -3(6(-19 + 3 \times 8)) = -3(6(-19 + 3 \times 8)) = -3 \times 30 = -90 \neq 0$, logo a matriz é invertível.

$$(xvii) \begin{vmatrix} 9 & 0 & -6 & 4 & -2 \\ 5 & -3 & 0 & -1 & 2 \\ 8 & 0 & 4 & 1 & 1 \\ -6 & 0 & 3 & -1 & 3 \\ 7 & 0 & 5 & 2 & 3 \end{vmatrix} = -3(-1)^4 \begin{vmatrix} 9 & -6 & 4 & -2 \\ 8 & 4 & 1 & 1 \\ -6 & 3 & -1 & 3 \\ 7 & 5 & 2 & 3 \end{vmatrix} = 3 \begin{vmatrix} -2 & -6 & 4 & 9 \\ 1 & 4 & 1 & 8 \\ 3 & 3 & -1 & -6 \\ 3 & 5 & 2 & 7 \end{vmatrix} =$$

$$-3 \begin{vmatrix} 1 & 4 & 1 & 8 \\ -2 & -6 & 4 & 9 \\ 3 & 3 & -1 & -6 \\ 3 & 5 & 2 & 7 \end{vmatrix} = -3 \begin{vmatrix} 1 & 4 & 1 & 8 \\ 0 & 2 & 6 & 25 \\ 0 & -9 & -4 & -30 \\ 0 & -7 & -1 & -17 \end{vmatrix} = -3 \begin{vmatrix} 2 & 6 & 25 \\ -7 & 2 & -5 \\ -7 & -1 & -17 \end{vmatrix} = -3 \begin{vmatrix} 2 & 6 & 25 \\ -7 & 2 & -5 \\ 0 & -3 & -12 \end{vmatrix} =$$

$-3 \begin{vmatrix} 2 & 0 & 1 \\ -7 & 2 & -5 \\ 0 & -3 & -12 \end{vmatrix} = -3 \begin{vmatrix} 2 & 0 & 1 \\ -7 & 2 & -13 \\ 0 & -3 & 0 \end{vmatrix} = -3(21 - 78) = 171 \neq 0$, logo a matriz é invertível.

(xviii) $\begin{vmatrix} 2 & 1 & 0 & 0 \\ -7 & 3 & 0 & -2 \\ \pi & 3 & 1 & 1 \\ 3 & 9 & 0 & 0 \end{vmatrix} = (-1)^6 \begin{vmatrix} 2 & 1 & 0 \\ -7 & 3 & -2 \\ 3 & 9 & 0 \end{vmatrix} = -2(-1)^5 \begin{vmatrix} 2 & 1 \\ 3 & 9 \end{vmatrix} = 30 \neq 0$, logo a matriz é invertível.

(xix) $\begin{vmatrix} 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{vmatrix} = (-1)(-1)^{4+2} \begin{vmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{vmatrix} \stackrel{(2314)}{=} \text{é par}$
-4.

$$(xx) \begin{vmatrix} n & n-1 & \vdots & 2 & 1 \\ n-1 & n-1 & \vdots & 2 & 1 \\ \dots & \dots & \ddots & \vdots & \vdots \\ 2 & 2 & \dots & 2 & 1 \\ 1 & 1 & \dots & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 2 & \dots & 2 & 2 \\ \vdots & \vdots & \ddots & \dots & \dots \\ 1 & 2 & \dots & n-1 & n-1 \\ 1 & 2 & \dots & n-1 & n \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \dots & \dots \\ 0 & 1 & \dots & n-2 & n-2 \\ 0 & 1 & \dots & n-2 & n-1 \end{vmatrix} = \dots = \begin{vmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1 \end{vmatrix} = 1 \neq 0, \text{ logo a matriz é invertível.}$$

5. (i) Seja $A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ e & 0 & b & 0 & 0 \\ 0 & f & 0 & c & 0 \\ 0 & 0 & g & 0 & d \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$, com $a, b, c, d, e, f, g, h \in \mathbb{R}$. Se $a = 0$ ou $h = 0$ então

$\det A = 0$, isto é, A não é invertível. Se $a \neq 0$ e $h \neq 0$ então

$$\det A = \begin{vmatrix} 0 & a & 0 & 0 & 0 \\ e & 0 & b & 0 & 0 \\ 0 & f & 0 & c & 0 \\ 0 & 0 & g & 0 & d \\ 0 & 0 & 0 & h & 0 \end{vmatrix} = \begin{vmatrix} 0 & a & 0 & 0 & 0 \\ e & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g & 0 & d \\ 0 & 0 & 0 & h & 0 \end{vmatrix} = 0,$$

isto é, A não é invertível. Logo, A não é invertível para quaisquer $a, b, c, d, e, f, g, h \in \mathbb{R}$.

Em alternativa, pelo teorema de Laplace

$$\det A = (-1)^5 f \begin{vmatrix} 0 & 0 & 0 & 0 \\ e & b & 0 & 0 \\ 0 & g & 0 & d \\ 0 & 0 & h & 0 \end{vmatrix} + (-1)^7 c \begin{vmatrix} 0 & a & 0 & 0 \\ e & 0 & b & 0 \\ 0 & 0 & g & d \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0.$$

(ii) Diga, para que valores de $a, b, c, d, e, f, g, h, i, j \in \mathbb{R}$, é invertível a seguinte matriz

$$\begin{aligned} \det \begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 \\ f & 0 & b & 0 & 0 & 0 \\ 0 & g & 0 & c & 0 & 0 \\ 0 & 0 & h & 0 & d & 0 \\ 0 & 0 & 0 & i & 0 & e \\ 0 & 0 & 0 & 0 & j & 0 \end{bmatrix} &= a(-1)^3 \det \begin{bmatrix} f & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & h & 0 & d & 0 \\ 0 & 0 & i & 0 & e \\ 0 & 0 & 0 & j & 0 \end{bmatrix} = \\ &= -ac(-1)^5 \det \begin{bmatrix} f & b & 0 & 0 \\ 0 & h & d & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & j & 0 \end{bmatrix} = ace(-1)^7 \det \begin{bmatrix} f & b & 0 \\ 0 & h & d \\ 0 & 0 & j \end{bmatrix} = -acefhj, \end{aligned}$$

pelo que a matriz será invertível se e só se $a, c, e, f, h, j \in \mathbb{R} \setminus \{0\}$.

6. Determinemos todos os valores do escalar λ para os quais a matriz $A - \lambda I$ não é invertível, isto é, todos os valores próprios de A .

$$(i) \det(A - \lambda I) = \begin{vmatrix} -\lambda & 3 \\ 2 & -1 - \lambda \end{vmatrix} = \lambda(1 + \lambda) - 6 = \lambda^2 + \lambda - 6.$$

Logo, $\det(A - \lambda I) = 0 \Leftrightarrow (\lambda = 2 \text{ ou } \lambda = -3)$.

$$(ii) \det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 0 & 2 \\ 0 & -1 - \lambda & -2 \\ 2 & -2 & -\lambda \end{vmatrix} = \lambda(1 - \lambda^2) + 4(1 + \lambda) - 4(1 - \lambda) = \\ = \lambda[(1 - \lambda^2) + 8]. \text{ Logo, } \det(A - \lambda I) = 0 \Leftrightarrow (\lambda = 0 \text{ ou } \lambda = -3 \text{ ou } \lambda = 3).$$

$$(iii) \det(A - \lambda I) = \det \begin{bmatrix} 1 - \lambda & 1 & 1 \\ 1 & 1 - \lambda & 1 \\ 1 & 1 & 1 - \lambda \end{bmatrix} = \det \begin{bmatrix} 1 - \lambda & 1 & 1 \\ \lambda & -\lambda & 0 \\ \lambda & 0 & -\lambda \end{bmatrix} = \\ = \det \begin{bmatrix} 2 - \lambda & 1 & 1 \\ \lambda & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = \det \begin{bmatrix} 3 - \lambda & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = (-1)^3(\lambda - 3)\lambda^2.$$

Logo, $\det(A - \lambda I) = 0 \Leftrightarrow (\lambda = 0 \text{ ou } \lambda = 3)$.

(iv)

$$\det(A - \lambda I) = \det \begin{bmatrix} 1 - \lambda & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 1 \\ 1 & \cdots & 1 & 1 - \lambda \end{bmatrix}_{n \times n} \stackrel[-L_1 + L_i \rightarrow L_i]{i=2, \dots, n} = \det \begin{bmatrix} 1 - \lambda & 1 & 1 & \cdots & 1 \\ \lambda & -\lambda & 0 & \cdots & 0 \\ \vdots & 0 & -\lambda & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \lambda & 0 & \cdots & 0 & -\lambda \end{bmatrix} =$$

$$\underset{i=2,\dots,n}{C_i+C_1\rightarrow C_1} = \det \begin{bmatrix} n-\lambda & 1 & 1 & \cdots & 1 \\ 0 & -\lambda & 0 & \cdots & 0 \\ \vdots & \ddots & -\lambda & \ddots & \vdots \\ & & & \ddots & 0 \\ 0 & \cdots & & 0 & -\lambda \end{bmatrix} = (-1)^n (\lambda - n) \lambda^{n-1}.$$

Logo, $\det(A - \lambda I) = 0 \Leftrightarrow (\lambda = 0 \text{ ou } \lambda = n)$.

7. $\begin{bmatrix} 0 & \pi \\ -1 & \pi \end{bmatrix}, \begin{bmatrix} \frac{\pi}{2} & \pi^2 \\ \frac{1}{4} - \frac{1}{\pi} & \frac{\pi}{2} \end{bmatrix}, \begin{bmatrix} \pi & 1 \\ -\pi & 0 \end{bmatrix}.$

8. (i) $A^{-1} = \frac{1}{\det A} (\text{cof } A)^T = \frac{1}{-2} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

(ii) $A^{-1} = \frac{1}{\det A} (\text{cof } A)^T = \frac{1}{1} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(iii) $A^{-1} = \frac{1}{\det A} (\text{cof } A)^T = \frac{1}{6} \begin{bmatrix} 18 & 0 & 6 \\ 24 & 0 & 6 \\ 4 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 3 & 4 & 2/3 \\ 0 & 0 & 1/6 \\ 1 & 1 & 1/6 \end{bmatrix}$

9.

$$\det A = 2(-1)^{3+4} \det \begin{bmatrix} 1 & -3 & 0 & 1 \\ -2 & 0 & -1 & 1 \\ 0 & 0 & -2 & 1 \\ 2 & 0 & -1 & 1 \end{bmatrix} = -2(-3)(-1)^{1+2} \det \begin{bmatrix} -2 & -1 & 1 \\ 0 & -2 & 1 \\ 2 & -1 & 1 \end{bmatrix} = -24 \neq 0$$

logo A é invertível e

$$(A^{-1})_{(2,3)} = \left(\frac{1}{\det A} (\text{cof } A)^T \right)_{(2,3)} = \frac{1}{\det A} (\text{cof } A)_{(3,2)} = -\frac{1}{24} (-1)^{3+2} \det \begin{bmatrix} 1 & 0 & 0 & 1 \\ -2 & -1 & 0 & 1 \\ 0 & -2 & 0 & 1 \\ 2 & -1 & 0 & 1 \end{bmatrix} = 0.$$

10. $I - \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} A = A \det \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -2 & 1 \\ 2 & 0 \end{bmatrix} A = I \Leftrightarrow A = \begin{bmatrix} -2 & 1 \\ 2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 1 \end{bmatrix}.$

11.

$$\begin{aligned} \det(A^T A + A \det A - 5A) &= \det((A^T + I \det A - 5I) A) = \det(A^T + I \det A - 5I) \det A \underset{\det A=4}{=} \\ &= 4 \det \left(\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} - I \right) = 4 \det \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} = 0, \end{aligned}$$

logo a matriz $A^T A + A \det A - 5A$ é não invertível.

12.

$$\det [-2A^{-1}A^T \det(A^{-2})] = [-2 \det(A^{-2})]^3 \frac{1}{\det A} \det(A^T) = (-2)^3 \frac{1}{(\det A)^6} = \frac{(-2)^3}{(-2)^6} = -\frac{1}{8}.$$

13.

$$\begin{aligned} \det \begin{bmatrix} 0 & a & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & a & 0 \end{bmatrix} &= 1 \Leftrightarrow a(-1)^{1+2} \det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & a & 0 \end{bmatrix} + (-1)^{1+4} \det \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 0 \\ 1 & 0 & a \end{bmatrix} = 1 \Leftrightarrow \\ &\Leftrightarrow -a(-1)^{2+3} \det \begin{bmatrix} 1 & 2 \\ 1 & a \end{bmatrix} + (-1)(-2)(-1)^{2+2} \det \begin{bmatrix} 1 & 2 \\ 1 & a \end{bmatrix} = 1 \Leftrightarrow \\ &\Leftrightarrow a(a-2) + 2(a-2) = 1 \Leftrightarrow a^2 - 4 = 1 \Leftrightarrow (a = \sqrt{5} \text{ ou } a = -\sqrt{5}). \end{aligned}$$

14.

$$\begin{aligned} \det \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 2 \\ 1 & 1 & 2 & -2 \\ -1 & 0 & -2 & 1 \end{bmatrix} &= \det \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ -1 & 0 & -2 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \\ &= \det \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} = -2. \end{aligned}$$

Logo

$$\begin{aligned} \det(A^3 + A^4) &= (\det A)^3 \det(I + A) = -8 \det \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 3 & -2 \\ -1 & 0 & -2 & 2 \end{bmatrix} = \\ &= -8 \det \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 3 & -2 \\ -1 & 0 & -2 & 2 \end{bmatrix} = -8 \times 2 \det \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \\ -1 & 0 & -2 \end{bmatrix} = 48. \end{aligned}$$

15.

$$\det(A^6 - A^5) = (\det A)^5 \det(A - I) = \left(\det \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right)^5 \det \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = 3.$$

$$\mathbf{16.} \quad AXB - B = AX \det(A^T A) \Leftrightarrow AX \left(B - \underbrace{I \det(A^T A)}_{=1} \right) = B \Leftrightarrow$$

$$\Leftrightarrow X = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

$$\mathbf{17. (i)} \quad \text{Tem-se } \det A = \begin{vmatrix} -3 & 2 & 0 & 2 \\ -1 & 0 & 0 & 3 \\ 0 & 9 & -2 & 0 \\ 3 & -1 & 0 & 2 \end{vmatrix} = (-2)(-1)^{3+3} \begin{vmatrix} -3 & 2 & 2 \\ -1 & 0 & 3 \\ 3 & -1 & 2 \end{vmatrix} =$$

$$= (-2)(2 + 18 - (-4) - 9) = -30 \neq 0 \text{ pelo que } A \text{ é invertível. Logo,}$$

$$(A^{-1})_{(2,2)} = \frac{1}{\det A} ((\text{cof } A)^T)_{(2,2)} = \frac{1}{\det A} (\text{cof } A)_{(2,2)} = \frac{1}{-30} (-1)^{2+2} \begin{vmatrix} -3 & 0 & 2 \\ 0 & -2 & 0 \\ 3 & 0 & 2 \end{vmatrix} = -\frac{4}{5}.$$

$$\det B = \begin{vmatrix} -1 & 0 & 0 & 2 \\ 4 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & -1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & -1 \end{vmatrix} = (-1)(-1)^{1+1} \begin{vmatrix} 0 & 1 & 8 \\ 1 & 0 & -3 \\ 0 & 2 & -1 \end{vmatrix} = -17 \neq 0$$

pois que B é invertível. Logo,

$$(B^{-1})_{(2,3)} = \frac{1}{\det B} ((\text{cof } B)^T)_{(2,3)} = \frac{1}{\det B} (\text{cof } B)_{(3,2)} = \frac{1}{-17} (-1)^{3+2} \begin{vmatrix} -1 & 0 & 2 \\ 4 & 1 & 0 \\ 0 & 2 & 2 \end{vmatrix} = \frac{14}{17}.$$

$$\mathbf{(ii)} \quad \det(A + B) = \det \begin{bmatrix} -4 & 2 & 0 & 4 \\ 3 & 0 & 1 & 3 \\ 0 & 10 & -2 & -3 \\ 3 & -2 & 2 & 4 \end{bmatrix} = -134 \neq -47 = -30 - 17 = \det A + \det B$$

$$\det(A - B) = \det \begin{bmatrix} -2 & 2 & 0 & 0 \\ -5 & 0 & -1 & 3 \\ 0 & 8 & -2 & 3 \\ 3 & 0 & -2 & 0 \end{bmatrix} = 138 \neq -13 = -30 + 17 = \det A - \det B$$

$$\mathbf{18. (i)} \quad x = \frac{\begin{vmatrix} 1 & 3 \\ 3 & 7 \\ 2 & 3 \\ 5 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix}} = \frac{-2}{-1} = 2 \quad \text{e} \quad y = \frac{\begin{vmatrix} 2 & 1 \\ 5 & 3 \\ 2 & 3 \\ 5 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix}} = \frac{1}{-1} = -1$$

(ii)

$$x = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 2 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 2 & 2 \end{vmatrix}} = \frac{-5}{-5} = 1, \quad y = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 2 & 2 \end{vmatrix}} = 0 \quad \text{e} \quad z = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 2 & 2 \end{vmatrix}} = \frac{5}{-5} = -1$$

19. $\det C = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ 0 & 1 & -2 \end{vmatrix} = -6 \neq 0$, logo C é invertível. $\det D = \begin{vmatrix} 9 & -8 & -1 \\ -7 & -3 & 0 \\ -2 & 0 & 0 \end{vmatrix} = 6 \neq 0$, logo D é invertível.

(i) $\det(2C^{-1}) = 2^3 \frac{1}{\det C} = -\frac{4}{3}$

(ii) $\det(C^3(2C)^{-1}) = (\det C)^3 \frac{1}{2^3 \det C} = (\det C)^2 \frac{1}{8} = \frac{9}{2}$

(iii) $\det((C^T 2C)^{-1}) = \det\left(C^{-1} \frac{1}{2} (C^{-1})^T\right) = \frac{1}{2^3} \det(C^{-1}) \det(C^{-1}) = \frac{1}{2^3} \left(\frac{1}{\det C}\right)^2 = \frac{1}{288}$

(iv) $\det(C^T \frac{1}{2} C^{-2}) = \frac{1}{2^3} \det(C^T) \det(C^{-2}) = \frac{1}{2^3} \det C \frac{1}{(\det C)^2} = \frac{1}{2^3 \det C} = -\frac{1}{48}$

(v)

$$\det(C^2 + 2D) = \det\left(\begin{bmatrix} 1 & 1 & -1 \\ 8 & 11 & 4 \\ 2 & 1 & 6 \end{bmatrix} + 2 \begin{bmatrix} 9 & -8 & -1 \\ -7 & -3 & 0 \\ -2 & 0 & 0 \end{bmatrix}\right) = \det\begin{bmatrix} 19 & -15 & -3 \\ -6 & 5 & 4 \\ -2 & 1 & 6 \end{bmatrix} = 62$$

20. $\det(2A^T B^{-3}) = 2^3 \det(A^T) \det(B^{-3}) = 8 \det A \frac{1}{(\det B)^3} = 64\sqrt{3}$.

21. (i) $\begin{vmatrix} 1 & 1 & a \\ 2 & b & 1 \\ c & 2 & 2 \end{vmatrix} = -2$ (ii) $\begin{vmatrix} a+3 & -1 & -2 \\ b+3 & b-2 & -2 \\ c+3 & 2-c & -c \end{vmatrix} = -2$ (iii) $\det(\frac{1}{2}B^T B) = \frac{1}{2}$

22. $\begin{vmatrix} a & b & 0 & c \\ d & e & 0 & f \\ g & h & 0 & i \\ x & y & -1 & z \end{vmatrix} = 5 \Leftrightarrow \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$

(i) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = 5$ (ii) $\begin{vmatrix} -a & -b & -c \\ 2d & 2e & 2f \\ -g & -h & -i \end{vmatrix} = 10$ (iii) $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix} = 5$

$$(iv) \begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = -5$$

$$23. (i) \begin{vmatrix} a & b & c \\ 6 & 3 & 0 \\ -\frac{1}{2} & -1 & -\frac{1}{2} \end{vmatrix} = -\frac{3}{2}$$

$$(ii) \begin{vmatrix} a & b & c \\ 2a+2 & 2b+1 & 2c \\ a+1 & b+2 & c+1 \end{vmatrix} = 1$$

$$(iii) \begin{vmatrix} a-1 & b-2 & c-1 \\ 3 & 3 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 1$$

$$(iv) \begin{vmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 3a+1 & 3b+2 & 3c+1 \end{vmatrix} = - \begin{vmatrix} -1 & 1 & 1 \\ 2 & 1 & 0 \\ 3a+1 & 3b+2 & 3c+1 \end{vmatrix} =$$

$$= - \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 3a+1 & 3b+2 & 3c+1 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 3a & 3b & 3c \end{vmatrix} = 3 \begin{vmatrix} a & b & c \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 3$$

$$24. \begin{vmatrix} 1 & 2 & \alpha \\ \beta & \beta\alpha + \beta^2 & 2\beta \\ \beta\alpha & \alpha & \alpha \end{vmatrix} = -\alpha\beta.$$

$$25. \begin{vmatrix} \lambda & 1 & 1 & 1 & 1 & 1 \\ \lambda & \lambda+1 & 2 & 2 & 2 & 2 \\ \lambda & \lambda+1 & \lambda+2 & 3 & 3 & 3 \\ \lambda & \lambda+1 & \lambda+2 & \lambda+3 & 4 & 4 \\ \lambda & \lambda+1 & \lambda+2 & \lambda+3 & \lambda+4 & 5 \\ \lambda & \lambda+1 & \lambda+2 & \lambda+3 & \lambda+4 & \lambda+5 \end{vmatrix} = \begin{vmatrix} \lambda & 1 & 1 & 1 & 1 & 1 \\ 0 & \lambda & 1 & 1 & 1 & 1 \\ 0 & 0 & \lambda & 1 & 1 & 1 \\ 0 & 0 & 0 & \lambda & 1 & 1 \\ 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{vmatrix} = \lambda^6.$$

$$26. \begin{vmatrix} \lambda & \lambda & \lambda & \dots & \lambda \\ 1 & \lambda+1 & 1 & \dots & 1 \\ \vdots & 1 & \lambda+1 & \ddots & \\ & \vdots & 1 & \ddots & 1 & \vdots \\ & & \vdots & \ddots & \lambda+1 & 1 \\ 1 & 1 & 1 & \dots & 1 & \lambda+1 \end{vmatrix} = \begin{vmatrix} \lambda & 0 & 0 & \dots & 0 \\ 1 & \lambda & 0 & \dots & 0 \\ \vdots & 0 & \lambda & \ddots & \\ & \vdots & 0 & \ddots & 0 & \vdots \\ & & \vdots & \ddots & \lambda & 0 \\ 1 & 0 & 0 & \dots & 0 & \lambda \end{vmatrix} = \lambda^n.$$

27. Sejam $\beta \neq 0$ e $A = (a_{ij})_{n \times n}$.

$$\det(\beta^{i-j}a_{ij}) = \begin{vmatrix} a_{11} & \frac{1}{\beta}a_{12} & \frac{1}{\beta^2}a_{13} & \dots & \frac{1}{\beta^{n-2}}a_{1 \ n-1} & \frac{1}{\beta^{n-1}}a_{1n} \\ \beta a_{21} & a_{22} & \frac{1}{\beta}a_{23} & \dots & \frac{1}{\beta^{n-3}}a_{2 \ n-1} & \frac{1}{\beta^{n-2}}a_{2n} \\ \vdots & \beta a_{32} & a_{33} & \ddots & & \\ & \vdots & \beta a_{43} & \ddots & \frac{1}{\beta}a_{n-2 \ n-1} & \vdots \\ & & \vdots & \ddots & a_{n-1 \ n-1} & \frac{1}{\beta}a_{n-1 \ n} \\ \beta^{n-1}a_{n1} & \beta^{n-2}a_{n2} & \beta^{n-3}a_{n3} & \dots & \beta a_{n \ n-1} & a_{nn} \end{vmatrix} =$$

$$\begin{aligned}
&= \frac{1}{\beta} \cdots \frac{1}{\beta^{n-2}} \frac{1}{\beta^{n-1}} \begin{vmatrix} \beta^{n-1}a_{11} & \beta^{n-2}a_{12} & \beta^{n-3}a_{13} & \cdots & \beta a_{1\ n-1} & a_{1n} \\ \beta^{n-1}a_{21} & \beta^{n-2}a_{22} & \beta^{n-3}a_{23} & \cdots & \beta a_{2\ n-1} & a_{2n} \\ \vdots & \beta^{n-2}a_{32} & \beta^{n-3}a_{33} & \ddots & & \\ & \vdots & \beta^{n-3}a_{43} & \ddots & \beta a_{n-2\ n-1} & \vdots \\ & & \vdots & \ddots & \beta a_{n-1\ n-1} & a_{n-1\ n} \\ \beta^{n-1}a_{n1} & \beta^{n-2}a_{n2} & \beta^{n-3}a_{n3} & \cdots & \beta a_{n\ n-1} & a_{nn} \end{vmatrix} = \\
&= \frac{1}{\beta} \cdots \frac{1}{\beta^{n-2}} \frac{1}{\beta^{n-1}} \beta^{n-1} \beta^{n-2} \cdots \beta \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1\ n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2\ n-1} & a_{2n} \\ \vdots & a_{32} & a_{33} & \ddots & & \\ & \vdots & a_{43} & \ddots & a_{n-2\ n-1} & \vdots \\ & & \vdots & \ddots & a_{n-1\ n-1} & a_{n-1\ n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n\ n-1} & a_{nn} \end{vmatrix} = \det A.
\end{aligned}$$

28. Seja

$$A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix},$$

com $a, b, c \in \mathbb{R}$. A matriz A é invertível se e só se $\det A \neq 0$. Tem-se

$$\det A = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = 0$$

se $a = b$ ou $a = c$. Se $a \neq b$ e $a \neq c$ então

$$\begin{aligned}
\det A &= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & c^2-a^2 - (c-a)(b+a) \end{vmatrix} = \\
&= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c-a)[(c+a)-(b+a)] \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c-a)(c-b) \end{vmatrix} = 0
\end{aligned}$$

se $b = c$.

Logo, a matriz A é invertível se e só se $a \neq b$, $a \neq c$ e $b \neq c$.

29. (i)

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & y_1 & y_1 \\ x_2 & x_2 & y_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x_1 & y_1 - x_1 & y_1 - x_1 \\ x_2 & 0 & y_2 - x_2 \end{vmatrix} = (y_1 - x_1)(-1)^{2+2} \det \begin{vmatrix} 1 & 0 \\ x_2 & y_2 - x_2 \end{vmatrix} = (y_1 - x_1)(y_2 - x_2).$$

(ii)

$$\det \begin{bmatrix} (x_1)^3 & (x_1)^2 & x_1 & 1 \\ (x_2)^3 & (x_2)^2 & x_2 & 1 \\ (x_3)^3 & (x_3)^2 & x_3 & 1 \\ (x_4)^3 & (x_4)^2 & x_4 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & x_1 & (x_1)^2 & (x_1)^3 \\ 1 & x_2 & (x_2)^2 & (x_2)^3 \\ 1 & x_3 & (x_3)^2 & (x_3)^3 \\ 1 & x_4 & (x_4)^2 & (x_4)^3 \end{bmatrix} =$$

$$\begin{aligned}
&= \det \begin{bmatrix} 1 & x_1 & (x_1)^2 & (x_1)^3 - (x_1)^3 \\ 0 & x_2 - x_1 & (x_2)^2 - (x_1)^2 & (x_2)^3 - (x_1)^3 \\ 0 & x_3 - x_1 & (x_3)^2 - (x_1)^2 & (x_3)^3 - (x_1)^3 \\ 0 & x_4 - x_1 & (x_4)^2 - (x_1)^2 & (x_4)^3 - (x_1)^3 \end{bmatrix} = \\
&= (x_4 - x_1)(x_3 - x_1)(x_2 - x_1) \det \begin{bmatrix} 1 & x_1 & (x_1)^2 & (x_1)^3 - (x_1)^3 \\ 0 & 1 & x_2 + x_1 & x_1^2 + x_1x_2 + x_2^2 \\ 0 & 1 & x_3 + x_1 & x_1^2 + x_1x_3 + x_3^2 \\ 0 & 1 & x_4 + x_1 & x_1^2 + x_1x_4 + x_4^2 \end{bmatrix} = \\
&= (x_4 - x_1)(x_3 - x_1)(x_2 - x_1) \det \begin{bmatrix} 1 & x_1 & (x_1)^2 & (x_1)^3 - (x_1)^3 \\ 0 & 1 & x_2 + x_1 & x_1^2 + x_1x_2 + x_2^2 \\ 0 & 0 & x_3 - x_2 & x_1(x_3 - x_2) + x_3^2 - x_2^2 \\ 0 & 0 & x_4 - x_2 & x_1(x_4 - x_2) + x_4^2 - x_2^2 \end{bmatrix} = \\
&= (x_4 - x_1)(x_3 - x_1)(x_2 - x_1)(x_3 - x_2)(x_4 - x_2) \det \begin{bmatrix} 1 & x_1 & (x_1)^2 & (x_1)^3 - (x_1)^3 \\ 0 & 1 & x_2 + x_1 & x_1^2 + x_1x_2 + x_2^2 \\ 0 & 0 & 1 & x_1 + x_3 + x_2 \\ 0 & 0 & 1 & x_1 + x_4 + x_2 \end{bmatrix} = \\
&= (x_4 - x_1)(x_3 - x_1)(x_2 - x_1)(x_3 - x_2)(x_4 - x_2) \det \begin{bmatrix} 1 & x_1 & (x_1)^2 & (x_1)^3 - (x_1)^3 \\ 0 & 1 & x_2 + x_1 & x_1^2 + x_1x_2 + x_2^2 \\ 0 & 0 & 1 & x_1 + x_3 + x_2 \\ 0 & 0 & 0 & x_4 - x_3 \end{bmatrix} = \\
&= (x_4 - x_1)(x_3 - x_1)(x_2 - x_1)(x_3 - x_2)(x_4 - x_2)(x_4 - x_3) = \\
&= (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)
\end{aligned}$$

$$\mathbf{30. (i)} \quad \begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\mathbf{(ii)} \quad \begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 2a_1 & a_1-b_1 & c_1 \\ 2a_2 & a_2-b_2 & c_2 \\ 2a_3 & a_3-b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 2a_1 & -b_1 & c_1 \\ 2a_2 & -b_2 & c_2 \\ 2a_3 & -b_3 & c_3 \end{vmatrix} = -2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\mathbf{(iii)} \quad \begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{aligned}
\mathbf{31.} \quad & \begin{vmatrix} a_1+b_1 & c_1+d_1 \\ a_2+b_2 & c_2+d_2 \end{vmatrix} = (a_1+b_1)(c_2+d_2) - (a_2+b_2)(c_1+d_1) = \\
&= a_1c_2 - a_2c_1 + a_1d_2 - b_2c_1 + b_1c_2 - a_2d_1 + b_1d_2 - b_2d_1 = \\
&= \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 \\ b_2 & d_2 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 \\ a_2 & c_2 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}.
\end{aligned}$$

$$\mathbf{32.} \quad \begin{vmatrix} 0 & 0 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0 \quad \text{e} \quad \begin{vmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{vmatrix} = 0$$

$$\mathbf{33. (i)} \quad \begin{vmatrix} 1 & x & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} 1 & x & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0 \Leftrightarrow -1 + x = 0 \Leftrightarrow x = 1$$

$$\mathbf{(ii)} \quad \begin{vmatrix} x & x & x & x \\ x & 4 & x & x \\ x & x & 4 & x \\ x & x & x & 4 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} x & x & x & x \\ 0 & 4-x & 0 & 0 \\ 0 & 0 & 4-x & 0 \\ 0 & 0 & 0 & 4-x \end{vmatrix} = 0 \Leftrightarrow x(4-x)^3 = 0 \Leftrightarrow (x = 0 \text{ ou } x = 4)$$

$$\mathbf{(iii)} \quad \begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} x & 1 & 1 & 1 \\ 0 & x-1 & 0 & 1-x \\ 0 & 0 & x-1 & 1-x \\ 1 & 1 & 1 & x \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{vmatrix} x & 1-x & 1-x & 1-x \\ 0 & x-1 & 0 & 1-x \\ 0 & 0 & x-1 & 1-x \\ 1 & 0 & 0 & x-1 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} 0 & 1-x & 1-x & 1-x^2 \\ 0 & x-1 & 0 & 1-x \\ 0 & 0 & x-1 & 1-x \\ 1 & 0 & 0 & x-1 \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{vmatrix} 0 & 0 & 0 & 3-2x-x^2 \\ 0 & x-1 & 0 & 1-x \\ 0 & 0 & x-1 & 1-x \\ 1 & 0 & 0 & 0 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} 0 & 0 & 0 & 3-2x-x^2 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow - \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & 3-2x-x^2 \end{vmatrix} = 0 \Leftrightarrow -(x-1)^2(3-2x-x^2) = 0 \Leftrightarrow (x = 1 \text{ ou } x = -3)$$

$$\mathbf{34.} \quad \begin{vmatrix} 2 & 1 & 8 \\ -1 & 0 & 10 \\ 3 & -7 & 4 \end{vmatrix} \xrightarrow{-3C_2+C_3 \rightarrow C_3} \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 10 \\ 3 & -7 & 25 \end{vmatrix} = 5 \begin{vmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & -7 & 5 \end{vmatrix}.$$

35. Sendo $A \in \mathcal{M}_{5 \times 6}(\mathbb{R})$, tem-se $A^T A \in \mathcal{M}_{6 \times 6}(\mathbb{R})$. Como $\mathcal{N}(A) \subseteq \mathcal{N}(A^T A)$ e

$$\dim \mathcal{N}(A) = 6 - \text{car } A \geq 6 - 5 = 1$$

então

$$1 \leq \dim \mathcal{N}(A) \leq \dim \mathcal{N}(A^T A).$$

Logo $\dim \mathcal{N}(A^T A) \neq 0$ pelo que $A^T A$ não é invertível, isto é,

$$\det(A^T A) = 0.$$