

Resolução da 2^a ficha de exercícios

$$1. \text{ (ii) } \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}^{1000} = \left(\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}^2 \right)^{500} = I^{500} = I$$

$$\begin{aligned} \text{(iv) } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{222} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{220} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{220} \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 + I \right) = \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{220} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

(xi) se $ad - bc \neq 0$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) = \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = I$$

$$2. \text{ (i) } [1]^{-1} = [1] \quad \text{(ii) } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{(iii) } \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{L_1 \leftrightarrow L_2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right] \cdot \text{Logo} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{(iv) } \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array} \right] \xrightarrow{-3L_1 + L_2 \rightarrow L_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array} \right] \xrightarrow{L_2 + L_1 \rightarrow L_1}$$

$$\xrightarrow{L_2 + L_1 \rightarrow L_1} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array} \right] \xrightarrow{-\frac{1}{2}L_2 \rightarrow L_2} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array} \right].$$

$$\text{Logo} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

$$\text{(v) } \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right] \xrightarrow{L_1 \leftrightarrow L_2} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array} \right] \xrightarrow{-2L_1 + L_2 \rightarrow L_2}$$

$$\xrightarrow{-2L_1 + L_2 \rightarrow L_2} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array} \right] \xrightarrow{L_2 + L_1 \rightarrow L_1} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & -2 \end{array} \right] \xrightarrow{-\frac{1}{2}L_2 \rightarrow L_2}.$$

$$\xrightarrow{-L_2 \rightarrow L_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array} \right] \cdot \text{Logo} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

(vi) $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right] \xrightarrow{-L_1+L_2 \rightarrow L_2} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array} \right]$. Logo, $\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right]$ não é invertível.

(vii) Nesta alínea só se apresenta a solução: $\left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array} \right]^{-1} = \left[\begin{array}{ccc} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array} \right]$.

(viii) $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array} \right] \xrightarrow[-7L_1+L_3 \rightarrow L_3]{-4L_1+L_2 \rightarrow L_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array} \right] \xrightarrow{-2L_2+L_3 \rightarrow L_3}$
 $\xrightarrow{-2L_2+L_3 \rightarrow L_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array} \right]$. Logo, $\left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array} \right]$ não é invertível.

Nas alíneas (ix) e (x) só se apresentam as soluções:

(ix) $\left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array} \right]^{-1} = \left[\begin{array}{ccc} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array} \right]$ (x) $\left(\frac{1}{3} \left[\begin{array}{ccc} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{array} \right] \right)^{-1} = \left[\begin{array}{ccc} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{array} \right]$

(xi) $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 5 & 0 & 0 & 1 \end{array} \right] \xrightarrow{-4L_1+L_3 \rightarrow L_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & -4 & 0 & 1 \end{array} \right] \xrightarrow{\frac{2}{3}L_3+L_1 \rightarrow L_1}$
 $\xrightarrow{\frac{2}{3}L_3+L_1 \rightarrow L_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{3} & 0 & \frac{2}{3} \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & -4 & 0 & 1 \end{array} \right] \xrightarrow[-\frac{1}{3}L_3 \rightarrow L_3]{\frac{1}{3}L_2 \rightarrow L_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{4}{3} & 0 & -\frac{1}{3} \end{array} \right]$.

Logo

$$\left[\begin{array}{ccc} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 4 & 0 & 5 \end{array} \right]^{-1} = \left[\begin{array}{ccc} -\frac{5}{3} & 0 & \frac{2}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{4}{3} & 0 & -\frac{1}{3} \end{array} \right].$$

(xii) Sejam $k_1, k_2, k_3, k_4 \neq 0$.

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow[L_2 \leftrightarrow L_3]{L_1 \leftrightarrow L_4} \left[\begin{array}{cccc|cccc} k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \end{array} \right] \xrightarrow[\frac{1}{k_1}L_4 \rightarrow L_4]{\frac{1}{k_4}L_1 \rightarrow L_1, \frac{1}{k_3}L_2 \rightarrow L_2, \frac{1}{k_2}L_3 \rightarrow L_3}$$

$$\begin{array}{l} \xrightarrow{\frac{1}{k_4} L_1 \rightarrow L_1} \\ \frac{1}{k_3} L_2 \rightarrow L_2 \\ \frac{1}{k_2} L_3 \rightarrow L_3 \\ \frac{1}{k_1} L_4 \rightarrow L_4 \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{k_1} & 1 & 0 & 0 \end{array} \right]. \text{ Logo}$$

$$\left[\begin{array}{cccc} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{array} \right]^{-1} = \left[\begin{array}{cccc} 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ \frac{1}{k_1} & 0 & 0 & 0 \end{array} \right].$$

3. Como $A^2 + 2A + 2I = \mathbf{0} \Leftrightarrow A(-\frac{1}{2}A - I) = (-\frac{1}{2}A - I)A = I$ então A é invertível e

$$A^{-1} = -\frac{1}{2}A - I.$$

$$\begin{aligned} \mathbf{4.} \quad 2(A - I) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A - \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}^T \Leftrightarrow \\ &\Leftrightarrow \left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}^T \Leftrightarrow \\ &\Leftrightarrow A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}. \end{aligned}$$

$$\mathbf{5.} \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix} \xrightarrow{-2L_1 + L_2 \rightarrow L_2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ logo}$$

$$CS = \{(-z, y, z, w) \in \mathbb{R}^4 : y, z, w \in \mathbb{R}\}.$$

Como

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

então o conjunto solução de $AX = B$ é dado por:

$$(2, 0, 0, 0) + CS = \{(2 - z, y, z, w) \in \mathbb{R}^4 : y, z, w \in \mathbb{R}\}.$$

6. Sendo $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ e $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ duas soluções do sistema linear não homogéneo $AX = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$,
então

$$\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \{X : AX = \mathbf{0}\},$$

pelo que

$$\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

é uma solução de $AX = B$ e é distinta das anteriores.

7. Seja $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ a 3ª coluna de A . Como $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 2 & 3 \end{bmatrix}$ é invertível então $b \neq 0$.

Pretende-se determinar $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ tal que

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 0 & b \\ 2 & 3 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Como

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 0 & b \\ 2 & 3 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} y + \begin{bmatrix} a \\ b \\ c \end{bmatrix} z$$

então

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} y + \begin{bmatrix} a \\ b \\ c \end{bmatrix} z = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

pelo que

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2/3 \\ 0 \end{bmatrix}.$$

Logo a 1ª coluna de A^{-1} é dada por:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2/3 \\ 0 \end{bmatrix}.$$