Resolução da ficha de preparação para o 3^o teste de Álgebra Linear - LEGM - MEC

1) $\begin{vmatrix} 2 & 4 & 0 & 0 & 4 \\ 2 & -3 & 0 & 0 & 2 \\ \sqrt{3} & \sqrt{5} & 1 & 1 & -1 \\ -\sqrt{5} & 2 & -2 & 2 & \sqrt{2} \\ 5 & 3 & 0 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 0 & 0 & 4 \\ 2 & -3 & 0 & 0 & 2 \\ \sqrt{3} & \sqrt{5} & 1 & 2 & -1 \\ -\sqrt{5} & 2 & -2 & 0 & \sqrt{2} \\ 5 & 3 & 0 & 0 & 4 \end{vmatrix} = 2(-1)^{3+4} \begin{vmatrix} 2 & 4 & 0 & 4 \\ 2 & -3 & 0 & 2 \\ -\sqrt{5} & 2 & -2 & \sqrt{2} \\ 5 & 3 & 0 & 4 \end{vmatrix} =$ $-2(-2)(-1)^{3+3} \begin{vmatrix} 2 & 4 & 4 \\ 2 & -3 & 2 \\ 5 & 3 & 4 \end{vmatrix} = 224.$ $(A^{-1})_{(5,2)} = \frac{1}{|A|}(-1)^{2+5}|A_{2\,5}| = -\frac{1}{224} \begin{vmatrix} 2 & 4 & 0 & 0 \\ \sqrt{3} & \sqrt{5} & 1 & 1 \\ -\sqrt{5} & 2 & -2 & 2 \\ 5 & 3 & 0 & 0 \end{vmatrix} =$ $-\frac{1}{224} \begin{vmatrix} 2 & 4 & 0 & 0 \\ \sqrt{3} & \sqrt{5} & 1 & 2 \\ -\sqrt{5} & 2 & -2 & 0 \\ 5 & 3 & 0 & 0 \end{vmatrix} = -\frac{1}{224} 2(-1)^{2+4} \begin{vmatrix} 2 & 4 & 0 \\ -\sqrt{5} & 2 & -2 \\ 5 & 3 & 0 \end{vmatrix} = \frac{1}{4}.$

2) $-6 = \begin{vmatrix} a & b & 0 & c \\ x & y & -1 & z \\ d & e & 0 & f \\ g & h & 0 & i \end{vmatrix} = (-1)(-1)^{2+3} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = - \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} = \begin{vmatrix} h & g & i \\ e & d & f \\ b & a & c \end{vmatrix} = -\frac{1}{2}\frac{1}{3} \begin{vmatrix} 2i & 2g & 2h \\ 3f & 3d & 3e \\ c & a & b \end{vmatrix} =$ $-\frac{1}{2}\frac{1}{3}\frac{1}{3} \begin{vmatrix} 2i & 6g & 2h \\ 3f & 9d & 3e \\ c & 3a & b \end{vmatrix} = -\frac{1}{18} \begin{vmatrix} 2i & 6g & 2h \\ 3f-2c & 9d-3a & 3e-2b \\ c & 3a & b \end{vmatrix}$. Logo $\begin{vmatrix} 2i & 6g & 2h \\ 3f-2c & 9d-6a & 3e-2b \\ c & 3a & b \end{vmatrix} = 108.$

3) $|A| = 0$ logo 0 é valor próprio de A. Como $\operatorname{tr} A = 12$ e $m_a(0) \geq m_g(0) = \dim \mathcal{N}(A) = \operatorname{nul}\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} = 2$ então os valores próprios de A são 0 e 12 tendo-se $|A - \lambda I| = (-\lambda)^2(12-\lambda)$ e $\mathbb{R}^3 = \mathcal{N}(A) \oplus \mathcal{N}(A - 12I)$.

Como $\mathcal{N}(A) = L(\{(1,0,-1),(2,-1,0)\})$ e por A ser simétrica $\mathcal{N}(A-12I) = (\mathcal{N}(A))^\perp = L(\{(1,2,1)\})$ então considerando a seguinte base ortogonal de $\mathcal{N}(A)$

$$\left\{(1,0,-1),(2,-1,0) - \operatorname*{proj}_{(1,0,-1)}(2,-1,0)\right\} = \{(1,0,-1),(1,-1,1)\}$$

e tendo-se $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 12 \end{bmatrix}$, com $P^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \end{bmatrix}$ tem-se $D = PAP^T$.

Observação. Podia ter-se calculado directamente $\mathcal{N}(A-12I)$.

4) Como $m_a(0) = m_a(2) = m_a(4) = 1$ então $m_g(0) + m_g(2) + m_g(4) = 3$ e assim A é diagonalizável, isto é,

$$\mathbb{R}^3 = \mathcal{N}(A) \oplus \mathcal{N}(A-2I) \oplus \mathcal{N}(A-4I).$$

Logo

$$A = P^{-1}DP = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 3 \end{bmatrix}.$$

5) a) Tem-se $(1,0,3,2),(0,1,0,0) \in V$ e $\langle (1,0,3,2),(0,1,0,0) \rangle = 0$. Por outro lado, como $V^\perp = L(\{(1,0,-1,1)\})$ então $\{(1,0,3,2),(0,1,0,0),(1,0,-1,1)\}$ é um conjunto ortogonal. Logo

$$\{(1,0,3,2),(0,1,0,0),(1,0,-1,1),w\}$$

é uma base ortogonal para $\mathbb{R}^4$ com

$$w \in \mathcal{N}\left(\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & -1 \end{bmatrix}\right) = L(\{(5,0,1,-4)\}).$$

Assim $\{(1,0,3,2),(0,1,0,0),(1,0,-1,1),(5,0,1,-4)\}$ é uma base ortogonal para $\mathbb{R}^4$ que inclui o vector $(1,0,3,2)$.

Resolução alternativa.

$$\langle (1,0,3,2),(0,1,0,0) \rangle = 0 \quad \text{e} \quad \{(1,0,3,2),(0,1,0,0),(0,0,1,0),(0,0,0,1)\} \quad \text{é uma base de } \mathbb{R}^4.$$

Atendendo a

$$(0,0,1,0) - \operatorname*{proj}_{(1,0,3,2)}(0,0,1,0) - \operatorname*{proj}_{(0,1,0,0)}(0,0,1,0) = (0,0,1,0) - \frac{3}{14}(1,0,3,2) = \left(-\frac{3}{14}, 0, \frac{5}{14}, -\frac{3}{7}\right),$$

seja $u = (-3,0,5,-6)$. Logo $\{(1,0,3,2),(0,1,0,0),(-3,0,5,-6),v\}$ é uma base ortogonal para $\mathbb{R}^4$ com

$$v \in \mathcal{N}\left(\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 5 & -6 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\right) = L(\{(-2,0,0,1)\}).$$

Assim $\{(1,0,3,2),(0,1,0,0),(-3,0,5,-6),(-2,0,0,1)\}$ é uma base ortogonal para $\mathbb{R}^4$ que inclui o vector $(1,0,3,2)$.

b)

$$d\left((1,0,1,1), U^\perp \cap V^\perp\right) = \left\| P_{\left(U^\perp \cap V^\perp\right)^\perp}(1,0,1,1) \right\| = \|P_{U+V}(1,0,1,1)\| \underset{U \subset V}{=}$$

$$= \|P_V(1,0,1,1)\| = \|(1,0,1,1) - P_{V^\perp}(1,0,1,1)\| = \left\| (1,0,1,1) - \operatorname*{proj}_{(1,0,-1,1)}(1,0,1,1) \right\| =$$

$$= \left\| (1,0,1,1) - \frac{1}{3}(1,0,-1,1) \right\| = \frac{2}{3}\sqrt{6}.$$

6)

$$\langle a_0 + a_1 t, b_0 + b_1 t\rangle = \left([a_0 + a_1 t]_{\mathcal{B}}\right)^T G_{\mathcal{B}} [b_0 + b_1 t]_{\mathcal{B}} =$$

$$= \left([a_0 + a_1 t]_{\mathcal{B}}\right)^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} [b_0 + b_1 t]_{\mathcal{B}} =$$

$$= \left(S_{\mathcal{B}_c \to \mathcal{B}} [a_0 + a_1 t]_{\mathcal{B}_c}\right)^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} S_{\mathcal{B}_c \to \mathcal{B}} [b_0 + b_1 t]_{\mathcal{B}_c} =$$

$$= \left(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}\right)^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} =$$

$$= \begin{bmatrix} a_0 & a_1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} =$$

$$= \begin{bmatrix} a_0 & a_1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

$$\left(L\left(\{2+3t\}\right)\right)^{\perp} = \{b_0 + b_1 t \in \mathcal{P}_1 : \langle 2+3t, b_0 + b_1 t\rangle = 0\} =$$

$$= \left\{ b_0 + b_1 t \in \mathcal{P}_1 : \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = 0 \right\} =$$

$$= \{b_0 + b_1 t \in \mathcal{P}_1 : 11 b_1 - 4 b_0 = 0\} = L\left(\{11 + 4t\}\right).$$

Logo $\{11 + 4t\}$ é uma base ortogonal para $\left(L\left(\{2+3t\}\right)\right)^{\perp}$. Como

$$\|11 + 4t\| = \sqrt{\langle 11 + 4t, 11 + 4t\rangle} = \sqrt{\begin{bmatrix} 11 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 11 \\ 4 \end{bmatrix}} = 5$$

então $\left\{ \frac{11}{5} + \frac{4}{5} t \right\}$ é uma base ortonormada para $\left(L\left(\{2+3t\}\right)\right)^{\perp}$.