2016 AMS von Neumann Symposium

Topological Recursion and its Influence in Analysis, Geometry, and Topology

Hilton Charlotte University Place, Charlotte, NC
July 4–8, 2016

Hopf Algebras and Topological Recursion

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Fancy way of defining an algebra

Definition 4.1.1 *An* algebra *over a commutative ring k is a k-module A equipped with k-module maps $\mu^A : A \underset{k}{\otimes} A \to A$, the* multiplication, *and $\imath^A : A \to A$, the* unit, *such that the following diagrams commute:*

$$
\begin{array}{ccc}
A \otimes k & \xrightarrow{\mathrm{id}\otimes\imath} & A \otimes A \\
\cong\downarrow & & \downarrow\mu \\
A & \xrightarrow[\mathrm{id}]{} & A
\end{array}
\qquad
\begin{array}{ccc}
k \otimes A & \xrightarrow{\imath\otimes\mathrm{id}} & A \otimes A \\
\cong\downarrow & & \downarrow\mu \\
A & \xrightarrow[\mathrm{id}]{} & A
\end{array}
$$

$$
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{\mu\otimes\mathrm{id}} & A \otimes A \\
\mathrm{id}\otimes\mu\downarrow & & \downarrow\mu \\
A \otimes A & \xrightarrow[\mu]{} & A
\end{array}
$$

Chari and Presley, *A Guide to Quantum Groups*; Kassel, *Quantum Groups*; Sweedler, *Hopf Algebras*...

Dualizing...

Definition 4.1.2 *A* coalgebra *over a commutative ring k is a k-module A equipped with k-module maps $\Delta^A : A \to A \otimes A$, the* comultiplication, *and $\epsilon^A : A \to k$, the* counit, *such that the following diagrams commute:*

$$\begin{array}{ccc} A\otimes k & \xleftarrow{\mathrm{id}\otimes\epsilon} & A\otimes A \\ \cong\uparrow & & \uparrow\Delta \\ A & \xleftarrow[\mathrm{id}]{} & A \end{array} \qquad \begin{array}{ccc} k\otimes A & \xleftarrow{\epsilon\otimes\mathrm{id}} & A\otimes A \\ \cong\uparrow & & \uparrow\Delta \\ A & \xleftarrow[\mathrm{id}]{} & A \end{array}$$

$$\begin{array}{ccc} A\otimes A\otimes A & \xleftarrow{\Delta\otimes\mathrm{id}} & A\otimes A \\ \mathrm{id}\otimes\Delta\uparrow & & \uparrow\Delta \\ A\otimes A & \xleftarrow[\Delta]{} & A \end{array} .$$

Chari and Presley, *A Guide to Quantum Groups*; Kassel, *Quantum Groups*; Sweedler, *Hopf Algebras*...

A Hopf algebra *over a commutative ring k is a k-module A such that*

(i) A is both an algebra and a coalgebra over k;

(ii) the comultiplication $\Delta : A \to A \otimes A$ and the counit $\epsilon : A \to k$ are homomorphisms of algebras;

(iii) the multiplication $\mu : A \otimes A \to A$ and the unit $\imath : k \to A$ are homomorphisms of coalgebras;

(iv) A is equipped with a bijective k-module map $S^A : A \to A$, called the antipode, *such that the following diagrams commute:*

$$\begin{array}{ccc} A \otimes A & \xrightarrow{S \otimes \mathrm{id}} & A \otimes A \\ {\scriptstyle\Delta}\uparrow & & \downarrow{\scriptstyle\mu} \\ A & \xrightarrow[\imath \circ \epsilon]{} & A \end{array} \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{\mathrm{id} \otimes S} & A \otimes A \\ {\scriptstyle\Delta}\uparrow & & \downarrow{\scriptstyle\mu} \\ A & \xrightarrow[\imath \circ \epsilon]{} & A \end{array}$$

Chari and Presley, *A Guide to Quantum Groups*; Kassel, *Quantum Groups*; Sweedler, *Hopf Algebras*...

Spectral curve $C(x,y)=0$ with branching points α_i such that $dx(\alpha_i)=0$

$$B(p,q) \sim \frac{dpdq}{(p-q)^2} + \text{analytic, near } \alpha_i$$

$K_p(q,\bar{q}) = \frac{1}{2}\frac{1}{(y(q)-y(\bar{q}))dx(q)} \otimes \int_{z=q}^{z=\bar{q}} B(p,z)$ for $q \in U_i$ and $\bar{q}$ the local Galois conjugate of q such that $x(q)=x(\overline{q})$

Topological Recursion:

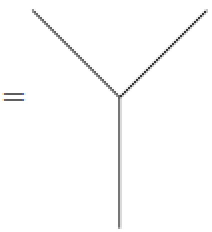
$$W_{k+1}^g(p,K) = \sum_{\text{branch points } \alpha_i} \text{Res}_{p\to\alpha_i} K_p(q,\bar{q}) \Bigg(W_{k+2}^{g-1}(q,\bar{q},K) + \sum_{L\cup M=K} \sum_{\text{stable},h=0}^{g} W_{|L|+1}^h(q,L) W_{|M|+1}^{g-h}(\bar{q},M) \Bigg), K=(p_1,...,p_k)$$

Jean-Louis Loday, Maria O. Ronco, *Hopf Algebra of the Planar Binary Trees*, Advances in Mathematics, Volume 139, Issue 2, 1998, Pages 293-309

S^n is the group of permutations of n elements. A permutation $\sigma \in S^n$ is represented here by its image. For instance (1234) is the identity in S^4

Y^n is the set of planar binary trees of order n. $\#Y^n = C_n$ the n^{th} order Catalan number

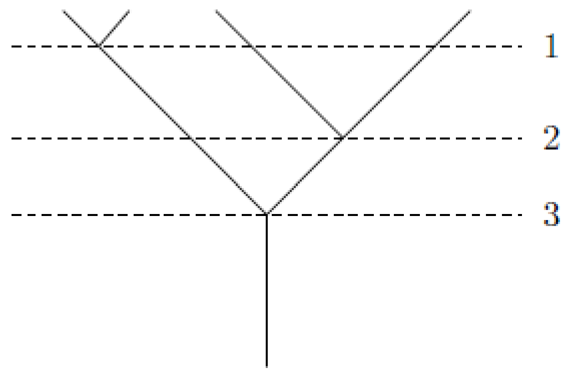
$1 =$

$(1) =$ 

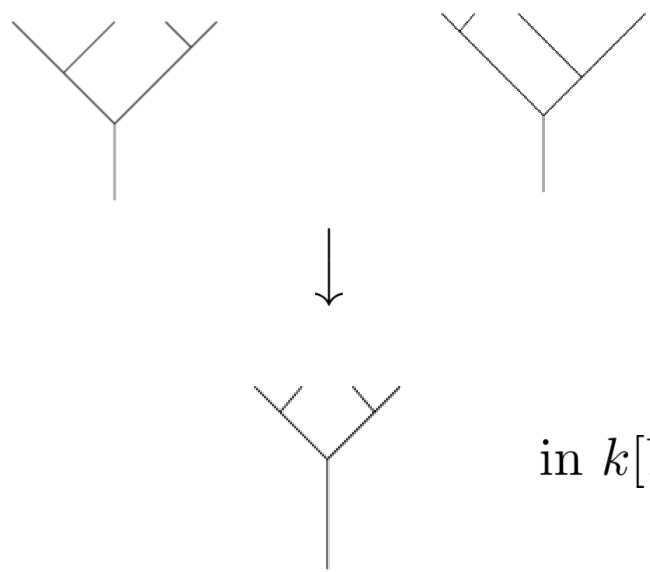
$k[Y^n]$ is the vector space generated by Y^n over a field k of char. 0

$k[Y^\infty] = \bigoplus_n k[Y^n]$ is a Hopf Algebra, the Loday-Ronco Hopf Algebra of planar binary trees

There is a one-to-one correspondence between planar binary trees with levels and permutations: the levels are read from left to right



Planar binary tree with levels that is the image of (**132**)



in $k[Y^\infty]$

Hopf algebra structure of Loday-Ronco: $k[Y^\infty]$ is a graded connected Hopf Algebra

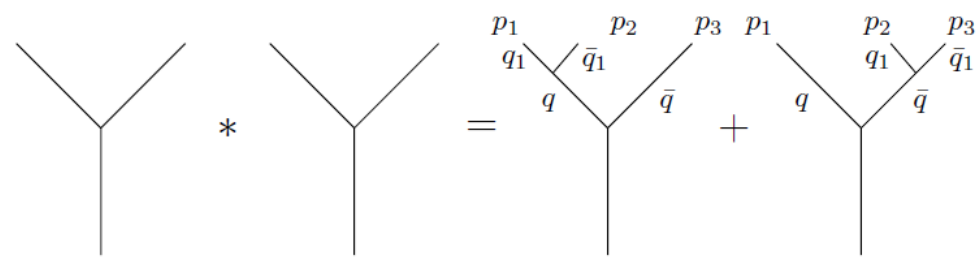
- grafting operation $\vee$: take two trees $t \in Y^p, t' \in Y^q$ and attach their two roots to the left and right branches of $(\mathbf{1})$ to produce a new tree $t \vee t' \in Y^{p+q+1}$. Every tree admits such a decomposition that is unique

- product: take $t = t_1 \vee t_2 \in Y^p$ and $t' = t_1^{'} \vee t_2^{'} \in Y^q$

$$t * t' = t_1 \vee (t_2 * t') + (t * t_1^{'}) \vee t_2^{'}$$

example:

$$\begin{aligned}(\mathbf{1}) * (\mathbf{1}) &= (| \vee |) * (| \vee |) \\ &= | \vee (| * (\mathbf{1})) + ((\mathbf{1}) * |) \vee | \\ &= | \vee (\mathbf{1}) + (\mathbf{1}) \vee | \\ &= (\mathbf{21}) + (\mathbf{12})\end{aligned}$$

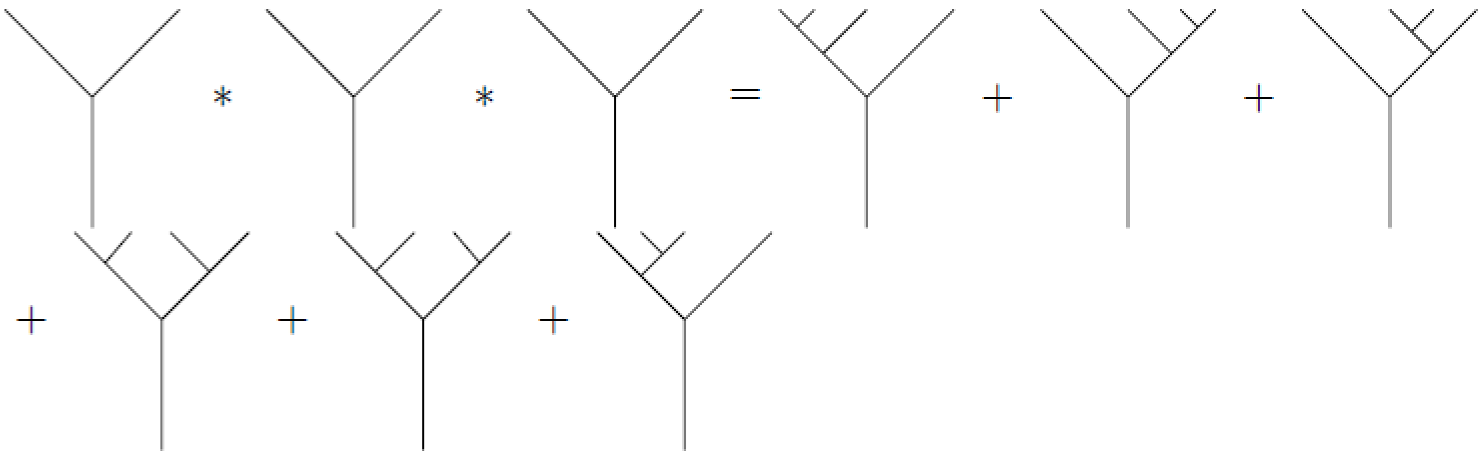


- co-product: take $t = t_1 \vee t_2, t_1 \in Y^p, t_2 \in Y^q$

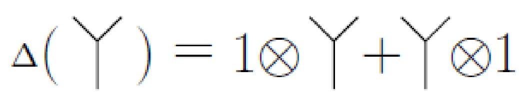
$$\Delta(t) = \sum_{j,k} \Big((t_1^{'})_j * (t_2^{'})_k\Big) \otimes \Big((t_1^{''})_{p-j} \vee (t_2^{''})_{q-k}\Big) + t \otimes |$$

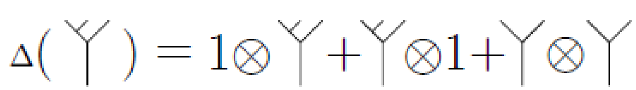
with Sweedler notation $\Delta(t_1) = \sum_j (t_1^{'})_j \otimes (t_1^{''})_{p-j}$ and the initial condition $\Delta(|) = | \otimes |$

Examples:

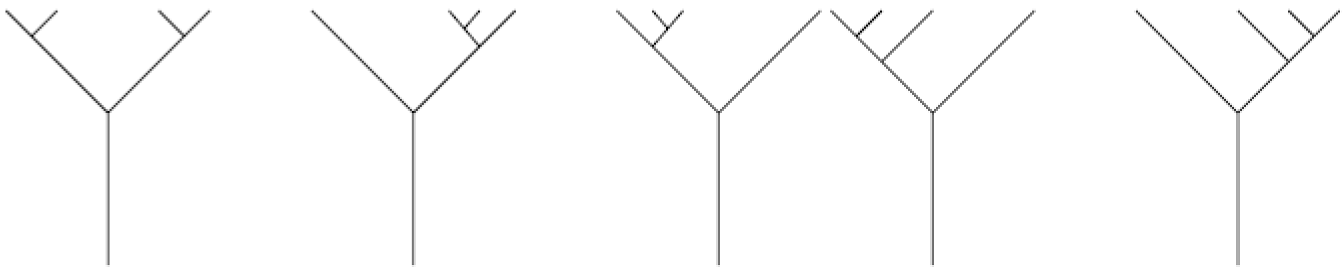


$(\mathbf{1}) * (\mathbf{1}) * (\mathbf{1}) = (\mathbf{123}) + (\mathbf{321}) + (\mathbf{312}) + (\mathbf{132}) + (\mathbf{231}) + (\mathbf{213})$ computed in $k[S^\infty]$. Note that in $k[Y^\infty]$ the fourth and the fifth trees are the same.

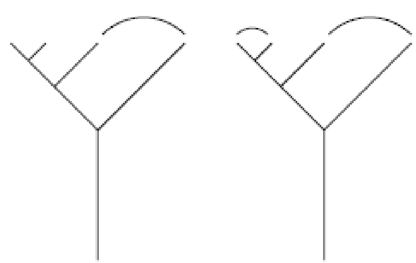




JNE, *Hopf Algebras and Topological Recursion*, Journal of Physics A: Mathematical and Theoretical, Volume 48, Number 44, arXiv:1503.02993v3 [math-ph]

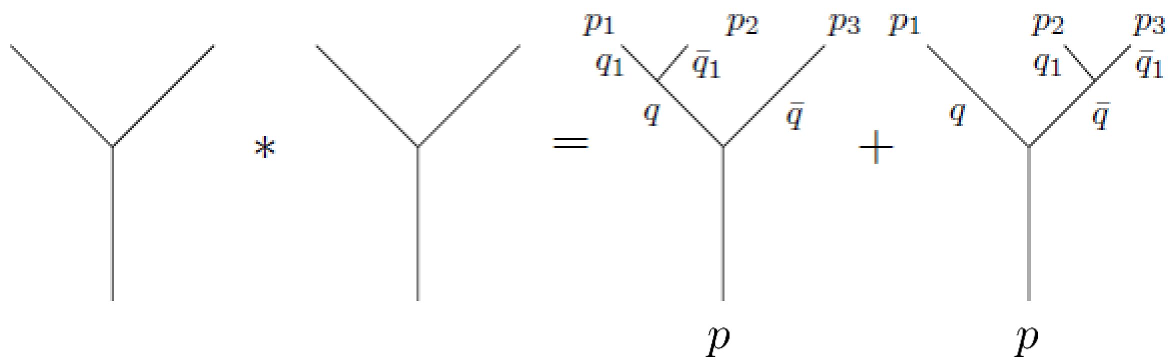


Planar binary trees of order 3 as generators of the correlation functions W_5^0, W_3^1 and W_1^2 with $\chi = -3$.



Example:

$$W_4^0(p, p_1, p_2, p_3)$$



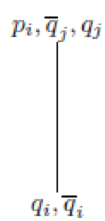
$$(\mathbf{1}) * (\mathbf{1}) = (\mathbf{12}) + (\mathbf{21})$$

$$\begin{aligned}
W_4^0(p, p_1, p_2, p_3) &= K_p(q, \bar{q}) \big(W_3^0(q, p_1, p_2) W_2^0(\bar{q}, p_3) \\
&+ W_2^0(q, p_1) W_3^0(\bar{q}, p_2, p_3) + \text{perm. of } \{p_1, p_2, p_3\} \big) \\
&= K_p(q, \bar{q}) K_q(q_1, \bar{q}_1) W_2^0(q_1, p_1) W_2^0(\bar{q}_1, p_2) W_2^0(\bar{q}, p_3) \\
&+ K_p(q, \bar{q}) K_{\bar{q}}(q_1, \bar{q}_1) W_2^0(q_1, p_2) W_2^0(\bar{q}_1, p_3) W_2^0(q, p_1) \\
&+ \text{perm. of } \{p_1, p_2, p_3\}
\end{aligned}$$

Definition. The propagator or cylinder (also called Bergman kernel in the literature) $W_2^0(q,\bar{q})$ is represented through ψ by the empty permutation $|$ and the recursion kernel is represented through ψ by $\curlyvee$ when in an internal vertex of some tree. Then each planar binary tree of order n is a representation of an instance of some correlation function in genus 0 with each vertex identified with a recursion kernel and each left leaf identified with the cylinder $W_2^0(q_i,p_j)$ or each right leaf identified with the cylinder $W_2^0(\bar{q}_i,p_k)$. Finally the image under ψ of a correlation function $W_{n+2}^0(p,p_1,\ldots,p_{n+1})$ with $\chi=-n$ is the sum of all planar binary trees of order n considering all permutations of their leaf labels and with the identifications mentioned above,

$$\psi\left(W_{n+2}^0(p,p_1,\ldots,p_{n+1})\right)=\sum_{\substack{t_i\in Y^n\\ \text{perm. of leaf labels } \{p_1,\ldots,p_{n+1}\}}} t_i.$$

$$W_2^0(q_i,\overline{q}_j) \qquad W_2^0(q_i,p_j) \quad \xrightarrow{\psi}$$

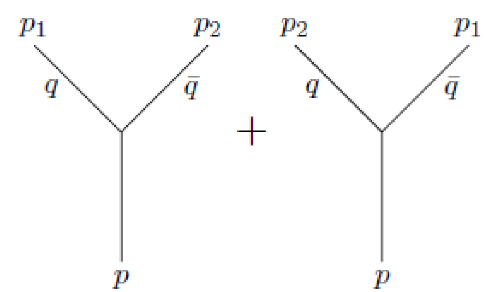


$$K_p(q_j,\overline{q}_j) \qquad K_{q_i}(q_j,\overline{q}_j) \quad \xrightarrow{\psi}$$

Example. *Consider the planar binary tree with one vertex. The 3-point correlation function $W_3^0(p, p_1, p_2)$ is represented by the sum of two planar binary trees with one vertex, obtained by the permutation of the leaf labels p_1 and p_2.*

$$\begin{aligned}\psi\left(W_3^0(p,p_1,p_2)\right) &= \psi\left(K_p(q,\bar{q})W_2^0(q,p_1)W_2^0(\bar{q},p_2)\right) + \text{ perm. of } \{p_1,p_2\} \\ &= \sum_{\text{perm. of }\{p_1,p_2\}} |\vee| \\ &= \sum_{\text{perm. of }\{p_1,p_2\}} (\mathbf{1})\end{aligned}$$

$$\psi(W_3^0(p,p_1,p_2)) =$$



Proposition. *If* $W^0_{n+2}(p, p_1, \ldots, p_{n+1})$ *is a correlation function with Euler characteristic* $\chi = -n$ *that is a solution of the Top. Rec. formula then we have*

$$\psi\left(W^0_{n+2}(p, p_1, \ldots, p_{n+1})\right) = \sum_{\substack{p+q+1=n \\ |t_1|=p, |t_2|=q}} t_1 \vee t_2$$

$$+ \text{ perm. of leaf labels } \{p_1, \ldots, p_{n+1}\}$$

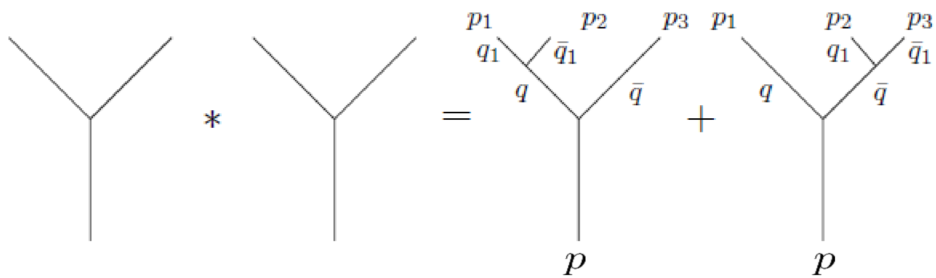
Theorem. *The* n*-order solution* $W^0_{n+2}(p_1, \ldots, p_{n+1})$ *of the topological recursion in genus 0 is represented by the linear combination*

$$\sum t = (\mathbf{1}) * (\mathbf{1}) * \cdots * (\mathbf{1})$$

with n *factors of* $(\mathbf{1})$ *followed by the sum over all permutations of its labels.*

Idea of the proof: by induction on the order n of $\chi = -n$ of $W^0_{n+2}(p, p_1, \ldots, p_{n+1})$

- $n = 2 : \psi\left(W^0_4(p, p_1, p_2, p_3)\right) = (\mathbf{1}) * (\mathbf{1}) = (\mathbf{12}) + (\mathbf{21})$



$$W^0_4(p, p_1, p_2, p_3) = K_p(q, \bar{q}) \left(W^0_3(q, p_1, p_2) W^0_2(\bar{q}, p_3) + W^0_2(q, p_1) W^0_3(\bar{q}, p_2, p_3) + \text{perm. of } \{p_1, p_2, p_3\}\right)$$

- assume for $n-1 : \psi\left(W^0_{n+1}(p, p_1, \ldots, p_n)\right) = \sum_{t \in Y^{n-1}} t = (\mathbf{1}) * (\mathbf{1}) * \cdots * (\mathbf{1})$ with $n-1$ factors

$$\sum_{t \in Y^{n-1}} (\mathbf{1}) * t = \sum_{t \in Y^{n-1}} | \vee (| * t) + \sum_{\substack{t_1 \in Y^a, t_2 \in Y^b \\ a+b+1 = n-1}} ((\mathbf{1}) * t_1) \vee t_2 = \sum_{t \in Y^{n-1}} | \vee t + \sum_{\substack{t_1 \in Y^a, t_2 \in Y^b \\ a+b+1 = n-1}} ((\mathbf{1}) * t_1) \vee t_2$$

$$\psi^* \left(\sum_{t \in Y^{n-1} \text{perm. of leaf labels}} (\mathbf{1}) * t \right) =$$

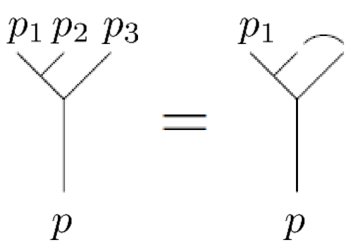
$$\sum_{L \cup M = K, |L| = 1} K_p(q, \bar{q}) W^0_2(q, L) W^0_{n+1}(\bar{q}, M) + \sum_{L \cup M = K, |L| > 1} K_p(q, \bar{q}) W^0_{l+1}(q, L) W^0_{m+1}(\bar{q}, M)$$

$$= \sum_{L \cup M = K} K_p(q, \bar{q}) W^0_{l+1}(q, L) W^0_{m+1}(\bar{q}, M)$$

$$= W^0_{n+2}(p, p_1, \ldots, p_{n+1})$$

Definition. *Starting with a planar binary tree of order n and $n+2$ labels (including the root label p) the operation ${}_i\leftrightarrow_{i+1}$ consists in erasing the labels of the leaves i and $i+1$ then connecting them by an edge and finally relabeling the remaining leaves, now numbered j with $j=0,\ldots,n-2$, with the p_{j+1} labels, producing in this way a graph with one loop.*

${}_1\leftrightarrow_2$



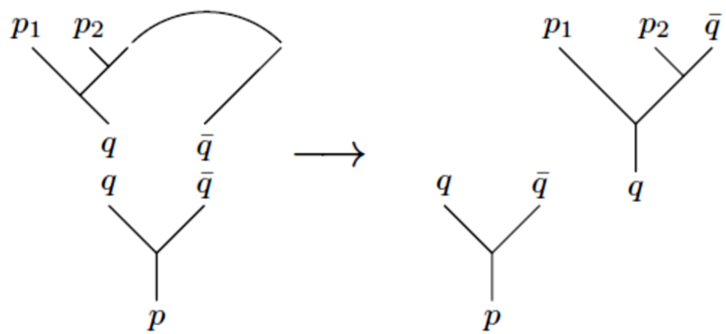
$$\leftrightarrow_{\text{same branches}} (t_1 \vee t_2) = (t_1)^1 \vee t_2 \text{ or } t_1 \vee (t_2)^1$$

$$\leftrightarrow_{\text{opp. branches}} (t_1 \vee t_2) = t_1 \mathbin{\widehat{\vee}} t_2$$

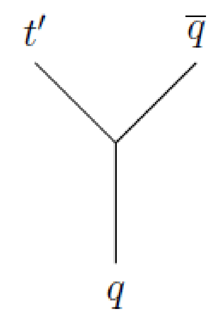
Definition. *A correlation function $W_k^g(p,p_1,\ldots,p_{k-1})$ of genus g and Euler characteristic $\chi = 2-2g-k$ is represented by a sum of all different graphs with loops $t^g \in (Y^n)^g$ for $n=-\chi$:*

$$\psi\left(W_k^g(p,p_1,\ldots,p_{k-1})\right) = \sum_{t^g \in (Y^n)^g} t^g$$

Definition. The ungrafting operation $\forall$ is defined by removing from t the tree $(\mathbf{1})$ that contains the root producing a forest with two trees t_1 and t_2. When applied to a graph t^g it can produce a forest $t^{g_1}, t^{g_2}, g_1 + g_2 = g$ or a graph t^{g-1}. When t represents an instance of a correlation function then the roots of t_1 and t_2 are labeled by q and $\bar{q}$ and as before the tree $(\mathbf{1})$ represents $K_p(q, \bar{q})$.



$$\leftrightarrow_{opp.branches} (t_1 \vee t_2) = t_1 \widehat{\vee}\ t_2 \Longrightarrow \forall (t_1 \widehat{\vee}\ t_2) =$$



This will take care of the first term in Top. Rec. for $W_n^1(p, p_1, \ldots, p_{n-1})$:

$$W_n^1(p, p_1, \ldots, p_n) = \sum_\alpha \mathrm{Res}_\alpha K_p(q, \overline{q}) W_{n+1}^0(q, \overline{q}, p_1, \ldots, p_{n-1}) + \ldots$$

$g = 2:$

$$\sum_{(t)^2 \in (Y^n)^2} (t)^2 = \sum_{\substack{(t_1)^2 \in (Y^p)^2, t_2 \in Y^q \\ p+q+1=n}} (t_1)^2 \vee t_2 + \sum_{\substack{(t_1)^1 \in (Y^p)^1, (t_2)^1 \in (Y^q)^1 \\ p+q+1=n}} (t_1)^1 \vee (t_2)^1$$

$$+ \sum_{\substack{t_1 \in Y^p, (t_2)^2 \in (Y^q)^2 \\ p+q+1=n}} t_1 \vee (t_2)^2 + \sum_{t_1 \in (Y^p)^1, t_2 \in Y^q} (t_1)^1 \widehat{\vee}\, t_2 + \sum_{t_1 \in Y^p, t_2 \in (Y^q)^1} t_1 \widehat{\vee}\, (t_2)^1$$

g arbitrary:

$$\sum_{(t)^g \in (Y^n)^g} (t)^g = \sum_{k=0}^{g} \sum_{\substack{(t_1)^k \in (Y^p)^k, (t_2)^{g-k} \in (Y^q)^{g-k} \\ p+q+1=n}} \big((t_1)^k \vee (t_2)^{g-k}\big)$$

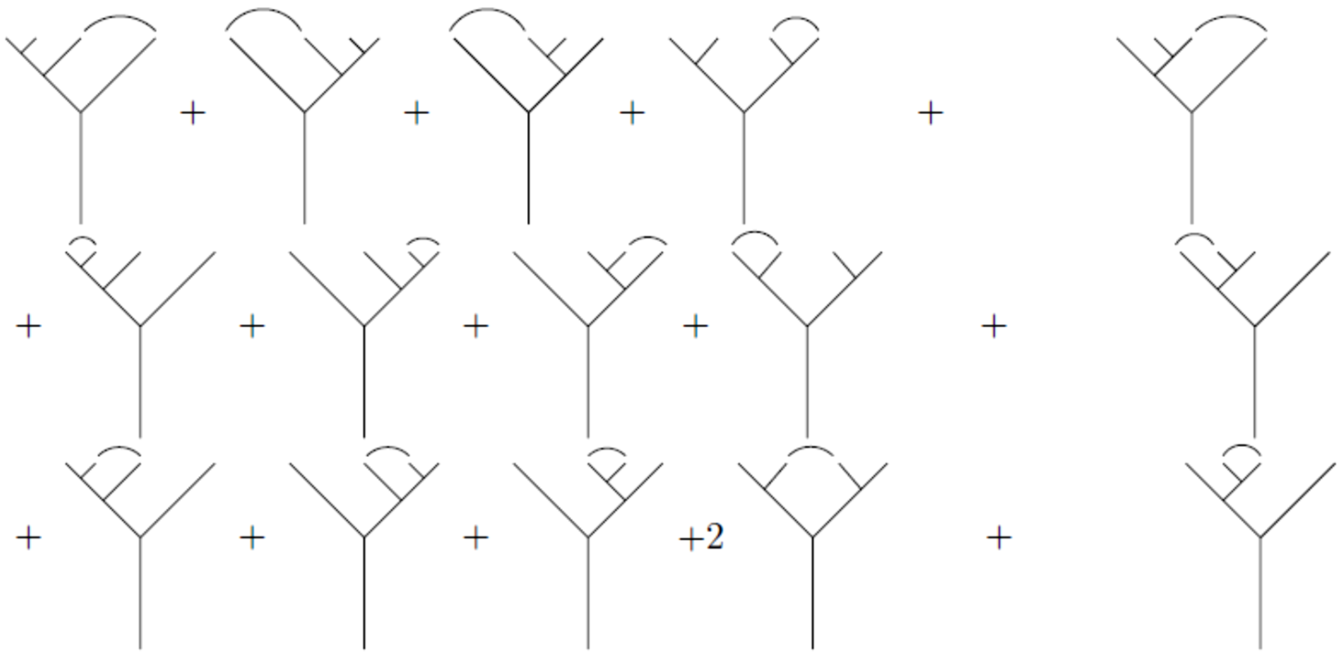
$$+ \sum_{k=0}^{g-1} \sum_{\substack{(t_1)^k \in (Y^p)^k, (t_2)^{g-k} \in (Y^q)^{g-k} \\ p+q+1=n}} \big((t_1)^{g-1-k} \widehat{\vee}\, (t_2)^k\big)$$

Theorem. *The n order solution W^g_{2-2g+n} of topological recursion in genus $g > 0$ and $k = 2-2g+n > 0$ variables is given by $(\mathbf{1}) * (\mathbf{1}) * \cdots * (\mathbf{1})$, with n factors, followed by the identification of pairs of nearest neighbor leaves producing graphs with loops and finally by summing over all permutations of $p_1, p_2, \ldots, p_{1-2g+n}$.*

$$W^g_k(p, p_1, \ldots, p_{k-1}) = \psi^* \left(\sum_{\substack{(t)^g \in (Y^n)^g \\ \text{perm. of leaf labels } K=\{p_1,\ldots,p_{k-1}\}}} (t)^g \right) =$$

$$\sum_{h=0}^{g} \sum_{\substack{L \cup M = K \\ \text{perm. of } K}} K_p(q, \bar{q}) \Big(W^{g-1}_{k+1}(q, \bar{q}, K) + W^h_{l+1}(q, L) W^{g-h}_{m+1}(\bar{q}, M) \Big)$$

Example: $W_3^1(p, p_1, p_2)$



$+$ perm. of $\{p_1, p_2\}$