IST - Álgebra Linear - 2^o semestre 2018/2019

Resolução da 2^a ficha de exercícios

1. (ii) $\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}^{1000} = \left(\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}^2 \right)^{500} = I^{500} = I$

(iv) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{222} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{220} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{220} \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 + I \right) =$

$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{220} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(xi) se $ad - bc \neq 0$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = I$$

2. (i) $[1]^{-1} = [1]$ **(ii)** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(iii) $\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \underset{L_1 \leftrightarrow L_2}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$. Logo $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(iv) $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \underset{-3L_1+L_2 \to L_2}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \underset{L_2+L_1 \to L_1}{\longrightarrow}$

$\underset{L_2+L_1 \to L_1}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right] \underset{-\frac{1}{2}L_2 \to L_2}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]$.

Logo $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

(v) $\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \underset{L_1 \leftrightarrow L_2}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right] \underset{-2L_1+L_2 \to L_2}{\longrightarrow}$

$\underset{-2L_1+L_2 \to L_2}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right] \underset{L_2+L_1 \to L_1}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & -2 \end{array}\right] \underset{-\frac{1}{2}L_2 \to L_2}{\longrightarrow}$.

$\underset{-L_2 \to L_2}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right]$. Logo $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

(vi) $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow[-L_1+L_2 \to L_2]{} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right]$. Logo, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ não é invertível.

(vii) Nesta alínea só se apresenta a solução: $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

(viii) $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{-4L_1+L_2 \to L_2 \\ -7L_1+L_3 \to L_3}]{} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right] \xrightarrow[-2L_2+L_3 \to L_3]{}$

$\xrightarrow[-2L_2+L_3 \to L_3]{} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$. Logo, $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ não é invertível.

Nas alíneas (ix) e (x) só se apresentam as soluções:

(ix) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ **(x)** $\left(\frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}\right)^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$

(xi) $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-4L_1+L_3 \to L_3]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & -4 & 0 & 1 \end{array}\right] \xrightarrow[\frac{2}{3}L_3+L_1 \to L_1]{}$

$\xrightarrow[\frac{2}{3}L_3+L_1 \to L_1]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{3} & 0 & \frac{2}{3} \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & -4 & 0 & 1 \end{array}\right] \xrightarrow[\substack{\frac{1}{3}L_2 \to L_2 \\ -\frac{1}{3}L_3 \to L_3}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{4}{3} & 0 & -\frac{1}{3} \end{array}\right]$.

Logo

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 4 & 0 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{5}{3} & 0 & \frac{2}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{4}{3} & 0 & -\frac{1}{3} \end{bmatrix}.$$

(xii) Seja $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ do tipo 2×2. Suponhamos que $a \neq 0$ e $ad - bc \neq 0$. Logo, tem-se:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow[-\frac{c}{a}L_1+L_2 \to L_2]{} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right] \xrightarrow[-\frac{ab}{ad-bc}L_2+L_1 \to L_2]{}$$

$$\xrightarrow[-\frac{ab}{ad-bc}L_2+L_1 \to L_2]{} \left[\begin{array}{cc|cc} a & 0 & 1+\frac{bc}{ad-bc} & -\frac{ab}{ad-bc} \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right] \xrightarrow[\substack{\frac{1}{a}L_1 \to L_1 \\ \frac{a}{ad-bc}L_2 \to L_2}]{}$$

$$\xrightarrow[\substack{\frac{1}{a}L_1\to L_1 \\ \frac{a}{ad-bc}L_2\to L_2}]{} \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

Logo,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

com $a \neq 0$ e $ad-bc \neq 0$.

Se $a = 0$ então $b \neq 0$ e $c \neq 0$, caso contrário A não seria invertível. Logo

$$\left[\begin{array}{cc|cc} 0 & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow[L_1\longleftrightarrow L_2]{} \left[\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{array}\right] \xrightarrow[-\frac{d}{b}L_2+L_1\to L_1]{}$$

$$\xrightarrow[-\frac{d}{b}L_2+L_1\to L_1]{} \left[\begin{array}{cc|cc} c & 0 & -\frac{d}{b} & 1 \\ 0 & b & 1 & 0 \end{array}\right] \xrightarrow[\substack{\frac{1}{c}L_1\to L_1 \\ \frac{1}{b}L_2\to L_2}]{} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{d}{bc} & \frac{1}{c} \\ 0 & 1 & \frac{1}{b} & 0 \end{array}\right]$$

e assim

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{-bc}\begin{bmatrix} d & -b \\ -c & 0 \end{bmatrix}$$

está incluída na fórmula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

bastando considerar $a = 0$.

Logo se $ad - bc \neq 0$ então $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ é invertível e

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

(xiii) Sejam k_1, k_2, k_3, k_4 tais que $k_1k_2k_3k_4 \neq 0$.

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{L_1\leftrightarrow L_4 \\ L_2\leftrightarrow L_3}]{} \left[\begin{array}{cccc|cccc} k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\substack{\frac{1}{k_4}L_1\to L_1 \\ \frac{1}{k_3}L_2\to L_2 \\ \frac{1}{k_2}L_3\to L_3 \\ \frac{1}{k_1}L_4\to L_4}]{}$$

$$\xrightarrow[\substack{\frac{1}{k_4}L_1\to L_1 \\ \frac{1}{k_3}L_2\to L_2 \\ \frac{1}{k_2}L_3\to L_3 \\ \frac{1}{k_1}L_4\to L_4}]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{k_1}1 & 0 & 0 & 0 \end{array}\right]. \text{ Logo } \begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ \frac{1}{k_1} & 0 & 0 & 0 \end{bmatrix}.$$

3. Como $A^2+2A+2I=\mathbf{0} \Leftrightarrow A\left(-\frac{1}{2}A-I\right)=\left(-\frac{1}{2}A-I\right)A=I$ então A é invertível e

$$A^{-1}=-\frac{1}{2}A-I.$$

4. a) $2\left(A-I\right)=\begin{bmatrix}1&1\\0&1\end{bmatrix}A-\begin{bmatrix}2&-1\\1&0\end{bmatrix}^T \Leftrightarrow$

$$2A-\begin{bmatrix}1&1\\0&1\end{bmatrix}A=2I-\begin{bmatrix}2&-1\\1&0\end{bmatrix}^T \Leftrightarrow$$

$$\Leftrightarrow \left(\begin{bmatrix}2&0\\0&2\end{bmatrix}-\begin{bmatrix}1&1\\0&1\end{bmatrix}\right)A=\begin{bmatrix}2&0\\0&2\end{bmatrix}-\begin{bmatrix}2&-1\\1&0\end{bmatrix}^T \Leftrightarrow$$

$$\Leftrightarrow A=\begin{bmatrix}1&-1\\0&1\end{bmatrix}^{-1}\begin{bmatrix}0&-1\\1&2\end{bmatrix}=\begin{bmatrix}1&1\\1&2\end{bmatrix}.$$

b) $\left(\begin{bmatrix}1&1&0\\0&1&1\\1&0&1\end{bmatrix}A^T\begin{bmatrix}1&0&0\\0&1&1\\0&1&0\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&1&0\end{bmatrix}\right)^T=\begin{bmatrix}1&0&0\\0&1&1\\0&1&0\end{bmatrix}A \Leftrightarrow$

$$\Leftrightarrow \begin{bmatrix}1&0&0\\0&1&1\\0&1&0\end{bmatrix}A\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&1\\0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&1\\0&1&0\end{bmatrix}A \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix}1&0&0\\0&1&1\\0&1&0\end{bmatrix}A\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&1\\0&1&0\end{bmatrix}A=\begin{bmatrix}1&0&0\\0&1&1\\0&0&0\end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix}1&0&0\\0&1&1\\0&1&0\end{bmatrix}A\left(\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}-I\right)=\begin{bmatrix}1&0&0\\0&1&1\\0&0&0\end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow A=\begin{bmatrix}1&0&0\\0&1&1\\0&1&0\end{bmatrix}^{-1}\begin{bmatrix}1&0&0\\0&1&1\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}^{-1}=\begin{bmatrix}0&1&0\\0&0&0\\1&0&1\end{bmatrix}.$$

5. $\begin{bmatrix}1&0&1&0\\2&0&2&0\end{bmatrix}\underset{-2L_1+L_2\to L_2}{\longrightarrow}\begin{bmatrix}1&0&1&0\\0&0&0&0\end{bmatrix}$, logo

$$CS_{Au=\mathbf{0}}=\left\{(-z,y,z,w)\in\mathbb{R}^4: y,z,w\in\mathbb{R}\right\}.$$

Como

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

então o conjunto solução de $Au = b$ é dado por:

$$(2,0,0,0) + CS_{Au=\mathbf{0}} = \left\{(2-z,y,z,w) \in \mathbb{R}^4 : y,z,w \in \mathbb{R}\right\}.$$

6. Sendo $(3,2,1)$ e $(4,4,4)$ duas soluções do sistema linear não homogéneo $Au = b$, com $b = (1,2,3)$. Então

$$(4,4,4) - (3,2,1) = (1,2,3) \in \left\{u \in \mathbb{R}^3 : Au = \mathbf{0}\right\},$$

pelo que

$$(4,4,4) + (1,2,3) = (5,6,7)$$

é uma solução de $Au = b$ e é distinta das anteriores.

7. Seja $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ a 3^a coluna de A. Como $A = \begin{bmatrix} 1 & 0 & \\ 0 & 0 & \\ 2 & 3 & \end{bmatrix}$ é invertível então $b \neq 0$.

Pretende-se determinar $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ tal que

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 0 & b \\ 2 & 3 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Como

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 0 & b \\ 2 & 3 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} y + \begin{bmatrix} a \\ b \\ c \end{bmatrix} z$$

então

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} y + \begin{bmatrix} a \\ b \\ c \end{bmatrix} z = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

pelo que

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2/3 \\ 0 \end{bmatrix}.$$

Logo a 1^a coluna de A^{-1} é dada por:

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 0 & b \\ 2 & 3 & c \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2/3 \\ 0 \end{bmatrix}.$$

8. $A = 2A^T = 2\left(2A^T\right)^T = 4A \Leftrightarrow 3A = \mathbf{0} \Leftrightarrow A = \mathbf{0}.$