Instituto Superior Técnico - Álgebra Linear - 2^o semestre 2017/2018

Resolução da 11^a ficha de exercícios

1. Sejam $(x_1,x_2),(y_1,y_2)\in\mathbb{R}^2$. Consideremos a aplicação $\langle,\rangle:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$, definida por

$$\langle(x_1,x_2),(y_1,y_2)\rangle=\begin{bmatrix} x_1 & x_2\end{bmatrix}\begin{bmatrix}3&2\\2&3\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}=3x_1y_1+2x_2y_1+2x_1y_2+3x_2y_2.$$

Atendendo a que a matriz $\begin{bmatrix}3&2\\2&3\end{bmatrix}$ é simétrica e tem os seus valores próprios (1 e 5) todos positivos, então esta aplicação define em $\mathbb{R}^2$ um produto interno. Além disso, verifica-se $\langle(1,0),(0,1)\rangle=2$, uma vez que

$$\begin{bmatrix}3&2\\2&3\end{bmatrix}=\begin{bmatrix}\langle(1,0),(1,0)\rangle & \langle(1,0),(0,1)\rangle\\ \langle(0,1),(1,0)\rangle & \langle(0,1),(0,1)\rangle\end{bmatrix}.$$

2. a)

$$U=\mathcal{N}\begin{bmatrix}1&0&1&1\\1&1&1&1\end{bmatrix}=\mathcal{N}\begin{bmatrix}1&0&1&1\\0&1&0&0\end{bmatrix}=L\left(\{(-1,0,1,0)\,,(-1,0,0,1)\}\right).$$

Logo

$$U^\perp=L\left(\{(1,0,1,1)\,,(0,1,0,0)\}\right).$$

Como o conjunto $\{(1,0,1,1)\,,(0,1,0,0)\}$ é ortogonal é então uma base ortogonal para $U^\perp$.

b)

$$d\left((1,1,1,1)\,,U\right)=\left\|\operatorname*{proj}_{U^\perp}(1,1,1,1)\right\|\underset{(1,1,1,1)\in U^\perp}{=}\|(1,1,1,1)\|=2.$$

c) Como $V=L\left(\{(1,0,1,1)\,,(0,1,0,0)\}\right)=U^\perp$ então

$$(1,0,0,0)=\underbrace{P_U(1,0,0,0)}_{u}+\underbrace{P_{U^\perp}(1,0,0,0)}_{v}=u+\operatorname*{proj}_{(1,0,1,1)}(1,0,0,0)+\operatorname*{proj}_{(0,1,0,0)}(1,0,0,0)=$$

$$=u+\frac{\langle(1,0,0,0)\,,(1,0,1,1)\rangle}{\|(1,0,1,1)\|^2}(1,0,1,1)+\frac{\langle(1,0,0,0)\,,(0,1,0,0)\rangle}{\|(0,1,0,0)\|^2}(0,1,0,0)=u+\left(\frac{1}{3},0,\frac{1}{3},\frac{1}{3}\right).$$

Logo

$$(1,0,0,0)=\left(\frac{2}{3},0,-\frac{1}{3},-\frac{1}{3}\right)+\left(\frac{1}{3},0,\frac{1}{3},\frac{1}{3}\right)$$

com

$$\left(\frac{1}{3},0,\frac{1}{3},\frac{1}{3}\right)\in V\qquad\text{e}\qquad\left(\frac{2}{3},0,-\frac{1}{3},-\frac{1}{3}\right)=-\frac{1}{3}(-1,0,1,0)-\frac{1}{3}(-1,0,0,1)\in U.$$

3. Como $V=L\left(\{(1,0,0),(0,1,1)\}\right)$ então $V^{\perp}=\mathcal{N}\left(\begin{bmatrix}1&0&0\\0&1&1\end{bmatrix}\right)=L\left(\{(0,-1,1)\}\right)$.

Logo $\{(1,0,0),(0,1,1),(0,-1,1)\}$ é uma base ortogonal para $\mathbb{R}^3$ que inclui um vector de $V^{\perp}$. Como os planos U e V são paralelos, tem-se:

$$d\left(U,V\right)=\left\|P_{V^{\perp}}\left((0,-1,1)-(0,0,0)\right)\right\| \underset{(0,-1,1)\in V^{\perp}}{=} \|(0,-1,1)\|=\sqrt{2}.$$

4. Um produto interno em $\mathbb{R}^3$ em relação ao qual a base $\mathcal{B}=\{(1,0,-1),(0,1,0),(-1,0,2)\}$ é ortonormada, é dado por:

$$\langle(x_1,x_2,x_3),(y_1,y_2,y_3)\rangle=\begin{bmatrix}x_1&x_2&x_3\end{bmatrix}_{\mathcal{B}}G_{\mathcal{B}}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}_{\mathcal{B}}=$$

$$=\begin{bmatrix}2x_1+x_3&x_2&x_1+x_3\end{bmatrix}\begin{bmatrix}2y_1+y_3\\y_2\\y_1+y_3\end{bmatrix}=$$

$$=5x_1y_1+3x_1y_3+x_2y_2+3x_3y_1+2x_3y_3$$

com $G_{\mathcal{B}}=I$. Relativamente a ele, tem-se

$$\left(L\left(\{(-2,0,3),(0,1,0)\}\right)\right)^{\perp}=\mathcal{N}\left(\begin{bmatrix}-2&0&3\\0&1&0\end{bmatrix}\underbrace{\begin{bmatrix}5&0&3\\0&1&0\\3&0&2\end{bmatrix}}_{G_{\mathcal{B}_c}}\right)=$$

$$=\mathcal{N}\left(\begin{bmatrix}-1&0&0\\0&1&0\end{bmatrix}\right)=L\left(\{(0,0,1)\}\right).$$

Logo, o conjunto $\{(0,0,1)\}$ é uma base para $\left(L\left(\{(-2,0,3),(0,1,0)\}\right)\right)^{\perp}$.

5. Em P_2:

$$\langle p(t),q(t)\rangle=p(-1)q(-1)+p(0)q(0)+p(1)q(1).$$

Considere também o seguinte subespaço de P_2:

$$U=\left\{p(t)\in P_2:p(0)=0\right\}.$$

(i) Em P_2, para $p(t)=a_0+a_1t+a_2t^2$ e $q(t)=b_0+b_1t+b_2t^2$ tem-se

$$\langle p(t),q(t)\rangle=p(-1)q(-1)+p(0)q(0)+p(1)q(1)=$$

$$=\left(a_0-a_1+a_2\right)\left(b_0-b_1+b_2\right)+a_0b_0+\left(a_0+a_1+a_2\right)\left(b_0+b_1+b_2\right)=$$

$$=3a_0b_0+2a_0b_2+2a_1b_1+2a_2b_0+2a_2b_2=$$

$$= \begin{bmatrix} a_0 & a_1 & a_1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_1 \end{bmatrix}.$$

Assim, relativamente à base canónica ordenada $\{1, t, t^2\}$ de P_2:

$$\langle p(t), q(t) \rangle = \begin{bmatrix} a_0 & a_1 & a_1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_1 \end{bmatrix}$$

onde

$$\begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \langle 1, 1 \rangle & \langle 1, t \rangle & \langle 1, t^2 \rangle \\ \langle t, 1 \rangle & \langle t, t \rangle & \langle t, t^2 \rangle \\ \langle t^2, 1 \rangle & \langle t^2, t \rangle & \langle t^2, t^2 \rangle \end{bmatrix}$$

Como $\begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$ é simétrica e os seus valores próprios ($\frac{1}{2}\sqrt{17} + \frac{5}{2}$, $\frac{5}{2} - \frac{1}{2}\sqrt{17}$ e 2) são todos positivos, logo, a aplicação $\langle , \rangle$ define um produto interno em P_2.

(ii) Tem-se:

$$U^\perp = \left\{ p(t) \in P_2 : \langle p(t), t \rangle = 0 \ \text{ e } \ \left\langle p(t), t^2 \right\rangle = 0 \right\}.$$

Logo,

$$\begin{cases} (a_0 - a_1 + a_2)(-1)^2 + a_0 0 + a_0 + a_1 + a_2 = 0 \\ \\ (a_0 - a_1 + a_2)(-1) + a_0 0 + a_0 + a_1 + a_2 = 0 \end{cases} \Leftrightarrow \begin{cases} a_0 = -a_2 \\ \\ a_1 = 0. \end{cases}$$

Logo,

$$U^\perp = \left\{ -a_2 + a_2 t^2 : a_2 \in \mathbb{R} \right\} = L(\{-1 + t^2\}).$$

Como $\|-1 + t^2\| = 1$ então $\{-1 + t^2\}$ é uma base ortonormada de $U^\perp$.

Observação. Note que $P_2 = U \oplus U^\perp$, tendo-se, neste caso, $\dim U = 2$ e $\dim U^\perp = 1$.

(iii) Escolhendo um ponto de U, por exemplo t, a distância entre $1 + t$ e U é dada por:

$$d(1 + t, U) = \|P_{U^\perp}(1 + t - t)\| = \|P_{U^\perp}(1)\| = \left\| \left\langle 1, -1 + t^2 \right\rangle (-1 + t^2) \right\| = 1.$$

6. Considere no espaço linear $\mathcal{M}_{2\times 2}(\mathbb{R})$ o produto interno definido da seguinte forma:

$$\langle A, B \rangle = \operatorname{tr}(AB^T).$$

Considere também o subespaço U de $\mathcal{M}_{2\times 2}(\mathbb{R})$ constituído por todas as matrizes simétricas reais do tipo 2×2:

$$U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}) : b = c \right\}.$$

(i) Tem-se:

$$U = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} : a, b, d \in \mathbb{R} \right\} =$$

$$= L\left(\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}\right)$$

pois

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

O conjunto $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ é uma base de U, uma vez que gera U, e é linearmente independente pois se tivermos:

$$\lambda_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \lambda_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \lambda_3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

então

$$\begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Logo, $\lambda_1 = \lambda_2 = \lambda_3$ e como tal, o conjunto $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ é linearmente independente. Vamos aplicar agora a este conjunto o método de ortogonalização de Gram-Schmidt. Sejam

$$\begin{aligned} A_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \mathrm{proj}_{A_1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \mathrm{proj}_{A_1}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \mathrm{proj}_{A_2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Logo,

$$\begin{aligned} A_2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \mathrm{proj}_{A_1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_1 \right\rangle A_1}{\|A_1\|^2} = \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{\mathrm{tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A_1^T\right) A_1}{\|A_1\|^2} = \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{\mathrm{tr}\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) A_1}{\|A_1\|^2} = \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{0A_1}{\|A_1\|^2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

e

$$
\begin{aligned}
A_3 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \operatorname{proj}_{A_1} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \operatorname{proj}_{A_2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_1 \right\rangle A_1}{\|A_1\|^2} - \frac{\left\langle \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_2 \right\rangle A_2}{\|A_2\|^2} = \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \frac{\operatorname{tr}\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} A_1^T \right) A_1}{\|A_1\|^2} - \frac{\operatorname{tr}\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} A_2^T \right) A_2}{\|A_2\|^2} = \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \frac{\operatorname{tr}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) A_1}{\|A_1\|^2} - \frac{\operatorname{tr}\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) A_2}{\|A_2\|^2} = \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \frac{0 A_1}{\|A_1\|^2} - \frac{0 A_2}{\|A_2\|^2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Logo, o conjunto $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ é uma base ortogonal de U. Como:

$$
\begin{aligned}
\|A_1\| &= \sqrt{\langle A_1, A_1 \rangle} = \sqrt{\operatorname{tr}\left(A_1 A_1^T\right)} = \sqrt{\operatorname{tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right)} = 1, \\
\|A_2\| &= \sqrt{\langle A_2, A_2 \rangle} = \sqrt{\operatorname{tr}\left(A_2 A_2^T\right)} = \sqrt{\operatorname{tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)} = \sqrt{2}, \\
\|A_3\| &= \sqrt{\langle A_3, A_3 \rangle} = \sqrt{\operatorname{tr}\left(A_3 A_3^T\right)} = \sqrt{\operatorname{tr}\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)} = 1,
\end{aligned}
$$

então o conjunto

$$
\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
$$

é uma base ortonormada de U.

(ii) Note-se que

$$
\begin{aligned}
U^\perp &= \left\{ A : \left\langle A, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\rangle = \left\langle A, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle = \left\langle A, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle = 0 \right\} = \\
&= L\left(\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\} \right)
\end{aligned}
$$

e $\left\{ \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix} \right\}$ é uma base ortonormada de $U^\perp$ uma vez que $\left\| \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\| = \sqrt{2}$. A projecção ortogonal da matriz $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ sobre $U^\perp$ é dada por:

$$\begin{aligned}
P_{U^\perp}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) &= \operatorname{proj}_{\begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \\
&= \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix} \right\rangle \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix} = \\
&= \operatorname{tr}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}^T \right) \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix} = \\
&= \operatorname{tr}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \right) \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix} = \\
&= \operatorname{tr}\left(\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \right) \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix} = \\
&= \frac{\sqrt{2}}{2} \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}.
\end{aligned}$$

Como se tem:

$$\mathcal{M}_{2\times 2}(\mathbb{R}) = U \oplus U^\perp,$$

então para todo $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R})$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = P_U\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + P_{U^\perp}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).$$

Logo,

$$\begin{aligned}
P_U\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - P_{U^\perp}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} = \\
&= \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}.
\end{aligned}$$

(iii) A matriz simétrica mais próxima da matriz $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ é a matriz

$$P_U\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}.$$

(iv) A distância entre $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ e U é dada por:

$$\begin{aligned}
d\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, U\right) &= \left\| P_{U^\perp}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right\| = \\
&= \left\| \operatorname{proj}_{\begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\| = \\
&= \left\| \left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix} \right\rangle \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix} \right\| = \\
&= \left| \left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix} \right\rangle \right| = \\
&= \left| \operatorname{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \right) \right| = \\
&= \left| \operatorname{tr}\left(\begin{bmatrix} \frac{1}{2}\sqrt{2}b & -\frac{1}{2}\sqrt{2}a \\ \frac{1}{2}\sqrt{2}d & -\frac{1}{2}\sqrt{2}c \end{bmatrix} \right) \right| = \frac{\sqrt{2}}{2}\,|b-c|\,.
\end{aligned}$$

7. a) Os vectores $u_1 = (1,1,0,0), u_2 = (1,0,1,-1)$ formam uma base de V. Seja $\{v_1, v_2\}$ a base ortogonal de V que se obtém aplicando o processo de ortogonalização de Gram-Schmidt aos vectores u_1 e u_2. Portanto,

$$v_1 = u_1 = (1,1,0,0)$$

e

$$v_2 = u_2 - \frac{\langle u_2, u_1\rangle}{\langle u_1, u_1\rangle}\, u_1 = (1,0,1,-1) - \frac{1}{2}(1,1,0,0) = \left(\frac{1}{2}, -\frac{1}{2}, 1, -1\right).$$

Assim,

$$u = P_V(2,-2,1,-1) \underset{\{v_1,v_2\}\text{ base ortogonal de } V}{=} \frac{\langle (2,-2,1,-1), v_1\rangle}{\langle v_1, v_1\rangle}\, v_1 + \frac{\langle (2,-2,1,-1), v_2\rangle}{\langle v_2, v_2\rangle}\, v_2 =$$

$$= \frac{0}{2}(1,1,0,0) + \frac{4}{5/2}\ \left(\frac{1}{2}, -\frac{1}{2}, 1, -1\right) = \left(\frac{4}{5}, -\frac{4}{5}, \frac{8}{5}, -\frac{8}{5}\right)$$

e

$$v = P_{V^\perp}(2,-2,1,-1) = (2,-2,1,-1) - P_V(2,-2,1,-1) = \left(\frac{6}{5}, -\frac{6}{5}, -\frac{3}{5}, \frac{3}{5}\right).$$

Finalmente

$$d((1,1,1,1), V) = ||P_{V^\perp}(1,1,1,1)|| = ||(1,1,1,1) - P_V(1,1,1,1)|| =$$

$$\underset{\{v_1,v_2\}\text{ base ortogonal de } V}{=} ||(1,1,1,1) - \frac{\langle (1,1,1,1), v_1\rangle}{\langle v_1, v_1\rangle}\, v_1 - \frac{\langle (1,1,1,1), v_2\rangle}{\langle v_2, v_2\rangle}\, v_2|| =$$

$$||(1,1,1,1) - \ (1,1,0,0)|| = ||(0,0,1,1)|| = \sqrt{2}.$$

b) Como

$$x = y - 2z - 3w$$

e

$$(y-2z-3w, y, z, w) = y(1,1,0,0) + z(-2,0,1,0) + w(-3,0,0,1)$$

concluí-se que

$$W = L((1,1,0,0),(-2,0,1,0),(-3,0,0,1)).$$

Portanto,

$$W^{\perp} = \mathcal{N}(A)$$

com $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}$.

c) $(1,1,0,0),(1,0,1,-1) \in W$, pelo que qualquer combinação linear entre eles também pertence a W (porque W é um espaço linear). Portanto $V \subset W$. Assim, $W^{\perp} \subset V^{\perp}$, donde

$$W^{\perp} \cap V^{\perp} = W^{\perp}.$$

Mais, $\{(1,-1,2,3)\}$ é uma base de $W^{\perp}$ pois

$$W^{\perp} = \mathcal{L}\left(\begin{bmatrix} 1 & -1 & 2 & 3 \end{bmatrix}\right).$$

d) Não, pois pela alínea c) podemos concluir que $W^{\perp} \subset V^{\perp}$ e portanto

$$W^{\perp} + V^{\perp} = V^{\perp} \neq \mathbb{R}^4.$$

8. a)

$$V^{\perp} = \left(\mathcal{N}\left(\begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix}\right)\right)^{\perp} = \left(\left(\mathcal{L}\left(\begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix}\right)\right)^{\perp}\right)^{\perp} = \mathcal{L}\left(\begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix}\right).$$

Logo $\{(4,3,2,1)\}$ é uma base ortogonal para $V^{\perp}$.

b)

$$d\left((1,2,3,4),V^{\perp}\right) = \|(1,2,3,4) - P_{V^{\perp}}(1,2,3,4)\| = \left\|(1,2,3,4) - \frac{2}{3}(4,3,2,1)\right\| = \frac{5\sqrt{6}}{3}.$$

c) Os vectores $(0,1,1,1)$ e $(1,-1,1,0)$ geram U e são ortogonais. Faltará encontrar uma base ortogonal para $U^{\perp}$. Como

$$U^{\perp} = \mathcal{N}\left(\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}\right) = L\left(\{(-2,-1,1,0),(-1,-1,0,1)\}\right).$$

Atendendo ao método de ortogonalização de Gram-Schmidt,

$$\left\{(-2,-1,1,0), (-1,-1,0,1) - \underset{(-2,-1,1,0)}{\operatorname{proj}} (-1,-1,0,1)\right\} =$$

$$=\left\{(-2,-1,1,0),(-1,-1,0,1)-\frac{1}{2}(-2,-1,1,0)\right\}=$$

$$=\left\{(-2,-1,1,0),\left(0,-\frac{1}{2},-\frac{1}{2},1\right)\right\}$$

é uma base para $U^{\perp}$. Logo,

$$\left\{(0,1,1,1),(1,-1,1,0),(-2,-1,1,0),\left(0,-\frac{1}{2},-\frac{1}{2},1\right)\right\}$$

é uma base ortogonal de $\mathbb{R}^4$ que inclui os vectores $(0,1,1,1)$ e $(1,-1,1,0)$.

d)

$$U=L\left(\{(0,1,1,1),(1,-1,1,0)\}\right)=\left(\mathcal{N}\left(\begin{bmatrix}0&1&1&1\\1&-1&1&0\end{bmatrix}\right)\right)^{\perp}=\left(\mathcal{C}\left(\begin{bmatrix}-2&-1\\-1&-1\\1&0\\0&1\end{bmatrix}\right)\right)^{\perp}.$$

9. $W=\{(x,y,z)\in\mathbb{R}^3:\ x-2y-3z=0\}$.

a) Como $W=L\left(\{(2,1,0),(3,0,1)\}\right)$, então

$$(x-1,y-1,z-1)\in W^{\perp}\Leftrightarrow$$

$$\Leftrightarrow\left(\langle(x-1,y-1,z-1),(2,1,0)\rangle=0\ \text{ e }\ \langle(x-1,y-1,z-1),(3,0,1)\rangle=0\right)$$

$$\Leftrightarrow\left(2(x-1)+(y-1)=0\ \text{ e }\ 3(x-1)+(z-1)=0\right)\Leftrightarrow\left(2x+y=3\ \text{ e }\ 3x+z=4\right)$$

pelo que, as equações cartesianas da recta que passa pelo ponto $u=(1,1,1)$ e é perpendicular ao plano W são:

$$2x+y=3\ \text{ e }\ 3x+z=4.$$

b) $W=\{(x,y,z)\in\mathbb{R}^3:\langle(x,y,z),(1,-2,-3)\rangle=0\}$. Logo, a equação cartesiana do plano que passa pelo ponto $u=(1,1,1)$ e é paralelo ao plano W é:

$$\langle(x-1,y-1,z-1),(1,-2,-3)\rangle=0\Leftrightarrow x-2y-3z=-4.$$

10. Seja $A=M(T;\mathcal{B}_c^3;\mathcal{B}_c^3)$. Tem-se $A=\begin{bmatrix}0&0&0\\1&1&0\\0&0&1\end{bmatrix}$, uma vez que

$$T(1,0,0)=(0,1,0),\ \ T(0,1,0)=(0,1,0)\ \text{ e }\ T(0,0,1)=(0,0,1).$$

Como

$$\det(A-\lambda I)=\begin{vmatrix}-\lambda&0&0\\1&1-\lambda&0\\0&0&1-\lambda\end{vmatrix}=-\lambda(\lambda-1)^2$$

então os valores próprios de T são: 0 e 1. Como

$$\mathcal{N}(T)=\mathcal{N}(A)=L\left(\{(-1,1,0)\}\right),\quad \mathcal{I}(T)=\mathcal{C}(A)=L\left(\{(0,1,0),(0,0,1)\}\right)$$

e $\langle(-1,1,0),(0,1,0)\rangle=1\neq0$ então $(\mathcal{N}(T))^{\perp}\neq\mathcal{I}(T)$, ou seja, T não é uma projecção ortogonal.