Cap pairings and spectral sequences

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Abstract

Let $I$ be a small category. For an $I$-diagram $X$ and $I^{op}$-diagrams $A$ and $B$ of pointed spaces, each pairing $X \land A \to B$ satisfying the projection formula induces a pairing $\text{holim}_i X_i \land \text{holim}_i A_i \to \text{holim}_i B_i$. In this note we show that there is an induced pairing $E^{p,q}_r(X) \otimes E^{s,t}_t(A) \to E^{r-p,q+t}_s(B)$ of homotopy spectral sequences compatible with abutments in the sense that

$$F^p\pi_q \text{holim}_i X_i \land F^q\pi_t \text{holim}_i A_i \to F^p\pi_{q+t} \text{holim}_i B_i.$$

1 Introduction

In this note we follow a suggestion of Pete Bousfield and present a series of results which can be used to confirm an expected property of the higher $K$-groups with compact support that was defined by Gillet and Soulé [7], 5.3 and page 170. Specifically, for a pointed cosimplicial space $X$ and pointed simplicial spaces $A$ and $B$, one readily deduces, from op. cit., Proposition 8, that each pairing

$$X^n \land A_n \xrightarrow{\mu_n} B_n,$$

which satisfies the usual projection formula (5), induces a pairing

$$\text{Tot} X \land \text{Diag} A \xrightarrow{\Delta} \text{Diag} B.$$

For $X$ a fibrant pointed cosimplicial space, consider the homotopy spectral sequence $E^{p,q}_r(X)$, for $r \geq 1$ and $q \geq p \geq 0$, abutting to $\pi_{q-p} \text{Tot} X$ defined by Bousfield and Kan [5], X.6, and for $A$ a termwise connected pointed simplicial

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space, consider the homotopy spectral sequence $E^r_{s,t}(A)$, for $r \geq 1$ and $s, t \geq 0$, abutting to $\pi_{s+t} \text{Diag} A$. The latter spectral sequence is derived from the work of Bousfield and Friedlander [4], B.5. Using the formulation of the cap product pairing of singular cohomology with singular homology (e.g. [16], page 250), one easily constructs for (1) a pairing of normalized chain complexes

$$N^p\pi_q X \otimes N_s \pi_t A \to N_{s-p} \pi_{q+t} B.$$  

We deduce the existence of an induced pairing, at least in positive dimensions, of homotopy spectral sequences

$$E_p^r (X) \otimes E^r_{s,t}(A) \xrightarrow{\mu} E^r_{s-p,q+t}(B)$$

which is compatible with (2), i.e.,

$$F_p \pi_q \text{Tot} X \otimes F_s \pi_t \text{Diag} A \to F_{s-p} \pi_{q+t} \text{Diag} B.$$  

by showing that (2) factors as $\text{Tot} X \wedge \text{Diag} A \to \text{Diag}(X \sharp A) \to \text{Diag} B$ and exhibiting a pairing

$$E_p^r (X) \otimes E^r_{s,t}(A) \to E^r_{s-p,q+t}(X \sharp A).$$

Since Gillet and Soulé formulate higher $K$-theory with compact supports in terms of spectra (see [4], [10]) and we restrict our focus to pointed spaces, we do not actually address their query. Nevertheless, these results provide an affirmative answer. Indeed, let $\text{Sp}^{\Sigma}$ denote the category of symmetric spectra [10], and let $\text{Sp}^\ast$ denote the category of pointed spaces. Using the adjoint functors $F_0 : \text{Sp}^\ast \rightleftarrows \text{Sp}^{\Sigma} : \text{Ev}_0$, where $F_0$ is left adjoint to the ”evaluation at zero” functor, one readily obtains what is needed. Thus for a variety $X$ over a field of characteristic zero and $\bar{X}$ a compactification of $X$, the cap product pairing $K^c(X) \wedge K'(X) \to K'(\bar{X})$ induces a pairing of weighted filtrations

$$F_p K^c_m(X) \otimes F_q K'_n(X) \to F_{q-p} K'_{m+n}(\bar{X}),$$

where $\pi_* K^c(X) = K^c_0(X)$ is the $K$-theory with compact supports of $X$.

A second application of these results can be obtained using the counit

$$\text{map}_* (U, V) \wedge U \to V$$

(e.g. [5], VIII.4.8), for pointed spaces $U$ and $V$.

**Example 1** For a pointed simplicial space $X$ and a pointed space $A$, let

$$\mu : \text{map}_*(X_p, A) \wedge X_p \to X_p \wedge A$$

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denote the pairing given by \( \mu(f, y) = (y, f(y)) \). For \( X \) termwise connected and \( A \) group-like, we obtain a pairing

\[
E^{p,q}_r(\text{map}_*(X, A)) \otimes E^{s,t}_r(X) \to E^{s-p+q+t}_r(X \land A)
\]

which is compatible with the abutments

\[
[\Sigma^{q-p}\text{Diag } X, A] \otimes \pi_{s+t}\text{Diag } X \to \pi_{s-p+q+t}\text{Diag } X \land A.
\]

Using the functor \( F_0 \) of the previous paragraph we obtain

**Example 2** For a ring spectrum \( E \) and a simplicial space \( X \), there is a pairing

\[
\mu : \text{map}_*(X_p, E) \land X_p \land E \to X_p \land E,
\]

as in the previous example. The associated pairing of spectral sequences is compatible with the standard cap pairing

\[
E^{p-q}(\text{Diag } X) \otimes E_{s+t}(\text{Diag } X) \to E_{s-p+q+t}(\text{Diag } X).
\]

Our last application was motivated in part by the work of Blanc and Thompson [1]. For pointed spaces \( M, Y \) and \( n \geq 0 \), let \( \pi_n(Y; M) = [\Sigma^n M, Y] \). Let \( X \) be a pointed simplicial space \( X \) satisfying the \( \pi_*\)-Kan condition [4], B.3., and assume that \( X \) is termwise fibrant. By op.cit. B.5, there is a tower of fibrations

\[
\text{Diag } X \to \cdots \to \text{Diag } P_t X \to \text{Diag } P_{t-1} X \to \cdots \text{Diag } P_{-1} X = *
\]

where \( P_t X \) the simplicial space obtained by the termwise application of the Postnikov sections functor. Thus there is first quadrant spectral sequence

\[
\pi_p \pi_q(X; M) = E^2_{p,q}(X; M) \Rightarrow \pi_{p+q}(\text{Diag } X; M)
\]

which converges if \( M \) is finite. Replacing the zero sphere \( S^0 \) by \( M \) in the formation of the models \( D^{r,t}_{s,t} \) [12] we obtain

**Example 3** The standard composition pairing induces a pairing

\[
[\Sigma^n M, M] \otimes E_{s+t}^r(X; M) \to E^r_{s,q+t}(X; M).
\]

This note is organized as follows: Section 2 contains the definition of the \( \sharp \)-product for pointed cosimplicial and simplicial spaces. Here we show (Theorem 5) that for a pointed cosimplicial space \( X \) and a pointed simplicial space \( A \), the pointed simplicial space \( X \sharp A \) is an initial object of the category of cap pairings, i.e., pairings satisfying the projection formula (5), going out of \( X \) and \( A \). This result is due to Pete Bousfield (personal communication).
In section 3 we discuss homotopy spectral sequences for pointed cosimplicial and simplicial spaces. Using the underlying structure of the standard cap product of singular cohomology with singular homology, we obtain a pairing (Theorem 12) of homotopy cosimplicial and simplicial spectral sequences for pointed spaces. In section 4 we use the results of the previous sections to deduce Theorem 17 which asserts that a pairing between suitable diagrams of pointed spaces induces a cap pairing of the homotopy limit and colimit spectral sequences.

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Throughout this note we use the term “space” to mean “simplicial set,” and we assume that the reader is familiar with model categories as defined by Quillen [13].

2 Cap pairings

Let $\Delta$ denote the skeletal subcategory of finite ordered sets and nondecreasing maps consisting of the objects $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$, and let $c.S_*$ (resp. $s.S_*$) denote the category of pointed cosimplicial (resp. simplicial) spaces. We follow the terminology of Hollender and Vogt [9] and consider pointed cosimplicial spaces as right $\Delta^{\text{op}}$-modules and pointed simplicial spaces as left $\Delta^{\text{op}}$-modules. For $X \in c.S_*$, $A \in s.S_*$ and $[k] \xrightarrow{\theta} [m] \in \Delta^{\text{op}}$, we denote the induced map on simplices of $X$ by $x \mapsto x \cdot \theta$ and the induced map on simplices of $A$ by $a \mapsto \theta \cdot a$ or simply $a \mapsto \theta \cdot a$.

For $X \in c.S_*$ and $A \in s.S_*$, the coend of $X$ and $A$, denoted $X \wedge_{\Delta^{\text{op}}} A \in S_*$, is obtained from the wedge $\bigvee_{n \geq 0} X^n \wedge A_n$ by identifying $(x \cdot \theta \wedge a) \in X^k \wedge A_k$ with $(x \wedge \theta \cdot a) \in X^m \wedge A_m$ for each $\theta$: $[k] \rightarrow [m] \in \Delta^{\text{op}}$. We also use the term balanced smash product to refer to the coend of $X$ and $A$. Using the latching maps $L^n X \rightarrow X^n$, $L_n A \rightarrow A_n$ for $n \geq 0$, where

$$L^n X = \colim_{[n] \xrightarrow{\varphi} [k] \in \Delta^{\text{op}}} X^k \quad L_n A = \colim_{[k] \xrightarrow{[\theta]} [n] \in \Delta^{\text{op}}} A_k$$

with $\theta$ a composition of facial operators and $\varphi$ a composition of degeneracy maps.
operators, one can form inductively push-out squares

\[ (L^nX \land A_n) \coprod_{L^nX \land L_nA}(X^n \land L_nA) \twoheadrightarrow X^n \land A_n \]

\[ F_{n-1}(X \land _{\Delta_{op}} A) \twoheadrightarrow F_n(X \land _{\Delta_{op}} A) \]

starting with \( n = 0 \) and \( F_{-1}(X \land _{\Delta_{op}} A) = * \).

Each latching map \( L_nA \to A_n \) is a cofibration, and \( L^nX \to X^n \) is a cofibration whenever \( n > 1 \) (see \[2\], Lemma 6.4). If the maximal augmentation

\[ a(X) = \{ x \in X^0 \mid x \cdot d_0 = x \cdot d_1 \} \]

is the base point of \( X^0 \), then \( L^1X = X^0 \lor X^0 \to X^1 \) is a cofibration, and we obtain a collection of cofiberings

\[ F_{n-1}(X \land _{\Delta_{op}} A) \twoheadrightarrow F_n(X \land _{\Delta_{op}} A) \to (X^n/L^nX) \land (A_n/L_nA), \] (4)

with \( \operatorname{colim}_n F_n(X \land _{\Delta_{op}} A) = X \land _{\Delta_{op}} A \).

**Remark 4** A basic example of the balanced smash product is \( \Delta_n^* \land _{\Delta_{op}} A \), where \( \Delta_n^* \in S \) is the standard \( n \)- simplex. There is a canonical identification \( \Delta_n^* \land _{\Delta_{op}} A = \operatorname{Diag}A \in S_* \) with \( \operatorname{Diag}A \in A_n \) (see \[4\], Proposition B.1), and \( (4) \) has the familiar form (e.g. \[15\], Section 5)

\[ F_{n-1} \operatorname{Diag}A \twoheadrightarrow F_n \operatorname{Diag}A \to S^n \land (A_n/L_nA). \]

Let \( X \in c.S_* \) and \( A,B \in s.S_* \). Motivated by the work of Gillet and Soulé \[7\], Appendix A.2, we call a collection of pointed maps \( \{ \mu_n \colon X^n \land A_n \to B_n \}_{n \geq 0} \), such that

\[ \theta \cdot \mu_m(x \cdot \theta \land a) = \mu_n(x \land \theta \cdot a) \] (5)

for \( \theta \colon [m] \to [n] \in \Delta_{op}, x \in X^n \) and \( a \in A_m \), a cap pairing of \( X \) and \( A \) to \( B \). As pointed out by Pete Bousfield (personal communication) the class of cap pairings going out of \( X \) and \( A \) factor through a universal target \( X \land A \in s.S_* \), where

\[ (X \land A)_n = (\Delta_n^* \land X) \land _{\Delta_{op}} A \]

and \( \Theta_n \colon X^n \land A_n \to (X \land A)_n \) is the cap pairing determined by

\[ (\{i^n\} \land X^n) \land A_n \to (\Delta_n^* \land X^n) \land A_n \to (X \land A)_n. \]

For simplices \( x \in X^n \) and \( a \in A_n \), let

\[ x \land a = \Theta_n(x \land a) \in (X \land A)_n. \]

**Theorem 5** For \( X \in c.S_* \) and \( A,B \in s.S_* \), the collection of cap pairings of \( X \) and \( A \) to \( B \) is naturally isomorphic to the set \( \operatorname{Hom}_{s.S_*}(X \land A,B) \).
PROOF. For a cap pairing \( \{ \mu_n : X^n \wedge A_n \to B_n \}_{n \geq 0} \), the composition of maps

\[
\Delta^m_n \times X^m \wedge A_m \xrightarrow{\sim} \Delta^m_n \times (X^m \wedge A_m) \xrightarrow{id \otimes \mu_m} \Delta^m_n \times B_m \to B_n
\]
determines a well defined map \( \mu^\sharp : X^\sharp A \to B \in s.S_s \), and a straightforward calculation shows that \( \mu_n = \mu^\sharp \circ \Theta_n \).

Using the natural isomorphisms

\[
\text{Hom}_{s.S_s}(X^\sharp A, B) = \int_{[n] \in \Delta^{op}} \text{Hom}_{s.s_s}(X^\sharp A)_n, B_n)
= \int_{[m],[n] \in \Delta^{op}} \text{Hom}_{s.s_s}(\Delta^m_n \times X^m \wedge A_m, B_n)
\cong \int_{[m],[n] \in \Delta^{op}} \text{Hom}(\Delta^m_n, \text{Hom}_{s.s_s}(X^m \wedge A_m, B_n))
\]
we readily deduce that the function \( \{ \mu_n \}_{n \geq 0} \mapsto \mu^\sharp \) is a bijection. □

Recall from [5], X.3, that the total space of \( X \in c.S \) is given by the mapping space \( \text{Tot} X = \text{map}_{c.S}(\Delta^\bullet, X) \). Now let \( X \in c.S_s \). Using the simplicial adjunction \( \text{maps}_s(K_+, L) = \text{maps}_s(K, L) \), where \( K \in S \) and \( L \in S_s \), we obtain a canonical identification \( \text{Tot} X = \text{map}_{c.S_s}(\Delta^\bullet_+, X) \). Let \( A, B \in s.S_s \), and let \( \mu_n : X^n \wedge A_n \to B_n, n \geq 0 \) be a cap pairing. By the proof of [7], Proposition 7, the composition of maps

\[
\text{maps}_s(\Delta^n_+, X^n) \wedge \Delta^n_+ \wedge A_n \xrightarrow{\epsilon \wedge id} X^n \wedge A_n \xrightarrow{\mu_n} B_n,
\]
where \( \epsilon : \text{maps}_s(\Delta^n_+, X^n) \wedge \Delta^n_+ \to X^n \) is the evaluation map, induces a pairing

\[
\text{Tot} X \wedge \text{Diag} A \xrightarrow{\mu^\sharp} \text{Diag} B.
\]

Combining this result with Theorem 5 yields

**Corollary 6** For a cap pairing \( \mu_n : X^n \wedge A_n \to B_n \), the induced pairing (6) factors as

\[
\begin{array}{ccc}
\text{Tot} X \wedge \text{Diag} A & \xrightarrow{\Theta} & \text{Diag} (X^\sharp A) \\
\mu & \downarrow \mu^\sharp & \downarrow \\
& & \text{Diag} B.
\end{array}
\]

**3 The \( \sharp \)-product and spectral sequences**

For \( X \in c.S_s \), let \( \text{Tot}_n X = \text{map}_{c.S_s}(\text{sk}_n \Delta^\bullet_s, X) \), where \( \text{sk}_n \Delta^\bullet \) denotes the cosimplicial space given in codimension \( s \) by the \( n \)-skeleton of \( \Delta^\bullet \). The analog
of (3) is the sequence of pull-back squares

\[
\begin{align*}
\text{Tot}_n X \ar[r] & \text{maps}_*(\Delta^n_+, X^n) \\
\text{Tot}_{n-1} X \ar[r] & \text{maps}_*(\partial \Delta^n_+, X^n) \times_{\text{maps}_*(\partial \Delta^n_-, M^n)} \text{maps}_*(\Delta^n_+, M^n X)
\end{align*}
\]

obtained using the inclusions \( \partial \Delta^n \to \Delta^n \) and the matching maps \( X^n \to M^n X \)

where

\[
M^n X = \lim_{[k] \to [n] \in \Delta^{op}} X^k
\]

with \( \theta \) ranging over the degeneracy operators. If each matching map is a fibration, it follows that each vertical map of (7) is a fibration, and hence the Tot tower

\[
* \leftarrow \text{Tot}_0 X \leftarrow \cdots \leftarrow \text{Tot}_m X \leftarrow \cdots \leftarrow \lim_n \text{Tot}_n X = \text{Tot} X
\]

is a sequence of fibrations under \( \text{Tot} X \). Let \( \text{Fib}^n X \) denote the fiber of the matching map \( X^n \to M^n X \). The analog of (4) is the fibering

\[
\text{maps}_*(S^n, \text{Fib}^n X) \to \text{Tot}_n X \to \text{Tot}_{n-1} X.
\]

Note that for the Bousfield and Kan model category structure on \( \text{c.S}_* \) [5], \( \text{X.5} \), a pointed cosimplicial space \( X \) is cofibrant precisely when each latching map is a cofibration and fibrant precisely when each matching map is a fibration.

Let \( X \in \text{c.S}_* \) be fibrant. Using the Tot tower Bousfield and Kan op.cit., IX.4, construct the homotopy spectral sequence \( \{ E_1^{s,t}(X) \}_{r \geq 1, t \geq s \geq 0} \) abutting to \( \pi_{t-s} \text{Tot} X \) with

\[
E_1^{s,t}(X) = \pi_{t-s} \text{maps}_*(S^s, \text{Fib}^s X) \cong \pi_t \text{Fib}^s X.
\]

By op.cit., Proposition X.6.3, there is a natural isomorphism

\[
\pi_t \text{Fib}^s X \cong N^s \pi_t X = \{ x \in \pi_t X^s : x \cdot s_i = 0 \text{ for } 0 \leq i \leq s - 1 \}
\]

of groups when \( t > s \geq 0 \) and of pointed sets when \( t = s \geq 0 \).

In positive dimensions the homotopy spectral sequence coincides with the Bousfield and Kan second quadrant homotopy spectral sequence [6] which is constructed from an array of differential relations on \( \{ N^s \pi_t X \}_{t \geq s \geq 0} \) using the universal examples of pointed cosimplicial spaces

\[
D_r^{s,t} = \Sigma^{t-s} \text{sk}_{s+r-1} \Delta^*/\text{sk}_{s-1} \Delta^*, \quad r \geq 1, t \geq s \geq 0,
\]

\[
E_{q,t} = \Sigma^t \text{sk}_q C^*/\Delta^*, \quad q, t \geq 0,
\]

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where $C: S \to S_s$ is the cone functor given by $CK = \text{colim}_{\Delta^r \to K} \Delta^{n+1}$ (see [8], III.5). Specifically, one uses the natural identification $[D_1^{s,t}, X] = N^s\pi_tX$ together with homotopy cofiberings

$$D_1^{s+r,t+r-1} \xrightarrow{i} D_r^{s,t} \xrightarrow{k} D_{r+1}^{s,t} \quad D_1^{s,t} \xrightarrow{i} D_r^{s,t} \xrightarrow{j} D_{r-1}^{s+1,t+1}$$

and defines pointed differential relations $d_r: N^s\pi_tX \to N^{s+r}\pi_{t-r-1}X$ by letting $d_r a = b$ whenever there is a map $f: D_r^{s,t} \to X \in \text{Ho c.S}_s$ with $a = f \circ i$ and $b = f \circ j$. Let $R\text{Tot}: \text{Ho c.S}_s \to \text{Ho S}_s$ denote the total right derived functor of $\text{Tot}$. By [2], Section 4.7, there is an isomorphism $R\text{Tot} E_{q,t} \cong S^t \in \text{Ho S}_s$, and one readily verifies that there is a homotopy cofibering

$$E_{q+1,t-1} \to D_1^{q,t+q} \xrightarrow{i} E_{q,t} \in \text{Ho c.S}_s$$

with $i \in [D_1^{q,t+q}, E_{q,t}] \cong \mathbb{Z}$ a generator cf. [6], page 313. Let $e_q: \pi_t R\text{Tot} X \to N^q\pi_{t+q}X$ denote the pointed relation obtained by the zig-zag of obvious maps

$$\pi_t R\text{Tot} X \xrightarrow{[E_{q,t}, X]} \xrightarrow{i^t} N^q\pi_tX.$$

By [5], Proposition X.6.3 and [6], Theorem 7.1, these relations coincide with those obtained from the Tot tower, and hence, for a fibrant pointed cosimplicial space $X$, one obtains a decreasing filtration $F^s\pi_{t-s}X \xrightarrow{\text{Tot}} X = \pi_{t-s}X \cap \ker e_{s-1}$ as well as an induced homomorphism $F^s\pi_{t-s}X \to E_{\infty}^s(X)$ whose kernel is $F^{s+1}\pi_{t-s}X$. For a detailed discussion of convergence of the homotopy spectral sequence we refer the reader to [3], Section 4.

For a termwise connected pointed simplicial space $A$, there is an analogous homotopy spectral sequence [12] defined using universal pointed simplicial spaces $D^t_s$ and $S^s_t$. For $X, Y \in S_s$, let $X \wedge Y$ denote the pointed simplicial space with $(X \wedge Y)_n = X_n \wedge Y$. For $s, t \geq 0$, let $S^{s,t} = S^s \wedge S^t$, and for $s > 0$, let $D_1^{s,t} = (\Delta^s/V_0^s) \wedge S^t$, where $V_0^s \subset \Delta^s$ is the 0-horn spanned by the faces $d_1 t^s, \ldots, d_s t^s$. We put $D_0^{s,t} = S^0 \wedge S^t$. By [12] Proposition 3.3, there is a natural identification $N_s \pi_t A = [D_1^{s,t}, A]$ and the differential $\partial: N_s \pi_t A \to N_{s-1} \pi_t A$, $t > 0$ corresponds to the map $j: D_1^{s-1,t} \to D_1^{s,t-1}$ induced by the facial operator $d_0$. For $s \geq r > 1$, $D_{s,t}^r$ is assembled in accordance with the familiar zig-zag of the differential $d_r$ of a first quadrant homology spectral sequence and is determined, up to unique isomorphism of $\text{Ho s.S}_s$, by the homotopy cofibering

$$D_1^{s-r+1,t+r-2} \xrightarrow{i} D_{s,t}^{r-1} \to D_{s,t}^r. \quad (8)$$

Here $\text{Ho s.S}_s$ is the homotopy category of pointed simplicial spaces obtained using the Reedy model category structure [14], and

$$N_s \pi_t A = \{ a \in \pi_t A_s \mid d_i \cdot a = 0 \text{ for } 0 < i \leq s \}.$$
Using (8) and the homotopy cofiberings
\[
D_{s,t}^1 \xrightarrow{i} D_{s,t}^r \rightarrow D_{s-1,t+1}^{r-1}, \quad S^{s-r,t+r-1} \rightarrow D_{s,t}^r \xrightarrow{i'} S^{s,t},
\]
one assembles differential relations
\[
d_r : N_s \pi_t A \xrightarrow{\sim} N_{s-r} \pi_{t+r-1} A, \quad e_s : N_s \pi_t A \xrightarrow{\sim} \pi_{s+t} \text{Diag } A \quad (9)
\]
by letting \(d_r a = b\) whenever there is a map \(f : D_{s,t}^r \rightarrow A \in \text{Hos}.S_s\) with \(a = f \circ i\) and \(b = f \circ j\) and letting \(e_s a = x\) whenever there is a map \(g : S^{s,t} \rightarrow A \in \text{Hos}.S_s\) with \(a = g \circ i'\) and \(x = \text{Diag } g \in [\text{Diag } S^{s,t}, \text{Diag } A] = \pi_{s+t} \text{Diag } A\).

These relations satisfy the standard properties as those derived from an exact couple (see [17], XIII.2). For \(A\) termwise connected, we obtain a spectral sequence \(\{E_r^{s,t}(A)\}_{r \geq 1}\) converging strongly to \(\pi_{s+t} \text{Diag } A\) with differentials
\[
d_r : E_r^{s,t}(A) \rightarrow E_r^{s-r,t+r-1}(A)
\]
induced by (9), where
\[
E_r^{s,t}(A) = N_s \pi_t A \cap \ker d_{r-1}/N_s \pi_t A \cap \text{im } d_{r-1}
\]

\[
F_s \pi_{s+t} \text{Diag } A = \text{image } (e_s : N_s \pi_t A \xrightarrow{\sim} \pi_{s+t} \text{Diag } A).
\]

For a cosimplicial simplicial abelian group \(C\), let \((F^n TC, \partial_T)\) denote the filtered total chain complex with
\[
(TC)_n = \bigoplus_{k \geq 0} N^k N_{n+k} C, \quad (F^n TC)_n = \bigoplus_{k \geq m} N^k N_{n+k} C
\]
and \(\partial_T = \partial + (-1)^{n+1} \delta : (TC)_n \rightarrow (TC)_{n-1}\) obtained from the normalized double complex \(N^m N_n C\)
\[
\{c \in C^m_n \mid c \cdot s_i = 0 \text{ and } d_j \cdot c = 0 \text{ for } 0 \leq i \leq m-1 \text{ and } 0 < j \leq n\}
\]
with commuting differentials
\[
N^m N_n C \xrightarrow{\delta} N^{m+1} N_n C \xrightarrow{\partial} N^m N_{n-1} C
\]
given by \(\delta(c) = \sum_{i=0}^{m+1} (-1)^i c \cdot d_i\) and \(\partial(c) = d_0 \cdot c\). The associated spectral sequence
\[
E_1^{s,t}(C) = H_{t-s}(F^s TC/F^{s+1} TC) \cong N^s H_t C \\
d_r : E_r^{s,t}(C) \rightarrow E_r^{s+r,t+r-1}(C)
\]
converges conditionally to $H_{t-s} \hat{T}C$, where $(\hat{T}C)_n = \prod_{k \geq 0} N^k N_{k+n} C$ is the completion of $TC$.

For a bisimplicial abelian group $B$, let $(F_m TB, \partial_T)$ denote the filtered total chain complex with

$$(TB)_n = \bigoplus_{k \geq 0} N_k N_{n-k} B; \quad (F_m TB)_n = \bigoplus_{k \leq m} N_k N_{n-k} B$$

and $\partial_T|_{N_k N_{n-k} B} = \partial^h + (-1)^k \partial^v$ obtained from the normalized double complex $N m N_n B = \{ B \in C_{m,n} \mid d_i^h \cdot b = 0 \text{ and } d_j^v \cdot b = 0 \text{ for } 0 < i \leq m \text{ and } 0 < j \leq n \}$ with commuting differentials

$$N_{m-1} N_n B \xleftarrow{\partial^h} N_m N_n B \xrightarrow{\partial^v} N_{m} N_{n-1} B.$$

given by $\partial^h(b) = d_0^h \cdot b$ and $\partial^v(b) = d_0^v \cdot b$. The associated spectral sequence

$$E_{s,t}^r(B) = H_{s+t}(F_s TB/F_{s-1} TB) \cong N_s H_t B$$

$$d_r: E_{s,t}^r(B) \to E_{s-r,t+r-1}^r(B)$$

converges strongly to $H_{s+t} TB$.

For a cosimplicial simplicial abelian group $C$ and a bisimplicial abelian group $B$, let $C \sharp B$ denote the bisimplicial abelian group with

$$(C \sharp B)_n = (\mathbb{Z} \Delta^* n \otimes C) \otimes_{\Delta^* \text{op}} B.$$

Here $\mathbb{Z} \Delta^* n$ is the cosimplicial simplicial free abelian group generated by simplices of the cosimplicial discrete space $\Delta^* n$. For simplices $c \in C^n$ and $b \in B_n$, let

$$c \sharp b = [1 \cdot \iota^n \otimes c \otimes b] \in (C \sharp B)_n.$$

The bisimplicial abelian group $C \sharp B$ has the obvious universal property with respect to the class of cap pairings $\{ \mu_n: C^n \otimes B_n \to L_n \}_{n \geq 0}$, where $L$ ranges over the collection of bisimplicial abelian groups.

Let

$$C_q^p \otimes B_{s,t} \xrightarrow{\sim} (C \sharp B)_{s-p,q+t}$$

denote the homomorphism determined by sending $c \in C_q^p$ and $b \in B_{s,t}$ to

$$(-1)^{(s-p)q} \sum_{(\mu, \nu)} \text{sign}(\mu, \nu) \cdot d_0 \cdots d_{p-1} (s_{\nu_1} \cdots s_{\nu_1} \cdot c \cdot d_{p+1} \cdots d_{s-\sharp s_{\mu_q}^v} \cdots s_{\mu_1}^v \cdot b),$$

10
where \((\mu, \nu)\) ranges over the set of \((q, t)\)-shuffles. Thus (10) is the composition of the Eilenberg–Zilber and (graded) Alexander–Whitney maps.

Now consider the chain complex \((\hat{T}C \otimes TB, \partial)\) with

\[
(TC \otimes TB)_n = \bigoplus_k (TC)_k \otimes (TB)_{n-k}
\]

\[
\partial|_{(TC)_k \otimes (TB)_{n-k}} = \partial_T \otimes id + (-1)^k id \otimes \partial_T.
\]

Using the familiar simplicial identities (see [11], Definitions 1.1) one readily verifies

\[
d_{b}^i \cdot (c \triangle b) = \begin{cases} 
(-1)^q(c \cdot d_{p+1}) \triangle b, & \text{if } i = 0, \\
(-1)^q c \triangle (d_{p+i}^b \cdot b), & \text{if } 0 < i \leq n,
\end{cases}
\]

and it follows that we obtain induced homomorphisms

\[
N_p N_q C \otimes N_s N_t B \triangle \rightarrow N_{s-p} N_{q+t}(C \sharp B), \quad s \geq p.
\]  

(11)

**Proposition 7** The collection of homomorphisms (11) induces a pairing of spectral sequences

\[
E_{r}^{p,q}(C) \otimes E_{s,t}^r(B) \triangle \rightarrow E_{s-p,q+t}^r(C \sharp B)
\]

satisfying the Leibniz rule

\[
d_r (c \triangle b) = (d_r c \triangle b) + (-1)^q (c \triangle d_r b)
\]

for \(c \in E_{r}^{p,q}(C)\) and \(b \in E_{s,t}^r(B)\).

**PROOF.** Using the familiar properties of the Eilenberg-Zilber and Alexander-Whitney chain homomorphisms (see [11], Propositions 29.8 and 29.9) one readily verifies that the homomorphism \((\hat{T}C) \otimes (TB)_n \triangle \rightarrow T(C \sharp B)_{m+n}\) determined by sending \((c_j) \in (TC)_m\) with \(c_j \in N_j N_{m-j} C\) and \(b_k \in N_k N_{n-k} B\) to

\[
(c_j) \triangle b_k = \sum_{j=0}^k c_j \triangle b_k \in T(T \sharp B)_{m+n}
\]

satisfies \(\partial_T (c \triangle b) = (\partial_T c \triangle b) + (-1)^m (c \triangle \partial b)\). Thus we obtain a chain homomorphism

\[
\hat{T}C \otimes TB \triangle \rightarrow T(C \sharp B)
\]  

(12)

which clearly carries \(F_p \hat{T}C \otimes F_s TB\) to \(F_{s-p} T(C \sharp B)\). The proof of the proposition is now straightforward. \(\square\)

In order to facilitate the proof of the analog of Proposition 7 for pointed spaces, we first pause and take time to point out (cf. [6], Section 9.4) that (12) factors naturally through a pairing

\[
\hat{T}C \otimes TB \triangle \rightarrow T(C \sharp B),
\]  

(13)
where \( C^*_B \subseteq C^*_B \) is the bisimplicial abelian subgroup with \( (C^*_B)_n \) generated by simplices of the form

\[
d_0 \cdots d_{p-1} (c \cdot d_{p+1} \cdots d_{p+n} \leq b) \quad \text{for } c \in C^p \text{ and } b \in B_{p+n}.
\]

**Proposition 8** For a cosimplicial simplicial abelian group \( C \) and a bisimplicial abelian group \( B \), the inclusion (13) induces a pairing of spectral sequences

\[
E^r_{p,q}(C) \otimes E_{s,t}^r(B) \xrightarrow{\sim} E_{s-p,q+t}^r(C \leq B);
\]

moreover, this pairing is compatible with the pairing of Proposition 7.

Recall that \( X \in \text{c.S}_* \) is cofibrant when its maximal augmentation is a point. Clearly, the functor \( \Delta^r \cdot \) is cofibrant, we obtain

\[
\text{from cofibrant cosimplicial spaces. Since each pointed simplicial space is a pointed cofibrant cosimplicial spaces. Since each pointed simplicial space is cofibrant, we obtain}
\]

**Lemma 9** The functor \( \text{c.S}_* \times \text{s.S}_* \xrightarrow{\cdot} \text{s.S}_* \) has a total left derived functor

\[
\text{Ho c.S}_* \times \text{Ho s.S}_* \xrightarrow{\cdot \text{Ho s.S}_*} \text{Ho s.S}_*;
\]

and the canonical map \( X \leq_C A \rightarrow X \leq A \in \text{Ho s.S}_* \) is an isomorphism.

Using the identities \([D^p_{1,q}, X] = N^p \pi_q X, [D^1_{s,t}, A] = N_s \pi_t A\) and the previous lemma, we see that each element \( f \in N_{s-p} \pi_{q+t}(D^p_{1,q} \leq D^1_{s,t})\) induces a pairing

\[
\mu_f \colon N^p \pi_q X \wedge N_s \pi_t A \rightarrow N_{s-p} \pi_{q+t}(X \leq A).
\]

By [6], Section 5.3, \( N^p \pi_q D^p_{1,q} \equiv \mathbb{Z} \) and is generated by the nondegenerate \( q \)-simplex \( \sigma^{p,q} \) of \( \Sigma^{q-p} \Delta^p / \partial \Delta^p \). Similarly, \( N_s \pi_t D^1_{s,t} \equiv \mathbb{Z} \) and is generated by the nondegenerate \( t \)-simplex \( i^{s,t} \) of \( (\Delta^s/V^0_s) \wedge S^t \). Since \( s = (D^1_{s,t})_m, m < s - 1 \) and \( L_n D^1_{s,t} = (D^1_{s,t})_n, s < n \), we have a homotopy cofibring

\[
(\Delta^s \leq D^p_{1,q})_T^{s-1} \wedge S^t \rightarrow (D^p_{1,q} \leq D^1_{s,t})_n \rightarrow (\Delta^s \leq D^p_{1,q})_T^s \wedge S^t,
\]

and we readily deduce that \( (D^p_{1,q} \leq D^1_{s,t})_n \) is \( (q + t - 1) \)-connected. Here we write \( X^n = X^n/L^n X \) for \( X \in \text{c.S}_* \).

For a pointed space \( K \), let \( \tilde{K} \) denote the simplicial free abelian group generated by the non-base simplices of \( K \). Recall that \( \tilde{H}_*(K) = \pi_* \tilde{Z} \) and the Hurewicz homomorphism (e.g. [11], Definition 13.2) is induced by the obvious inclusion \( K \rightarrow \tilde{K} \).

**Lemma 10** For \( X \in \text{c.S}_* \) and \( A \in \text{s.S}_* \), there is a natural isomorphism of bisimplicial abelian groups

\[
\tilde{Z}(X \leq A) \cong \tilde{Z}X \leq \tilde{Z}A.
\]
PROOF. The functor $\tilde{Z}$ is a left adjoint of the forgetful functor. Thus $\tilde{Z}$ preserves colimits. The proof of the lemma follows by induction using suitable latching objects. \(\square\)

Let $X \in \mathbf{c.S.}$ and $A \in \mathbf{s.S.}$. Using the natural isomorphisms $E^p_q(X;\tilde{Z}) \cong N^p \tilde{H}_q(X)$, $E^1_{s,t}(A;\tilde{Z}) \cong N_s \tilde{H}_t(A)$, Lemma 10 and Proposition 7, we obtain a pairing

$$N^p \tilde{H}_q(X) \otimes N_s \tilde{H}_t(A) \xrightarrow{\sim} N_{s-p} \tilde{H}_{q+t}(X \uplus A).$$

By exactness of the normalization functor and the Hurewicz isomorphism theorem there is a unique map

$$f : D^1_{s-p,q+t} \rightarrow D^1_{0} \uplus D^1_{s,t} \in \mathbf{Ho.S.S.}$$

such that $h(f) = [\sigma^{p,q}] \sim [s^{s,t}] \in N_{s-p} \tilde{H}_{q+t}(D^1_{0} \uplus D^1_{s,t})$, and hence we obtain an induced pairing

$$N^p \tilde{\pi}_q X \wedge N_s \tilde{\pi}_t A \xrightarrow{\sim} N_{s-p} \tilde{\pi}_{q+t}(X \uplus A) \quad \text{with } s \geq p \text{ and } q, t > 0 \quad (14)$$

which clearly factors through a pairing

$$N^p \tilde{\pi}_q X \wedge N_s \tilde{\pi}_t A \xrightarrow{\sim} N_{s-p} \tilde{\pi}_{q+t}(X \uplus A). \quad (15)$$

Here $X \uplus A \subseteq X \uplus A$ denotes the pointed simplicial subspace with $(X \uplus A)_n$ consisting of simplices of the form $d_0 \cdots d_{p-1}(x \cdot d_{p+1} \cdots d_{p+n} \uplus a)$, where $x \in X^p, a \in A_{p+n}$ and $p \geq 0$. Note that $s_id_0 \cdots d_{p-1}(x \cdot d_{p+1} \cdots d_{p+n} \uplus a) = d_0 \cdots d_{p-1}(x \cdot d_{p+1} \cdots d_{p+n+1} \uplus s\cdot a)$ for $0 \leq i \leq n-1$, and hence $X \uplus A$ is closed under degeneracy operators.

For $X \in \mathbf{c.S.}$ and $A \in \mathbf{s.S.}$ consider the geometric normalizations $X_N^0 = X^0, X_N^m = X^m/\text{im } d_0 \cup \cdots \cup \text{im } d_{m-1}$ for $m > 0$, and $A_N^0 = A_0$ and $A_N^N = A_n/\text{im } s_0 \cup \cdots \cup s_{n-1}$ for $n > 0$. Using the Alexander-Whitney maps we obtain maps $\text{aw}_i : X^i \wedge A_{n+i} \rightarrow (X \uplus A)_n^N$ with

$$\text{aw}_i(x \wedge a) = d_0 \cdots d_{i-1}(x \cdot d_{i+1} \cdots d_{n+i} \uplus a).$$

Consider the increasing filtration of $(X \uplus A)_n^N$ with

$$F_k(X \uplus A)_n^N = \bigcup_{i=0}^{k} \text{im } (X^i \wedge A_{n+i} \xrightarrow{\text{aw}_i} (X \uplus A)_n^N).$$

Lemma 11 For a pointed cosimplicial space $X$ and a pointed simplicial space $A$, there is a natural cofibering

$$F_{k-1}(X \uplus A)_n^N \rightarrow F_k(X \uplus A)_n^N \rightarrow X_N^k \wedge A_N^{k+n}.$$
The pairings (14) and (15) induce pairings of homotopy spectral sequences

\[ E_{r}^{p,q}(X) \otimes E_{s,t}^{r}(A) \xrightarrow{\sim} E_{r-s+p,q+t}^{r}(X \wedge A) \]

\[ F^{p} \pi_{q} \text{Tot} X \otimes F^{s} \pi_{t} \text{Diag} A \xrightarrow{\sim} F^{s-p} \pi_{q+t} \text{Diag}(X \wedge A) \]

\[ E_{r}^{p,q}(X) \otimes E_{s,t}^{r}(A) \xrightarrow{\sim} E_{r-s+p,q+t}^{r}(X \wedge A) \]

\[ F^{p} \pi_{q} \text{Tot} X \otimes F^{s} \pi_{t} \text{Diag} A \xrightarrow{\sim} F^{s-p} \pi_{q+t} \text{Diag}(X \wedge A) \]

with \( d_{r}(x \wedge a) = (d_{r}x) \wedge a + (-1)^{q-p}x \wedge d_{r}a. \)

**Theorem 12** The pairings (14) and (15) induce pairings of homotopy spectral sequences

**Proof.** The existence of the former pairing follows directly from the existence of the latter pairing, for which it suffices to consider the following two...
cases: \( X = D_{r}^{p,q}, A = D_{s,t}^{r} \) and \( X = E_{p,q-p}, A = S^{s,t} \).

For the first case there are weak equivalences

\[
X_{N}^{k} \simeq \begin{cases} 
S^{q} & \text{if } k = p, \\
\ast & \text{if } p < k < p + r, \\
S^{q+r-1} & \text{if } k = p + r
\end{cases}
\]

\[
A_{N}^{k+n} \simeq \begin{cases} 
S^{t} & \text{if } n = s - k, \\
\ast & \text{if } s - k < n < s - k \\
S^{t+r-1} & \text{if } n = s - k - r.
\end{cases}
\]

Hence

\[
(X \bowtie A)_{N}^{n} \simeq \begin{cases} 
S^{q} \wedge S^{t} & \text{if } n = s - p \\
\ast & \text{if } s - p - r < n < s - p
\end{cases}
\]

and there is a homotopy cofibration sequence

\[
S^{q} \wedge S^{t+r-1} \rightarrow (X \bowtie A)_{s-p-r}^{N} \rightarrow S^{q+r-1} \wedge S^{t}.
\]

Since \( N_{n}\tilde{H}_{a}(X \bowtie A) = \tilde{H}_{a}(X \bowtie A)_{N}^{N} \), we see that the Hurewicz map \( N_{n}\pi_{m}(X \bowtie A) \rightarrow N_{n}\tilde{H}_{m}(X \bowtie A) \) is an isomorphism when \((n,m) = (p-s,q+t)\) and when \((n,m) = (s-p-r,q+t+r-1)\).

We now consider the second case. Using the weak equivalences

\[
X_{N}^{k} \simeq \begin{cases} 
S^{q} & \text{if } k = p, \\
\ast & \text{if } k \neq p
\end{cases}
\]

\[
A_{k+n}^{N} \simeq \begin{cases} 
S^{t} & \text{if } n = s - k, \\
\ast & \text{if } n \neq s - k
\end{cases}
\]

we see that \((X \bowtie A)_{N}^{N} \simeq \ast\) for \( n \neq s - p \) and \((X \bowtie A)_{s-p}^{N} \simeq S^{q} \wedge S^{t}\). The proof of the theorem is an immediate consequence of Proposition 8.

Let \( X \) be a space. For a commutative ring \( R \) and an \( R \)-module \( G \), consider homology with coefficients in \( G \) and cohomology with coefficients in \( R \). Using the formula

\[
(c \bowtie d) \wedge \sigma = c \bowtie (d \wedge \sigma) \in H_{s-p-q}(X;G),
\]

where \( c \in H^{p}(X;R) \), \( d \in H^{q}(X;R) \) and \( \sigma \in H_{s}(X;G) \), we see that \( H_{s}(X;G) \) is a graded \( H^{*}(X;R) \)-module.

**Proposition 13** For \( X, Y \in c.S_{*} \) and \( A \in s.S_{*} \) there is a canonical identification

\[
X \bowtie (Y \bowtie A) = (X \wedge Y) \bowtie A.
\]

**Proof.** The map \((\Delta_{k}^{i} \bowtie X^{j}) \wedge (\Delta_{k}^{j} \bowtie Y^{i}) \wedge A_{i} \rightarrow \Delta_{k}^{i} \bowtie X^{i} \wedge Y^{i} \wedge A_{i} \) determined by \((\theta \wedge x \wedge \phi \wedge y \wedge a) \mapsto (\theta \cdot \phi \wedge x \cdot \phi \wedge y \wedge a)\) induces a canonical
map Φ: \(X \sharp(Y \sharp A) \to (X \wedge Y) \sharp A\). Here \(\theta \cdot \phi\) means the composition of maps \([k] \xrightarrow{\theta} [j]\). An explicit calculation shows that Φ is an isomorphism whenever \(A = (\Delta^n \wedge \Delta^q)_+\). The proof now follows by a direct limit argument.

Bousfield and Kan [6], Theorem 10.4, show that the Alexander–Whitney and Eilenberg–Zilber maps determine a pairing of spectral sequences

\[
E^r_{p,q}(X) \otimes E^s,t_r(Y) \xrightarrow{\sim} E^{p+s,q+t}_r(X \wedge Y).
\]

Using the associative properties of the Alexander–Whitney and Eilenberg–Zilber map (see [11], Propositions 29.8 and 29.9) and Proposition 13, a lengthy computation reveals

\[
(x \smile y) \smile a = (x \smile (y \smile a)), \quad x \in N^p\pi_qX, \quad y \in N^p\pi_qY, \quad a \in N_s\pi_tA,
\]

and we leave to the reader the exercise of proving

**Proposition 14** For \(X, Y \in \text{c} S_*\) fibrant and \(A \in \text{s} S_*\) termwise connected, there is a commutative diagram of induced pairings

\[
\begin{array}{ccc}
E^p,q_r(X) \otimes E^p,q_r(Y) \otimes E^p,q_r(A) & \xrightarrow{\sim} & E^p,q_r(X) \otimes E^p,q_r(A) \\
\downarrow & & \downarrow \\
E^{p+\delta,q+t}(X \wedge Y) \otimes E^p,q_r(A) & \xrightarrow{\sim} & E^{p+\delta,q+t}(X \wedge Y) \otimes E^p,q_r(A).
\end{array}
\]

\section{Homotopy limits, colimits and \(\sharp\)-products}

Let \(I\) be a small category. We continue to follow the terminology of 2.1 and call covariant functors from \(I\) to \(S_*\) left \(I\)-modules and contravariant functors right \(I\)-modules. We denote the categories of left and right \(I\)-modules by \(\text{I-Mod}\) and \(\text{Mod}_I\). For \(X \in \text{Mod}_I\) and \(A \in \text{I-Mod}\), we often write \(X^i = X(i)\) and \(A_i = A(i)\).

For \(i \in I\), let \(B(i \downarrow I)\) denote the classify space of the category of objects under \(i\). Following Bousfield and Kan [5], XI.3.2 and XII.2.1, the homotopy limit of \(X \in \text{Mod}_I\) is the end

\[
\text{holim}_I X = \text{hom}_I(B(- \downarrow I), X) \in S_*
\]

and the homotopy colimit of \(A \in \text{I-Mod}\) is the coend

\[
\text{hocolim}_I A = B(- \downarrow I) \ltimes_I A \in S_*.
\]
Let $X \in \text{Mod}_I$ and $A, B \in \text{IMod}$. As in 2.3 a cap pairing of $X$ and $A$ to $B$ consists of a collection of pointed maps $\{\mu_i: X^i \land A_i \to B_i\}_{i \in I}$ satisfying a projection formula that is analogous to (5). The class of cap pairings going out of $X$ and $A$ factor through a universal target $X \# A \in \text{IMod}$, where

$$(X \# A)_i = \left(\text{Hom}_I(\bullet, i) \ltimes X\right) \land_A A \in S,$$ 

and $\theta_i: X^i \land A_i \to (X \# A)_i$ is the cap pairing determine by $\theta_i(x \land a) = [id_i \land x \land a]$. The proof of the following is similar to that of Theorem 5.

**Theorem 15** For $X \in \text{Mod}_I$ and $A, B \in \text{IMod}$, the collection of cap pairings of $X$ and $A$ to $B$ is naturally isomorphic to the set $\text{Hom}_{\text{IMod}}(X \# A, B)$.

By the proof of [7], Proposition 7, each cap pairing $\{\mu_i: X^i \land A_i \to B_i\}_{i \in I}$ induces a pairing $\mu: \text{holim}_I X \land \text{hocolim}_I A \to \text{hocolim}_I B$. Combining this result with Theorem 15 yields

**Corollary 16** For a cap pairing $\mu_i: X^i \land A_i \to B_i, i \in I$ the induced pairing $\mu$ factors as

$$\text{holim}_I X \land \text{hocolim}_I A \xrightarrow{\Theta} \text{holim}_I (X \# A) \xrightarrow{\mu^i} \text{hocolim}_I B.$$ 

For a small category $I$, consider the functors $\text{IMod} \xrightarrow{\cap^*} \text{c.S}_s, \text{Mod}_I \xrightarrow{\vee^*} \text{s.S}_s$, where

$$\cap^n X = \prod_{i_0 \to \cdots \to i_n} X^{i_0} \quad \text{and} \quad \vee^n A = \bigvee_{i_0 \to \cdots \to i_n} A_{i_0}.$$ 

For $i \in I$, let $N_*(i \downarrow I)$ denote the nerve of the comma category of objects under $i \in I$. Viewing $N_n(i \downarrow I)$ as a *discrete space*, we obtain a right $I$-module $N_*(\downarrow I)_+$. Using the identities

$$\cap^n X = \text{hom}_I(N_n(\downarrow I), X)$$

$$\vee_n A = N_n(\downarrow I) \ltimes_I A$$

$$B(i \downarrow I) = \Delta^* \times_{\Delta^*_{\text{op}}} N_*(i \downarrow I)$$

we obtain natural isomorphisms

$$\text{holim}_I X \cong \text{Tot} \text{hom}_I(N_*(\downarrow I), X)$$

$$\text{hocolim}_I A \cong \text{Diag} (N_*(\downarrow I) \otimes_I A).$$

Let $A \in \text{IMod}$ be termwise connected, and let $X \in \text{c.S}_s$ be termwise fibrant.
By [5], XI.7.1, and 3.3, there are homotopy spectral sequences

\[ E^r_{s,t}(X) = E^r_{s,t}(\text{top}X) \Longrightarrow \pi_{t-s} \text{holim}_I X \]
\[ E^r_{s,t}(A) = E^r_{s,t}(\vee_A) \Longrightarrow \pi_{t+s} \text{hocolim}_I A. \]

One verifies readily that the collection of pointed maps

\[ \text{hom}_I(N_n(- \downarrow I), X) \wedge (N_n(- \downarrow I) \wedge I A) \xrightarrow{\theta_n} N_n(- \downarrow I) \wedge I (X \# A), \quad n \geq 0 \]

determine by \( \theta_n(\phi \wedge (u \wedge a)) = [u \wedge \theta(\phi(u) \wedge a)] \) is a cap pairing. By the universal property of Theorem 15 there is an induced map

\[ (\text{top}X) \# (\vee_A) \to \vee_A (X \# A), \]

and one deduces

**Theorem 17** For \( X \in \text{Mod}_I \) termwise fibrant and \( A \in \text{iMod} \) termwise connected, the pairing (14) induces a pairing of homotopy spectral sequences

\[ E^p_{\pi q}(X) \otimes E^r_{s,t}(A) \xrightarrow{\sim} E^r_{s-p,q+t}(X \# A) \]

\[ F^p_{\pi q} \text{holim}_I X \otimes F^r_{s,\pi t} \text{hocolim}_I A \xrightarrow{\sim} F^r_{s-p,\pi q+t} \text{hocolim}_I (X \# A). \]

We leave to the conscientious reader the formulation of the analog of Proposition 14 for \( X, Y \in \text{Mod}_I \) and \( A \in \text{iMod} \).

**References**


