On the Maslov correction in Quantum Mechanics from a geometric point of view

João N. Esteves, CAMGSD, IST

Harmonic Oscilator à la Physicist

Potencial: $V(x) = \frac{1}{2}kx^2$ Newton's law: $m\ddot{x} = -\frac{dV(x)}{dx} = -kx$

Oscilatory motion with frequency $\omega = \sqrt{\frac{k}{m}}$

Hamiltonian: $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$

Quantization (Schrödinger picture):

$p, x \to P, X$ Hermitian operators in some Hilbert space with $[X, P] = i\hbar$

$\psi(x)$ a L^2 eigenfunction of X: $X\psi(x,t) = x\psi(x,t) \Longrightarrow P = -i\hbar\frac{\partial}{\partial x}$

Schrödinger equation: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + \frac{1}{2}m\omega^2 x^2 = i\hbar\frac{\partial}{\partial t}\psi(x,t)$

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Matricial or Heisenberg picture

Normalized operators: $[\hat{X},\hat{P}] = i$

$$\hat{X} = \sqrt{\frac{mw}{\hbar}}X \qquad H = \hbar\omega\hat{H}$$
$$\hat{P} = \frac{1}{\sqrt{mw\hbar}}P \qquad \hat{H} = \frac{1}{2}\left(\hat{X}^2 + \hat{P}^2\right)$$

Creation and annihilation operators:

$$a = \frac{1}{\sqrt{2}}\left(\hat{X} + i\hat{P}\right)$$
$$a^\dagger = \frac{1}{\sqrt{2}}\left(\hat{X} - i\hat{P}\right) \qquad [a, a^\dagger] = 1$$

From $\hat{X} = \frac{1}{\sqrt{2}}(a + a^\dagger)$ and $\hat{P} = \frac{i}{\sqrt{2}}(a^\dagger - a)$ one gets

$$\hat{H} = N + \tfrac{1}{2} \text{ with } N = a^\dagger a$$

$$N\left|\nu\right\rangle = \nu\left|\nu\right\rangle \Leftrightarrow \hat{H}\left|\nu\right\rangle = (\nu + \tfrac{1}{2})\left|\nu\right\rangle$$

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$$\|a\,|\nu\rangle\|^2 \geq 0 \Leftrightarrow \left\langle \nu \,\middle|\, a^\dagger a \,\middle|\, \nu \right\rangle = \nu \langle \nu \,|\, \nu \rangle \geq 0 \Longrightarrow \nu \geq 0 \text{ and } a\,|\nu\rangle = 0 \text{ iff } \nu = 0$$

I

$$[N, a] = -a \Longrightarrow N\,(a\,|\nu\rangle) = (\nu - 1)a\,|\nu\rangle$$

II

In the same way $\|a^\dagger\,|\nu\rangle\,\|^2 = (\nu + 1)\,\langle\nu|\nu\rangle$

From I and II is can be seen that $\nu \in \mathbb{N} \cup \{0\}$

This means that the energy spectrum is discrete and with a nonzero ground state:

$$H = \hbar\omega(\nu + \tfrac{1}{2}),\ \nu = 0, 1, 2, \ldots$$

Schrödinger Equation

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \qquad\qquad H\psi = \frac{i\partial\psi}{\partial t}$$

The wave function ψ belongs to a suitable Hilbert space. $|\psi(x)|^2$ has the meaning of a probability density

Stationary solutions (probability independent of time):

$\psi(x,t) = \phi(x)e^{i\omega t} \Longrightarrow$ time independent Schrödinger equation:

$$H\phi(x) = E\phi(x), \text{ with } E = \hbar\omega$$

Schrödinger Equation

First ansatz for ϕ: $\phi(x) = e^{ikx}$ (momentum eigenfunction). If $V(x)$ is constant,

$$H\phi(x) = E\phi(x) \Longrightarrow p^2 = (\hbar k)^2 = 2m(E - V)$$

Not square integrable and gives a state with total indetermination in x

For nonconstant V try solutions of the type $\phi(x) = e^{iS(x)/\hbar}$ for some smooth function S (phase function).

$$(H - E)\,\phi(x) = \left(\frac{S'(x)^2}{2m} + V - E - \frac{i\hbar}{2m} S''(x)\right) e^{iS(x)/\hbar}$$

Hamilton-Jacobi: $H\phi(x) = E\phi(x) \Longrightarrow H(x, S'(x)) = \frac{S'(x)^2}{2m} + V(x)) = E$

$$\Leftrightarrow S'(x) = \pm\sqrt{2m(E - V(x))}$$

See dS as a section of $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ so that $p = S'(x)$. Then S satisfies HJ iff graph$(dS) \subset H^{-1}(E)$

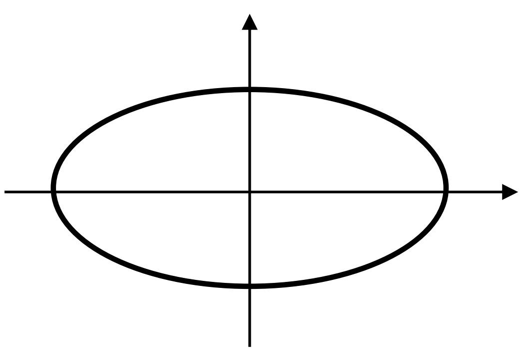
Schrödinger Equation

Set $L = $graph$(dS)$. It has 3 properties:

1. L is an n-dimensional submanifold of $H^{-1}(E)$
2. The pull-back to L of the form $\alpha = \sum_j p_j dq_j$ is exact
3. The restriciton of $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ to L is a diffeo. (L projectable)

L is a projectable Lagrangian manifold

A generalization is immediatly needed: Harmonic oscillator



L not projectable nor exact

The WKB aproximation

Ansatz $\phi(x) = e^{iS(x)/\hbar}$ is too restrictive: $P(x) = 1, \forall x$

Try $\phi(x) = a(x)e^{iS(x)/\hbar}$, with $a(x)$ smooth

$$\Downarrow \quad S \text{ solution of HJ}$$

In n dim...

$$(H - E)\phi(x) = -\frac{1}{2m}\left(i\hbar\left(a\Delta S + 2\sum_j \frac{\partial a}{\partial x_j}\frac{\partial S}{\partial x_j}\right) + \hbar^2\Delta a\right)e^{iS(x)/\hbar}$$

Homogeneous transport equation:

$$a\Delta S + 2\sum_j \frac{\partial a}{\partial x_j}\frac{\partial S}{\partial x_j} = 0$$

(in 1 dim. $aS^{''} + 2a'S' = 0 \Longrightarrow a = \frac{c}{\sqrt{S'}}$, for some $c \in \mathbb{R}$)

Geometry of the transport equation

Next order aproximation in $\hbar$: $\phi(x) = (a_0(x) + a_1(x)\hbar)\, e^{iS(x)/\hbar}$

Inhomogeneous transport equation

$$a_1 \Delta S + 2\sum_j \frac{\partial a_1}{\partial x_j}\frac{\partial S}{\partial x_j} = i\Delta a_0$$

$$a\Delta S + 2\sum_j \frac{\partial a}{\partial x_j}\frac{\partial S}{\partial x_j} = 0 \Longrightarrow \sum_j \frac{\partial}{\partial x_j}\left(a^2 \frac{\partial S}{\partial x_j}\right) = 0$$

$$H(q,p) = \sum_i p_i^2/(2m) + V(q_i) \Rightarrow X_H = \sum_j \left(\frac{p}{m}\frac{\partial}{\partial x_j} - \frac{\partial V}{\partial x_j}\frac{\partial}{\partial p_j}\right)$$

$$\imath : L \hookrightarrow T^*M \text{ gives } X_H|_L = \sum_j \left(\frac{1}{m}\frac{\partial S}{\partial x_j}\frac{\partial}{\partial x_j} - \frac{\partial V}{\partial x_j}\frac{\partial}{\partial p_j}\right)$$

$\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$, $\pi_*\left(X_H|_L\right) = \sum_j \frac{1}{m}\frac{\partial S}{\partial x_j}\frac{\partial}{\partial x_j} \Rightarrow a^2(x)\pi_*\left(X_H|_L\right)$ is divergence free. Can be reformulated as $\mathcal{L}_{\pi_\star(X_H|_L)}(a^2\left|dx^1 \wedge \cdots \wedge dx^n\right|) = 0$

Semi-classical approximation

L projectable $\Leftrightarrow \pi \circ \imath : L \to \mathbb{R}^n$ is a diffeo. Pullback this condition to L to get $(\pi \circ \imath)^* \mathcal{L}_{(\pi_\star(X_{H|L}))}(a(x)|dx|^{1/2}) = 0$. In conclusion:

As a first attempt to obtain a semi-classical approximation $I_\hbar$ to the Schrödinger equation, we build this approximation giving an exact projectable Lagrangian manifold L and a half density $a(x)|dx|^{1/2}$ on L such that $\mathcal{L}_{X_{H|L}} a(x)|dx|^{1/2} = 0$. Then $I_\hbar(q) = ((\pi \circ \imath)^{-1})^* \left(a(x)e^{iS/\hbar}|dx|^{1/2}\right)$ where S parametrizes L through $L = \mathrm{graph}(dS)$

<u>Generalization to non-exact L</u>

Let $\imath : L \hookrightarrow T^*M$ be an immersion of a Lagrangian manifold L not necessarily exact (L may not be given globally as $\mathrm{graph}(dS)$)

Cover L with opens sets L_i. Since L is Lagrangian, $\imath^*\alpha$ is closed. Then

$$\imath^*\alpha|_{L_i} = d\phi_i \text{ by Poincaré}$$

Phase functions e^{ϕ_i} must agree on intersections $\Longrightarrow \phi_i - \phi_j = 2\pi i n\hbar, n \in \mathbb{Z}$ on $L_i \cap Ł_j$

Generalization to non-projectable L

$$\Longrightarrow \quad \alpha \in H^1(M, \mathbb{Z}_\hbar), Z_\hbar = 2\pi\hbar\mathbb{Z} \quad \text{(Čech-de Rham)}$$

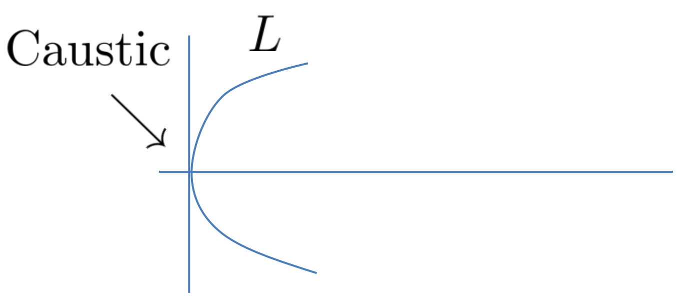
Example: $M = S^1, T^*M = S^1 \times \mathbb{R}$ and consider $\alpha = pd\theta$

$$\int pd\theta = 2\pi n\hbar, n \in \mathbb{Z} \Longrightarrow p = n\hbar$$

Generalization to non-projectable L

Take $U \subset M$ open such that $p \in U$ is non-caustic.

$$(\pi \circ \imath)^{-1} U = \cup_i L_i$$



Define $I_\hbar$ as

$$I_\hbar = \sum_i \left(\left(\pi \circ \imath|_{L_i}\right)^{-1} \right)^* a(x) e^{\phi_i(x)}$$

Generalization to non-projectable L

Example (L exact but not projectable)

Particle on a constant force field: $H = \frac{1}{2}\left(p^2 - q\right)$

$$H^{-1}(0) = \{(q,p) : p^2 = q\} \Longrightarrow \imath : L \hookrightarrow T^*M, \imath(x) = (x^2, x)$$

$$X_H = p\frac{\partial}{\partial q} + \frac{\partial}{\partial p}. \text{ Phase function: } \phi(x) = 2x^3/3, \imath^*\alpha = d\phi$$

L_+, L_- upper and lower branches of L. Take a half-density $a(x)|dx|^{1/2}$ on L

On L_+: $\left((\pi \circ i)^{-1}\right)^* \left(a(x)|dx|^{1/2}\right) = a(q^{1/2}) \left|\frac{dq}{dx}\right|^{-1/2} |dq|^{1/2} = 2^{-1/2}q^{-1/4}a(q^{1/2})|dq|^{1/2}$

On L_-: $\left((\pi \circ i)^{-1}\right)^* \left(a(x)|dx|^{1/2}\right) = 2^{-1/2}q^{-1/4}a(-q^{1/2})|dq|^{1/2}$

On L_+, $\left((\pi \circ \imath)_*\right)^{-1}(\pi_* X_H) = \frac{1}{2}\frac{\partial}{\partial x}$. Then $\mathcal{L}_{X_H} a(x)|dx|^{1/2} = 0 \Longrightarrow a(x) \equiv a$

$$I_\hbar = \left(e^{2iq^{3/2}/3\hbar} + e^{-2iq^{3/2}/3\hbar}\right) 2^{-1/2}q^{-1/4}a|dq|^{1/2}$$

Generalization to non-projectable L

L not exact nor projectable $\Longrightarrow$ same requirement that $\alpha \in H^1(M, \mathbb{Z}_\hbar)$

Harmonic oscillator

$$H = \frac{1}{2}(p^2 + q^2), 2\pi n\hbar = \int_{S^1} pdq = \int_{D^2} dpdq = 2\pi E$$

$$\Longrightarrow E = n\hbar. \text{ Not the correct quantization condition.}$$

More work ahead $\Longrightarrow$ The Maslov correction

The Maslov correction

Motivation: Take the cylinder $S^1 \times \mathbb{R}$ foliated by $\mathbb{R}$

$$\imath : L \hookrightarrow T^*S^1, \imath(x) = (q_0, x), x \in \mathbb{R}$$

L not projectable on q but projectable on p. Quantization should describe a particle localized at q_0 (delta function).

Do a symplectic transf. $(q,p) \to (q',p') = (-p,q) \Longrightarrow \alpha = pdq \to \alpha = -qdp$

Given $a(p)|dp|^{1/2}$ and $J_\hbar(p) = e^{iT(p)/\hbar}a(p)|dp|^{1/2}$ set

$$J_\hbar(q) = \mathcal{F}_\hbar^{-1}\left(a(p)e^{iT(p)/\hbar}\right)|dq|^{1/2} \text{ with } \mathcal{F}_\hbar f(p) = \frac{1}{(2\pi\hbar)^{1/2}}\int f(q)e^{-iqp/\hbar}dq$$

Take the phase function $\phi = -q_0 x$. Then $J_\hbar = e^{-iq_0 p}a|dp|^{1/2}$($a$ const.) and

$$J_\hbar(q) = \mathcal{F}_\hbar^{-1}\left(e^{-iq_0 p/\hbar}a\right)(q)|dq|^{1/2} \sim \delta(q - q_0)$$

Method of Stationary Phase

(See Guillemin and Sternberg)

Let $\phi(y)$ and $a(y)$ be smooth function in $\mathbb{R}^n$. $a(y)$ has compact support.

Evaluate $\int a(y) e^{ik\phi(y)} dy$ for large $k \in \mathbb{R}$

First show that the integral is dominated by the values of the exponential where $\phi'(p) = 0$

Then use a lemma by Morse showing that there is a coord. transf. $z = \varphi(y)$ such that in some neighborhood U of p

$$\phi(z) = \phi(p) + Q(z) \text{ with } Q \text{ a canonical quadr. form of index } l$$

Then notice that

$$Q = \left(\frac{\partial y}{\partial z}\right)^T \left(\frac{\partial^2 \phi}{\partial y^i y^j}\right) \left(\frac{\partial y}{\partial z}\right) \text{ so that } \left|\det \frac{\partial y}{\partial z}(p)\right| = \left|\det \frac{\partial^2 \phi}{\partial y^i \partial y^j}(p)\right|^{-1/2}$$

Method of Stationary Phase

Make a change of variables to get $\int b(z)e^{ikQ(z)}dz$ for some $b(z)$ which may not have compact support. Show that the integral is also dominated by terms with the constant values $b(p)$

Then we are lead to compute $\Pi_i \int e^{\pm ikz_i^2} dz_i$

The Gaussian has an analytic continuation to imaginary values of the exponent:

$$\int e^{\pm iku^2/2} du = \left(\frac{2\pi}{k}\right)^{1/2} e^{\pm \pi i/4}$$

Finally,

$$\int a(y)e^{ik\phi(y)}dy = \left(\frac{2\pi}{k}\right)^{n/2} \sum_{p|\phi'(p)=0} e^{i\pi \operatorname{sgn} H(p)/4} \frac{e^{ik\phi(p)}a(p)}{\sqrt{|\det H(p)|}} + O(k^{-n/2-1})$$

The Maslov correction

Example: comparison of the two methods in a bi-projectable case

$$\imath : L \hookrightarrow T^*\mathbb{R}, \imath(x) = (x, kx), x \in \mathbb{R}$$

Two phase functions $\phi(x) = kx^2/2, \tau(x) = -kx^2/2$ such that

$$d\phi = \imath^*(pdq), d\tau = \imath^*(-qdp)$$

Take a constant half-density on L, $a|dx|^{1/2}$. Then

$$\left((\pi_q \circ \imath)^{-1}\right)^* a|dx|^{1/2} = a|dq|^{1/2}, \quad \left((\pi_p \circ \imath)^{-1}\right)^* a|dx|^{1/2} = a|k|^{-1/2}|dp|^{1/2}$$

$$I_\hbar(q) = e^{ikq^2/2\hbar} a|dq|^{1/2} \text{ and } J_\hbar(q) = \mathcal{F}_\hbar^{-1}\left(|k|^{-1/2} a e^{-ip^2/2k\hbar}\right)|dq|^{1/2}$$

Compute the Fourier transform by the method of the stationary phase (see for instance Hörmander or Guillemin and Sternberg)

$$J_\hbar(q) = e^{-i\pi.sgn(k)/4} I_\hbar(q)$$

The Maslov correction

Another example (L not q-projectable)

$$\imath : L \hookrightarrow T^*\mathbb{R}, \imath(x) = (x^2, x), x \in \mathbb{R}, \text{ phase function } \tau(x) = -x^3/3,\ d\tau = \imath^*(-q dp)$$

Take $a(x)|dx|^{1/2}$ on L and quantize through Maslov

We have $\left((\pi_p \circ \imath^{-1}\right)^* (a(x)|dx|^{1/2}) = a(p)|dp|^{1/2}$

$$J_\hbar(q) = \mathcal{F}^{-1}\left(e^{-ip^3/3\hbar} a(p)\right)|dq|^{1/2}$$

Stationary phase: critical point of $pq - p^3/3$ is $q = p^2 \implies p = \pm\sqrt{q}$

Then

$$J_\hbar(q) = \left(e^{-i\pi/4} e^{i2q^{3/2}/3\hbar} a(q^{1/2}) + e^{i\pi/4} e^{-i2q^{3/2}/3\hbar} a(-q^{1/2})\right) 2^{-1/2} q^{-1/4} |dq|^{1/2} + O(\hbar)$$

The Maslov correction

The relative phase factor $e^{i\pi/2}$ between the 2 terms has its origin in the saddle point $x = 0$ of $\tau(x) \sim x^3$:

$$\mathrm{sgn}(\tau'') = -1 \text{ for } x > 0 \Longrightarrow \text{phase iqual to } e^{-i\pi/4}$$
$$\mathrm{sgn}(\tau'') = 1 \text{ for } x < 0 \Longrightarrow \text{phase iqual to } e^{i\pi/4}$$

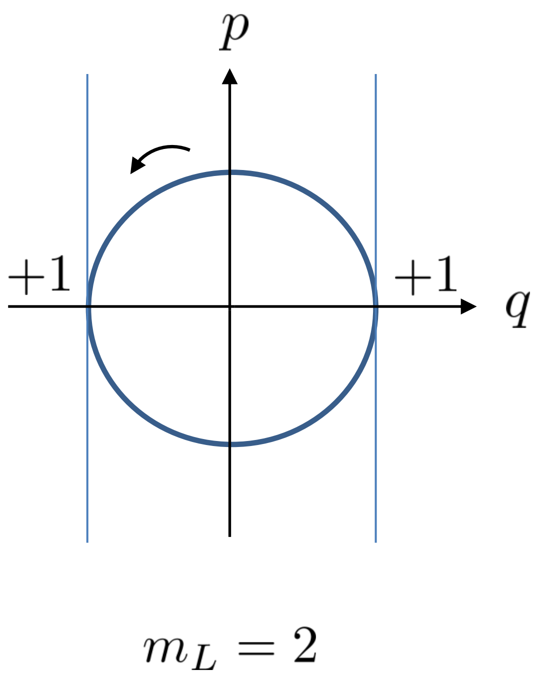
$$\mathrm{sgn}(\tau'') \text{ changes by 2 when crossing a caustic}$$

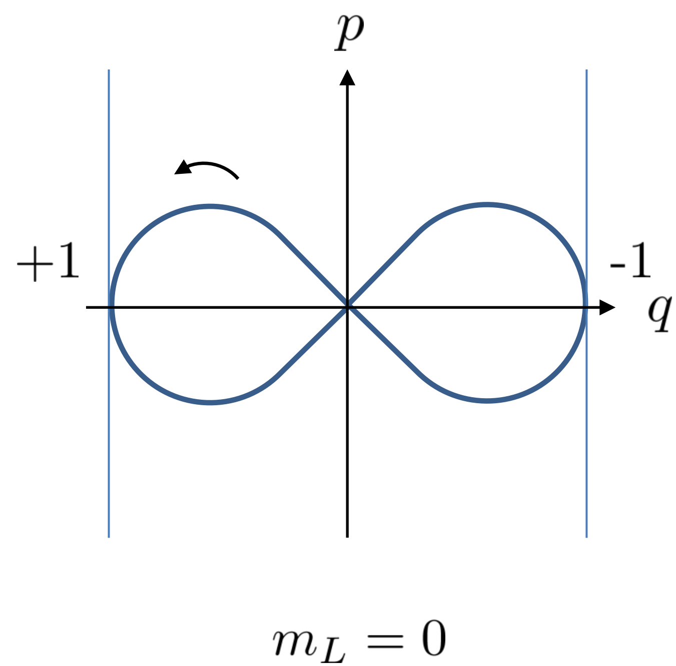
$$\Downarrow$$

Rule - Observing that the non-degeneracy condition implies that each curve remains on the same side relative to the fiber on a neighborhood of a caustic, assign an integer to any closed curve in phase space as follows:

Going counterclockwise assign 1 when crossing a caustic if the fibre is on the right with respect to the motion and -1 on the reverse situation.

The Maslov correction





Maslov index: $m_L = \sum$ integers on the curve

The Maslov correction

Quantization condition: cover L by open sets L_i and quantize by pullback or by Maslov. Then

$$I_\hbar(q) = \sum_j e^{-i\pi s_j/4} e^{i\phi_j(q)/\hbar} a(q_j)|dq_j|^{1/2}$$

where $s_j = 0$ if L_j is q-projectable or $s_j = \mathrm{sgn}(\tau_j'')$ for a suitable phase function $\tau_j(p)$ on L_j

This should be consistent on overlaps $L_i \cap L_j$ so that

$$\pi(s_j - s_i)/4 + (\phi_i - \phi_j)/\hbar = 2\pi n, \quad n \in \mathbb{Z}$$

Now the sum of $s_i - s_j$ on a closed curve equals $2m_L$. We then have the quantization condition

$$\boxed{-\frac{\pi\hbar}{2} m_L + \int_L \imath^* \alpha = 2\pi n\hbar}$$

This gives immediatly for the Harmonic Oscillator $E = (n + \frac{1}{2})\hbar\omega$

Metalinear structures

M a smooth manifold and B the principal bundle of frames of TM

- n-form: $\quad w : B(M, GL(n,\mathbb{R}), n) \longrightarrow \mathbb{R}$

Transf. law: given a frame $v = (v_1, \ldots, v_n)$, $w(Av) = (\det A)w(v)$ for $A \in GL(n,\mathbb{R})$

- s-density: $\mu : B(M, GL(n,\mathbb{R}), n) \longrightarrow \mathbb{R}$

Transf. law: $\mu(Av) = |\det A|^s \mu(v)$ for $A \in GL(n,\mathbb{R})$

Is it possbile to have a transf. law for $(\det A)^{1/2}$ with A in $GL(n,\mathbb{R})$?

Only outside $GL(n,\mathbb{R})$, for instance seeing it as a subgroup of $GL(n,\mathbb{C})$

Metalinear structures

First try: $n = 1$

$$\mathbb{Z} \circlearrowright \mathbb{C} : k \cdot u = u + 2\pi i k$$

Well known exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow 1$$

$$\mathbb{C}^* \cong \mathbb{C}/\mathbb{Z}$$

General case

Given $A \in GL(n, \mathbb{C})$ take $B = (\det A)^{-1/n} A \in SL(n, \mathbb{C})$ and then identify $GL(n, \mathbb{C})$ with $SL(n, \mathbb{C}) \times \mathbb{C}/\mathbb{Z}$ through the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow SL(n, \mathbb{C}) \times \mathbb{C} \xrightarrow{\pi} GL(n, \mathbb{C}) \longrightarrow 1$$

$$\pi(A, u) = e^u A$$

Action of $\mathbb{Z}$ on $SL(n, \mathbb{C}) \times \mathbb{C}$:

$$k \cdot (A, z) = (\exp{(2\pi i k/n)} A, z + 2\pi i k/n)$$

Metalinear structures

$\det : GL(n,\mathbb{C}) \longrightarrow \mathbb{C}^*$ pulls-back to $SL(n,\mathbb{C}) \times \mathbb{C}$ $\quad (\det \circ \pi)(A,z) = e^{nz}$

and admits a well defined holomorphic square root $\quad u \longrightarrow e^{nz/2}$

which is invariant by the action of $2\mathbb{Z}$ and so is defined on $SL(n,\mathbb{C}) \times \mathbb{C}/2\mathbb{Z}$

Call this the metalinear group: $ML(n,\mathbb{C}) \simeq SL(n,\mathbb{C}) \times \mathbb{C}/2\mathbb{Z}$

It is a double cover of $GL(n,\mathbb{C})$

Let $r : ML(n,\mathbb{C}) \to GL(n,\mathbb{C})$. See $GL(n,\mathbb{R}) \subset GL(n,\mathbb{C})$ and set

$ML(n,\mathbb{R}) = r^{-1}GL(n,\mathbb{R})$. $ML(n,\mathbb{R})$ is a double cover of $GL(n,\mathbb{R})$

$$\begin{array}{ccccc} MB(V) & \times & ML(n,\mathbb{R}) & \longrightarrow & MB(V) \\ \rho \downarrow & & r \downarrow & & \rho \downarrow \\ B(V) & \times & GL(n,\mathbb{R}) & \longrightarrow & B(V) \end{array}$$

Half-form $\nu : MB(V) \to \mathbb{C}$ s.t.

$\nu(fA) = \chi(A)\nu(f)$
with $\chi(A)^2 = \det(rA)$

Harmonic Oscillator - Quantization

Phase-space $M = \mathbb{R}^2 \backslash (0,0)$ $\qquad$ $H = \frac{1}{2}(p^2 + q^2)$

symplectic form $\omega = dq \wedge dp$ $\qquad$ $\omega(X_H, \cdot) = -dH \Longrightarrow X_H = -p\frac{\partial}{\partial q} + q\frac{\partial}{\partial p}$

Complex structure $z = p - iq$ $\quad \left\{ \begin{array}{l} H = \frac{1}{2}|z|^2 \qquad \omega = \frac{i}{2} dz \wedge d\bar{z} \\ X_H = i\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right) \end{array} \right.$

Complex polarization $\mathcal{F} =< X_H >$

$$H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2 \Longrightarrow 2 \text{ metalinear structures}$$

Harmonic Oscilator - Quantization

One metalinear struture is just the trivial bundle $M \times ML(1, \mathbb{C})$. It dos not give the correct energy quantization

Second metalinear structure:

1. $\mathbb{R}^2 \backslash (0,0) \simeq \mathbb{R}^+ \times \mathbb{R}/\mathbb{Z}$ with $\mathbb{Z} \circlearrowright \mathbb{R}^+ \times \mathbb{R} : k \cdot (r, \theta) = (r, \theta + 2\pi k)$
2. $\mathbb{Z} \circlearrowright \mathbb{R}^+ \times \mathbb{R} \times ML(1, \mathbb{C}) : k \cdot (r, \theta, \lambda) = (r, \theta + 2\pi k, \epsilon^k \lambda)$

   with ϵ the non-trivial element of $p : ML(1, \mathbb{C}) \to GL(1, \mathbb{C})$
3. Set $MB(\mathcal{F}) = \mathbb{R}^+ \times \mathbb{R} \times ML(1, \mathbb{C})/\mathbb{Z}$ and define $ML(1, \mathbb{C}) \circlearrowright MB(\mathcal{F})$

   as $[r, \theta, \lambda]\lambda' = [r, \theta, \lambda\lambda']$

Harmonic Oscilator - Quantization

$\rho : MB(\mathcal{F}) \to B(\mathcal{F}), \rho[r, \theta, \lambda] = X_H(re^{i\theta})p(\lambda)$

$MB(\mathcal{F})$ is a metalinear bundle over $\mathcal{F}$

Cover M by two opens sets V_1 for $0 < [\theta] < 2\pi$ and V_2 for $-\pi < [\theta] < \pi$ and take 2 sections $u_1, u_2 : M \to MB(\mathcal{F})$:

$$u_1(r,\theta) = [r, \theta, 1] \text{ on } V_1$$
$$u_2(r,\theta) = [r, \theta, 1] \text{ on } V_2$$

Then

$$u_1(r,\theta) = u_2(r,\theta) \text{ on } 0 < [\theta] < \pi$$
$$u_1(r,\theta) = [r, \theta, 1] = [r, \theta - 2\pi, \epsilon]) = [r, \theta - 2\pi, 1]\epsilon = u_2(r,\theta)\epsilon \text{ on } \pi < [\theta] < 2\pi$$

Harmonic Oscilator - Quantization

Take our quantum bundle $L \otimes \Lambda^{1/2}$ with L a Hilbert line bundle over M. Recall that $\nu \in \Lambda^{1/2}$ is such that $\nu : MB(\mathcal{F}) \to \mathbb{C}$

For $\phi \otimes \nu$ a section of $L \otimes \Lambda^{1/2}$ the quantization condition is

$$\nabla_{X_H} \phi \otimes \nu + \phi \otimes \mathcal{L}_{X_H} \nu = 0$$

with $\nabla_{X_H} = X_H + 2\pi i \left(\imath(X_H) \cdot \alpha\right)$ for $d\alpha = \omega$

Fixing a gauge, $\alpha = \frac{i}{2} z d\bar{z}$. $X_H \sim \frac{\partial}{\partial \theta}$ and with $\phi(r, \theta) = f(r) e^{iK\theta}, K \in \mathbb{N}$

$$X_H \phi + 2\pi i (i(X_H) \cdot \alpha)\phi = 0 \Leftrightarrow \left(K - \frac{2\pi r^2}{2}\right) f(r) = 0$$

$$\Downarrow$$

Distributional solution: $f(r) \sim \delta(r - \sqrt{K/\pi})$

Harmonic Oscilator - Quantization

Effect of the metalinear structure: recall that

$u_1(r,\theta) = u_2(r,\theta)$ on $0 < [\theta] < \pi$
$u_1(r,\theta) = [r,\theta,1] = [r,\theta-2\pi,\epsilon]) = [r,\theta-2\pi,1]\epsilon = u_2(r,\theta)\epsilon$ on $\pi < [\theta] < 2\pi$

Take $\nu_1, \nu_2 : MB(\mathcal{F}) \longrightarrow \mathbb{C}$ such that $\nu_i(u_i) = 1, i = 1, 2.$ Then

$$f(r)e^{iK\theta} \otimes \nu_1 \text{ on } V_1 \qquad \qquad \nu_2 = \nu_1 \quad \text{for } 0 < [\theta] < \pi,$$
and
$$f(r)e^{iK\theta} \otimes \nu_2 \text{ on } V_2 \qquad \qquad \nu_2 = -\nu_1 \quad \text{for } \pi < [\theta] < 2\pi$$

Do a complete tour from $3\pi/2$ to $-\pi/2$

$$\underset{V_1 \cap V_2}{e^{iK3\pi/2} \otimes \nu_1} \underset{V_1}{\longrightarrow} \underset{V_1 \cap V_2}{e^{iK\pi/2} \otimes \nu_1 = e^{iK\pi/2} \otimes \nu_2} \underset{V_2}{\longrightarrow} \underset{V_1 \cap V_2}{e^{-iK\pi/2} \otimes \nu_2 = -e^{-iK\pi/2} \otimes \nu_1}$$

$$e^{iK3\pi/2} = -e^{-iK\pi/2} \Longrightarrow K = N + \tfrac{1}{2}, N \in \mathbb{Z} \qquad H = \tfrac{1}{2}r^2, r = \sqrt{K/\pi}$$

References

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