Análise Complexa e Equações Diferenciais
$2^{\underline{o}}$ Semestre 2015/2016
Cursos: MEBiol, MEAmbi, MEQ

Ficha de Problemas nº 1
Números complexos

1. Calcule o valor dos números complexos apresentando o resultado na forma algébrica, isto é na forma $a+b\mathrm{i}$, a, $b \in \mathbb{R}$.

(a) $\dfrac{1}{\mathrm{i}}$ (b) $\dfrac{1-\mathrm{i}}{1+\mathrm{i}}$ (c) $\dfrac{2}{1-3\mathrm{i}}$ (d) $\left(1+\mathrm{i}\sqrt{3}\right)^3$ (e) $\overline{2\mathrm{i}\left(\dfrac{1}{2}-\mathrm{i}\right)}$

(f) $\dfrac{20+10\mathrm{i}}{(1+\mathrm{i})(2-\mathrm{i})}$ (g) $(1+2\mathrm{i})^2+4\mathrm{i}^3$ (h) $\dfrac{1-\alpha\mathrm{i}}{1+\alpha\mathrm{i}}$ com $\alpha \in \mathbb{R}$

2. Escreva uma expressão da forma $re^{i\theta}$, para cada um dos números complexos

(a) i^3 (b) $\sqrt{2}(1+\mathrm{i})$ (c) $\sqrt{3}-\mathrm{i}$ (d) $2-2\sqrt{3}\mathrm{i}$ (e) $(1-\mathrm{i})^{-1}$

(f) $(\sqrt{3}-\mathrm{i})(1+\mathrm{i})$ (g) $(1+\sqrt{3}\mathrm{i})^3$ (h) $\dfrac{i\sqrt{2}}{4+4i}$ (i) $\dfrac{3\sqrt{2}+2i}{-\sqrt{2}-2i/3}$

3. Escreva uma expressão da forma $x+\mathrm{i}y$, $(x,\ y \in \mathbb{R})$, para cada um dos números complexos

(a) $e^{\pi\,\mathrm{i}/4}$ (b) $5e^{-\pi\,\mathrm{i}}$ (c) $2e^{3\,\pi\,\mathrm{i}/2}$ (d) $e^{4\,\pi\,\mathrm{i}/3}$ (e) $e^{7\,\pi\,\mathrm{i}/6}$

4. Se $z_1 = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$ determine os valores de $z_1\bar{z}_1$, $(\bar{z}_1)^4$ e de $\sqrt[5]{z_1}$.

5. Calcule, para $n = 1, 2, 3, \ldots$,

(a) i^n (b) $\left(\frac{1-\mathrm{i}}{1+\mathrm{i}}\right)^n$ (c) $(1+\mathrm{i})^n+(1-\mathrm{i})^n$

6. Mostre que para $x \in \mathbb{R}$ se tem

$$\frac{\sqrt{1+x^2}+ix}{x-i\sqrt{1+x^2}} = i$$

7. Mostre que, para todos os complexos z_1, z_2

$$\overline{\left(z_1+z_2\right)} = \overline{z_1}+\overline{z_2} \qquad , \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

8. Encontre todos os valores da raiz

(a) $\sqrt[3]{\mathrm{i}}$ (b) $\sqrt{2-2\sqrt{3}\mathrm{i}}$ (c) $\sqrt[4]{-1}$ (d) $\sqrt[3]{\sqrt{2}+i\sqrt{2}}$

9. Mostre que os pontos do plano de Argand representados pelos números complexos $z_1 = 2+i$, $z_2 = 4+3i$, $z_3 = 2+5i$ e $z_4 = 3i$ representam os vértices de um quadrado.

10. Determine as soluções das seguintes equações:

(a) $(1-z)^6 = (1+z)^6$

(b) $z^2 + 2z + 5 = 0$

(c) $z^4 - 3(1+2i)z^2 - 7 + 9i = 0$

(d) $1 - z^2 + z^4 - z^6 = 0$

(e) $1 + z + z^2 + ... + z^7 = 0$

(f) $z^2 + \bar{z} - 2 = 0$

(g) $z^4 - 4z^3 + 6z^2 - 4z - 15 = 0$

11. Esboce os subconjuntos de $\mathbb{C}$ dados por:

(a) $|z+2| = 6$ (b) $|z-3\mathrm{i}| = |z+\mathrm{i}|$ (c) $\mathrm{Im}(z+\mathrm{i}) < 2$

(d) $|z+2\mathrm{i}| \geq 2$ (e) $|z-1| \geq |z-1-\mathrm{i}|$ (f) $\mathrm{Im}[(z+\mathrm{i})/2\mathrm{i}] < 0$

(g) $\mathrm{Re}\, z \neq 0$ (h) $1 < |z-1| < 2$ (i) $|z|^2 > z + \overline{z}$

Soluções

1. **(a)** $-\mathrm{i}$ **(b)** $-\mathrm{i}$ **(c)** $\frac{1}{5}/1+3\mathrm{i})$ **(d)** -8 **(e)** $2-\mathrm{i}$ **(f)** $7+\mathrm{i}$ **(g)** -3 **(h)** $\frac{1-\alpha^2-2\alpha\mathrm{i}}{1+\alpha^2}$

2. **(a)** $e^{3\pi\mathrm{i}/2}$ **(b)** $2e^{\mathrm{i}\pi/4}$ **(c)** $2e^{-\mathrm{i}\pi/6}$ **(d)**$4e^{-\mathrm{i}\pi/3}$ **(e)** $\frac{1}{\sqrt{2}}e^{\mathrm{i}\pi/4}$ **(f)** $2\sqrt{2}e^{\mathrm{i}\pi/12}$

(g) $8e^{\mathrm{i}\pi}$ **(h)** $\frac{1}{4}e^{i\pi/4}$ **(i)** $3\,e^{i\pi}$.

3. **(a)** $\frac{\sqrt{2}}{2}(1+\mathrm{i})$ **(b)**-5 **(c)** $-2\mathrm{i}$ **(d)**$-\frac{1}{2}\left(1+\sqrt{3}\mathrm{i}\right)$ **(e)**$-\frac{1}{2}\left(\sqrt{3}+\mathrm{i}\right)$.

4. $z_1\bar{z}_1 = 1$, $(\bar{z}_1)^4 = e^{4\pi i/3}$ e $\sqrt[5]{z_1} = e^{i\frac{\frac{2\pi}{3}+2k\pi}{5}}$ com $k = 0,\ 1,\ 2,\ 3,\ 4$.

5. **(a)** $\cos\frac{n\pi}{2} + \mathrm{i}\,\mathrm{sen}\,\frac{n\pi}{2} = \begin{cases} 1 & \text{se} \quad n = 4k \\ \mathrm{i} & \text{se} \quad n = 4k+1 \\ -1 & \text{se} \quad n = 4k+2 \\ -\mathrm{i} & \text{se} \quad n = 4k+3 \end{cases}$ para $k \in \mathbb{N}_0$

(b)$(-\mathrm{i})^n$ **(c)**$2(\sqrt{2})^n \cos\frac{n\pi}{4}$

6.

7.

8. **(a)** $\{e^{\mathrm{i}\pi/6},\ e^{5\pi\mathrm{i}/6},\ -\mathrm{i}\}$ **(bb)**$\{\pm\,2e^{-\pi\mathrm{i}/6}\}$ **(c)**$\{e^{-\mathrm{i}\pi/4},\ e^{\mathrm{i}\pi/4},\ e^{3\pi\mathrm{i}/4},\ e^{5\pi\mathrm{i}/4}\}$
(d)$\{\sqrt[3]{2}\,e^{i\pi/12},\ \sqrt[3]{2}\,e^{3i\pi/4}, \sqrt[3]{2}\,e^{17\mathrm{i}\pi/12}\}$

9. Verifique que os comprimentos dos lados do polígono e das suas diagonais são iguais; em alternativa, considere os pontos $w_j = z_j - (2+3\mathrm{i})$, $j = 1, 2, 3, 4$ e verifique que w_j^4 tem o mesmo valor para $j = 1, 2, 3, 4$.

10. **(a)** $z \in \{0.\,\mathrm{i}\sqrt{3}, -\,\mathrm{i}\sqrt{3},\ \frac{\mathrm{i}}{\sqrt{3}},\ -\frac{\mathrm{i}}{\sqrt{3}}\}$ **(b)**$z = -1 \pm 2\mathrm{i}$ **(c)** $\pm\sqrt{1+3i},\ \pm\sqrt{2+3i}$
(d) $z \in \{-1,\ 1,\ e^{-\mathrm{i}\pi/4},\ e^{\mathrm{i}\pi/4},\ e^{3\pi\mathrm{i}/4},\ e^{5\pi\mathrm{i}/4}\}$ **(e)** $z \in \{-1,\ \mathrm{i},\ -\mathrm{i},\ e^{-\mathrm{i}\pi/4},\ e^{\mathrm{i}\pi/4},\ e^{3\pi\mathrm{i}/4},\ e^{5\pi\mathrm{i}/4}\}$
(f) $z \in \{1, -2\}$ **(g)** $z \in \{3, 1+2\mathrm{i}, -1, 1-2\mathrm{i}\}$

11. **(a)** $(x+2)^2 + y^2 = 6^2$ **(b)** $\mathrm{Im}\,z = 1$ **(c)** $\mathrm{Im}\,z < 1$ **(d)**$x^2 + (y+2)^2 \geq 4$

(e) $\mathrm{Im}\,z \geq \frac{1}{2}$ **(f)** $\mathrm{Re}\,z > 0$ **(g)** $\mathrm{Re}\,z \neq 0$

(h)A *região anular* compreendida entre as circunferências $(x-1)^2+y^2 = 1$ e $(x-1)^2+y^2 = 4$ (não inclui as circunf.). **(i)** $(x-1)^2 + y^2 > 1$