# Variational multiscale problems and applications to thin films

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**To the memory of my uncle Manuel.**

**To my parents Lucília and Justino**

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# Abstract

The main objective of this dissertation is to study the asymptotic behavior of two kinds of multiple scale problems by $\Gamma$-convergence: Relaxation problems involving families of multiple scale integral functionals, and 3D-2D reduction problems for heterogeneous thin domains with periodic microstructure. Periodicity, standard growth conditions and nonconvexity are assumed whereas a stronger uniform continuity with respect to the macroscopic variable, normally required in the existing literature, is avoided.

**Key words:** Integral functionals, periodic integrands, $\Gamma$-convergence, two-scale convergence, quasiconvexity, equi-integrability, dimension reduction, thin films.

# LIST OF NOTATIONS

- $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$;
- $[[a]]$: Integer part of $a$;
- $\nabla_i := \partial/\partial x_i$;
- $x_\alpha := (x_1, x_2)$;
- $\nabla_\alpha = (\nabla_1, \nabla_2)$;
- $\mathbb{R}^{d\times N}$(resp. $\mathbb{Q}^{d\times N}$) $\equiv$ {set of real (resp. rational)-valued $d \times N$ matrices};
- $(\overline{\xi}|z)$ with $\overline{\xi} \in \mathbb{R}^{3\times 2}$ and $z \in \mathbb{R}^3$: Matrix whose first two columns are those of $\overline{\xi}$ and the last one is $z$;
- $B(a,\delta) := \{x \in \mathbb{R}^N : |x-a| < \delta\}$, $a \in \mathbb{R}^N$, $\delta > 0$;
- $Q = (0,1)^N$;
- $Q(a,\delta) := a + \delta Q \equiv (a,\delta)^N$, $a \in \mathbb{R}^N$, $\delta > 0$;
- $Q^{'} = (0,1)^2$;
- $Q'(a,\delta) := a + \delta Q'$, $a \in \mathbb{R}^2$, $\delta > 0$;
- $\chi_A$ : Characteristic function of a set $A$;
- $\overline{A}$: Closure of A;
- $\partial A$: Boundary of $A$;
- $A \subset\subset B$: $\overline{A} \subset B$, $\overline{A}$ compact;

- $\mathcal{L}^N(E)$ or $|E|$: Lebesgue measure of $E \subset \mathbb{R}^N$;
- $\mathcal{A}(\Omega)$: Open subsets of $\Omega$;
- $\mathcal{A}_0(\Omega)$ : Open and bounded subsets of $\Omega$;
- $\mathcal{A}_\infty(\Omega)$: Lipschitz subsets of $\Omega$;
- $C^\infty(X;\mathbb{R}^d)$: $\mathbb{R}^d$-valued functions defined in $X$ with derivatives of any order in $X$ ($C^\infty(X)$ if $d=1$);
- $\mathrm{supp}(u)$: Support of $u$;
- $C_c(X;\mathbb{R}^d)$: $\mathbb{R}^d$-valued functions defined in $X$ with compact support in $X$ ($C_c(X)$ if $d=1$);
- $C_c^\infty(X;\mathbb{R}^d) := C^\infty(X;\mathbb{R}^d) \cap C_c(X;\mathbb{R}^d)$;
- $C_0(X;\mathbb{R}^d) := \overline{C_c(X;\mathbb{R}^d)}$ with respect to the supremum norm;
- $C_0^\infty(X;\mathbb{R}^d) := C^\infty(X;\mathbb{R}^d) \cap C_0(X;\mathbb{R}^d)$;
- $C_{\mathrm{per}}(Q;\mathbb{R}^d)$: $Q$- periodic continuous functions defined in $\mathbb{R}^N$ with values in $\mathbb{R}^d$ ($C_{\mathrm{per}}(Q)$ if $d=1$);
- $W^{1,p}_{\mathrm{per}}(kQ;\mathbb{R}^d)$: $W^{1,p}$-closure of all $kQ$- periodic and $C^1$-functions defined on $\mathbb{R}^N$ with values in $\mathbb{R}^d$ ($W^{1,p}_{\mathrm{per}}(kQ)$ if $d=1$);
- $L^p(X,\mu)$ or $L^p(X,\mu;\mathbb{R}^d)$: Usual scalar and vectorial Lebesgue spaces $\big(L^p(X;\mathbb{R}^d)$ if $\mu = \mathcal{L}^N$ or even $L^p(X)$ if also $d=1\big)$;
- $\operatorname*{ess\,sup}_{x\in X} |u(x)| = ||u||_{L^\infty(X)}$;
- $W^{1,p}(X;\mathbb{R}^d)$ or $W^{1,p}(X)$ if $d=1$: Usual Sobolev spaces;
- $W^{1,p}(\omega;\mathbb{R}^3)$: $u \in W^{1,p}(\Omega;\mathbb{R}^3)$ such that $\nabla_3 u(x) = 0$ for a.e. $x \in \Omega$, $\Omega := \omega \times I$, $I := (-1,1)$;
- $\rightharpoonup$ ... Weak convergence in $L^p$ or $W^{1,p}$;
- $\overset{\star}{\rightharpoonup}$ ... Weak$\star$ convergence in the sense of measures; also weak$\star$ convergence in $L^\infty$ or in $W^{1,\infty}$;

- $s.l.s.c$: Sequential lower semicontinuous;
- $s.w.l.s.c$ on $W^{1,p}(\Omega;\mathbb{R}^d)$ and $s.w\star.l.s.c$ on $W^{1,\infty}(\Omega;\mathbb{R}^d)$: $s.l.s.c$ with respect to the weak or weak$\star$ convergence of $W^{1,p}(\Omega;\mathbb{R}^d)$ and $W^{1,\infty}(\Omega;\mathbb{R}^d)$;
- $slsc\mathcal{I}$: Sequential lower semicontinuous envelope of $\mathcal{I}$;
- $swlsc\mathcal{I}$: Sequential lower semicontinuous envelope of $\mathcal{I}$ with respect to a weak topology;
- $Qf$: Quasiconvex envelope of $f$;
- $\mathrm{Ker}(T)$: Kernel of an operator $T$;
- $\Gamma(L^p(\Omega))$-limit: $\Gamma$-convergence with respect to the usual metric in $L^p(\Omega;\mathbb{R}^d)$;
- $\lim\limits_{k,m,n} := \lim\limits_{k} \lim\limits_{m} \lim\limits_{n}$, with obvious generalizations.

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# 1. INTRODUCTION

The main objective of this dissertation is to study the effective behavior of elastic (thin) bodies with multiple scales and periodic microstructure. This study is undertaken from a variational point of view through an asymptotic analysis based on $\Gamma$-convergence arguments. The asymptotic analysis of media with multiple scale of homogenization is referred to in the literature as *Reiterated Homogenization*.

Roughly speaking, the aim of homogenization theory is to describe the behavior of microscopically heterogeneous composite physical structures by means of homogeneous structures with global characteristics equivalent to the initial ones. In many physical situations the heterogeneities are very small in comparison with the region in which the structure is to be studied and the heterogeneities are evenly distributed, so that they can be modelled by a periodic distribution of period a small parameter. In practice, one is interested in the global behavior of these structures when the heterogeneities are very, very small. From the mathematical point of view, we are led to characterizing the asymptotic behavior of (systems of) ordinary or partial differential equations with oscillating periodic coefficients of period a small parameter $\varepsilon$, as $\varepsilon$ tends to zero.

A well-known model problem in periodic homogenization, used frequently to describe thermal as well as electrical or linear elasticity properties in a periodic composite medium has as underlying the following linear second-order partial differential equation

$$-\mathrm{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = g \quad \text{on } \Omega. \tag{1.1}$$

Here $\Omega$ is the (material) domain in $\mathbb{R}^N$ ($N \geqslant 1$), $A$ is a scalar or tensor-valued function with periodic coefficients, and $u_\varepsilon$ and $g$ are scalar or vector-valued functions in some appropriate functional spaces. One wishes to know the asymptotic behavior of the solutions $u_\varepsilon$ as $\varepsilon \to 0$.

They converge, under appropriate hypotheses, to a solution of an "homogenized" differential equation of the form

$$-\mathrm{div}(A_{\mathrm{hom}}(\nabla u)) = g \quad \text{on} \quad \Omega.$$

Starting with the use of asymptotic expansions methods (see Bensoussan, Lions and Papanicolau [14], Jikov, Kozlov and Oleinik [58] and Sanchez-Palencia [72]) adapted to the study of periodic problems like (1.1), homogenization techniques evolved toward more general situations through the concepts of $G$-convergence due to Spagnolo (see [74]), of $H$-convergence due to Murat and Tartar (see [68] [67], and [76]), of $\Gamma$-convergence due to De Giorgi (see [38] and [40]), and of two-scale convergence due to Nguetseng (see [61], [69] and [70]), further developed by Allaire and Briane (see [4] and [5]), and generalized by many other authors. We refer to the book of Cioranescu and Donato [32] for an introduction to homogenization and for an overview of different homogenization methods.

From a variational point of view, for instance in the context of elasticity, the theory of periodic homogenization rests on the study of a family of minimum problems

$$\min\left\{\int_\Omega f_\varepsilon(x, u(x), \nabla u(x))\, dx + \int_\Omega ug\, dx : \ u = \varphi \text{ on } \partial\Omega\right\}, \tag{1.2}$$

where the functions $f_\varepsilon$ (the elastic density energy) are increasingly oscillating in the first variable as $\varepsilon$ tends to zero, and $u$ (the deformation), $g$ (the density of applied body forces) and $\varphi$ are scalar or vector-valued functions in some Sobolev space. In the example (1.1) if $A = (A_{ij})$ and $u$ is a scalar function, $f_\varepsilon(x, \nabla u) = \sum A_{ij}(\frac{x}{\varepsilon})\nabla_i u \nabla_j u$, where $\nabla_i = \partial/\partial x_i$. More general minimum problems can be considered but in this Introduction we restrict to this case for simplicity. The homogenization of the family of minimum problems (1.2) leads to an "effective homogenized minimum problem" (not depending on $\varepsilon$)

$$\min\left\{\int_\Omega f_{\mathrm{hom}}(x, u(x), \nabla u(x))\, dx + \int_\Omega ug\, dx : \ u = \varphi \text{ on } \partial\Omega\right\} \tag{1.3}$$

such that a sequence of minimizers of (1.2) converges, as $\varepsilon$ tends to zero, to a limit $u$, which is a minimizer of (1.3). The fundamental property of De Giorgi's notion of $\Gamma$-convergence, and its main link to the other homogenization techniques, is that, under certain growth and compactness properties on $f_\varepsilon$ and some regularity on $g$, it implies a sequence of minimizers of (1.2) has this convergence property.

Due to the the properties of $\Gamma$-convergence (see Theorem 2.5.11 and Propositions 2.5.6 and 2.5.13 below), the convergence of minimizers (or almost minimizers) of (1.2) to minima of

(1.3) can be derived from the Γ-convergence of the family

$$\mathcal{I}_\varepsilon(u) = \int_\Omega f_\varepsilon(x, u(x), \nabla u(x))\, dx \tag{1.4}$$

to the homogenized functional

$$\mathcal{I}_{\text{hom}}(u) = \int_\Omega f_{\text{hom}}(x, u(x), \nabla u(x))\, dx.$$

This functional provides the macroscopic, or average description, of the periodic body by capturing the limiting behavior of the equilibrium states of $\{\mathcal{I}_\varepsilon\}_\varepsilon$. The effective energy density $f_{\text{hom}}$ is to be determined.

In this work we seek to approximate, in a Γ-convergence sense, the behavior of elastic (thin) bodies whose microstructure is periodic of period $\varepsilon$ and $\varepsilon^2$ (Figure. 1.1).

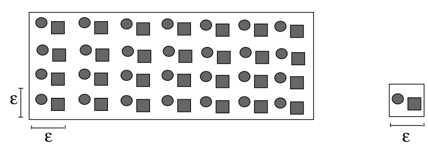
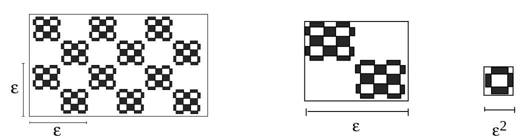



*Fig. 1.1:* Domains with periodic structure: Examples




To describe this behavior, let us introduce some notation. We identify $\mathbb{R}^{d\times N}$ (resp. $\mathbb{Q}^{d\times N}$) with the set of real (resp. rational)-valued $d \times N$ matrices, with $d, N \geqslant 1$. For $\overline{\xi} \in \mathbb{R}^{3\times 2}$ and $z \in \mathbb{R}^3$, let $(\overline{\xi}|z)$ denote the matrix whose first two columns are those of $\overline{\xi}$ and the last one is $z$. Let $x_\alpha := (x_1, x_2)$, with $x_1$, $x_2 \in \mathbb{R}$ and let $\nabla_\alpha = (\nabla_1, \nabla_2)$. We will consider two families of energies:

$$\mathcal{I}_\varepsilon(u) = \int_\Omega f\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \nabla u(x)\right)\, dx$$

for $\Omega \subset \mathbb{R}^N$, and

$$\mathcal{W}_\varepsilon(u) := \int_\Omega W\left(x, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}, \nabla_\alpha u(x)\Big|\frac{1}{\varepsilon}\nabla_3 u(x)\right)\, dx,$$

for $\Omega := \omega \times I$, $\omega \subset \mathbb{R}^2$ and $I := (-1, 1)$.

The functional $\mathcal{I}_\varepsilon(u)$ can be interpreted as the energy of a deformation $u$ of an elastic body whose microstructure is periodic of period $\varepsilon$ and $\varepsilon^2$. Similarly, as it will be seen later (Section 3.2), the functional $\mathcal{W}_\varepsilon(u)$ can be interpreted as the rescaled energy of a deformation $u$ of a cylindrical thin film of thickness $\varepsilon$ whose microstructure is periodic of period $\varepsilon$ in the in-plane direction $x_\alpha$, and periodic of period $\varepsilon^2$ in all directions. The variable $x$ is called the macroscopic or slow variable, whereas the variables $y = x/\varepsilon$ and $z = x/\varepsilon^2$ (respectively $z = x_\alpha/\varepsilon^2$) are called the microscopic or fast variables. Roughly speaking, the dependence of the energy on $x$ captures its macroscopic variation while the dependence on $y$ and $z$ captures its microscopic or local variations. One could generalize the study to a higher number of scales by iterating the argument.

The overall plan of this dissertation in the ensuing chapters is as follows. Chapter 2 is of introductory nature. Its objective is to set up the basic notations and background results that are used later. The aim of Chapter 3 is to present the previous developments in the asymptotic analysis of this type of problems. This serves as a motivation for our work and highlights the novelties. Chapter 4 is a collection of two works, obtained in collaboration with J-F Babadjian [10] and with I. Fonseca [12], respectively. We consider $\Omega \subset \mathbb{R}^N$ ($N \geqslant 1$) open and bounded, $1 < p < \infty$, and use the notation $\Gamma(L^p(\Omega))$-limit to refer to the $\Gamma$-limit with respect to the usual metric in $L^p(\Omega; \mathbb{R}^d)$ with $d \geqslant 1$. We set $Q := (0,1)^N$ and $\mathcal{L}^N$ stands for the Lebesgue measure in $\mathbb{R}^N$.

*Main problem of Chapter 4* (Theorem 4.2.1): Characterize the behavior, as $\varepsilon$ tends to zero, of a family of integral functionals defined on $L^p(\Omega; \mathbb{R}^d)$ by

$$\mathcal{I}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \nabla u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^d), \\ \\ \infty & \text{otherwise,} \end{cases} \tag{1.5}$$

where the integrand $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d \times N} \to \mathbb{R}$ satisfies:

- $f(x, \cdot, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$;
- $f(\cdot, y, z, \xi)$ is $\mathcal{L}^N$-measurable for all $(y, z, \xi) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d \times N}$;
- $f(x, \cdot, z, \xi)$ is $Q$-periodic for all $(z, \xi) \in \mathbb{R}^N \times \mathbb{R}^{d \times N}$ and for a.e. $x \in \Omega$; $f(x, y, \cdot, \xi)$ is $Q$-periodic for all $(y, \xi) \in \mathbb{R}^N \times \mathbb{R}^{d \times N}$ and for a.e. $x \in \Omega$;

- there exists $\beta > 0$ such that for all $(y,z,\xi) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and for a.e. $x \in \Omega$

$$\frac{1}{\beta}|\xi|^p - \beta \leqslant f(x,y,z,\xi) \leqslant \beta(1+|\xi|^p).$$

This kind of asymptotic problems can be seen as a generalization of the *Iterated Homogenization Theorem* for linear integrands, proved by Bensoussan, Lions and Papanicolau [14], in which the homogenized operator is derived by a formal two-scale asymptotic expansion method. In the $\Gamma$-convergence setting it is customary to assume that

$$|f(x,y,z,\xi) - f(x',y',z,\xi)| \leqslant \omega(|x-x'|+|y-y'|)\big[b(z)+f(x,y,z,\xi)\big], \tag{1.6}$$

for some $b \in L^1_{\text{loc}}(\mathbb{R}^N)$ and some continuous positive real function $\omega$ with $\omega(0) = 0$ (see Braides and Defranceschi [19], Braides and Lukkassen [21] and Lukkassen [60]). As remarked by Allaire (Section 5 in [4]), the natural regularity on $f$ for the integral (1.5) to be well defined is not clear. The measurability of the function $x \mapsto f(x, x/\varepsilon, x/\varepsilon^2, \xi)$, for fixed $\xi$, is assured whenever $f$ is continuous in its second and third variables. The originality of this work is that we do not require any strong uniform continuity hypotheses on $f$ with respect to the first and second variables. In particular, we will recover the results of Fonseca and Zappale in [51], where the authors were also able to weaken hypothesis (1.6) in the convex case when $f = f(y,z,\xi)$, but by using multiscale arguments that however cannot be adapted to the nonconvex setting.

As previous results show, the natural candidate for the $\Gamma(L^p(\Omega))$-limit functional of the family $\{\mathcal{I}_\varepsilon\}_\varepsilon$ is the functional obtained by iterating twice the homogenization formula derived for functionals of the type

$$\int_\Omega f\left(x, \frac{x}{\varepsilon}, \nabla u(x)\right)\, dx.$$

Therefore, we expect that

$$\mathcal{I}_{\text{hom}}(u) := \begin{cases} \displaystyle\int_\Omega \overline{f}_{\text{hom}}(x, \nabla u(x))\, dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\ \infty & \text{otherwise,} \end{cases} \tag{1.7}$$

where $\overline{f}_{\text{hom}}$ is defined, for all $\xi \in \mathbb{R}^{d\times N}$ and for a.e. $x \in \Omega$, by

$$\overline{f}_{\text{hom}}(x,\xi) := \lim_{T\to\infty} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f_{\text{hom}}(x,y,\xi+\nabla\phi(y))\, dy : \ \phi \in W_0^{1,p}\big((0,T)^N;\mathbb{R}^d\big) \right\},$$

and where, for all $(y,\xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and for a.e. $x \in \Omega$,

$$f_{\hom}(x,y,\xi) := \lim_{T\to\infty} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(x,y,z,\xi + \nabla\phi(z))\, dz :\ \phi \in W_0^{1,p}\big((0,T)^N;\mathbb{R}^d\big) \right\}.$$

The analysis we do calls for a new look into the following auxiliary problem.

*Auxiliary problem* (Theorem 4.1.1): Study the asymptotic behavior of the functional $\mathcal{I}_\varepsilon : L^p(\Omega;\mathbb{R}^d) \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, given by

$$\mathcal{I}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\left(x, \frac{x}{\varepsilon}, \nabla u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\ \infty & \text{otherwise.} \end{cases} \tag{1.8}$$

This analysis undertaken in Section 4.1 by means of two-scale convergence arguments allows us to weaken the hypotheses considered in previous works (see Braides [15], Braides and Defranceschi [19] and Braides and Lukkassen [21]), and consequently derive (1.7).

Chapter 5 collects parts of two joint works with J-F Babadjian [9, 10]. Here we study the asymptotic behavior of cylindrical heterogeneous thin domains whose microscopic heterogeneities vary periodically. More precisely, given $\omega \subset \mathbb{R}^2$ open and bounded, we consider thin microstructures of the form $\Omega_\varepsilon := \omega \times (-\varepsilon, \varepsilon)$, whose heterogeneities are periodic of period $\varepsilon$ in the in-plane direction and of period $\varepsilon^2$ in all directions. Two simultaneous features occur in this case: a reiterated homogenization and a dimension reduction phenomena. As is usual, in order to study this asymptotic problem we rescale the thin body into a reference domain of unit thickness $\Omega := \omega \times (-1,1)$ (see e.g. Acerbi, Buttazzo and Percivale [2], Le Dret and Raoult [55]), and we study the rescaled family of functionals defined on $\Omega$, whose dependence on $\varepsilon$ turns out to be explicit in the transverse derivative.

*Main problem of Chapter 5* (Theorem 5.2.4): Characterize the behavior as $\varepsilon$ tends to zero of the family of rescaled functionals defined on $L^p(\Omega;\mathbb{R}^d)$ by

$$\mathcal{W}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega W\left(x, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}, \nabla_\alpha u(x) \Big| \frac{1}{\varepsilon}\nabla_3 u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^3), \\ \infty & \text{otherwise,} \end{cases} \tag{1.9}$$

where $W : \Omega \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfies:

- $W(x, \cdot, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$;

- $W(\cdot, \cdot, \cdot, \xi)$ is $\mathcal{L}^3 \otimes \mathcal{L}^3 \otimes \mathcal{L}^2$-measurable for all $\xi \in \mathbb{R}^{3\times 3}$;

- $y_\alpha \mapsto W(x, y_\alpha, y_3, z_\alpha, \xi)$ is $Q'$-periodic for all $(z_\alpha, y_3, \xi) \in \mathbb{R}^3 \times \mathbb{R}^{3\times 3}$ and for a.e. $x \in \Omega$, where $Q' := (0,1)^2$;

- $(z_\alpha, y_3) \mapsto W(x, y_\alpha, y_3, z_\alpha, \xi)$ is $Q$ -periodic for all $(y_\alpha, \xi) \in \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$ and for a.e. $x \in \Omega$, where $Q := (0,1)^3$;

- there exists $\beta > 0$ such that for all $(y, z_\alpha, \xi) \in \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$ and for a.e. $x \in \Omega$
$$\frac{1}{\beta}|\xi|^p - \beta \leqslant W(x, y, z_\alpha, \xi) \leqslant \beta(1 + |\xi|^p).$$

We identify $W^{1,p}(\omega;\mathbb{R}^3)$ with the set of functions $u \in W^{1,p}(\Omega;\mathbb{R}^3)$ such that $D_3 u(x) = 0$ for a.e. $x \in \Omega$. Under the above hypotheses, the $\Gamma(L^p(\Omega))$-limit of the family $\{\mathcal{W}_\varepsilon\}_\varepsilon$ is given by the functional

$$\mathcal{W}_{\mathrm{hom}}(u) := \begin{cases} 2\displaystyle\int_\omega \overline{W}_{\mathrm{hom}}(x_\alpha, \nabla_\alpha u(x_\alpha))\, dx_\alpha & \text{if } u \in W^{1,p}(\omega;\mathbb{R}^3), \\ \\ \infty & \text{otherwise,} \end{cases}$$

where $\overline{W}_{\mathrm{hom}}$ is defined, for all $\overline{\xi} \in \mathbb{R}^{3\times 2}$ and for a.e. $x_\alpha \in \omega$, by

$$\begin{aligned} \overline{W}_{\mathrm{hom}}(x_\alpha, \overline{\xi}) \quad := \quad & \lim_{T\to\infty} \inf_\phi \left\{ \frac{1}{2T^2} \int_{(0,T)^2 \times I} W_{\mathrm{hom}}(x_\alpha, y_3, y_\alpha, \overline{\xi} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y))\, dy : \right. \\ & \left. \phi \in W^{1,p}((0,T)^2 \times I;\mathbb{R}^3), \quad \phi = 0 \text{ on } \partial(0,T)^2 \times I \right\} \end{aligned} \tag{1.10}$$

and where, for all $(y_\alpha, \xi) \in \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$ and for a.e. $x \in \Omega$,

$$\begin{aligned} W_{\mathrm{hom}}(x, y_\alpha, \xi) \quad := \quad & \lim_{T\to\infty} \inf_\phi \left\{ \frac{1}{T^3} \int_{(0,T)^3} W(x, y_\alpha, z_3, z_\alpha, \xi + \nabla\phi(z))\, dz : \right. \\ & \left. \phi \in W_0^{1,p}((0,T)^3;\mathbb{R}^3) \right\}. \end{aligned}$$

Our main contribution is that we are able to homogenize this material in the reducing direction. As far as we know, there have not been previous results in this direction (see Braides, Fonseca and Francfort [20] and Shu [75]). Let us outline the idea for the derivation of the formula (1.10). In a first step, since the volume of $\Omega_\varepsilon$ is of order $\varepsilon$ and $\varepsilon^2 \ll \varepsilon$, we

can think of $\varepsilon$ as being a fixed parameter and let $\varepsilon^2$ tend to zero. At this point, dimension reduction is not occurring, and (1.9) can be seen as a single one-scale homogenization problem, as in (1.5). This leads to the stored energy density $W_{\text{hom}}(x, y_\alpha, \xi)$. In a second step, $W_{\text{hom}}(x, y_\alpha, \xi)$ is used as the integrand for the following reduction dimension problem.

*Auxiliary problem* (Theorem 5.1.1): Characterize the asymptotic behavior of a family of functionals $\mathcal{I}_\varepsilon : L^p(\Omega; \mathbb{R}^3) \to \overline{\mathbb{R}}$ given by

$$\mathcal{I}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega W\left(x, \frac{x_\alpha}{\varepsilon}, \nabla_\alpha u(x) \Big| \frac{1}{\varepsilon}\nabla_3 u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^3), \\ \\ \infty & \text{otherwise.} \end{cases}$$

This problem will be studied in Section 5.1. Two features differentiate our approach from what is available in most of the literature on the subject (see Braides, Fonseca and Francfort [20] and Shu [75]; see also Chapter 3). The first feature is the use of a two-scale convergence argument as in problem (1.8). The second feature is a decoupling argument, motivated by the work in Babadjian and Francfort [11], to take into account the different nature of the variables that appear in the structure of the limit functional.

We note that if $\Omega$ is assumed to be Lipschitz, as $p > 1$, the $\Gamma$-limit of the previous functionals for $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ would be the same if the weak $W^{1,p}$-topology had been considered in place of the strong $L^p$-topology. For $p = 1$ our argument fails to characterize this $\Gamma$-limit for $u \in W^{1,1}(\Omega; \mathbb{R}^d)$, either with the strong $L^1$-topology or with the weak $W^{1,1}$-topology, since sequences whose gradients are bounded in $L^1$ are not necessarily relatively compact in $W^{1,1}(\Omega; \mathbb{R}^d)$. They are relatively compact only in the space of functions of bounded variation.

We finally remark that, from the applications point of view, it would be interesting to prove similar results to the ones addressed in Chapters 4 and 5 for integrands that are only measurable with respect to (some of) the oscillating variables. This is what is relevant in the case of mixtures. In Chapter 6 we conclude with some generalizations in this direction and we address some open problems for future research.

# Part I

# PRELIMINARIES AND PREVIOUS RESULTS

# 2. PRELIMINARIES

The purpose of this introductory chapter is to give a survey of the concepts and results that are used throughout this dissertation. Almost all these results are stated without proofs as they can be readily found in the references given below.

## 2.1 A short review of Measure Theory

In this section we recall well known results in Measure Theory (see e.g Ambrosio, Fusco and Pallara [7], Evans and Gariepy [46] and Fonseca and Leoni [47], as well as the bibliography therein; see also Brezis [29] and Dugundji [44] for a reference on functional analysis and topological notions).

### 2.1.1 Measures and integration

A *measurable space* is a pair $(X, \mathcal{M})$ where $X$ is a nonempty set and $\mathcal{M}$ is a $\sigma$-algebra in $X$. A set $E \subset X$ is said to be *measurable* if $E \in \mathcal{M}$. If $X$ is a topological space and if not otherwise said, then $\mathcal{M}$ is taken to be the Borel $\sigma$-algebra in $X$, that we denote by $\mathcal{B}(X)$, i.e. the smallest $\sigma$-algebra that contains all open subsets of $X$.

**Definition 2.1.1.** (Measure) A *measure* on $(X, \mathcal{M})$ is a set function $\mu : \mathcal{M} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu$ is $\sigma$-additive, i.e.

$$\mu\Big( \bigcup_{n=0}^{\infty} E_n \Big) = \sum_{n=0}^{\infty} \mu(E_n)$$

for any sequence $\{E_n\}_n$ of pairwise disjoint elements of $\mathcal{M}$.

The triple $(X, \mathcal{M}, \mu)$ is called a *measure space*; it is said to be *$\sigma$-finite* if $X$ is the union of an increasing sequence of sets with finite $\mu$-measure, and it is said to be finite if $\mu(X) < \infty$.

**Definition 2.1.2.** (Borel measure) Let $X$ be a topological space and let $(X, \mathcal{M}, \mu)$ be a measure space. The measure $\mu$ is said to be *Borel* if $\mathcal{B}(X) \subseteq \mathcal{M}$.

*Remark* 2.1.3. Any measure $\mu$ on $(X, \mathcal{M})$ is monotone with respect to set inclusion and continuous along monotone sequences, that is, if $\{E_n\}_n$ is an increasing sequence of sets (respectively a decreasing sequence of sets with $\mu(E_1)$ finite), then

$$\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} \mu(E_n), \quad \text{resp.} \quad \mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} \mu(E_n).$$

**Definition 2.1.4.** Let $(X, \mathcal{M}, \mu)$ be a measure space.

i) A set $N \subset X$ is said to be *$\mu$-negligible* if there exists $E \in \mathcal{M}$ such that $N \subset E$ and $\mu(E) = 0$.

ii) A property $P(x)$, depending on the point $x \in X$, is said to *hold $\mu$-a.e.* (or simply *a.e.* ) in $X$ if the set where $P$ fails is a $\mu$-negligible set.

**Proposition 2.1.5.** *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\mathcal{M}_\mu$ be the collection of all the subsets of $X$ of the form $F = E \cup N$, with $E \in \mathcal{M}$ and $N$ $\mu$-negligible. Then $\mathcal{M}_\mu$ is a $\sigma$-algebra and it is called the* $\mu$-completion *of $\mathcal{M}$.*

**Definition 2.1.6.** ($\mu$-measurable set) Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \subset X$ is said to be *$\mu$-measurable* if $E \in \mathcal{M}_\mu$.

Given $(X, \mathcal{M}, \mu)$ a measure space the measure $\mu$ extends to $\mathcal{M}_\mu$ by setting, for $F$ as above, $\mu(F) = \mu(E)$, and $\mu$ is said to be *complete* if $\mathcal{M} = \mathcal{M}_\mu$. Throughout this work any Borel measure is tacitly understood to be extended to its completion.

We denote by $\mathcal{L}^N$ the usual Lebesgue measure in $\mathbb{R}^N$ as well as its restriction to $\mathcal{B}(\mathbb{R}^N)$. The set $\mathfrak{L}^N$ stands for the $\sigma$-algebra of all Lebesgue measurable sets (that is $\mathfrak{L}^N = \mathcal{B}(\mathbb{R}^N)_{\mathcal{L}^N}$). Given $E \in \mathfrak{L}^N$, we will write indifferently $\mathcal{L}^N(E)$ or $|E|$.

**Definition 2.1.7.** Let $(X, \mathcal{M})$ and $Y$ be, respectively, a measurable and a topological space. Let $u : X \to Y$.

i) (Measurable function) The function $u$ is said to be *$\mathcal{M}$-measurable*, or simply *measurable*, if $u^{-1}(B) \in \mathcal{M}$ for every open set $B \subset Y$.

ii) (Borel function) Assuming that $X$ is also a topological space, the function $u$ is said to be *Borel* if $u^{-1}(B) \in \mathcal{B}(X)$ for every open set $B \subset Y$.

**Definition 2.1.8.** ($\mu$- measurable function) Let $(X, \mathcal{M}, \mu)$ and $Y$ be, respectively, a measure and topological space. A function $u : X \to Y$ is said to be *$\mu$-measurable* if it is $\mathcal{M}_\mu$-measurable.

The following theorem provides conditions guaranteeing the existence of a measurable selection of a given multifunction. It can be found in Castaing and Valadier [30] (Theorem III.30) and it is important for the analysis undertaken in Subsection 4.2.4 below.

**Theorem 2.1.9.** *Let $(X, \mathcal{M}, \mu)$ be a finite complete measure space, and let $Y$ be a complete and separable metric space. Let $F : X \to \{C \subset Y : C \neq \emptyset \text{ and } C \text{ is a closed set}\}$ be a multifunction such that $\{(x, y) \in X \times Y : y \in F(x)\} \in \mathcal{M} \otimes \mathcal{B}(Y)$.[1] Then there exists a sequence of measurable functions $u_n : X \to Y$ such that*

$$F(x) = \overline{\{u_n(x) : \ n \in \mathbb{N}\}}$$

*for $\mu$ a.e. $x \in X$.*

Let $(X, \mathcal{M}, \mu)$ be a measure space and let us denote by $L^p(X, \mu)$, with $1 \leqslant p \leqslant \infty$ , the usual Lebesgue spaces, that is the set (of the equivalence classes) of all $\mu$-measurable functions $u : X \to \mathbb{R}$ such that the (Lebesgue) integral

$$||u||_{L^p(X,\mu)} := \left( \int_X |u|^p \, d\mu \right)^{1/p} < \infty$$

for $p < \infty$, or

$$||u||_{L^\infty(X,\mu)} := \inf\{C \in [0, \infty] : \ |u(x)| \leqslant C \text{ for } \mu\text{-a.e. } \ x \in X\} < \infty.^2$$

We abbreviate $L^p(X, \mu)$ by $L^p(X)$ when this will cause no confusion (e.g. when $\mu$ is the Lebesgue measure in $\mathbb{R}^N$). Is is well known that $L^p(X, \mu)$ is a Banach space with the norm $|| \cdot ||_{L^p(X,\mu)}$ for $1 \leqslant p \leqslant \infty$ (Hilbert when $p = 2$); it is a reflexive space for $1 < p < \infty$, and in this case its dual space is (identified with) $L^q(X)$ with $q = p/(p - 1)$. If $(X, \mathcal{M}, \mu)$ is $\sigma$-finite then $L^\infty(X, \mu)$ is the dual space of $L^1(X, \mu)$; if, in addition, $X$ is separable[3] then $L^p(X, \mu)$ is separable for $1 \leqslant p < \infty$.

[1] The set $\mathcal{M} \otimes \mathcal{B}(Y)$ denotes the usual product $\sigma$-algebra of $\mathcal{M}$ and $\mathcal{B}(Y)$.

[2] $||u||_{L^\infty(X,\mu)}$ is sometimes called the essential supremum of $u$ and written $||u||_{L^\infty(X,\mu)} = \operatorname{ess\,sup}_{x \in X} |u(x)|$; usual convention: $\inf\{\emptyset\} = \infty$.

[3] We recall that a measurable space $(X, \mathcal{M})$ is said to be *separable* if there exists a sequence $\{E_n\}_n \subset \mathcal{M}$ such that the smallest $\sigma$-algebra that contains all the sets $E_n$ is $\mathcal{M}$. If $X$ is a metric space and $\mathcal{M}$ is a Borel $\sigma$-algebra, then $(X, \mathcal{B}(X))$ is a separable space.

**Lemma 2.1.10.** (Chebyshev inequality) *Let* $(X, \mathcal{M}, \mu)$ *be a measure space. If* $u \in L^p(X; \mu)$, *with* $1 \leqslant p < \infty$, *then for any* $t > 0$

$$\mu(\{x \in X : \ |u(x)| > t\}) \leqslant \frac{1}{t^p} \int_X |u|^p \, d\mu.$$

We assume that the reader is familiar with the properties of integrals, measurable functions and $L^p$-spaces. For the sake of completeness we state here the fundamental convergence results in the theory of integration on abstract measure spaces.

**Theorem 2.1.11.** (Levi's Theorem or Monotone Convergence Theorem) *Let* $(X, \mathcal{M}, \mu)$ *be a measure space, and let* $u_n : X \to \mathbb{R}$ *be an increasing sequence of* $\mu$*-measurable functions. Assume that* $u_n \geqslant v$ *for any* $n \in \mathbb{N}$, *with* $v \in L^1(X, \mu)$, *then*

$$\lim_{n\to\infty} \int_X u_n \, d\mu = \int_X \lim_{n\to\infty} u_n \, d\mu.$$

**Lemma 2.1.12.** (Fatou's Lemma) *Let* $(X, \mathcal{M}, \mu)$ *be a measure space and let* $u_n : X \to \mathbb{R}$ *be a sequence of* $\mu$*-measurable functions.*

*i) If there exists* $v \in L^1(X, \mu)$ *such that* $u_n \geqslant v$ *for any* $n \in \mathbb{N}$, *then*

$$\int_X \liminf_{n\to\infty} u_n \, d\mu \leqslant \liminf_{n\to\infty} \int_X u_n \, d\mu.$$

*ii) If there exists* $v \in L^1(X, \mu)$ *such that* $u_n \leqslant v$ *for any* $n \in \mathbb{N}$, *then*

$$\int_X \limsup_{n\to\infty} u_n \, d\mu \geqslant \limsup_{n\to\infty} \int_X u_n \, d\mu.$$

As a consequence we get the following result.

**Corollary 2.1.13.** *Let* $(X, \mathcal{M}, \mu)$ *be a measure space and let* $\{u_n\} \subset L^p(X, \mu)$ *with* $1 \leqslant p < \infty$, *be such that* $u_n \to u$ $\mu$*-a.e. as* $n \to \infty$, *for some function* $u \in L^p(X, \mu)$. *Then* $||u_n - u||_{L^p(X,\mu)} \to 0$ *if and only if* $||u_n||_{L^p(X,\mu)} \to ||u||_{L^p(X,\mu)}$.

**Theorem 2.1.14.** (Dominated Convergence Theorem) *Let* $u$, $u_n : X \to \mathbb{R}$ *be* $\mu$*-measurable functions, and assume that* $u_n \to u$ $\mu$*-a.e. as* $n \to \infty$. *If*

$$\int_X \sup_{n\in\mathbb{N}} |u_n| \, d\mu < \infty$$

*then*

$$\lim_{n\to\infty} \int_X u_n \, d\mu = \int_X u \, d\mu.$$

The following variant of the Dominated Convergence Theorem can be found in Evans and Gariepy [46] and it will be of use in the sequel.

**Proposition 2.1.15.** *Let $v$, $v_n \in L^1(X,\mu)$ and let $u$, $u_n$ be $\mu$-measurable, for $n \in \mathbb{N}$. Suppose that $|u_n| \leqslant v_n$ for all $n \in \mathbb{N}$, and that $u_n$ and $v_n$ converge $\mu$-a.e. to $u$ and $v$, respectively. If in addition*

$$\lim_{n\to\infty} \int_X v_n \, d\mu = \int_X v \, d\mu,$$

*then*

$$\lim_{n\to\infty} \int_X |u_n - u| d\mu = 0.$$

**Theorem 2.1.16.** (Fubini-Tonelli Theorem) *Let $(X_1, \mathcal{M}_1, \mu_1)$ and $(X_2, \mathcal{M}_2, \mu_2)$ be two $\sigma$-finite measure spaces. Then, there is a unique positive $\sigma$-finite measure $\mu$ on $(X_1 \times X_2, \mathcal{M}_1 \otimes \mathcal{M}_2)$ such that*

$$\mu(E_1 \times E_2) = \mu_1(E_1)\mu_2(E_2)$$

*for all $E_1 \in \mathcal{M}_1$ and $E_2 \in \mathcal{M}_2$.*

i) (Tonelli) *In addition, for any measurable function $u : X_1 \times X_2 \to [0, \infty]$ we have that*

$$x \to \int_{X_2} u(x,y) d\mu_2(y) \quad \text{and} \quad y \to \int_{X_1} u(x,y) d\mu_1(x)$$

*are $\mathcal{M}_1$-measurable and $\mathcal{M}_2$-measurable respectively, and*

$$\begin{aligned} \int_{X_1\times X_2} u \, d\mu &= \int_{X_1} \left( \int_{X_2} u(x,y) d\mu_2(y) \right) d\mu_1(x) \qquad (2.1) \\ &= \int_{X_2} \left( \int_{X_1} u(x,y) d\mu_1(x) \right) d\mu_2(y). \end{aligned}$$

ii) (Fubini) *If $u \in L^1(X_1 \times X_2; \mu)$, then $u(x, \cdot) \in L^1(X_2, \mu_2)$ for $\mu_1$-a.e. $x \in X_1$, $u(\cdot, y) \in L^1(X_1, \mu_1)$ for $\mu_2$-a.e. $y \in X_2$, the a.e. defined functions*

$$x \to \int_{X_2} u(x,y) d\mu_2(y) \quad \text{and} \quad y \to \int_{X_1} u(x,y) d\mu_1(y)$$

*are in $L^1(X_1, \mu_1)$ and $L^1(X_2, \mu_2)$, respectively, and equality (2.1) holds.*

We present here another version of Fubini-Tonelli's Theorem which deals with complete measures and is relevant to Lebesgue integration on $\mathbb{R}^N$.

**Theorem 2.1.17.** *Suppose that* $(X_1, \mathcal{M}_1, \mu_1)$ *and* $(X_2, \mathcal{M}_2, \mu_2)$ *are two complete and* $\sigma$-*finite measure spaces. If* $u : X_1 \times X_2 \to [0, \infty]$ *is* $\mu$-*measurable (where* $\mu$ *is the measure given by theorem 2.1.16)*[4], *then the functions*

$$x \to \int_{X_2} u(x,y)d\mu_2(y) \quad \text{and} \quad y \to \int_{X_1} u(x,y)d\mu_1(y)$$

*are respectively* $\mu_1$-*measurable and* $\mu_2$-*measurable and*

$$\begin{aligned}\int_{X_1 \times X_2} u\, d\mu &= \int_{X_1} \left( \int_{X_2} u(x,y)d\mu_2(y) \right) d\mu_1(x) \\ &= \int_{X_2} \left( \int_{X_1} u(x,y)d\mu_1(x) \right) d\mu_2(y).\end{aligned}$$

### 2.1.2 Radon measures and Vitali's Covering Theorem

**Definition 2.1.18.** (Radon Measure on $(X, \mathcal{M})$) Let $X$ be a topological space. A *Radon measure* on a measurable space $(X, \mathcal{M})$ is a Borel measure, finite on compact sets, and such that for every open set $E \subset X$

$$\mu(E) = \sup\{\mu(K) : \; K \subset E, \, K \text{ compact}\} \; (\textit{inner-regularity}),$$

and for every set $E \in \mathcal{M}$

$$\mu(E) = \sup\{\mu(A) : \; E \subset A, \, A \text{ open}\} \; (\textit{outer-regularity}).$$

*Remark* 2.1.19. If $X$ is a locally compact Hausdorff space the following two properties hold.[5]

i) (see Theorem 2.7 in Rudin [71]) Let $E, K \subset X$ be an open and compact set, respectively, with $K \subset E$. Then there is an open set $A$ with compact closure such that

$$K \subset A \subset \overline{A} \subset E.$$

[4] By Definition 2.1.6 this means measurable with respect to the completion $\sigma$-algebra of $\mathcal{M}_1 \otimes \mathcal{M}_2$; if $\mu$ is a Borel measure then it is understood to be extended to $(\mathcal{M}_1 \otimes \mathcal{M}_2)_\mu$. In the case of the Lebesgue measure $\mathcal{B}(\mathbb{R}^N) = \mathcal{B}(\mathbb{R}) \overset{\text{N-times}}{\otimes \cdots \otimes} \mathcal{B}(\mathbb{R})$.

[5] A topological space $X$ is said to be *Hausdorff* if given two distinct points $x$ and $y$, there are disjoint open sets $E_1$ and $E_2$ such that $x \in E_1$ and $y \in E_2$; $X$ is said to be *locally compact* if for each $x \in X$ there is an open set $E$ containing $x$ such that $\overline{E}$ is compact. Metric spaces are Hausdorff spaces and $\mathbb{R}^N$ with the usual metric is Hausdorff and locally compact.

ii) Let $\mu$ be a Radon measure on $X$ and let $E$ be an open set of $X$. Then

$$\mu(E) = \sup\{\mu(K) : \ K \subset E, \ K \text{ compact}\} = \sup\{\mu(A) : \ A \subset\subset E, A \text{ open}\},$$

where $A \subset\subset E$ means that $\overline{A}$ is a compact set with $\overline{A} \subset E$.

Given a topological space $X$ we denote by $\mathcal{A}(X)$ the family of all its open subsets. The following lemma provides sufficient conditions for a set function $\Pi : \mathcal{A}(X) \to [0, \infty)$ to be the restriction of a Radon measure on $\mathcal{A}(X)$. It is close in spirit to De Giorgi-Letta's criterion (see [39]) and it is of importance to apply the Direct Method of $\Gamma$-convergence (see Chapters 4 and 5 below) as well as for the use of relaxation methods that strongly rely on the structure of Radon measures.

**Lemma 2.1.20.** (see Fonseca and Malý [48]; also Fonseca and Leoni [47]) *Let $X$ be a locally compact Hausdorff space, let $\Pi : \mathcal{A}(X) \to [0, \infty)$, and let $\mu$ be a finite Radon measure $\mu$ on $X$ satisfying*

i) *(nested-subadditivity)* $\Pi(D) \leq \Pi(D \backslash \overline{B}) + \Pi(C)$ *for all* $B, C, D \in \mathcal{A}(X)$ *with* $B \subset\subset C \subset D$;

ii) *Given* $D \in \mathcal{A}(X)$, *for all* $\epsilon > 0$ *there exists* $D_\varepsilon \in \mathcal{A}(X)$ *such that* $D_\varepsilon \subset\subset D$ *and* $\Pi(D \backslash \overline{D}_\varepsilon) \leq \varepsilon$;

iii) $\Pi(X) \geq \mu(X)$;

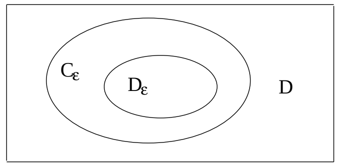
iv) $\Pi(D) \leq \mu(\overline{D})$ *for all* $D \in \mathcal{A}(X)$.

*Then* $\Pi = \mu|_{\mathcal{A}(X)}$.

*Proof.* Fix $D \in \mathcal{A}(X)$. We start by proving that the inequality $\Pi(D) \leqslant \mu(D)$ holds. Let $\varepsilon > 0$ and by condition ii) choose $D_\varepsilon \in \mathcal{A}(X)$ such that $\overline{D}_\varepsilon$ is a compact set, $\overline{D}_\varepsilon \subset D$ and $\Pi(D \backslash \overline{D}_\varepsilon) \leq \varepsilon$. As $X$ is a locally compact Hausdorff space, we can find $C_\varepsilon \in \mathcal{A}(X)$ such that $\overline{D}_\varepsilon \subset C_\varepsilon \subset \overline{C}_\varepsilon \subset D$ (see Remark 2.1.19).

By hypotheses i), ii) and iv) we have

$$\Pi(D) \leqslant \Pi(D \setminus \overline{D}_\varepsilon) + \Pi(C_\varepsilon) \leqslant \varepsilon + \mu(\overline{C}_\varepsilon) \leqslant \varepsilon + \mu(D),$$




and then letting $\varepsilon \to 0$ it follows that

$$\Pi(D) \leqslant \mu(D).$$

To prove the reverse inequality, using the inner regularity property of the measure $\mu$ (see Remark 2.1.19), for every $\varepsilon > 0$ we may find $B \in \mathcal{A}(X)$ with $B \subset\subset D$ and such that

$$\mu(D) \leqslant \varepsilon + \mu(B).$$

Therefore

$$\mu(D) \leqslant \varepsilon + \mu(X) - \mu(X \setminus \overline{B})$$

and, consequently, by iii) and the previous step

$$\mu(D) \leqslant \varepsilon + \Pi(X) - \Pi(X \setminus \overline{B}).$$

Hence by i) we have

$$\mu(D) \leqslant \varepsilon + \mu(D)$$

and therefore letting $\varepsilon \to 0$ we get $\mu(D) \leqslant \Pi(D)$.

■

Given $a \in \mathbb{R}^N$ and $\delta > 0$ we denote by $B(a, \delta) := \{x \in \mathbb{R}^N : |x - a| < \delta\}$.

**Theorem 2.1.21.** (Vitali's Covering Theorem) (see Braides and Defranceschi [19]) *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with $N \geqslant 1$, and let $\mathcal{F}$ be a family of closed subsets of $\Omega$. If there exists a positive number $M > 1$ such that for each $F \in \mathcal{F}$, $B(x, \delta) \subseteq F \subseteq B(x, M\delta)$ for some $x \in \Omega$ and $\delta > 0$, and if*

$$\inf\{\text{diam}F : \; x \in F, \; F \subset \mathcal{F}\} = 0$$

*for a.e. $x \in \Omega$, then there exists a disjoint countable subfamily $\{F_j\}_j$ of $\mathcal{F}$ such that*

$$\Big|\Omega \setminus \bigcup_j F_j\Big| = 0.$$

### 2.1.3 Decomposition and differentiation of measures

**Definition 2.1.22.** Let $(X, \mathcal{M}, \mu)$ and $(X, \mathcal{M}, \nu)$ be two measure spaces. The measure $\nu$ is said to be *absolutely continuous* with respect to $\mu$, and we write $\nu << \mu$, if for every $E \in \mathcal{M}$ the following implication holds:

$$\mu(E) = 0 \;\Rightarrow\; \nu(E) = 0.$$

The measures $\mu$ and $\nu$ are said to be *mutually singular*, and we write $\nu \perp \mu$, if there exists $E \in \mathcal{M}$ such that $\mu(E) = 0$ and $\nu(X \setminus E) = 0$.

**Theorem 2.1.23.** (Lebesgue-Radon-Nikodým Theorem) *Let $(X, \mathcal{M}, \mu)$ and $(X, \mathcal{M}, \nu)$ be two $\sigma$-finite measure spaces. Then there exists a unique pair of measures $\nu_a$ and $\nu_s$ such that $\nu_a << \mu$, $\nu_s \perp \mu$ and $\nu = \nu_a + \nu_s$. Moreover, there is a unique measurable function $u : X \to [0, \infty]$ such that for all $E \in \mathcal{M}$*

$$\nu_a(E) = \int_E u \, d\mu.$$

The decomposition $\nu = \nu_a + \nu_s$, where $\nu_a << \mu$ and $\nu_s \perp \mu$, is called the *Lebesgue decomposition* of $\nu$ with respect to $\mu$. In the case where $\nu << \mu$, Theorem 2.1.23 says that

$$\nu(E) = \int_E u \, d\mu$$

for all $E \in \mathcal{M}$. This result is known as the Radon-Nikodým Theorem, and $u$ is called the *Radon-Nikodým derivative* of $\nu$ with respect to $\mu$, $u = d\nu/d\mu$.

**Theorem 2.1.24.** (General version of the Besicovitch derivation Theorem)(see Proposition 2.2 in Ambrosio and Dal Maso [6]) *Let $\mu$ and $\nu$ be two Radon measures on $\mathbb{R}^N$ and let $\nu = \nu_a + \nu_s$ be the Lebesgue decomposition of $\nu$ with respect to $\mu$ ($d\nu_a = u d\mu$). There exists a Borel set $E \subset \mathbb{R}^N$, with $\mu(E) = 0$, such that, for every $x \in \mathbb{R}^N \setminus E$ and $C \subset \mathbb{R}^N$ open bounded convex set containing the origin, the limit*

$$\lim_{\delta \downarrow 0} \frac{\nu(x + \delta C)}{\mu(x + \delta C)}$$

*exists, is finite, and coincides with $u(x)$.*

Let $X$ be a topological space and let $\mu$ be a measure on $X$. A function $u$ is said to be in $L^1_{\mathrm{loc}}(X, \mu)$ if $u \in L^1(E, \mu)$ whenever $E \subset\subset X$. Theorem 2.1.24 implies the following result.

**Proposition 2.1.25.** (Lebesgue Differentiation Theorem) *Let $\mu$ be a Radon measure on $\mathbb{R}^N$ and let $u \in L^1_{\text{loc}}(\mathbb{R}^N, \mu)$. Then for a.e. $x \in \mathbb{R}^N$*

$$\lim_{\delta \downarrow 0} \frac{1}{\mu(B(x,\delta))} \int_{B(x,\delta)} |u(y) - u(x)| \, d\mu(y) = 0, \tag{2.2}$$

*and in particular*

$$u(x) = \lim_{\delta \downarrow 0} \frac{1}{\mu(B(x,\delta))} \int_{B(x,\delta)} u(y) \, d\mu(y).$$

**Definition 2.1.26.** Any point $x$ where (2.2) holds is called a *Lebesgue point of $u$.*

**Theorem 2.1.27.** (see Theorem 2.8 in Fonseca and Müller [49]) *Let $\mu$ be a Radon measure on $\mathbb{R}^N$ and $u \in L^1_{\text{loc}}(\mathbb{R}^n, \mu)$. Then there exists a Borel set $E \subset \mathbb{R}^N$, with $\mu(E) = 0$, such that for every $x \in \mathbb{R}^N \setminus E$*

$$\lim_{\delta \downarrow 0} \frac{1}{\mu(x + \delta C)} \int_{x+\delta C} |u(y) - u(x)| \, d\mu(y) = 0, \tag{2.3}$$

*for every $C \subset \mathbb{R}^N$ open bounded convex set containing the origin.*

**Definition 2.1.28.** (Signed measure) A *signed measure* on a measurable space $(X, \mathcal{M})$ is a set function $\mu : \mathcal{M} \to [-\infty, \infty]$ such that $\mu(\emptyset) = 0$, $\mu$ takes at most one of the values $\infty$ or $-\infty$, and for any family $\{E_n\}_n$ of pairwise disjoint elements of $\mathcal{M}$

$$\mu\Big( \bigcup_{n=0}^{\infty} E_n \Big) = \sum_{n=0}^{\infty} \mu(E_n).^6$$

In particular a measure is a signed measure.

**Definition 2.1.29.** (Total variation of a signed measure) Let $(X, \mathcal{M})$ be a measurable space and let $\mu : \mathcal{M} \to [-\infty, \infty]$ be a signed-measure. Its *total variation* $|\mu| : \mathcal{M} \to [0, \infty]$ is defined by

$$|\mu|(E) := \sup \left\{ \sum_{n=0}^{\infty} |\mu(E_n)| : \ E_n \in \mathcal{B}(\mathcal{M}) \text{ pairwise disjoint, } E = \bigcup_{n=0}^{\infty} E_n \right\}.$$

If $\mu$ is a positive measure then $\mu = |\mu|$, and if $\mu$ is a signed-measure then $|\mu|$ is a measure. The signed measure $\mu$ is said to be *$\sigma$-finite* if $|\mu|$ is a $\sigma$-finite measure on $X$; $\mu$ is said to be a *(signed) Radon measure* if its total variation $|\mu|$ is a Radon measure. Finally, given two signed measures $\mu$ and $\tau$ on a measurable space $(X, \mathcal{M})$, $\mu$ is absolutely continuous with respect to $\tau$ (respectively mutually singular) if $|\mu| << |\tau|$ (respectively $|\mu| \perp |\tau|$), and an analog of the Lebesgue-Radon-Nikodým Theorem holds for signed-measures.

[6] The absolutely convergence of this series is understood.

### 2.1.4 Weak$\star$ convergence of measures

Let $X$ be a locally compact Hausdorff space, and let $C_c(X)$ denote the set of continuous functions with compact support on $X$. We denote by $C_0(X)$ the completion of $C_c(X)$ with respect to the supremum norm.[7] By the Riesz-Representation Theorem the dual of the Banach space $C_0(X)$ is the space of finite (signed) Radon measures $\mu : \mathcal{B}(\mathbb{R}^N) \to \mathbb{R}$. This characterization leads to the following notion of convergence of a sequence of finite (signed) Radon measures.

**Definition 2.1.30.** (Weak$\star$ convergence of measures) Let $\{\mu_n\}_n$ be a sequence of finite (signed) Radon measures on $X$. This sequence is said to weak$\star$ converge to a finite (signed) Radon measure $\mu$ on $X$, and we write $\mu_n \overset{\star}{\rightharpoonup} \mu$, if for all $\phi \in C_0(X)$

$$\lim_{n\to\infty} \int_X \phi \, d\mu_n = \int_X \phi \, d\mu.$$

**Proposition 2.1.31.** (Weak$\star$ compactness property) *Let $X$ be a $\sigma$-compact metric space.*[8] *Then every sequence $\{\mu_n\}_n$ of finite (signed) Radon measures on $X$ with $\sup_{n\in\mathbb{N}}\{|\mu_n|(X)\} < \infty$ has a weak$\star$ converging subsequence.*

**Proposition 2.1.32.** *Let $X$ be a locally compact Hausdorff space and let $\{\mu_n\}_n$ be a sequence of finite (signed) Radon measures on $X$ such that $\mu_n \overset{\star}{\rightharpoonup} \mu$. Then*

*i) if $K \subset X$ is compact*

$$\mu(K) \geq \limsup_{n\to\infty} \mu_n(K);$$

*ii) if $A \subset X$ is open*

$$\mu(A) \leq \liminf_{n\to\infty} \mu_n(A);$$

*iii) if $A \subset\subset X$ is open and $\mu(\partial A) = 0$*

$$\mu(A) = \lim_{n\to\infty} \mu_n(A).$$

[7] The *support* of $u$ is by definition $\mathrm{supp}\,(u) := \overline{\{x \in X : \ u(x) \neq 0\}}$; the function $u$ is said to have *compact support* in $X$ if $\mathrm{supp}\,(u) \subset\subset X$.

[8] A metric space is said to be *$\sigma$-compact* if it is the union of a countable collection of compact subsets; for instance $\mathbb{R}^N$ is a $\sigma$-compact metric space.

## 2.2 Sobolev spaces

The aim of this section is to give the main properties needed throughout the text on weak derivatives and Sobolev spaces. We refer to the books of Adams [3], Brezis [29], Evans and Gariepy [46], Fonseca and Leoni [47], Giusti [57], and Ziemer [77] for a detailed analysis on this topic.

### 2.2.1 Definition and main properties

Let $\Omega$ be an open subset of $\mathbb{R}^N$ with $N \geqslant 1$, and let $1 \leqslant p \leqslant \infty$. In the sequel $W^{1,p}(\Omega)$ (respectively $W^{1,p}_{\text{loc}}(\Omega)$) stands for the usual Sobolev space, that is the space of functions $u \in L^p(\Omega)$ (respectively $L^p_{\text{loc}}(\Omega)$) with weak derivatives of order one in $L^p(\Omega)$ (respectively $L^p_{\text{loc}}(\Omega)$).[9] For any $u \in W^{1,p}(\Omega)$ we set $\nabla u := (\nabla_1 u, ..., \nabla_N u)$. The space $W^{1,p}_0(\Omega)$ stands for the closure of $C^\infty_0(\Omega)$ in $W^{1,p}(\Omega)$ for $1 < p < \infty$. $W^{1,\infty}_0(\Omega)$ denotes the closure of $C^\infty_0(\Omega)$ in the weak$\star$ topology of $W^{1,\infty}(\Omega)$. It is well known that $W^{1,p}(\Omega)$ is a Banach space (Hilbert for $p = 2$) when endowed with the norm

$$||u||_{W^{1,p}(\Omega)} := \left( ||u||^p_{L^p(\Omega)} + \sum_{i=1}^N ||\nabla_i u||^p_{L^p(\Omega)} \right)^{1/p}$$

for $1 \leqslant p < \infty$; for $p = \infty$ the norm is given by

$$||u||_{W^{1,\infty}(\Omega)} := ||u||_{L^\infty(\Omega)} + \sum_{i=1}^N ||\nabla_i u||_{L^\infty(\Omega)}.^{10}$$

Since for $1 < p < \infty$ the space $W^{1,p}(\Omega)$ is reflexive, for each bounded sequence $\{u_n\}_n \subset W^{1,p}(\Omega)$, with $1 < p < \infty$, there exists a subsequence $\{u_{n_k}\}_k \subset W^{1,p}(\Omega)$ and $u \in W^{1,p}(\Omega)$ such that $u_{n_k} \rightharpoonup u$ in $W^{1,p}(\Omega)$ (see Theorem III.27 in Brezis [29]). For $1 \leqslant p < \infty$ the space $W^{1,p}(\Omega)$ is separable. Finally we remark that the $W^{1,p}(\Omega)$-weak limit of a sequence in $W^{1,p}_0(\Omega)$ still belongs to $W^{1,p}_0(\Omega)$ since a convex subset of a Banach space is closed with respect to the weak topology if and only if it is closed with respect to the strong topology.

[9] For any $i \in \{1, ..., d\}$ we set $\nabla_i := \partial/\partial x_i$. Given $u, v \in L^p_{\text{loc}}(\Omega)$ we recall that $v$ is said to be the $i^{th}$-*derivative* of $u$ and we write $\nabla_i u = v$, provided

$$\int_\Omega u \nabla_i \varphi \, dx = -\int_\Omega v \varphi \, dx$$

for all functions $\varphi \in C^\infty_c(\Omega)$.

[10] $||u||_{W^{1,p}(\Omega)} \equiv ||u||_{L^p(\Omega)} + ||\nabla u||_{L^p(\Omega)}$.

For $d \geqslant 1$ we denote by

$$L^p(\Omega;\mathbb{R}^d) := \left\{ u : \Omega \to \mathbb{R}^d : \ u_i \in L^p(\Omega) \text{ for all } i \in \{1, ..., d\} \right\};$$

$$W^{1,p}(\Omega;\mathbb{R}^d) := \left\{ u \in L^p(\Omega;\mathbb{R}^d) : \nabla_j u \in L^p(\Omega;\mathbb{R}^d) \text{ for } j = 1, ..., N \right\},$$

where $\nabla_j u := (\nabla_j u_1, ..., \nabla_j u_d)$. If $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ we write $\nabla u := (\nabla_1 u|...|\nabla_N u)$.

The following result gives a sufficient condition for a function to belong to $W^{1,p}(\Omega)$. It is a straightforward consequence of the definition of the space and of the properties above.

**Proposition 2.2.1.** *Let $\Omega \subset \mathbb{R}^N$ be open, and let $\{u_n\}_n$ be a sequence in $W^{1,p}(\Omega)$ converging in $L^p(\Omega)$ to some function $u$. Then*

*i) if $1 \leqslant p \leqslant \infty$ and there exists a function $g \in L^p(\Omega;\mathbb{R}^N)$ such that $\nabla u_n \to g$ in $L^p(\Omega;\mathbb{R}^N)$, then $u \in W^{1,p}(\Omega)$ and $g = \nabla u$;*

*ii) if $1 < p \leqslant \infty$ and the sequence $\{\nabla u_n\}_n$ is bounded in $L^p(\Omega;\mathbb{R}^N)$ then $u \in W^{1,p}(\Omega)$ and $\nabla u_n \rightharpoonup \nabla u$ in $L^p(\Omega;\mathbb{R}^N)$ (weak$\star$ if $p = \infty$).*[11]

### 2.2.2 Extension, approximation and traces

**Theorem 2.2.2.** (Extension Theorem) *Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary and let $1 \leqslant p \leqslant \infty$. Let $\tilde{\Omega}$ be any open set such that $\Omega \subset\subset \tilde{\Omega}$. Then there exist a bounded linear (extension) operator*

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$$

*such that $Eu = u$ in $\Omega$ and* $\operatorname{supp}(Eu) \subset \tilde{\Omega}$ *for all $u \in W^{1,p}(\Omega)$.*

**Theorem 2.2.3.** (Approximation by smooth functions) *Let $\Omega$ be an open subset of $\mathbb{R}^N$ and let $1 \leqslant p < \infty$. Then*

*i) (Meyers-Serrin) $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$;*

*ii) if, in addition, $\Omega$ is a bounded Lipschitz set, then the restriction to $\Omega$ of functions in $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\Omega)$.*

[11] If $p = 1$ the function $u$ is in $BV(\Omega)$, the space of functions with bounded variation, but not necessarily in $W^{1,1}(\Omega)$.

Next we recall a Trace Theorem that makes it possible to assign "boundary values" along $\partial\Omega$ to a function $u \in W^{1,p}(\Omega)$.

**Theorem 2.2.4.** (Trace Theorem) *Let $\Omega \subset \mathbb{R}^N$ be a bounded set with Lipschitz boundary, and let $1 \leqslant p < \infty$. Then there exists a bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that*

*i) $Tu = u\lfloor\partial\Omega$ if $u \in C(\overline{\Omega})$;*

*ii) $||Tu||_{L^p(\partial\Omega)} \leqslant C||u||_{W^{1,p}(\Omega)}$ for each $u \in W^{1,p}(\Omega)$, for some constant $C = C(N, p, \Omega)$.*

*Moreover $Ker(T) = W_0^{1,p}(\Omega)$.*

### 2.2.3 Compactness and Poincaré inequalities

**Theorem 2.2.5.** (Sobolev-Rellich-Kondrachov Theorem) *Let $\Omega$ be an open bounded set in $\mathbb{R}^N$ with Lipschitz boundary and let $1 \leqslant p < \infty$. Then*

*i) for $1 \leqslant p < N$ and $1 \leqslant q < p^\star = \frac{Np}{N-p}$*

$$W^{1,p}(\Omega) \subset L^q(\Omega),$$

*and the imbedding is compact;*[12]

*ii) if $p = N$ then for every $1 \leqslant q < \infty$*

$$W^{1,p}(\Omega) \subset L^q(\Omega),$$

*and the imbedding is compact;*

*iii) if $p > N$ then*

$$W^{1,p}(\Omega) \subset C(\overline{\Omega}),$$

*and the imbedding is compact.*

*Remark* 2.2.6. Under hypotheses of Theorem 2.2.5:

[12] Recall that given $X$ and $Y$ two Banach spaces, $X \subset Y$, the space $X$ is said to be compactly embedded in $Y$ if $||x||_Y \leqslant C||x||_X$ for some constant $C$, and if each bounded sequence in $X$ has a convergent subsequence in $Y$.

a) condition *iii*) is still true for $p = \infty$;

b) the imbedding $W^{1,p}(\Omega) \subset L^p(\Omega)$ is compact for every $1 \leqslant p \leqslant \infty$ (when $p = \infty$ this follows from Morrey's inequality and the Arzela-Ascoli Theorem; see Brezis [29]);

c) if $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ with $1 \leqslant p \leqslant \infty$ ($\overset{\star}{\rightharpoonup}$ if $p = \infty$) then

- If $1 \leqslant p < N$, $u_n \to u$ in $L^q(\Omega)$ for every $1 \leqslant q < p^\star$;
- if $p = N$, $u_n \to u$ in $L^q(\Omega)$ for every $1 \leqslant q < \infty$;
- if $N < p \leqslant \infty$, then $u_n \to u$ in $L^\infty(\Omega)$.

In particular $u_n \to u$ in $L^p(\Omega)$ for every $1 \leqslant p \leqslant \infty$.

*Remark* 2.2.7. All the conclusions of Theorem 2.2.5 hold in $W_0^{1,p}(\Omega)$ for a general open bounded set $\Omega \subset \mathbb{R}^N$.

**Proposition 2.2.8.** (Poincaré-type inequalities)

*i) (Poincaré inequality) Let $\Omega$ be an open set in $\mathbb{R}^N$ with finite width, that is, a subset of $\mathbb{R}^N$ that lies between two parallel hyperplanes, and let $1 \leqslant p < \infty$. Then there exist a constant $C$ (depending only on p, $N$ and the distance between the two planes) such that for all $u \in W_0^{1,p}(\Omega)$*

$$\int_\Omega |u|^p \, dx \leqslant C \int_\Omega |\nabla u|^p \, dx.$$ [13]

*ii) Let $\Omega$ be a bounded and connected subset of $\mathbb{R}^N$ with Lipschitz boundary. Then there exists a constant $C = C(N, p, \Omega)$ such that*

$$||u - u_\Omega||_{L^p(\Omega)} \leqslant C||\nabla u||_{L^p(\Omega)},$$

*for all $u \in W^{1,p}(\Omega)$ with $1 \leqslant p \leqslant \infty$, where $u_\Omega = \frac{1}{|\Omega|} \int_\Omega u \, dx$.*

[13] Thus $\|u\|_{W^{1,p}}$ and $||\nabla u||_{L^p(\Omega)}$ are equivalent in the space $W_0^{1,p}(\Omega)$. In particular, this result is true for any open bounded set of $\mathbb{R}^N$.

### 2.2.4 Weak convergence and decomposition lemmas for sequences of gradients and of scaled-gradients

We conclude this section by recalling two decomposition lemmas that will allow us to characterize the $\Gamma$-limit of the functionals studied in Chapters 4 and 5 by considering recovering sequences whose gradients have equi-integrability properties. We start with a short review of equi-integrability. Throughout this part we assume that $\Omega$ is an open bounded subset of $\mathbb{R}^N$ with $N \geqslant 1$.

**Definition 2.2.9.** (Equi-integrability) A sequence of functions $\{u_n\}_n \subset L^1(\Omega)$ is said to be *equi-integrable* if for all $\varepsilon > 0$ there exist $\delta > 0$ such that

$$\sup_{n\in\mathbb{N}} \int_E |u_n|\, dx < \varepsilon$$

whenever $E \subset \Omega$ with $|E| < \delta$.

**Proposition 2.2.10.** *A sequence $\{u_n\}_n \subset L^1(\Omega)$ is equi-integrable if and only if one of the two following conditions holds:*

*i) for every $\varepsilon > 0$ there exists a constant $M > 0$ such that for all $n \in \mathbb{N}$*

$$\int_{\{x\in\Omega:\ |u_n|>M\}} |u_n|\, dx \leqslant \varepsilon;$$

*ii)* (De la Vallée Poussin Criterion) *there exists an increasing and continuous function $\varphi : [0,\infty) \to [0,\infty]$ satisfying*

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$$

*and such that*

$$\int_\Omega \varphi(|u_n|)\, dx \leqslant 1 \quad \text{for all } n \in \mathbb{N}.$$

**Theorem 2.2.11.** (Dunford-Pettis Theorem) *A sequence $\{u_n\}_n \subset L^1(\Omega)$ is weakly compact in $L^1(\Omega)$ if and only if*

*i) $\{u_n\}_n$ is bounded in $L^1(\Omega)$,*

*ii) $\{u_n\}_n$ is equi-integrable.*

In particular, if $u_n \rightharpoonup u$ in $L^1(\Omega)$ then $\{u_n\}_n$ is equi-integrable. In fact the following characterization holds.

**Proposition 2.2.12.** *A sequence $u_n \rightharpoonup u$ in $L^1(\Omega)$ if and only if*

*i)* $\sup_{n\in\mathbb{N}} ||u_n||_{L^1(\Omega)} < \infty$,

*ii)* $\int_C u_n \, dx \to \int_C u \, dx$ *for any cube* $C \subset \mathbb{R}^N$,

*ii)* $\{u_n\}_n$ *is equi-integrable.*

**Definition 2.2.13.** ($p$-equi-integrability) A sequence $\{u_n\}_n \subset L^p(\Omega)$, with $1 < p < \infty$, is said to be $p$-equi-integrable if $\{|u_n|^p\}_n$ is equi-integrable.

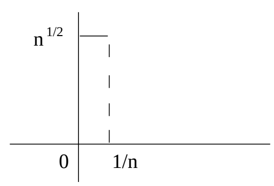
**Theorem 2.2.14.** (Vitali's Theorem) *Let $1 < p < \infty$. A sequence $\{u_n\}_n \subset L^p(\Omega)$ converges strongly to $u$ in $L^p(\Omega)$ if and only if*

*i)* $\{u_n\}_n$ *converges to $u$ in measure*[14],

*ii)* $\{u_n\}_n$ *is $p$-equi-integrable.*

In general $u_n \rightharpoonup u$ in $L^p$ with $1 < p < \infty$ does not imply that $\{u_n\}_n$ is $p$-equi-integrable.

*Example.* Let $u_n(x) = \sqrt{n}\chi_{[0,\frac{1}{n}]}(x)$ with $x \in \mathbb{R}$, where $\chi_{[0,\frac{1}{n}]}$ denotes the characteristic function of the interval $\left[0, \frac{1}{n}\right]$. Then $u_n \rightharpoonup 0$ in $L^2(\mathbb{R})$ but $\{|u_n|^2\}_n$ is not equi-integrable.




The following result characterizes weak convergent sequences in $L^p(\Omega)$ for $1 < p \leqslant \infty$.

**Proposition 2.2.15.** *A sequence $u_n \rightharpoonup u$ in $L^p(\Omega)$ with $1 < p \leqslant \infty$ ($\overset{\star}{\rightharpoonup}$ if $p = \infty$) if and only if*

*i)* $\sup_{n\in\mathbb{N}} ||u_n||_{L^p(\Omega)} < \infty$,

*ii)* $\int_C u_n \, dx \to \int_C u$ *for any cube* $C \subset \mathbb{R}^N$.

[14] This means that $|\{x \in \Omega : \ |u_n(x) - u(x)| > \varepsilon\}| \to 0$, as $n \to \infty$, for every $\varepsilon > 0$.

As a consequence of next theorem each sequence with bounded gradients in $L^p$, for $1 < p < \infty$, admits a subsequence that can be decomposed as a sum of a sequence with $p$-equi-integrable gradients and a remainder that converges to zero in measure. This property turns out to be an important tool for the asymptotic analysis of integral functionals relying on localization arguments.

**Theorem 2.2.16.** (Decomposition Lemma) (see Fonseca and Leoni [47]; see also Fonseca, Müller and Pedregal [50] and Kristensen [56]) *Let $1 < p < \infty$ and assume that $\partial\Omega$ is Lipschitz, and that $u_n \rightharpoonup v_0$ in $W^{1,p}(\Omega;\mathbb{R}^d)$. Then, there exists a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}_n$ and a sequence $\{v_k\}_k \subset W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^d)$ such that*

*i)* $v_k \rightharpoonup v_0$ *in* $W^{1,p}(\Omega;\mathbb{R}^d)$,

*ii)* $v_k = v_0$ *in a neighborhood of* $\partial\Omega$,

*iii)* $\{\nabla v_k\}_k$ *is p-equi-integrable,*

*iv)* $\lim_{k\to\infty} \mathcal{L}^N(\{x \in \Omega : v_k(x) \neq u_{n_k}(x)\}) = 0.$

The analog of this Theorem for sequences of scaled-gradients in a cylindrical domain of $\mathbb{R}^3$ will be relevant for the applications to thin films.

**Theorem 2.2.17.** (Theorem 1.1 in Bocea and Fonseca [22]) *Let $\Omega := \omega \times (-1,1)$, where $\omega \subset \mathbb{R}^2$ is an open bounded set with Lipschitz boundary. Let $\{\varepsilon_n\}_n$ be a sequence of positive real numbers converging to zero, and let $\{u_n\}_n$ be a bounded sequence in $W^{1,p}(\Omega;\mathbb{R}^3)$, with $1 < p < \infty$, satisfying*

$$\sup_{n\in\mathbb{N}} \int_\Omega \left| \left( \nabla_\alpha u_n \Big| \frac{1}{\varepsilon_n} \nabla_3 u_n \right) \right|^p dx_\alpha\, dx_3 < \infty.$$

*Suppose further that $u_n \rightharpoonup v_0$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ and $\frac{1}{\varepsilon_n}\nabla_3 v_n \rightharpoonup b$ in $L^p(\Omega;\mathbb{R}^3)$. Then there exists a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}_n$ and a sequence $\{v_k\}_k \subset W^{1,\infty}(\Omega;\mathbb{R}^3)$ such that*

*i)* $v_k \rightharpoonup v_0$ *in* $W^{1,p}(\Omega;\mathbb{R}^3)$;

*ii)* $\frac{1}{\varepsilon_{n_k}} D_3 v_k \rightharpoonup b$ *in* $L^p(\Omega;\mathbb{R}^3)$;

*iii)* $\left\{ \left( D_\alpha v_k \Big| \frac{1}{\varepsilon_{n_k}} D_3 v_k \right) \right\}_k$ *is p-equi-integrable;*

*iv)* $\lim_{k\to\infty} \mathcal{L}^3(\{x \in \Omega : \ v_k(x) \neq u_{n_k}(x)\}) = 0.$

## 2.3 An overview of the Direct Method of the Calculus of Variations

A typical problem in the Calculus of Variations is to minimize an integral functional (energy) of the form

$$\mathcal{I}(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx \tag{2.4}$$

in a class of admissible functions $u : \Omega \to \mathbb{R}^d$ with $d \geqslant 1$ (deformations in the context of elasticity) where $\Omega$ is an open bounded set in $\mathbb{R}^N$ with $N \geqslant 1$ (reference configuration) and the integrand is some Borel function $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$ satisfying appropriate growth and coercivity conditions. The objective of this section is to give an overview of Tonelli's Direct Method of the Calculus of Variations. This is the most classical way of proving the existence of a minimizer for this kind of variational problems.

### 2.3.1 The basic notions

We start by recalling the notions of semicontinuity and coercivity on a topological space $X$, the two main ingredients of Tonelli's Direct Method of the Calculus of Variations. For our purposes in this work it is sufficient to use the sequential versions of these topological notions and we refer to Dal Maso [35] and Fonseca and Leoni [47] for a more detailed description of the properties presented here.

**Definition 2.3.1.** (Sequential lower semicontinuity) A function $\mathcal{I} : X \to [-\infty, \infty]$ is said to be *sequentially lower semicontinuous*, *s.l.s.c* for short, if, whenever $\{u_n\}_n$ is a sequence converging to $u$

$$\mathcal{I}(u) \leqslant \liminf_{n\to\infty} \mathcal{I}(u_n).$$

*Example.* Let $X$ be a Banach space then $||\cdot||_X$ is sequentially lower semicontinuous for the weak topology of $X$.

The supremum of a family of *s.l.s.c* functions is *s.l.s.c* while the infimum of a finite family of *s.l.s.c* functions is *s.l.s.c* A function $\mathcal{I}$ is *sequentially upper semicontinuous* if -$\mathcal{I}$ is sequentially lower semicontinuous.

**Definition 2.3.2.** (Sequential coercivity) We say that $\mathcal{I} : X \to [-\infty, \infty]$ is *sequentially*

*coercive* if for any $t \in \mathbb{R}$ the closures of the sublevel sets of $\mathcal{I}$

$$\{u \in X \,|\, \mathcal{I}(u) \leqslant t\}$$

are sequentially compact in X.[15]

*Example 1.14 in Dal Maso [35].* If $X$ is a reflexive normed space (e.g. $W^{1,p}(\Omega;\mathbb{R}^d)$ with $1 < p < \infty$) and $\mathcal{I}(u) \to \infty$ as $||u||_X \to \infty$, then $\mathcal{I}$ is sequentially coercive with respect to the weak topology of $X$.

The Direct Method of the Calculus of Variations can be summarized in the following result.

**Theorem 2.3.3.** (Weierstrass Theorem) *Let $\mathcal{I} : X \to [-\infty, \infty]$ be sequentially coercive and s.l.s.c. Then $\mathcal{I}$ attains its minimum in $X$.*

*Proof:*

*Step 1.* (if $\mathcal{I} \not\equiv \infty$) take a minimizing sequence $\{u_n\}_n$ of $\mathcal{I}$ in $X$:

$$\inf_{u \in X} \mathcal{I}(u) = \lim_{n \to \infty} \mathcal{I}(u_n) < \infty;$$

*Step 2.* (compactness property) as $\mathcal{I}$ is sequentially coercive, this sequence has a convergent subsequence $\{u_{n_k}\}_k$;

*Step 3.* as $\mathcal{I}$ is sequentially lower semicontinuous, the limit $u_0$ of the subsequence $\{u_{n_k}\}_k$ is a minimum point of $\mathcal{I}$ on $X$ since

$$\mathcal{I}(u_0) \leqslant \lim_{k \to \infty} \mathcal{I}(u_{n_k}) = \inf_{u \in X} \mathcal{I}(u) \leqslant \mathcal{I}(u_0).$$

In general, the topology of $X$ needs to be weak enough to ensure that the previous compactness argument hold. In the applications of this method to functionals of the form (2.4), $X$ is typically a Sobolev Space endowed with the weak or weak$\star$ topology.

*Typical minimization problem for the functional (2.4).* To study:

$$\min_{u \in \mathcal{A}} \mathcal{I}(u)$$

[15] Recall that a subset $K$ of $X$ is said to be *sequentially compact* in X if every sequence in $K$ has a subsequence which converges (with respect to the topology of $X$) to a point of $K$. In a metric space sequential compactness and compactness are equivalent notions.

where

$$\mathcal{A} = \left\{ u \in W^{1,p}(\Omega;\mathbb{R}^d) : \; u - \varphi \in W_0^{1,p}(\Omega), \; \varphi \in W^{1,p}(\Omega) \right\}$$

or

$$\mathcal{A} = \left\{ u \in W^{1,p}(\Omega;\mathbb{R}^d) : \quad \int_\Omega u = C \right\},$$

$C \in \mathbb{R}$, given. In this example, if $1 < p < \infty$ and if we assume that $f(x,s,\xi) \geqslant C|\xi|^p - C$ for some positive constant $C$, then by virtue of Poincaré inequalities - modulo some regularity in the domain - and by the reflexivity of the space $W^{1,p}(\Omega;\mathbb{R}^d)$, the functional $\mathcal{I}$ given in (2.4) is sequentially coercive with respect to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^d)$ in the admissible (weakly closed) class $\mathcal{A}$. Generally, the main difficulty here is to ensure the sequential lower semicontinuity of the functional $\mathcal{I}$.

To simplify, we will write *s.w.l.s.c* on $W^{1,p}(\Omega;\mathbb{R}^d)$ and *s.w⋆.l.s.c* on $W^{1,\infty}(\Omega;\mathbb{R}^d)$ when we refer to functionals sequentially lower semicontinuous with respect to the weak or weak⋆ convergence of $W^{1,p}(\Omega;\mathbb{R}^d)$ and $W^{1,\infty}(\Omega;\mathbb{R}^d)$, respectively.

When sequentially lower semicontinuous properties fail, one usually tries to relax this condition.

**Definition 2.3.4.** (Sequential lower semicontinuous envelope) The *sequential lower semicontinuous envelope* (or *relaxed functional*) of $\mathcal{I} : X \to [-\infty, \infty]$ is defined by

$$slsc\mathcal{I} := \sup\{\mathcal{G} : X \to [-\infty,\infty], \; \mathcal{G} \textit{ sequentially lower semicontinuous}, \; \mathcal{G} \leqslant \mathcal{I}\}.$$

The functional $slsc\mathcal{I}$ is sequentially lower semicontinuous (it coincides with $\mathcal{I}$ if the function $\mathcal{I}$ is sequentially lower semicontinuous). If $\mathcal{I}$ is sequentially coercive, so is $slsc\mathcal{I}$. The next theorem describes the limits of minimizing sequences of a functional $\mathcal{I}$, not necessarily sequentially lower semicontinuous, in terms of minimum points of $slsc\mathcal{I}$.

**Theorem 2.3.5.** *Assume that the function $\mathcal{I} : X \to [-\infty,\infty]$ is sequentially coercive. Then $slsc\mathcal{I}$ has a minimum point in $X$ and*

$$\min_{u\in X} slsc\mathcal{I}(u) = \inf_{u \in X} \mathcal{I}(u).$$

If $X$ satisfies the first axiom of countability, that is if every point $x \in X$ has a countable base of open sets, then it is possible to derive a characterization of $slsc\mathcal{I}$ in terms of sequences.

**Proposition 2.3.6.** *Suppose that $X$ satisfies the first axiom of countability. Let $\mathcal{I} : X \to [-\infty, \infty]$ and let $u \in X$. Then $slsc\mathcal{I}(u)$ is characterized by the following properties*

*i) for every sequence $\{u_n\}_n$ converging to $u$ in $X$*

$$slsc\mathcal{I}(u) \leqslant \liminf_{n\to\infty} \mathcal{I}(u_n),$$

*ii) there exists a sequence $\{u_n\}_n$ converging to $u$ in $X$ such that*

$$slsc\mathcal{I}(u) \geqslant \limsup_{n\to\infty} \mathcal{I}(u_n).$$

*As a consequence, if $X$ is metrizable then*

$$slsc\mathcal{I}(u) = \inf_{n\in\mathbb{N}} \Big\{ \liminf_{n\to\infty} \mathcal{I}(u_n),\ u_n \to u \text{ in } X \Big\}.$$

We turn back to the study of functionals of the type (2.4). In the scalar case ($N = 1$ or $d = 1$), and under some standard growth, coercivity and regularity conditions on $f$, the convexity of $f(x, s, \cdot)$ turns out to be a necessary and sufficient condition for the *s.w.l.s.c* of $\mathcal{I}$ in $W^{1,p}(\Omega; \mathbb{R}^d)$ (*s.w⋆.l.s.c* if $p = \infty$). In the vectorial case ($N > 1$ and $d > 1$) this condition (still sufficient) is no longer necessary - quasiconvexity is.

### 2.3.2 Convex and quasiconvex functions: main properties

We refer to Fonseca and Leoni [47] for the proofs of the results presented here; see also Braides and Defranceschi [19], Dacorogna [34], Evans and Gariepy [46], as well as the references therein. Throughout this part $N, d \geqslant 1$.

**Definition 2.3.7.** (Convex function) A function $f : \mathbb{R}^N \to \mathbb{R}$ is said to be *convex* (resp. *strictly convex*) if

$$f(\theta\xi + (1-\theta)\eta) \leqslant \theta f(\xi) + (1-\theta) f(\eta)$$

(resp. $<$) for all $\xi$, $\eta \in \mathbb{R}^N$ and $\theta \in (0, 1)$.

**Definition 2.3.8.** The subdifferential $\partial f$ of a convex function $f : \mathbb{R}^N \to \mathbb{R}$ at a point $\xi \in \mathbb{R}^N$ is a set-valued function characterized by the property that $\eta \in \partial f(\xi)$ if and only if for all $\gamma \in \mathbb{R}^N$

$$f(\gamma) \geqslant f(\xi) + < \eta, \gamma - \xi > .$$

For all points $\xi \in \mathbb{R}^N$ the set $\partial f(\xi)$ is nonempty and convex. Moreover, $f$ is differentiable at a point $\xi$ if and only if $\partial f(\xi)$ contains a single element, which is then $\nabla f(\xi)$.

**Theorem 2.3.9.** (see Theorem 1 in section 6.3 of Evans and Gariepy [46]) *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a convex function. Then $f$ is locally Lipschitz in $\mathbb{R}^N$, and there exists a constant $C$ (depending only on $N$) such that for every ball $B(\xi, r) \subset \mathbb{R}^N$, $r > 0$,*

$$\sup_{\eta \in B\left(\xi, \frac{r}{2}\right)} |f(\eta)| \leqslant \frac{C}{|B(\xi, r)|} \int_{B(\xi,r)} |f(\eta)|\, d\eta.$$

As a consequence of Theorem 2.3.9 and Rademacher's Theorem[16], if $f$ is convex then it is differentiable almost everywhere. For all points $\xi$ where $f$ is differentiable

$$f(\eta) \geqslant f(\xi) + \nabla f(\xi) \cdot (\eta - \xi) \tag{2.5}$$

for all $\eta \in \mathbb{R}^N$, which expresses the geometrical fact that the graph of $f$ lies above its tangent hyperplane at the point $\xi$. If $f$ is convex and satisfies $0 \leqslant f(\xi) \leqslant C(1 + |\xi|^p)$, for some $C > 0$, $1 \leqslant p < \infty$ and for all $\xi \in \mathbb{R}^N$, then

$$|f(\xi) - f(\eta)| \leqslant C(1 + |\xi|^{p-1} + |\eta|^{p-1})|\xi - \eta|, \tag{2.6}$$

for all $\xi$ and $\eta$ in $\mathbb{R}^N$.

**Theorem 2.3.10.** (Theorem 1 in section 6.3 of Evans and Gariepy [46]) *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a convex function. Then for each ball $B(\xi, r) \subset \mathbb{R}^N$, $r > 0$, we have*

$$\operatorname*{ess\,sup}_{\eta \in B\left(\xi, \frac{r}{2}\right)} |\nabla f(\eta)| \leqslant \frac{C}{r|B(\xi, r)|} \int_{B(\xi,r)} |f(\eta)|\, d\eta.$$

**Theorem 2.3.11.** (Jensen inequality) (see also Lemma 23.2 in Dal Maso [35]) *A function $f : \mathbb{R}^N \to \mathbb{R}$ is convex if and only if given any measure $\mu$ on a measurable space $(E, \mathcal{M})$, with $E \subset \mathbb{R}^N$ containing at least two distinct points and such that $\mu(E) = 1$ (probability measure on $E$), and given any $g \in L^1(E, \mu; \mathbb{R}^N)$ then*

$$f\left(\int_E g\, d\mu\right) \leqslant \int_E f(g)\, d\mu.$$

[16] Rademacher's Theorem (see section 3.1.2 in Evans and Gariepy [46]): Every locally Lipschitz function $f : \mathbb{R}^N \to \mathbb{R}$ is differentiable a.e.

**Definition 2.3.12.** (Convex envelope) The *convex envelope* (or *convexification*) of a function $f : \mathbb{R}^N \to \mathbb{R}$ is the function $\mathcal{C}f : \mathbb{R}^N \to [-\infty, \infty]$ defined by

$$\mathcal{C}f(\xi) := \sup\{g(\xi) : \ g : \mathbb{R}^N \to \mathbb{R}, \ g \text{ convex}, \ g \leqslant f\}.^{17}$$

Clearly $\mathcal{C}f = f$ if the function $f$ is convex. We now recall Morrey's notion of quasiconvexity.

**Definition 2.3.13.** (Quasiconvex function; Morrey [65]) A Borel measurable function $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ is *quasiconvex* at a point $\xi \in \mathbb{R}^{d\times N}$ if

$$f(\xi) \leqslant \int_\Omega f(\xi + \nabla\phi(\eta))\, d\eta \tag{2.7}$$

for every $\phi \in W_0^{1,\infty}(\Omega; \mathbb{R}^d)$ (for which the integral (2.7) is well defined), and for every open bounded set $\Omega \subset \mathbb{R}^N$ with $|\partial\Omega| = 0$. The function $f$ is said to be quasiconvex if it is quasiconvex at every $\xi \in \mathbb{R}^{d\times N}$.

*Remark* 2.3.14.

i) A quasiconvex function is locally Lipschitz.

ii) Using the Vitali's Covering Theorem (Theorem 2.1.21), it can be seen that it suffices to check (2.7) for one fixed open bounded set $\Omega$ of $\mathbb{R}^N$, for instance $\Omega = Q = (0,1)^N$ (see also Remark 5.15 in Braides and Defranceschi [19]).

iii) In view of Jensen's inequality (Theorem 2.3.11), if a function $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ is convex then it is quasiconvex. Indeed for every $\xi \in \mathbb{R}^{d\times N}$

$$\begin{aligned} f(\xi) &= f\left(\int_Q \xi + \nabla\phi(\eta)\, d\eta\right) \\ &\leqslant \int_Q f\left(\xi + \nabla\phi(\eta)\right)\, d\eta, \quad \forall \phi \in W_0^{1,\infty}(Q). \end{aligned} \tag{2.8}$$

The converse is not generally true (e.g. $f(\xi) = |\det \xi|$ with $\xi \in \mathbb{R}^{2\times 2}$; see for instance Fonseca and Leoni [47]), but both notions are equivalent in the scalar case $N = 1$ or $d = 1$.

**Proposition 2.3.15.** (Marcellini [64]) *If $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ is quasiconvex and if it satisfies the growth condition $0 \leqslant f(\xi) \leqslant C(1 + |\xi|^p)$ for all $\xi \in \mathbb{R}^{d\times N}$ and some $1 \leqslant p < \infty$, then for all $\xi$, $\eta \in \mathbb{R}^{d\times N}$*

$$|f(\xi) - f(\eta)| \leqslant C(1 + |\xi|^{p-1} + |\eta|^{p-1})|\xi - \eta|. \tag{2.9}$$

[17] Usual convention: $\sup\{\emptyset\} = -\infty$.

**Definition 2.3.16.** (Quasiconvex envelope) The *quasiconvex envelope* (or *quasiconvexification*) of a function $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ is the function $\mathcal{Q}f : \mathbb{R}^{d\times N} \to [-\infty, \infty]$ defined by

$$\mathcal{Q}f(\xi) := \sup\{g(\xi) : \ g : \mathbb{R}^{d\times N} \to \mathbb{R}, \ g \text{ quasiconvex}, \ g \leqslant f\}.$$

*Remark* 2.3.17. Clearly $\mathcal{Q}f = f$ if the function $f$ is quasiconvex. In general $\mathcal{C}f \leqslant \mathcal{Q}f \leqslant f$.

As a consequence of this remark and the fact that the function $\xi \to |\xi|^p$, $\xi \in \mathbb{R}^{d\times N}$, is convex for $p \geqslant 1$, we get the following useful lemma.

**Lemma 2.3.18.** *If $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ satisfies*

$$\frac{|\xi|^p}{C} - C \leqslant f(\xi) \leqslant C(1 + |\xi|^p)$$

*for some positive constant $C$ and for all $\xi \in \mathbb{R}^{d\times N}$ with $1 \leqslant p < \infty$, then for all $\xi \in \mathbb{R}^{d\times N}$*

$$\frac{|\xi|^p}{C} - C \leqslant \mathcal{Q}f(\xi) \leqslant C(1 + |\xi|^p).$$

**Theorem 2.3.19.** *If $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ is locally bounded and Borel measurable then for any open bounded set $\Omega \subset \mathbb{R}^N$ with $|\partial\Omega| = 0$*

$$\mathcal{Q}f(\xi) = \inf\left\{\frac{1}{|\Omega|}\int_\Omega f(\xi + \nabla\phi(\eta))\, d\eta : \ \phi \in W_0^{1,\infty}(\Omega; \mathbb{R}^d)\right\}.$$

*In addition the function $\mathcal{Q}f$ is quasiconvex.*

Next we recall a stronger notion called $W^{1,p}$-quasiconvex introduced by Ball and Murat in [13] that allows the competing functions $\phi$ to belong to the Sobolev space $W_0^{1,p}(\Omega; \mathbb{R}^d)$ rather than to the smaller space $W_0^{1,\infty}(\Omega; \mathbb{R}^d)$.

**Definition 2.3.20.** ($W^{1,p}$-quasiconvexity) Let $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ be a Borel measurable function, and let $1 \leqslant p \leqslant \infty$. The function $f$ is said to be $W^{1,p}$*-quasiconvex* at $\xi \in \mathbb{R}^{d\times N}$ if

$$f(\xi) \leqslant \int_\Omega f(\xi + \nabla\phi(\eta))\, d\eta \tag{2.10}$$

for every $\phi \in W_0^{1,p}(\Omega; \mathbb{R}^d)$ (for which the integral (2.10) is well defined) and for every open bounded set $\Omega \subset \mathbb{R}^N$ with $|\partial\Omega| = 0$. The function $u$ is said $W^{1,p}$-quasiconvex if it is $W^{1,p}$-quasiconvex at every $\xi \in \mathbb{R}^{d\times N}$.

*Remark* 2.3.21.

i) As before, it is enough to check inequality (2.10) for $\Omega = Q$.

ii) When $p = \infty$ we recover the notion of quasiconvexity.

iii) It $f$ is $W^{1,p}$-quasiconvex then $f$ is $W^{1,q}$-quasiconvex for all $q$ with $p \leqslant q \leqslant \infty$. Thus $W^{1,1}$-quasiconvexity is the strongest condition and $W^{1,\infty}$-quasiconvexity the weakest.

**Proposition 2.3.22.** *(see Proposition 2.4 in Ball and Murat [13]) Let $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ be a Borel function and assume that*

$$f(\xi) \leqslant C(1 + |\xi|^p)$$

*for all $\xi \in \mathbb{R}^{d\times N}$, where $C$ is a positive constant and $1 \leqslant p < \infty$. Then $f$ is $W^{1,p}$-quasiconvex if and only if $f$ is quasiconvex.*

**Lemma 2.3.23.** *If $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ is a locally bounded and Borel measurable function such that*

$$f(\xi) \leqslant C(1 + |\xi|^p)$$

*for some positive constant $C$ and for all $\xi \in \mathbb{R}^{d\times N}$, with $1 \leqslant p < \infty$, then, for any open bounded set $\Omega \subset \mathbb{R}^N$ with $|\partial\Omega| = 0$,*

$$\mathcal{Q}f(\xi) = \inf \left\{ \frac{1}{|\Omega|} \int_\Omega f(\xi + \nabla\phi(\eta))\, d\eta : \ \phi \in W_0^{1,p}(\Omega; \mathbb{R}^d) \right\}.$$

There are very few explicit examples in the literature of quasiconvex envelopes due to the nonlocal character of its definition. A very interesting example in the nonlinear elasticity setting is the Saint-Venant-Kirchhoff stored energy function.

*Example.* (Le Dret and Raoult [54]) Let

$$f(\xi) = \frac{\mu}{8(1+\nu)} \left|\xi^T\xi - \mathbb{I}_3\right|^2 + \frac{\mu\nu}{8(1+\nu)(1-2\nu)} \left(|\xi|^2 - 3\right)^2,$$

where $\mathbb{I}^3$ denotes the identity matrix in $\mathbb{R}^3$, and $\mu > 0$ and $0 \leqslant \nu < \frac{1}{2}$ are respectively the Young modulus and the Poisson ratio of a hyperelastic material. Then

$$\mathcal{Q}f(\xi) = \Phi(\alpha_1(\xi), \alpha_2(\xi), \alpha_2(\xi)),$$

where $0 \leqslant \alpha_1(\xi) \leqslant \alpha_2(\xi) \leqslant \alpha_3(\xi)$ are the singular values of the matrix $\xi$, i.e. the square roots of the eigenvalues of $\xi^T\xi$, and where

$$\begin{aligned}\Phi(\alpha_1, \alpha_2, \alpha_2) := \quad & \tfrac{\mu}{8}\left[(\alpha_3^2 - 1)^+\right]^2 + \tfrac{\mu}{8(1-\nu^2)}\left[(\alpha_2^2 + \nu\alpha_3^2 - (1+\nu))^+\right]^2 \\ & + \tfrac{\mu\nu}{8(1-\nu^2)(1-2\nu)}\left[((1-\nu)\alpha_1^2 + \nu(\alpha_2^2 + \alpha_3^2) - (1+\nu))^+\right]^2.\end{aligned}$$

Here $(\cdot)^+$ stands for the positive part.

We conclude this Subsection with the following result.

**Lemma 2.3.24.** *(see Dal Maso, Fonseca, Leoni and Morini [36]) Let $\Omega \subset \mathbb{R}^N$ with $N \geqslant 1$ be an open set, and let $f : \Omega \times \mathbb{R}^{d\times N} \to \mathbb{R}$ with $d \geqslant 1$ be a lower semicontinuous function which satisfies the following hypotheses:*

*i) for every $x \in \Omega$ the function $f(x, \cdot)$ is continuous in $\mathbb{R}^{d\times N}$;*

*ii) there exist two locally bounded functions $a, b : \Omega \to [0, \infty)$, a lower semicontinuous function $c : \Omega \to (0, \infty)$, and a constant $1 < p < \infty$ such that for every $(x, \xi) \in \Omega \times \mathbb{R}^{d\times N}$*

$$c(x)|\xi|^p - b(x) \leqslant f(x, \xi) \leqslant a(x)(1 + |\xi|^p).$$

*For every $x \in \Omega$ let $\mathcal{Q}f(x, \cdot)$ be the quasiconvexification of the function $f(x, \cdot)$. Then $\mathcal{Q}f$ is lower semicontinuous on $\Omega \times \mathbb{R}^{d\times N}$.*

### 2.3.3 Lower semicontinuity characterization for integral functionals defined on Sobolev spaces

We start by recalling the definition of an important class of integrands.

**Definition 2.3.25.** (Carathéodory integrand) Let $\Omega \subset \mathbb{R}^N$ with $N \geqslant 1$ be an open set, and let $B$ be a Borel set of $\mathbb{R}^l$ with $l \geqslant 1$. A function $f : \Omega \times B \to \overline{\mathbb{R}}$ is said to be a *Carathéodory integrand* if

i) $x \to f(x, \xi)$ is measurable for every $\xi \in B$,

ii) $\xi \to f(x, \xi)$ is continuous for almost all $x \in \Omega$.

In this work we will deal with Carathéodory integrands $f : \Omega \times \mathbb{R}^l \to \mathbb{R}$ with $l \geqslant 1$, and we will use the following characterization.

**Theorem 2.3.26.** (Scorza-Dragoni Theorem) (see Ekeland and Teman [45]) *Let $\Omega \subset \mathbb{R}^N$ with $N \geqslant 1$ be an open set. A function $f : \Omega \times \mathbb{R}^l \to \mathbb{R}$, with $l \geqslant 1$, is Carathéodory if and only if given a compact set $K \subset \Omega$ and a positive number $\varepsilon$, there exists a compact set $K_\varepsilon \subset K$ such that $\mathcal{L}^N(K \setminus K_\varepsilon) \leqslant \varepsilon$ and the restriction of $f$ to $K_\varepsilon \times \mathbb{R}^l$ is continuous.*

The following result shows that every Carathéodory integrand is (equivalent to) a Borel function.

**Proposition 2.3.27.** (see Proposition 3.3 in Braides and Defranceschi [19] or Ekeland and Teman [45]) *Let $\Omega \subset \mathbb{R}^N$ with $N \geqslant 1$ be an open set, and let $B$ be a Borel set of $\mathbb{R}^l$ with $l \geqslant 1$. Every Carathéodory integrand $f : \Omega \times B \to \mathbb{R}$ is (equivalent to) a Borel function, that is there exists a Borel function $g : \Omega \times B \to \mathbb{R}$ such that $f(x, \cdot) = g(x, \cdot)$ for a.e. $x$.*

As a consequence of the Dominated Convergence Theorem (Proposition 2.1.15) a useful result follows.

**Lemma 2.3.28.** *Let $\Omega \subset \mathbb{R}^N$ with $N \geqslant 1$ be an open bounded set, and let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$ with $d \geqslant 1$ be a Carathéodory integrand. Let $1 \leqslant p < \infty$ and assume that*

$$|f(x, s, \xi)| \leqslant C(1 + |\xi|^p)$$

*for a.e. $x \in \mathbb{R}^N$ and for all $(s, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times N}$, for some positive constant $C$. Then the functional $\mathcal{I} : W^{1,p}(\Omega; \mathbb{R}^d) \to \mathbb{R}$ defined by*

$$\mathcal{I}(u) := \int_\Omega f(x, u(x), \nabla u(x))\, dx$$

*is continuous for the strong topology of $W^{1,p}(\Omega; \mathbb{R}^d)$.*

The following theorem shows that quasiconvexity is a necessary and sufficient condition for *s.w.l.s.c* on the Sobolev spaces $W^{1,p}(\Omega; \mathbb{R}^d)$ (*s.w$\star$.l.s.c* if $p = \infty$) for integral functionals $\mathcal{I} : W^{1,p}(\Omega; \mathbb{R}^d) \to \mathbb{R}$,

$$\mathcal{I}(u) := \int_\Omega f(x, u(x), \nabla u(x))\, dx. \tag{2.11}$$

**Theorem 2.3.29.** *(see Statement II.5 in Acerbi and Fusco [1]; see also Morrey [65]) Let $\Omega \subset \mathbb{R}^N$ with $N \geqslant 1$ be an open set, and let $f : \mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$ with $d \geqslant 1$ be a Carathéodory integrand satisfying*

*i)* $0 \leq f(x, s, \xi) \leq a(x) + C(|s|^p + |\xi|^p)$ *with $1 \leqslant p < \infty$, for every $(x, s, \xi) \in \mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$, where $C > 0$ and $a$ is a non-negative function in $L^1_{\text{loc}}(\mathbb{R}^N)$;*

*ii)* $0 \leq f(x, s, \xi) \leq b(x) + c(s, \xi)$, *for every $(x, s, \xi) \in \mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$, where $b$ is a non-negative function in $L^1_{\text{loc}}(\mathbb{R}^N)$, and $c$ is a non-negative and locally bounded function on $\mathbb{R}^d \times \mathbb{R}^{d\times N}$, if $p = \infty$.*

*Then the functional $\mathcal{I}$ given in (2.11) is s.w.l.s.c on $W^{1,p}(\Omega;\mathbb{R}^d)$ [or s.w$\star$.l.s.c on $W^{1,\infty}(\Omega;\mathbb{R}^d)$ if $p=\infty$] if and only if $f(x,s,\cdot)$ is quasiconvex for every $x\in\mathbb{R}^N$ and for every $s\in\mathbb{R}^d$.*

We conclude this part with a relaxation theorem obtained in Acerbi and Fusco [1] (see Statement III.7; see also Dacorogna [34]). We will denote the sequential lower semicontinuous envelope of $\mathcal{I}$ with respect to weak topology of $W^{1,p}$ (weak$\star$ topology if $p=\infty$) by $swlsc\mathcal{I}$.

**Theorem 2.3.30.** *Let $\Omega\subset\mathbb{R}^N$ with $N\geqslant 1$ be an open set, and let $f:\mathbb{R}^N\times\mathbb{R}^d\times\mathbb{R}^{d\times N}\to\mathbb{R}$ with $d\geqslant 1$ be a Carathéodory integrand such that*

*i)* $0\leq f(x,s,\xi)\leq a(x)+C(|s|^p+|\xi|^p),\ p\geq 1;$

*ii)* $0\leq f(x,s,\xi)\leq b(x)+c(s,\xi),\ p=\infty,$

*for every $(x,s,\xi)\in\mathbb{R}^N\times\mathbb{R}^d\times\mathbb{R}^{d\times N}$, where $C$ is a non-negative constant, $b$ is a non-negative function in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $c$ is a locally bounded and non-negative function on $\mathbb{R}^d\times\mathbb{R}^{d\times N}$. Then*

$$swlsc\mathcal{I}(u)=\int_\Omega \mathcal{Q}f(x,u(x),\nabla u(x))\,dx,$$

*where $\mathcal{I}$ is the functional given in (2.11) and $\mathcal{Q}f(x,s,\xi)$ stands for the quasiconvexification of $f(x,s,.)$ at $\xi$.*

## 2.4 Integral representation of nonlinear local functionals defined on Sobolev spaces

In this section we recall two integral representation theorems for local functionals depending on Sobolev functions and on open sets, that are useful in relaxation and $\Gamma$-convergence theories.

The first theorem was obtained by Buttazzo and Dal Maso and gives abstract conditions under which a local functional $\mathcal{I}$ admits an integral representation of the form

$$\mathcal{I}(u,A)=\int_A f(x,\nabla u(x))\,dx$$

for some Carathéodory integrand $f$ (see Theorem 1.1 in [26] and references therein).

**Theorem 2.4.1.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$ with $N \geqslant 1$. Let $\mathcal{I} : W^{1,p}(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega) \to \mathbb{R}$, with $d \geqslant 1$ and $1 \leqslant p \leqslant \infty$, satisfies the following properties*

i) *$\mathcal{I}$ is local on $\mathcal{A}(\Omega)$, i.e. $\mathcal{I}(u,A) = \mathcal{I}(v,A)$ whenever $A \in \mathcal{A}(\Omega)$, $u, v \in W^{1,p}(\Omega;\mathbb{R}^d)$ and $u = v$ a.e. on $A$;*

ii) *$\mathcal{I}$ is a measure on $\mathcal{A}(\Omega)$, i.e. for every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ the set function $\mathcal{I}(u,.)$ is the restriction to $\mathcal{A}(\Omega)$ of a finite Radon measure;*

iii) *$\mathcal{I}$ satisfies a growth condition of order $p$, i.e. when $p < \infty$ there exist $a \in L^1(\Omega)$ and $b \geqslant 0$ such that for every $A \in \mathcal{A}(\Omega)$ and every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$,*

$$|\mathcal{I}(u,A)| \leqslant \int_A [a(x) + b|\nabla u|^p]\, dx,$$

*and when $p = \infty$ for every $r \geqslant 0$ there exist $a_r \in L^1(\Omega)$ such that*

$$|\mathcal{I}(u,A)| \leqslant \int_A a_r(x)\, dx,$$

*for every $A \in \mathcal{A}(\Omega)$ and every $u \in W^{1,\infty}(\Omega;\mathbb{R}^d)$ with $|\nabla u| \leqslant r$ a.e. in $A$;*

iv) *$\mathcal{I}$ is translation invariant, i.e. for every $A \in \mathcal{A}(\Omega)$, $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, $c \in \mathbb{R}^d$,*

$$\mathcal{I}(u + c, A) = \mathcal{I}(u,A);$$

v) *for every $A \in \mathcal{A}(\Omega)$, the function $\mathcal{I}(\cdot, A)$ is s.w.l.s.c on $W^{1,p}$ (s.w$\star$.l.s.c if $p = \infty$).*

*Then, there exists a function $f : \Omega \times \mathbb{R}^{d\times N} \to \mathbb{R}$ such that*

a) *for every $A \in \mathcal{A}(\Omega)$ and every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ the integral representation formula holds*

$$\mathcal{I}(u,A) = \int_A f(x, \nabla u(x))\, dx;$$

b) *$f$ is a Carathéodory integrand;*

c) *$f(x,z)$ satisfies a growth condition of order $p$, that is, when $p < \infty$ there exist $a \in L^1(\Omega)$ and $b \geqslant 0$ such that*

$$|f(x,z)| \leqslant a(x) + b|z|^p,$$

*for a.e. $x \in \Omega$ and for all $z \in \mathbb{R}^{d\times N}$, and when $p = \infty$ for every $r \geqslant 0$ there exists $a_r \in L^1(\Omega)$ such that*

$$|f(x,z)| \leqslant a_r(x)$$

*for a.e. $x \in \Omega$ and for all $z \in \mathbb{R}^{d\times N}$ with $|z| \leqslant r$.*

*Remark* 2.4.2.

- Conditions a), b), c) imply i), ii), iii), iv) but not v). Nevertheless, the integral representation theorem does not hold if we drop hypothesis v) (see examples in Buttazzo and Dal Maso [26]).
- Conditions a), b), c) and v) imply that for a.e. $x \in \Omega$ the function $\xi \to f(x,\xi)$ is quasiconvex (see Theorem 2.3.29).

The second theorem was derived in Bouchitté, Fonseca, Leoni and Mascarenhas [24] (Theorem 1.1), and gives abstract conditions under which a local functional $\mathcal{I}$ admits an integral representation of the form

$$\mathcal{I}(u,A) = \int_A f(x,u(x),\nabla u(x))\,dx$$

for some Borel function $f$ (see references in [24]; see also Buttazzo and Dal Maso [27] where under additional uniform continuity hypothesis they derive a similar result in terms of a Carathéodory integrand).

**Theorem 2.4.3.** *Let $\Omega \subset \mathbb{R}^N$ with $N \geqslant 1$. Let $\mathcal{I} : W^{1,p}(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,\infty]$, with $1 < p < \infty$, be a functional satisfying hypotheses*

*(i) $\mathcal{I}(u;\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;*

*(ii) $\mathcal{I}(u;A) = \mathcal{I}(v;A)$ whenever $u = v$ $\mathcal{L}^N$-a.e. on $A \in \mathcal{A}(\Omega)$;*

*(iii) $\mathcal{I}(\cdot;A)$ is s.l.s.c with respect to the $L^1(\Omega;\mathbb{R}^d)$-topology;*

*(iv) there exists $C > 0$ such that*

$$\frac{1}{C}\int_A |\nabla u|^p\,dx \leqslant \mathcal{I}(u;A) \leqslant C\int_A (1+|\nabla u|^p)\,dx.$$

*Then, for every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$ we have*

$$\mathcal{I}(u;A) = \int_A f(x,u,\nabla u)\,dx$$

*where $f$ is the Borel function given by*

$$f(x,s,\xi) = \limsup_{\varepsilon\to 0} \frac{m(s+\xi(\cdot - x)); Q(x,\varepsilon))}{\varepsilon^N}$$

*for all $(x,s,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$, and where for $(v,A) \in W^{1,p}(\Omega;\mathbb{R}^d) \times \mathcal{A}_\infty(\Omega)$*

$$m(v;A) := \inf\{\mathcal{I}(w;A) \text{ with } w = v \text{ in a neighborhood of } \partial A\},$$

*and $Q(x,\varepsilon) := x + \varepsilon Q$ for $x \in \Omega$ and $\varepsilon > 0$. Here $\mathcal{A}_\infty(\Omega)$ denotes the class of Lipschitz subdomains of $\Omega$.*

## 2.5 Γ-convergence of a family of functionals

The aim of this section is to recall the notion of Γ-convergence introduced by De Giorgi and afterwards applied to a large number of different problems in the Calculus of Variations (see De Giorgi and Franzoni [40]; see also De Giorgi and Dal Maso [38]). Γ-convergence plays an important role in situations where, to understand properties of equilibrium states of a given functional, it is necessary to study the behavior of a family of minimum problems depending on a small parameter $\varepsilon$:

$$\min_{u\in X} \mathcal{I}_\varepsilon(u). \tag{2.12}$$

A typical example concerns periodic homogenization problems. Here one is led to consider functionals of the type

$$\mathcal{I}_\varepsilon(u) = \int_\Omega f_\varepsilon(x,u(x),\nabla u(x))\,dx \tag{2.13}$$

where the integrands $f_\varepsilon$ are increasingly oscillating in the first variable as the parameter $\varepsilon$ goes to zero.

In general to characterize the solutions of problem (2.12) one is led to consider an "effective minimum problem" (not depending on $\varepsilon$)

$$\min_{u\in X} \mathcal{I}(u),$$

that captures the relevant behavior of minimizers and for which a solution can be obtained more easily. The effectiveness of Γ-convergence is linked to the possibility of obtaining convergent sequences from minimizers (or almost-minimizers) of problem (2.12). This is possible in the class of integral functionals (2.13) under suitable conditions on the integrands $f_\varepsilon$.

We give the abstract definition of Γ-convergence and we collect some of its most useful properties for the analysis of functionals of the type (2.13). We refer to Braides [18] and Dal Maso [35] for a comprehensive treatment and detailed bibliography on this subject.

### 2.5.1 The notion of Γ-convergence and main results

Throughout this part $(X, d)$ denotes a metric space.

**Definition 2.5.1.** (Γ-convergence of a sequence of functionals) Let $\{\mathcal{I}_n\}_n$ be a sequence of functionals defined on $X$ with values on $\overline{\mathbb{R}}$. The functional $\mathcal{I} : X \to \overline{\mathbb{R}}$ is said to be the Γ-lim inf (resp. Γ-lim sup) of $\{\mathcal{I}_n\}_n$ with respect to the metric $d$ if for every $u \in X$

$$\mathcal{I}(u) = \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} \mathcal{I}_n(u_n) : \; u_n \in X,\, u_n \to u \text{ in } X \right\} \text{ (resp. } \limsup_{n\to\infty}).$$

In this case we write

$$\mathcal{I} = \Gamma\text{-}\liminf_{n\to\infty} \mathcal{I}_n \quad \left(\text{resp. } \; \mathcal{I} = \Gamma\text{-}\limsup_{n\to\infty} \mathcal{I}_n\right).$$

Moreover, the functional $\mathcal{I}$ is said to be the Γ-lim of $\{\mathcal{I}_n\}_n$ if

$$\mathcal{I} = \Gamma\text{-}\liminf_{n\to\infty} \mathcal{I}_n = \Gamma\text{-}\limsup_{n\to\infty} \mathcal{I}_n,$$

and in this case we write

$$\mathcal{I} = \Gamma\text{-}\lim_{n\to\infty} \mathcal{I}_n.$$

**Lemma 2.5.2.** *Let $\mathcal{I}_n, \mathcal{I} : X \to \overline{\mathbb{R}}$. Then $\mathcal{I} = \Gamma\text{-}\lim_{n\to\infty} \mathcal{I}_n$ if and only if for every $u$ in $X$ the following conditions hold:*

*i) for all sequences $\{u_n\}_n$ converging to $u$ in $X$,*

$$\mathcal{I}(u) \leqslant \liminf_{n\to\infty} \mathcal{I}_n(u_n); \tag{2.14}$$

*ii)* (recovering sequence) *there exists a sequence* $\{u_n\}_n$ *converging to* $u$ *in* $X$ *such that*

$$\mathcal{I}(u) \geqslant \limsup_{n\to\infty} \mathcal{I}_n(u_n). \tag{2.15}$$

Due to this result, Definition 2.5.1 can be also given pointwise: The family of functionals $\{\mathcal{I}_n\}_n$ Γ-converges to the value $\mathcal{I}(u)$ at a point $u \in X$ if inequalities (2.14) and (2.14) hold; in this case we write $\mathcal{I}(u) = \Gamma\text{-}\lim_{n\to\infty} \mathcal{I}_n(u)$. Thus, $\mathcal{I}$ is the Γ-lim of $\{\mathcal{I}_n\}_n$ with respect to the metric $d$ if and only if $\mathcal{I}(u) = \Gamma\text{-}\lim_{n\to\infty} \mathcal{I}_n(u)$ for all $u \in X$.

For every $\varepsilon > 0$ let $\mathcal{I}_\varepsilon$ be a functional over $X$ with values on $\overline{\mathbb{R}}$, $\mathcal{I}_\varepsilon : X \to \overline{\mathbb{R}}$.

**Definition 2.5.3.** (Γ-convergence of a family of functionals)

A functional $\mathcal{I} : X \to \overline{\mathbb{R}}$ is said to be the Γ-lim inf (resp. Γ-lim sup or Γ-lim) of $\{\mathcal{I}_\varepsilon\}_\varepsilon$ with respect to the metric $d$, as $\varepsilon \to 0$, if for every sequence $\varepsilon_n \downarrow 0$,

$$\mathcal{I} = \Gamma\text{-}\liminf_{n\to\infty} \mathcal{I}_{\varepsilon_n} \quad \left(\text{resp.} \quad \mathcal{I} = \Gamma\text{-}\limsup_{n\to\infty} \mathcal{I}_{\varepsilon_n} \ \text{ or } \ \mathcal{I} = \Gamma\text{-}\lim_{n\to\infty} \mathcal{I}_{\varepsilon_n}\right),$$

and we write

$$\mathcal{I} = \Gamma\text{-}\liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon \quad \left(\text{resp.} \quad \mathcal{I} = \Gamma\text{-}\limsup_{\varepsilon\to 0} \mathcal{I}_\varepsilon \ \text{ or } \ \mathcal{I} = \Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon\right).$$

The expressions "Γ-converge to $\mathcal{I}$" or "is the Γ-lim of $\mathcal{I}_\varepsilon$" are used interchangeably.

*Remark* 2.5.4.

i) If a family of functionals Γ-converges so does every sub-family (and to the same limit).

ii) As a consequence of Proposition 2.3.6, if $\mathcal{I}_\varepsilon \equiv \mathcal{I}$ then $\{\mathcal{I}_\varepsilon\}_\varepsilon$ Γ-converges to $slsc\mathcal{I}$ (hence a constant family may Γ-converges to a limit different from the constant itself).

iii) Obviously, the notions of Γ-convergence and pointwise or uniform convergence do not coincide.

*Examples* (see Dal Maso [35]). Let $X = \mathbb{R}$ with the usual metric.

- Let $\mathcal{I}_\varepsilon(u) := \frac{u}{\varepsilon} \exp[-2\left(\frac{u}{\varepsilon}\right)^2]$. Then $\{\mathcal{I}_\varepsilon\}_\varepsilon$ converges pointwise to 0 but it Γ-converges in $\mathbb{R}$ to the function

$$\mathcal{I}(u) := \begin{cases} \frac{-1}{2}\exp[\frac{1}{2}] & \text{if } u \neq 0, \\ \\ 0 & \text{if } u = 0. \end{cases}$$

- Let $\mathcal{I}_\varepsilon(u) := \sin\left(\frac{u}{\varepsilon}\right)$. Then $\{\mathcal{I}_\varepsilon\}_\varepsilon$ $\Gamma$-converges in $\mathbb{R}$ to the constant function $\mathcal{I} := -1$ but it does not converge pointwise, except at $u = 0$.

**Proposition 2.5.5.** *If $\mathcal{I} = \Gamma\text{-}\liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon$ (or $\Gamma\text{-}\limsup_{\varepsilon\to 0}$) then $\mathcal{I}$ is lower semicontinuous (with respect to the metric $d$). Consequently, if $\mathcal{I} = \Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon$ then $\mathcal{I}$ is lower semicontinuous.*

The following result studies the $\Gamma$-limit of a continuous perturbation of a family of functionals.

**Proposition 2.5.6.** *Let $\mathcal{G} : X \to \mathbb{R}$ be a continuous functional and let $\{\mathcal{I}_\varepsilon\}_\varepsilon$ be a family of functionals on $X$. Then*

*i)* $\Gamma\text{-}\liminf_{\varepsilon\to 0}(\mathcal{I}_\varepsilon + \mathcal{G}) = \Gamma\text{-}\liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon + \mathcal{G}$;

*ii)* $\Gamma\text{-}\limsup_{\varepsilon\to 0}(\mathcal{I}_\varepsilon + \mathcal{G}) = \Gamma\text{-}\limsup_{\varepsilon\to 0} \mathcal{I}_\varepsilon + \mathcal{G}$.

*In particular, if $\{\mathcal{I}_\varepsilon\}_\varepsilon$ $\Gamma$-converges to $\mathcal{I}$ in $X$, then $\{\mathcal{I}_\varepsilon + \mathcal{G}\}_\varepsilon$ $\Gamma$-converges to $\mathcal{I} + \mathcal{G}$ in $X$.*

The next result states that a metric space $(X, d)$ satisfies the Urysohn property with respect to $\Gamma$-convergence.

**Proposition 2.5.7.** *Given $\mathcal{I} : X \to \overline{\mathbb{R}}$ and $\varepsilon_n \downarrow 0$, $\mathcal{I} = \Gamma\text{-}\lim_{n\to\infty} \mathcal{I}_{\varepsilon_n}$ if and only if for every subsequence $\{\varepsilon_{n_j}\}_j$ there exists a further subsequence $\{\varepsilon_{j_k}\}_k$ such that $\{\mathcal{I}_{\varepsilon_{j_k}}\}_k$ $\Gamma$-converges to $\mathcal{I}$.*

If, in addition, $(X, d)$ is a separable metric space then the following compactness property holds.

**Theorem 2.5.8.** *Each sequence $\varepsilon_n \downarrow 0$ has a subsequence $\{\varepsilon_{n_j}\}_j \equiv \{\varepsilon_j\}_j$ such that $\Gamma\text{-}\lim_{j\to\infty} \mathcal{I}_{\varepsilon_j}$ exists.*

*Remark* 2.5.9.

i) If the space $X$ is not separable the conclusion of Theorem 2.5.8 may fail (see a counterexample in Braides and Defranceschi [19]).

ii) We remark that, as a consequence of Theorem 2.5.8 and Proposition 2.5.7, if $(X, d)$ is separable then to conclude that $\Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon = \mathcal{I}$ it suffices to prove that given $\varepsilon_n \downarrow 0$ there exists a subsequence $\{\varepsilon_{n_j}\}_j \equiv \{\varepsilon_j\}_j$ such that $\mathcal{I} = \Gamma\text{-} \liminf_{j\to\infty} \mathcal{I}_{\varepsilon_j}$.

**Definition 2.5.10.** A family of functionals $\{\mathcal{I}_\varepsilon\}_\varepsilon$ is said to be *equi-coercive* if for every real number $\lambda$ there exists a compact set $K_\lambda$ in $X$ such that for each sequence $\varepsilon_n \downarrow 0$,

$$\{u \in X : \ \mathcal{I}_{\varepsilon_n}(u) \leqslant \lambda\} \subseteq K_\lambda \text{ for every } n \in \mathbb{N}.$$

*Example.* Let $\Omega$ be an open, bounded subset of $\mathbb{R}^N$ with $N \geqslant 1$. Let $\mathcal{I}_\varepsilon$ and $f_\varepsilon$ be given as in (2.13), and assume that $C|\xi|^p - C \leqslant f_\varepsilon(x, s, \xi)$ for some positive constant $C$ and some $1 < p < \infty$. Then by Poincaré inequality the family of functionals $\mathcal{I}_\varepsilon$ is equi-coervice on $W_0^{1,p}(\Omega)$ with respect to the strong topology of $L^p$.

As mentioned before, one of the most important properties of $\Gamma$-convergence, and the reason why this kind of convergence is so important in the asymptotic analysis of variational problems, is that under appropriate compactness properties it implies the convergence of (almost) minimizers of a family of equi-coercive functionals to the minimum of the limiting functional (the minimum exists by virtue of Weierstrass's Theorem, Theorem 2.3.3). More precisely, we have the following result.

**Theorem 2.5.11.** *(Fundamental Theorem of $\Gamma$-convergence) If $\{\mathcal{I}_\varepsilon\}_\varepsilon$ is a family of equi-coercive functionals on $X$ and if*

$$\mathcal{I} = \Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon,$$

*then the functional $\mathcal{I}$ has a minimum on $X$ and*

$$\min_{u\in X} \mathcal{I}(u) = \lim_{\varepsilon\to 0} \inf_{u\in X} \mathcal{I}_\varepsilon(u).$$

*Moreover, given $\varepsilon_n \downarrow 0$ and $\{u_n\}_n$ a converging sequence such that*

$$\lim_{n\to\infty} \mathcal{I}_{\varepsilon_n}(u_n) = \lim_{n\to\infty} \inf_{u\in X} \mathcal{I}_{\varepsilon_n}(u), \tag{2.16}$$

*then its limit is a minimum point for $\mathcal{I}$ on $X$.*

If (2.16) holds, then $\{u_n\}_n$ is said to be a sequence of almost-minimizers for $\mathcal{I}$.

**Theorem 2.5.12.** *Let $\{\mathcal{I}_\varepsilon\}_\varepsilon$ be a family of functionals on $X$. The following equalities hold*

*i)* $\Gamma\text{-} \liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon = \Gamma\text{-} \liminf_{\varepsilon\to 0} slsc\, \mathcal{I}_\varepsilon,$

*ii)* $\Gamma\text{-}\limsup_{\varepsilon\to 0} \mathcal{I}_\varepsilon = \Gamma\text{-}\limsup_{\varepsilon\to 0} slsc\mathcal{I}_\varepsilon$,

*where $slsc\mathcal{I}_\varepsilon$ denotes the sequential lower semicontinuous envelope of $\mathcal{I}_\varepsilon$ (see Definition 2.3.4). In particular, $\{\mathcal{I}_\varepsilon\}_\varepsilon$ $\Gamma$-converges to $\mathcal{I}$ if and only if $\{slsc\mathcal{I}_\varepsilon\}_\varepsilon$ $\Gamma$-converges to $\mathcal{I}$.*

### 2.5.2 The Direct Method of Γ-convergence for a class of integral functionals

The purpose of this part is to give an overview of the Direct Method of Γ-convergence, first outlined by De Giorgi and later used by many other authors (see De Giorgi [37], Dal Maso and De Giorgi [38]; see also Braides [17] and Dal Maso [35]). It is an important tool to obtain the representation of the Γ-limit of a family of functionals, and is used in Chapters 4 and 5. We restrict ourselves to a $L^p$-setting and discuss the Direct Method for

$$\mathcal{I}_\varepsilon(u) := \int_\Omega f_\varepsilon\left(x, u(x), \nabla u(x)\right)\, dx$$

with $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, where $\Omega \subset \mathbb{R}^N$ with $N \geqslant 1$ is an open set, and $f_\varepsilon : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$, with $d \geqslant 1$, is a family of Borel functions satisfying standard coercivity and growth conditions. In what follows, we will use the notation $\Gamma(L^p(\Omega))$-limit to refer to the Γ-convergence with respect to the usual metric in $L^p(\Omega;\mathbb{R}^d)$ for $1 < p < \infty$. Denoting by $\mathcal{I} = \Gamma(L^p(\Omega))\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon$ (if it exists), the main questions are:

- Does there exist a Borel function $f : \mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$ such that for all $u \in W^{1,p}(\Omega;\mathbb{R}^d)$
$$\mathcal{I}(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx\ ?$$
- Is it possible to derive $f$ explicitly, or, at least, to determine some of its properties?

The main steps to arrive at an answer are:

i) Consider the dependence of the integrals on the integration set (localization), that is, consider the family of functionals $\mathcal{I}_\varepsilon : L^p(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to \mathbb{R}$ defined for $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ by
$$\mathcal{I}_\varepsilon(u, A) = \int_A f_\varepsilon\left(x, u(x); \nabla u(x)\right)\, dx,$$

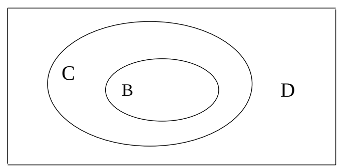
and then study the functional

$$\mathcal{I}(u,A) = \Gamma(L^p(\Omega))\text{-} \lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u,A).$$

ii) Examine the dependence of $\mathcal{I}$ on the variable $A$ in order to prove that the set function $A \to \mathcal{I}(u,A)$ is the trace of a Radon measure for every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, using, for instance, Lemma 2.1.20 above or another De Giorgi's-Letta type of argument.

We remark that, in general, the main difficulty in applying this lemma is to prove the (nested) subadditivity of $\mathcal{I}(u,\cdot)$:

$$\mathcal{I}(u,D) \leq \mathcal{I}(u,D\backslash\overline{B}) + \mathcal{I}(u,C),$$




for all $B,C,D \in \mathcal{A}(\Omega)$ with $B \subset\subset C \subset D$. By the definition of $\mathcal{I}$, this means that we should be able to construct recovering sequences for $\mathcal{I}(u,D)$ from recovering sequences for $\mathcal{I}(u,D\backslash\overline{B})$ and $\mathcal{I}(u,C)$ by some matching process. For families of functionals that satisfy standard growth and coerciveness conditions, this procedure is possible by a slicing argument introduced by De Giorgi in [37] described in Theorem 2.5.13 below.

iii) If the functional $\mathcal{I}$ has good continuity or convexity properties, the next step is to use convenient integral representation theorems (for instance Theorems 2.4.1 and Theorem 2.4.3) to prove that the measure $\mathcal{I}(u,A)$ can be written in the form

$$\mathcal{I}(u,A) = \int_A f(x,u(x),\nabla u(x))\,dx.$$

Due to Remark 2.5.9, one can use this strategy to prove both the existence of the $\Gamma$-limit of the family $\{\mathcal{I}_\varepsilon\}_\varepsilon$ and its integral representation:

*Step 1.* Establish a *compactness result* that guarantees each sequence $\{\mathcal{I}_{\varepsilon_n}\}_n$, with $\varepsilon_n \downarrow 0$, has a subsequence $\Gamma(L^p(\Omega))$-converging to an abstract limit functional.

*Step 2.* Establish an integral representation result.

*Step 3.* Prove the representation formula is well defined, i.e. does not depend on the subsequence.

This method involves testing on the linear functions only (*Step 3*), as opposed to working with general recovering sequences (Lemma 2.5.2). In addition, the two first steps can be carried out in a systematic way for a large class of situations, due to the properties of $\Gamma$-convergence.

The next result is an important tool for the application of $\Gamma$-convergence to variational problems for functionals of the form $\{\mathcal{I}_\varepsilon\}_\varepsilon$ with Dirichlet boundary conditions. It states that, under appropriate hypotheses on the integrand $f_\varepsilon$, the boundary conditions do not affect the limit functional. In particular, it provides appropriate conditions to accomplish point ii) above.

**Proposition 2.5.13.** *Let $\Omega$ be an open and bounded subset of $\mathbb{R}^N$ and let $p > 1$. Assume that $f_\varepsilon : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$ is a family of Carathéodory integrands such that*

$$\frac{1}{C}|\xi|^p - C \leqslant f_\varepsilon(x,s,\xi) \leqslant C(1+|s|^p+|\xi|^p) \tag{2.17}$$

*for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times N}$, for some constant $C > 0$. Consider the family of functionals $\mathcal{I}_\varepsilon : L^p(\Omega;\mathbb{R}^d) \to [0,\infty]$ defined by*

$$\mathcal{I}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f_\varepsilon\left(x,u(x),\nabla u(x)\right)\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\[2ex] \infty & \text{otherwise,} \end{cases}$$

*and for each $\varphi \in W^{1,p}(\Omega;\mathbb{R}^d)$ define the functional*

$$\mathcal{G}_\varepsilon^\varphi(u) := \begin{cases} \mathcal{I}_\varepsilon(u) & \text{if } u-\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^d), \\[2ex] \infty & \text{otherwise.} \end{cases}$$

*If $\{\mathcal{I}_\varepsilon\}_\varepsilon$ $\Gamma(L^p(\Omega))$-converges to a functional $\mathcal{I}$, then the family of functionals $\{\mathcal{G}_\varepsilon^\varphi\}_\varepsilon$ $\Gamma(L^p(\Omega))$-converges to the functional*

$$\mathcal{G}^\varphi(u) := \begin{cases} \mathcal{I}(u) & \text{if } u-\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^d), \\[2ex] \infty & \text{otherwise.} \end{cases}$$

*Proof.* (for an alternative proof see Proposition 11.7 in Braides and Defranceschi [19]; see also Theorem 21.1 in Dal Maso [35]) We start by remarking that there is no loss of generality in assuming that $f$ is positive. If not, we may replace $f$ by $f+C$ that, in view of hypotheses (2.17), is positive. As mentioned before, the proof relies on De Giorgi's slicing argument introduced in [37]. Let $u \in W^{1,p}(\Omega)$ and let $\varepsilon_n \downarrow 0$. If $u - \varphi \notin W_0^{1,p}(\Omega;\mathbb{R}^d)$ then, by (2.17),

$$\Gamma\text{-} \lim_{n\to\infty} \mathcal{G}^{\varphi}_{\varepsilon_n}(u) = \infty.$$

Let us assume that $u-\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^d)$. The conclusion will follow if we can find a sequence $\{w_n\}_n \subset W^{1,p}(\Omega)$ with $w_n = u$ on $\partial\Omega$ such that

$$\mathcal{I}(u) = \lim_{n\to\infty} \int_\Omega f_{\varepsilon_n}(x, w_n, Dw_n)\, dx \tag{2.18}$$

holds. Let $\{v_n\}_n \subset W^{1,p}(\Omega;\mathbb{R}^d)$ be a sequence such that $||u - v_n||_{L^p(\Omega)} \underset{n\to\infty}{\rightarrow} 0$ and

$$\mathcal{I}(u) = \lim_{n\longrightarrow\infty} \mathcal{I}_{\varepsilon_n}(v_n).$$

Set

$$\beta_0 := \sup_n \int_\Omega (1 + |Dv_n|^p)\, dx < \infty \text{ (by (2.17))},$$

and define for $n \in \mathbb{N}$

$$K_n := \left|\left[\left[\frac{1}{||v_n - u||^{\frac{1}{2}}_{L^p(\Omega)}}\right]\right]\right|,$$

$$M_n := \left|\left[\left[\sqrt{K_n}\right]\right]\right|,$$

and finally

$$\Omega_n := \left\{x \in \Omega : \ \operatorname{dist}(x,\partial\Omega) < \frac{M_n}{K_n}\right\},$$

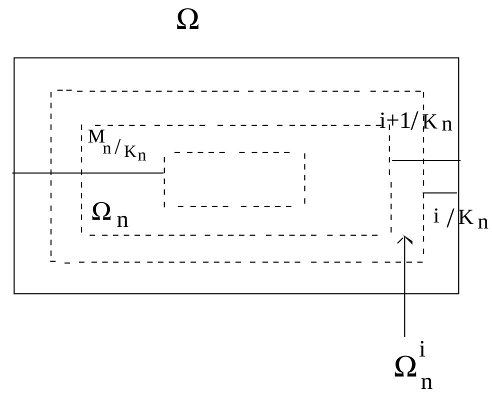
where $[[\cdot]]$ stands for the integer part function.

We observe that by definition $K_n \uparrow \infty$ and $\mathcal{L}^N(\Omega_n) \downarrow 0$ as $n \to \infty$. For each $n$,subdivide $\Omega_n$ (*) into $M_n$ disjoint subsets

$$\Omega_n^i := \left\{x \in \Omega_n : \ \operatorname{dist}(x,\partial\Omega) \in \left[\frac{i}{K_n}, \frac{i+1}{K_n}\right]\right\}, \quad i = 0, ..., M_n - 1,$$

and choose $i_n \in \{0, ..., M_n - 1\}$ such that

$$\int_{\Omega_n^{i_n}} (1 + |Dv_n|^p)\, dx \leqslant \int_{\Omega_n^i} (1 + |Dv_n|^p)\, dx$$




for all $i = 0, ..., M_n - 1$. Then

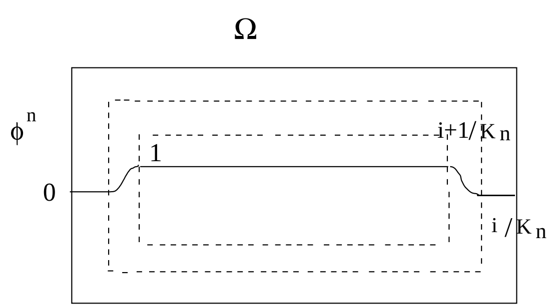
$$M_n \int_{\Omega_n^{i_n}} (1+|Dv_n|^p)\, dx \leqslant \int_{\Omega_n} (1+|Dv_n|^p)\, dx = \sum_{i=0}^{M_n-1} \int_{\Omega_n^i} (1+|Dv_n|^p)\, dx \leqslant \int_{\Omega} (1+|Dv_n|^p)\, dx \leqslant \beta_0,$$

or, equivalently,

$$\int_{\Omega_n^{i_n}} (1 + |Dv_n|^p)\, dx \leqslant \frac{\beta_0}{M_n}.$$

Let $\phi_n \in C_0^\infty(\Omega)$ be such that $0 \leqslant \phi_n \leqslant 1$, $||D\phi_n||_\infty \leqslant K_n$,

$$\phi_n := \begin{cases} 1 & \text{if } \operatorname{dist}(x, \partial\Omega) \geqslant \frac{i_n+1}{K_n}, \\ \\ 0 & \text{if } \operatorname{dist}(x, \partial\Omega) \leqslant \frac{i_n}{K_n}, \end{cases}$$




and define $w_n := \phi_n v_n + (1 - \phi_n)u \in W^{1,p}(\Omega; \mathbb{R}^d)$. Clearly $w_n \to u$ strongly in $L^p(\Omega; \mathbb{R}^d)$,

$w_n = u$ in $\Omega\backslash\mathcal{K}_n$, with $\mathcal{K}_n := \left\{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) \geqslant \frac{i_n}{K_n}\right\}$ and $|\Omega\backslash\mathcal{K}_n| \to 0$. Moreover

$$\begin{aligned}
\mathcal{I}(u) \quad &= \lim_{n\to\infty} \mathcal{I}_{\varepsilon_n}(v_n) \\
&\geqslant \limsup_{n\to\infty} \int_{\Omega\cap\left\{x:\ \mathrm{dist}(x,\partial\Omega)\geqslant\frac{i_n+1}{K_n}\right\}} f_{\varepsilon_n}\left(x, w_n, Dw_n\right)\, dx.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathcal{I}(u) \quad &\geqslant \limsup_{n\to\infty} \int_{\Omega} f_{\varepsilon_n}\left(x, w_n, Dw_n\right) - C \ \liminf_{n\to\infty} \int_{\Omega_n\cap\left\{x:\ \mathrm{dist}(x,\partial\Omega)<\frac{i_n}{K_n}\right\}} (1 + |u|^p + |Du|^p)\, dx \\
&\quad -C\beta \ \liminf_{n\to\infty} \int_{\Omega_n^{i_n}} (1 + |v_n|^p + |Dv_n|^p)\, dx - C\beta \ \liminf_{n\to\infty} |K_n|^p \int_{\Omega_n^{i_n}} |v_n - u|^p\, dx \\
&\quad -C\beta \ \liminf_{n\to\infty} \int_{\Omega_n^{i_n}} (|u|^p + |Du|^p)\, dx,
\end{aligned}$$

where $C$ is the constant given in hypothesis (2.17) and $\beta$ is some positive constant. Then

$$\begin{aligned}
\mathcal{I}(u) \quad &\geqslant \limsup_{n\to\infty} \int_{\Omega} f_{\varepsilon_n}\left(x, w_n, Dw_n\right) - C\beta\beta_0 \ \liminf_{n\to\infty} \frac{1}{M_n} - C\beta \ \liminf_{n\to\infty} ||v_n - u||^{\frac{p}{2}}_{L^p(\Omega)} \\
&= \limsup_{n\to\infty} \mathcal{I}_{\varepsilon_n}(w_n),
\end{aligned}$$

Accordingly (2.18) holds, that is

$$\mathcal{I}(u) = \lim_{n\to\infty} \int_{\Omega} f_{\varepsilon_n}\left(x, w_n, Dw_n\right)\, dx.$$

■

The next result shows that in many situations we can assume, without loss of generality, that our functionals $\mathcal{I}_\varepsilon$ have good convexity properties. More precisely, as a consequence of Theorem 2.3.30 and Theorem 2.5.12 we have the following result.

**Corollary 2.5.14.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with $N \geqslant 1$. For each $\varepsilon > 0$ let $f_\varepsilon : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$ be a Carathéodory integrand and suppose that*

$$0 \leq f_\varepsilon(x, s, \xi) \leq a(x) + C(|s|^p + |\xi|^p),$$

*for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times N}$, with $1 \leqslant p < \infty$, where $C$ is a positive constant, and $a$ is a non-negative function in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. For each $\varepsilon > 0$ consider the functional $\mathcal{I}_\varepsilon$ defined in $W^{1,p}(\Omega;\mathbb{R}^d)$ by*

$$\mathcal{I}_\varepsilon(u) = \int_\Omega f_\varepsilon(x, u(x), \nabla u(x))\, dx.$$

*Then $\{\mathcal{I}_\varepsilon\}_\varepsilon$ $\Gamma(L^p(\Omega))$-converge to a functional $\mathcal{I}$ if and only if the family of functionals $\mathcal{J}_\varepsilon$ defined in $W^{1,p}(\Omega;\mathbb{R}^d)$ by*

$$\mathcal{J}_\varepsilon(u) = \int_\Omega \mathcal{Q}f_\varepsilon(x, u(x), \nabla u(x))\, dx,$$

*$\Gamma(L^p(\Omega))$-converge to $\mathcal{I}$, where $\mathcal{Q}f_\varepsilon(x,s,\xi)$, stands for the quasiconvexification of $f_\varepsilon(x,s,.)$ at $\xi$.*

## 2.6 Two-Scale Convergence

The origins of two-scale convergence are in a paper by Nguetseng [69] (see [61] and also [70]) concerning the homogenization of linear elliptic problems with periodic coefficients of the form

$$\begin{cases} -\mathrm{div}(A(\frac{x}{\varepsilon})\nabla u_\varepsilon)) = g & \text{on } \Omega, \\ \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is some open, bounded and Lipschitz subset of $\mathbb{R}^N$ with $N \geqslant 1$, $u_\varepsilon \in W^1_0(\Omega)$, $g \in L^2(\Omega)$, and some ellipticity conditions on the coefficients of $A$ are assumed.

Nguetseng gives an alternative proof of the classical homogenization result previously obtained by two-scale asymptotic expansion and energy methods (see Bensoussan, Lions and Papanicolau [14] and Tartar [76]), by means of a detailed study of functionals of the form

$$\int_\Omega f_n(x)\phi\Big(x, \frac{x}{\varepsilon_n}\Big)\, dx.$$

The key point of his argument was to prove that from each bounded sequence $\{f_n\}_n$ in $L^2(\Omega)$ there exists a subsequence (still denoted by $\{f_n\}_n$) such that

$$\int_\Omega f_n(x)\phi\Big(x, \frac{x}{\varepsilon_n}\Big)\, dx \to \int_\Omega \int_Q f(x,y)\phi(x,y)\, dy\, dx,$$

and to derive a similar result for sequences of gradients. Allaire [3] called this two-scale convergence and developed further properties of this notion as a tool to study more general homogenization problems. Later this was extended to the notion of $n$-scale convergence by Allaire and Briane (see [5] and Lukkassen, Nguetseng and Wall [61]) to study reiterated homogenization problems of the type

$$\begin{cases} -\mathrm{div}(A(\frac{x}{\varepsilon}, \dots \frac{x}{\varepsilon^N}; \nabla u_\varepsilon)) = g & \text{on } \Omega, \\ \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Since then two-scale convergence is a well-known tool in the theory of homogenization and has been generalized by many authors. As explained later, the great advantage of using two-scale convergence techniques in our work is that it allows us to substantially weaken the continuity hypothesis required in the current literature when studying homogenization of integral functionals. The aim of this section is to present in a schematic way the main properties of two-scale convergence.

**Definition 2.6.1.** (Periodic function) A function $f : \mathbb{R}^N \to \mathbb{R}$, with $N \geqslant 1$, is

i) *$Q$- periodic* if $f(\cdot) = f(\cdot + le_i)$ for all $l \in \mathbb{Z}$, where $\{e_1, \dots, e_N\}$ is the canonical basis of $\mathbb{R}^N$;

ii) *$kQ$- periodic (or $k$- periodic)*, with $k \in \mathbb{N}$, if $f(k\,\cdot\,)$ is $Q$-periodic.

As for notation, we denote by $C_{\mathrm{per}}(Q)$ the Banach space of all $Q$-periodic continuous functions defined on $\mathbb{R}^N$ with values in $\mathbb{R}$ endowed with the supremum norm, and by $W^{1,p}_{\mathrm{per}}(kQ)$ the $W^{1,p}$-closure of all $kQ$- periodic and $C^1$-functions defined on $\mathbb{R}^N$ with values in $\mathbb{R}$ endowed with the $W^{1,p}$-norm.

Given $\Omega$ an open bounded subset of $\mathbb{R}^N$ and $1 \leqslant p < \infty$, we denote by $L^p(\Omega; C_{\mathrm{per}}(Q))$ (resp. $L^p(\Omega; W^{1,p}_{\mathrm{per}}(kQ))$) the space of all measurable functions $f : \Omega \to C_{\mathrm{per}}(Q)$ ($f : \Omega \to W^{1,p}_{\mathrm{per}}(kQ)$) such that

$$\int_\Omega ||f(x)||^p_{C_{\mathrm{per}}(Q)}\, dx < \infty, \quad \left(\text{resp.} \quad \int_\Omega ||f(x)||^p_{W^{1,p}_{\mathrm{per}}(kQ)}\, dx < \infty\right)$$

where

$$||f(x)||_{C_{\mathrm{per}}(Q)} := \sup_{y \in Q} |f(x, y)|$$

$$\Big(\text{resp. } ||f(x)||^p_{W^{1,p}_{\rm per}(kQ)} = \int_{kQ} |f(x,y)|^p\,dy + \int_{kQ} |\nabla_y f(x,y)|^p\,dy < \infty\Big).$$

Clearly a function $f \in L^p(\Omega; C_{\rm per}(Q))$ (resp. $L^p(\Omega; W^{1,p}_{\rm per}(kQ))$) may be identified with the function defined on $\Omega \times \mathbb{R}^N$ via $f(x,y) := f(x)(y)$ ($\nabla_y f$ denotes its derivative with respect to the second argument $y$).

### 2.6.1 Generalized Riemann-Lebesgue Lemmas

We start by recalling some facts about periodic oscillating functions of the form $f_\varepsilon(x) = f\big(x, \frac{x}{\varepsilon}\big)$, which play an essential role in homogenization theory (see Cioranescu and Donato [32]). When $f$ does not depend on the first variable, we have the following well known result.

**Lemma 2.6.2.** *(Riemann-Lebesgue Lemma) Let $f \in L^p_{\rm loc}(\mathbb{R}^N)$ with $1 \leqslant p \leqslant \infty$, and assume that $f$ is $kQ$-periodic. For $\varepsilon > 0$ define $f_\varepsilon(x) := f(\frac{x}{\varepsilon})$. Then $f_\varepsilon \rightharpoonup \overline{f}$ in $L^p_{loc}(\mathbb{R}^N)$ (weak$\star$ if $p = \infty$), where $\overline{f} = \frac{1}{k^N} \int_{kQ} f(y)\,dy$.*

A generalized version of the Riemann-Lebesgue Lemma holds for functions in $L^p(\Omega; C_{\rm per}(Q))$.

**Lemma 2.6.3.** (*see Lemma 5.2 in Allaire [4]; see also Bensoussan, Lions and Papanicolaou [14] and Donato [43]*) *Let $f \in L^p(\Omega; C_{\rm per}(Q))$ and let $\{\varepsilon_n\}_n$ be a fixed sequence of positive real numbers converging to zero. Then, for every $n \in \mathbb{N}$, the function $f(\cdot, \frac{\cdot}{\varepsilon_n})$ is measurable in $\Omega$,*

$$\left|\left| f\left(\cdot, \frac{\cdot}{\varepsilon_n}\right)\right|\right|_{L^p(\Omega)} \leqslant ||f||_{L^p(\Omega; C_{\rm per}(Q))} := \left(\int_\Omega ||f(x)||^p_{C_{\rm per}(\Omega)}\,dx\right)^{1/p}$$

*and*

$$\lim_n \int_\Omega \left| f\left(x, \frac{x}{\varepsilon_n}\right)\right|^p dx = \int_\Omega \int_Q |f(x,y)|^p\,dxdy.$$

We finish this part with a useful characterization of functions in $L^1(\Omega; C_{\rm per}(Q))$.

**Lemma 2.6.4.** (*see Lemma 5.3 in Allaire [4]*) *A function $f$ belongs to $L^1(\Omega; C_{\rm per}(Q))$ if and only if there exists a subset $E \subset \Omega$ of measure zero such that*

i) *the function $y \to f(x,y)$ is continuous and $Q$- periodic for any $x \in \Omega \setminus E$;*

ii) *the function $x \to f(x,y)$ is measurable for any $y \in Q$;*

iii) $x \to \sup_{y \in Q} |f(x,y)|$ *has finite $L^1(\Omega)$-norm.*

### 2.6.2 The notion of two-scale convergence and some properties

Let $p$ and $q$ be real numbers such that $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, and let $\{\varepsilon_n\}_n$ be a sequence of positive numbers converging to zero.

**Definition 2.6.5.** A sequence of functions $\{f_n\}_n$ in $L^p(\Omega)$ is said to two-scale converge to a limit $f \in L^p(\Omega \times Q)$, and we will write $f_n \overset{2s}{\rightharpoonup} f$, if

$$\int_\Omega f_n(x)\phi\Big(x, \frac{x}{\varepsilon_n}\Big)\,dx \to \int_\Omega \int_Q f(x,y)\phi(x,y)\,dy\,dx$$

for all $\phi \in L^q(\Omega; C_{\mathrm{per}}(Q))$.

*Examples.* (see e.g. Lukkassen, Nguetseng and Wall [61])

i) If $f_n \overset{L^p(\Omega)}{\longrightarrow} f$, then $f_n \overset{2s}{\rightharpoonup} f$.

ii) If $f_n \overset{2s}{\rightharpoonup} f$, then $f_n \rightharpoonup \int_Q f(\cdot, y)\,dy$ in $L^p(\Omega)$.

iii) If $f \in L^p(\Omega; C_{\mathrm{per}}(Q))$, then $f_n(\cdot,\cdot) := f\big(\cdot, \frac{\cdot}{\varepsilon_n}\big) \overset{2s}{\rightharpoonup} f$.

**Lemma 2.6.6.** *(see e.g. Lukkassen, Nguetseng and Wall [61]) For each sequence $\{f_n\}_n$ bounded in $L^p(\Omega)$ there exists a subsequence (still denoted by $\{f_n\}_n$) and $f \in L^p(\Omega \times Q)$ such that $f_n \overset{2s}{\rightharpoonup} f$.*

For sequences weakly convergent in $W^{1,p}(\Omega)$ the following compactness result holds.

**Theorem 2.6.7.** (*see Allaire [4] or Nguetseng [69]*) *Assume that $\{f_n\}_n$ weakly converges to a function $f$ in $W^{1,p}(\Omega)$. Then $f_n \overset{2s}{\rightharpoonup} f$, and there exist a subsequence* (*still denoted by* $\{f_n\}_n$ ) *and $f_1 \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Q))$ such that*

$$\nabla f_n \overset{2s}{\rightharpoonup} \nabla f + \nabla_y f_1.$$

We finally remark that analogous properties hold for the extended notion of $n$-scale convergence and we refer to Allaire and Briane [5].

# 3. VARIATIONAL PROBLEMS IN PERIODIC HOMOGENIZATION: PREVIOUS RESULTS

In this section we give a brief account of the developments on periodic homogenization of integral functionals, along with several references, that motivated the present work.

## 3.1 Pure periodic (iterated) homogenization

We turn our attention to the asymptotic analysis of a family of functionals defined on $L^p(\Omega;\mathbb{R}^d)$, with $1 < p < \infty$ and $d \geqslant 1$, by

$$\mathcal{I}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, u(x), \nabla u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\ \\ \infty & \text{otherwise,} \end{cases} \tag{3.1}$$

for some open bounded set $\Omega \subset \mathbb{R}^N$ with $N \geqslant 1$. The results below provide conditions on the integrands $f$ under which the $\Gamma$-limit of $\{\mathcal{I}_\varepsilon\}_\varepsilon$ can be obtained. We set $Q := (0,1)^N$.

*Main problem:* Study the $\Gamma(L^p(\Omega))$- $\lim\limits_{\varepsilon\to 0} \mathcal{I}_\varepsilon$.

### 3.1.1 The case where $\mathcal{I}_\varepsilon(u) = \displaystyle\int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u\right) dx$

Let $d = 1$, that is, assume that $u$ is a scalar-valued function, and let $f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a measurable function $Q$-periodic with respect to the first variable, strictly convex and of class $C^1$ with respect to the second one, and satisfying

i) $|\xi - \xi_0|^p \leqslant f(x,\xi) \leqslant C(1 + |\xi|^p)$, for all $x$, $\xi \in \mathbb{R}^N$ and some $\xi_0 \in \mathbb{R}^N$;

ii) $\left|f(x,\xi)^{\frac{1}{p}} - f(x,\xi')^{\frac{1}{p}}\right| \leqslant C|\xi - \xi'|$, for all $x$, $\xi, \xi' \in \mathbb{R}^N$,

for some positive constant $C$. Then Marcellini [63] showed that the $\Gamma(L^p(\Omega))$- limit of $\{\mathcal{I}_\varepsilon\}_\varepsilon$, with respect to the strong topology of $L^p(\Omega)$ is given by

$$\mathcal{I}_{\text{hom}}(u) = \int_\Omega f_{\text{hom}}(\nabla u(x))\, dx$$

for all $u \in W^{1,p}(\Omega)$, where $f_{\text{hom}}$ is defined by

$$f_{\text{hom}}(\xi) := \inf_\phi \left\{ \int_Q f(y, \xi + \nabla\phi(y))\, dy,\ \phi \in W^{1,p}_{\text{per}}(Q) \right\}. \tag{3.2}$$

The function $f_{\text{hom}}$ is strictly convex and satisfies the same growth conditions as $f$. We note that if $\Omega$ is assumed to be Lipschitz, then by Proposition 2.2.1, Remark 2.2.6 and the fact that $p > 1$, the $\Gamma$-limit of the previous functionals for $u \in W^{1,p}(\Omega)$ would be the same if the weak $W^{1,p}$-topology had been considered in place of the strong $L^p$-topology.

To illustrate the main idea in the convex case we give a sketch of Marcellini's proof.

*Step 1.* Use a compactness argument due to C. Sbordone [73] to obtain converging (sub)sequences $\{\mathcal{I}_{\varepsilon_j}\}_j$ to an integral $\Gamma$-limit functional whose integrand $f_{\{\varepsilon_j\}}$ is convex.

*Step 2.* Prove that $f_{\{\varepsilon_j\}}$ is independent on the variable $x$.

*Step 3.* Observe that by Jensen's inequality (see (2.8))

$$f_{\{\varepsilon_j\}}(\xi) = \inf \left\{ \int_Q f_{\{\varepsilon_j\}}(\xi + \nabla\phi(y))\, dy :\ \phi \in W^{1,p}_0(Q) \right\} \left(\text{resp. } \phi \in W^{1,p}(Q),\ \int_Q \nabla\phi = 0\right).$$

*Step 4.* Show that

$$\limsup_{\varepsilon\to 0}\ \min_\phi \left\{ \int_Q f\left(\frac{x}{\varepsilon}, \xi + \nabla\phi(y)\right) dy :\ \phi \in W^{1,p}_0(Q) \right\}$$

$$\leqslant f_{\text{hom}}(\xi)$$

$$\leqslant \liminf_{\varepsilon\to 0}\ \min_\phi \left\{ \int_Q f\left(\frac{x}{\varepsilon}, \xi + \nabla\phi(y)\right) dy :\quad \phi \in W^{1,p}(Q),\ \int_Q \nabla\phi = 0 \right\}.$$

*Step 4.* By Theorem 2.5.11 $\left(\text{resp. for } \phi \in W^{1,p}(Q),\ \int_Q \nabla\phi = 0\right)$

$$\lim_{j\to\infty} \min_\phi \left\{ \int_Q f\left(\frac{x}{\varepsilon_j}, \xi + \nabla\phi(y)\right) dy :\ \phi \in W^{1,p}_0(Q) \right\}$$

$$= \min\left\{\int_Q f_{\{\varepsilon_j\}}(\xi + \nabla\phi(y))\,dy:\ \phi \in W_0^{1,p}(Q)\right\}.$$

*Step 5.* Conclude that $f_{\{\varepsilon_j\}}(\xi) = f_{\mathrm{hom}}(\xi)$.

We refer to Carbone-Sbordone [31] and Cioranescu, Damlamian and De Arcangelis [33] for similar results in the convex case.

Müller shows in [66] that the homogenized formula (3.2) does not necessarily hold in the nonconvex case when $d > 1$, that is, when $u$ is assumed to be a vector-valued function; in fact for nonconvex $f$ and $d > 1$ it is necessary to consider variations which are periodic over an infinite ensemble of cells, instead of considering variations which are periodic over just the unit cell $Q$. Under the assumptions that $f : \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$, with $d \geqslant 1$, is a Borel measurable function not necessarily convex and $Q$-periodic in the first variable, and

i) there exist $C$, $\beta > 0$ such that

$$\beta|\xi|^p \leqslant f(x,\xi) \leqslant C(1 + |\xi|^p); \tag{3.3}$$

ii) there exists $L > 0$ such that the $p$-Lipschitz condition

$$|f(x,\xi) - f(x,\xi')| \leqslant L\big(1 + |\xi|^{p-1} + |\xi'|^{p-1}\big)|\xi - \xi'|$$

holds, Müller proved that

$$\Gamma(L^p(\Omega))\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u) = \int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,dx, \tag{3.4}$$

for all $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, where $f_{\mathrm{hom}}$ satisfies the same growth conditions as $f$, and is given by

$$f_{\mathrm{hom}}(\xi) := \inf_{T\in\mathbb{N}} \inf_{\phi} \left\{\frac{1}{T^N}\int_{(0,T)^N} f(y, \xi + \nabla\phi(y))\,dy, \quad \phi \in W_0^{1,p}\big((0,T)^N;\mathbb{R}^d\big)\right\}, \tag{3.5}$$

or, equivalently,

$$f_{\mathrm{hom}}(\xi) = \inf_{T\in\mathbb{N}} \inf_{\phi} \left\{\frac{1}{T^N}\int_{(0,T)^N} f(y, \xi + \nabla\phi(y))\,dy,\ \phi \in W_{\mathrm{per}}^{1,p}\big((0,T)^N;\mathbb{R}^d\big)\right\}. \tag{3.6}$$

The idea of the proof is to establish the lower bound for the $\Gamma$-limit for affine limit functions $u$ and to prove that this lower bound is achieved for an affine recovering sequence; the next

step is to use an approximation argument (for which $\Omega$ is required to be Lipschitz) to obtain the same bounds for a general function $u \in W^{1,p}(\Omega;\mathbb{R}^d)$.

When $f$ is convex, Müller (see Theorem 1.5 in [66]) recovered Marcellini's result under weakened growth conditions but assuming more regularity on the domain $\Omega$, showing in Section 4 of [66] that for convex integrands the expressions (3.5) or (3.6) are equivalent to (3.2). Müller remarks that when $d = 1$ this equivalence holds independently of any convexity assumption, because in this case

$$\inf_{T\in\mathbb{N}} \inf_{\phi} \left\{ \int_{(0,T)^N} f(y,\xi+\nabla\phi(y))\,dy,\ \phi \in W^{1,p}_{\text{per}}\big((0,T)^N\big) \right\}$$
$$= \inf_{T\in\mathbb{N}} \inf_{\phi} \left\{ \int_{(0,T)^N} \mathcal{C}f(y,\xi+\nabla\phi(y))\,dy,\ \phi \in W^{1,p}_{\text{per}}\big((0,T)^N\big) \right\}$$

where $\mathcal{C}f$ denotes the convex envelope of $f$ (see Definition 2.3.12). Finally, Müller showed with the example below that, when $d > 1$ and $f$ is nonconvex, expressions (3.5) or (3.6) are not necessarily equivalent to (3.2).

*Example.* Let $f^0 : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be given by $f^0(\xi) := |\xi|^4 + h(\det \xi)$, where

$$h(t) := \begin{cases} \frac{8(1+a)^2}{t+a} - 8(1+a) - 4 & \text{if } t > 0, \\[2ex] \frac{8(1+a)^2}{a} - 8(1+a) - 4 - \frac{8(1+a)^2}{a^2}t & \text{if } t \leqslant 0, \end{cases}$$

for $0 < a < \frac{1}{2}$. Let now $R := (0,\frac{1}{2}) \times (0,1)$, and define for $x \in Q := (0,1)^2$ and $\xi \in \mathbb{R}^{2\times 2}$

$$f(x,\xi) := \{\chi_R(x) + \delta\chi_{Q\setminus R}(x)\} f^0(\xi)$$

where $\delta$ is a small positive number. Extend $f$ by $Q$-periodicity in the first variable and let $\xi = \text{diag}(1,c)$, $\frac{\pi}{4} < c < 1$. Müller showed that

$$\inf_{\phi} \left\{ \int_Q f(y,\xi+\nabla\phi(y))\,dy,\ \phi \in W^{1,p}_{\text{per}}\big(Q;\mathbb{R}^2\big) \right\} > C > 0$$

and

$$\inf_{T\in\mathbb{N}} \inf_{\phi} \left\{ \frac{1}{T^2} \int_{(0,T)^2} f(y,\xi+\nabla\phi(y))\,dy,\ \phi \in W^{1,p}_0\big((0,T)^2;\mathbb{R}^2\big) \right\} \leqslant C\delta,$$

for some positive constant $C$. Thus

$$\inf_{T\in\mathbb{N}} \inf_{\phi} \left\{ \frac{1}{T^2} \int_{(0,T)^2} f(y,\xi+\nabla\phi(y))\,dy,\ \phi \in W^{1,p}_0\big((0,T)^2;\mathbb{R}^2\big) \right\}$$

$$< \inf_{\phi} \left\{ \int_Q f(y, \xi + \nabla\phi(y))\, dy,\ \phi \in W^{1,p}_{\text{per}}(Q; \mathbb{R}^2) \right\},$$

provided $\delta$ is sufficiently small. The function $f$ can be interpreted as the energy density of a composite material consisting of a strong and a very weak component (see Figure. 3.1), where $\delta$ represents the strength of the weak material. The first block (unit cell) can support compression while the second block (which shows $T \times T$ cells rescaled to the unit cell) can achieve very low energetic states as $T \to \infty$.

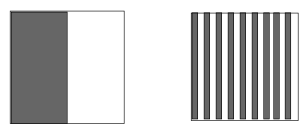


*Fig. 3.1:* Two different behaviors under compression

In an independent work Braides [16] also treated the vectorial and nonconvex case. Precisely, using a compactness $\Gamma$-convergence result of Fusco [53], Braides proved that equality (3.4) holds whenever $f : \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ is a Borel measurable function, almost periodic,[1] quasiconvex and satisfying the standard growth and coercivity conditions (3.3). Braides remarks that equality (3.5) is equivalent to

$$f_{\text{hom}}(\xi) := \lim_{T\to\infty} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(y, \xi + \nabla\phi(y))\, dy,\ \phi \in W^{1,p}_0((0,T)^N) \right\}, \tag{3.7}$$

and that $f_{\text{hom}}$ is a quasiconvex function. By Corollary 2.5.14 this result implies that (3.4) holds under the assumptions that $f$ is a Borel measurable function $Q$-periodic in the first variable and such that (3.3) holds (for a more direct proof see Braides and Defranceschi [19] where the authors derived a compactness result for functionals with this class of integrands and used the integral representation theorem of Buttazzo and Dal Maso [26]).

[1] A function $f : \mathbb{R}^N \to \mathbb{R}$ is said to be *almost periodic* if for every $\beta > 0$ there exists $L_\beta > 0$ such that for every $a \in \mathbb{R}^N$, there exists $\tau \in a + [0, L_\beta]^N$ such that, for a.e. $x \in \mathbb{R}^N$, $|f(x+\tau) - f(x)| < \varepsilon$. In particular, if $f$ is $Q$- periodic it is also almost periodic.

### 3.1.2 The case where $\mathcal{I}_\varepsilon(u) = \displaystyle\int_\Omega f\left(x, \frac{x}{\varepsilon}, u, \nabla u\right)\, dx$

In [15] Braides studied functionals of the form

$$\mathcal{I}_\varepsilon(u) = \int_\Omega f\Big(\, x, \frac{x}{\varepsilon}, u(x), \nabla u(x)\,\Big)\, dx,$$

for scalar-valued $u$ under the assumptions that the integrand $f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$, $f = f(x, y, s, \xi)$, is convex in $\xi$, and that there exist $b \in L^1_{\text{loc}}(\mathbb{R}^N)$ and a continuous positive real function $\omega$ with $\omega(0) = 0$ such that

$$|f(x, y, s, \xi) - f(x', y, s', \xi)| \leqslant \omega(|x - x'| + |s - s'|)\big[b(y) + f(x, y, s, \xi)\big] \tag{3.8}$$

for all $x$, $x'$, $y$, $\xi \in \mathbb{R}^N$ and all $s$, $s' \in \mathbb{R}^d$. In addition, $f$ is assumed to be measurable, $Q$-periodic with respect to $y$, continuous with respect to the variables $x$ and $s$, and to satisfy the growth condition

$$0 \leqslant f(x, y, s, \xi) \leqslant a(x)\big[b(y) + |s|^p + |\xi|^p\big]$$

for all $(x, y, s, \xi) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$, and for some continuous function $a : \mathbb{R}^N \to [0, \infty)$.

Using a compactness and representation result by Buttazzo and Dal Maso (Theorem 4.4 in [28]), Braides showed that

$$\Gamma(L^p(\Omega))\text{-}\lim_{\varepsilon \to 0} \mathcal{I}_\varepsilon(u) = \int_\Omega f_{\text{hom}}(x, u(x), \nabla u(x))\, dx,$$

for all $u \in W^{1,p}(\Omega)$, where $f_{\text{hom}}$ is the convex function given by

$$f_{\text{hom}}(x, s, \xi) := \inf_\phi \left\{ \int_Q f(x, y, s, \xi + \nabla\phi(y))\, dy, \quad \phi \in W^{1,p}_{\text{per}}(Q; \mathbb{R}^d) \right\}.$$

In a first step Braides proves the result for integrands of the form $f = f(y, \xi)$ by an argument similar to the one used by Marcellini. Then he uses this case to deduce the general one by an argument that takes into account some properties derived in [28].

A sketch of the proof of an analogous result in the vectorial setting for $f = f(x, y, \xi)$ can be found in Exercise 14.6 of Braides and Defranceschi [19] (convex and nonconvex case) and also in Theorem 1.3 of Braides and Lukkassen [21] (convex case).

Our goal in Theorem 4.1.1 is to prove a similar result under substantially weaker continuity hypothesis than (3.8). This will be possible by combining Γ-convergence and two-scale

convergence arguments. We treat the nonconvex case assuming that the integrand $f = f(x, y, \xi)$ is continuous with respect to the pair $(y, \xi)$, measurable in $x$, and $Q$-periodic as a function of $y$. We note that the improvement in (3.8) is done at the expense of requiring continuity in the variable $y$, as opposed to only measurability as in [15], [19] and [21]. Recently we were able to prove another version of this result for integrands that are continuous with respect to the first variable $x$ and measurable with respect to the second one $y$ (see Chapter 6). This case turns out to be more relevant for the applications to problems of mixtures. We still use a two-scale argument but the analysis is more delicate.

### 3.1.3 The case where $\mathcal{I}_\varepsilon(u) = \int_\Omega f\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \nabla u\right)\, dx$

In Theorem 1.1 of [21] Braides and Lukkassen (see also [60]) study the $\Gamma$-convergence of a family of functionals of the type

$$\mathcal{I}_\varepsilon(u) = \int_\Omega f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \nabla u(x)\right)\, dx,$$

where the integrand $f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0, \infty)$ is periodic in the first two oscillating variables and satisfies usual coercivity and growth conditions. In addition,

- $f(y, \cdot\,; \xi)$ is measurable for all $(y, \xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$;
- $f(y, z; \cdot)$ is convex for all $(y, z) \in \mathbb{R}^N \times \mathbb{R}^N$;
- there exist $b \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and a continuous positive real function $\omega$, with $\omega(0) = 0$, such that
$$|f(y, z, \xi) - f(y', z, \xi)| \leqslant \omega(|y - y'|)\big[b(z) + f(y, z, \xi)\big] \tag{3.9}$$
for all $y$, $y'$, $z \in \mathbb{R}^N$, and all $\xi \in \mathbb{R}^{d\times N}$.

A compactness and integral representation theorem by Fusco [53], an analogous argument to the one used by Marcellini, and the reiteration of the homogenization formula (3.2) are used in the proof. They showed that

$$\Gamma(L^p(\Omega))\text{-} \lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u) = \int_\Omega \overline{f}_{\mathrm{hom}}(\nabla u(x))\, dx$$

for all $u \in W^{1,p}(\Omega; \mathbb{R}^d)$, where $\overline{f}_{\mathrm{hom}}$ is defined by

$$\overline{f}_{\mathrm{hom}}(\xi) := \inf_\phi \left\{ \int_Q f_{\mathrm{hom}}(y; \xi + \nabla\phi(y))\, dy : \ \phi \in W^{1,p}_{\mathrm{per}}(Q; \mathbb{R}^d) \right\},$$

and

$$f_{\mathrm{hom}}(y;\xi) := \inf_{\phi} \left\{ \int_Q f(y,z;\xi+\nabla\phi(z))\,dz : \ \phi \in W^{1,p}_{\mathrm{per}}(Q;\mathbb{R}^d) \right\}$$

The analysis has been extended to the case of nonconvex integrands (see Theorem 22.1 in Braides and Defranceschi [19]) and to the case where $f$ depends explicitly on the macroscopic variable $x$, as in (3.1) (see Remark 22.8 of Braides and Defranceschi [19]), under the strong uniform continuity condition (3.9) or (1.6), respectively. Using techniques of multiscale convergence and restricting the argument to the convex and homogeneous case (no dependence on the variable $x$), Fonseca and Zappale were able to recover these results with weaker continuity conditions than (3.9). Namely, they only required $f$ to be continuous (see Theorem 1.9 in [51]).

As these results seem to show, it is not clear what is the natural regularity on $f$ for the integral (3.1) to be well defined. Motivated by Theorem 4.1.1 we treat in Theorem 4.2.1 the case where the integrand $f$ satisfies the following conditions:

- $f(x,\cdot,\cdot,\cdot)$ is continuous for a.e. $x \in \Omega$;
- $f(\cdot,y,z,\xi)$ is measurable for all $(y,z,\xi) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$;
- $f(x,\cdot,z,\xi)$ is $Q$-periodic for all $(z,\xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and for a.e. $x \in \Omega$; $f(x,y,\cdot,\xi)$ is $Q$-periodic for all $(y,\xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and for a.e. $x \in \Omega$;
- there exists $\beta > 0$ such that for all $(y,z,\xi) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and for a.e. $x \in \Omega$
$$\frac{1}{\beta}|\xi|^p - \beta \leqslant f(x,y,z,\xi) \leqslant \beta(1+|\xi|^p).$$

We recover Theorem 1.9 in Fonseca and Zappale [51].

We point out that our analysis follows the lines of the one in Braides and Defranceschi [19] (Theorem 22.1 and Remark 22.8), and that our main contribution is to use arguments that allow us to weaken the strong uniform continuity hypothesis (1.6).

## 3.2 Thin films with periodic microstructure in the nonlinear membrane theory

Reduction dimension arguments are used variationally to study minimization problems over domains whose dimension, in one or more directions, is small compared with the dimension

in the other direction. Membranes are 3-dimensional continuum bodies with a reference configuration with cylindrical shape, such that the height of the cylinder - the thickness - is small in comparison with the other dimensions. This feature suggests the possibility of deriving 2-dimensional models in membrane theory from the full 3-dimensional theory. The idea is to regard the thickness of these thin cylindrical bodies as a small parameter $\varepsilon$ and then to study the asymptotic behavior as $\varepsilon$ goes to zero.

Starting from the works of Acerbi, Buttazzo and Percivale [2], $\Gamma$-convergence has become an important tool to do this asymptotic analysis in nonlinear elasticity. We briefly discuss the main approach to this study.

Let $\omega$ be an open and bounded subset of $\mathbb{R}^2$. For each $0 < \varepsilon \ll 1$ define $\Omega_\varepsilon := \omega \times (-\varepsilon, \varepsilon)$ and denote $\Sigma_\varepsilon := \omega \times \{-\varepsilon, \varepsilon\}$ (Figure. 3.2).

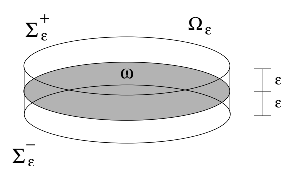



*Fig. 3.2:* Cylindrical thin domain of thickness $\varepsilon$

We assume that the body is pinned on the lateral boundary, that is $v(x) = x$ on $\partial\omega \times (-\varepsilon, \varepsilon)$, for all its admissible deformations, and that it is subjected to the action of regular surface traction densities $g(\varepsilon)$ on $\Sigma_\varepsilon$, and regular dead loads $f(\varepsilon)$. The total energy of this body under the action of this forces is the difference between the elastic energy and the work of external forces. More precisely,

$$\mathcal{W}(\varepsilon)(v) := \int_{\Omega_\varepsilon} W(\varepsilon)(x, Dv)\, dx - \int_{\Omega_\varepsilon} f(\varepsilon) \cdot v\, dx - \int_{\Sigma_\varepsilon} g(\varepsilon) \cdot v\, dS,$$

for $v \in \mathcal{V}(\varepsilon) := \{v \in W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3) : v(x) = x \text{ on } \partial\omega \times (-\varepsilon, \varepsilon)\}$.

*Main problem:* To study

$$\lim_{\varepsilon \to 0} \min_{v \in \mathcal{V}(\varepsilon)} \mathcal{W}(\varepsilon)(v)$$

by means of $\Gamma$-convergence. As usual, in order to study this problem as $\varepsilon \to 0$ we rescale the $\varepsilon$-thin body into a reference domain of unit thickness (see e.g. Acerbi, Buttazzo and Percivale [2], Anzellotti, Baldo and Percivale [8], Le Dret and Raoult [55], Braides, Fonseca and Francfort [20]), so that the resulting energy will be defined on a fixed body, while the

dependence on $\varepsilon$ turns out to be explicit in the transverse derivative. For this, we consider the change of variables

$$\Omega_\varepsilon \to \Omega := \omega \times I, \quad (x_1, x_2, x_3) \mapsto \left(x_1, x_2, \frac{1}{\varepsilon} x_3\right),$$

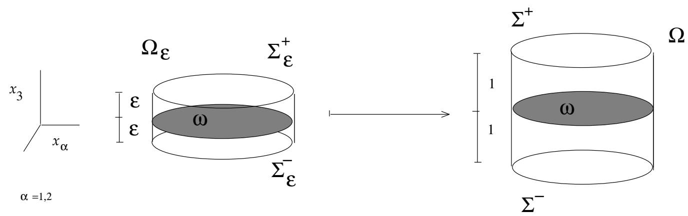



*Fig. 3.3:* Rescaled domain of unit thickness

and define $u(x_\alpha, x_3/\varepsilon) = v(x_\alpha, x_3)$ on the rescaled cylinder $\Omega$, where $I := (-1, 1)$ and $x_\alpha := (x_1, x_2)$ is the in-plane variable. We denote $\Sigma := \omega \times \{-1, 1\}$ (Figure. 3.3).

It is well known that membrane theory arises at the order $\varepsilon$ of a formal asymptotic expansion (see Fox, Raoult and Simo [52]), provided that the body forces are of order 1 and the surface loadings are of order $\varepsilon$. Since this energy is of order $\varepsilon$, we divide the total energy by $\varepsilon$ and in addition we assume that

$$\begin{cases} f(\varepsilon)(x_\alpha, \varepsilon x_3) &= f(x_\alpha, x_3), \\ g(\varepsilon)(x_\alpha, \varepsilon x_3) &= \varepsilon\, g(x_\alpha, x_3), \end{cases}$$

where $f \in L^{p'}(\Omega; \mathbb{R}^3)$, $g \in L^{p'}(\Sigma; \mathbb{R}^3)$ $(1/p + 1/p' = 1)$. If $W_\varepsilon(x_\alpha, x_3; \cdot) = W(\varepsilon)(x_\alpha, \varepsilon x_3; \cdot)$, for fixed $\varepsilon$ minimizing $\mathcal{W}(\varepsilon)$ on $\mathcal{V}(\varepsilon)$ is equivalent to minimizing

$$\mathcal{W}_\varepsilon(u) := \frac{\mathcal{W}(\varepsilon)(v)}{\varepsilon} = \int_\Omega W_\varepsilon \left(x, \nabla_\alpha u(x) \Big| \frac{1}{\varepsilon} \nabla_3 u(x)\right) dx - \int_\Omega f \cdot u \, dx - \int_\Sigma g \cdot u \, dS$$

on $\mathcal{V}_\varepsilon := \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : u(x) = (x_\alpha, \varepsilon x_3) \text{ on } \partial\omega \times I\}$. We recall that for $\overline{\xi} \in \mathbb{R}^{3\times 2}$ and $z \in \mathbb{R}^3$, $(\overline{\xi}|z)$ denote the matrix whose first two columns are those of $\overline{\xi}$ and the last one is $z$; $\nabla_\alpha = (\nabla_1, \nabla_2)$.

*Main step:* To study

$$\Gamma(L^p(\Omega))\text{-}\lim_{\varepsilon \to 0} \mathcal{W}_\varepsilon.$$

### 3.2.1 The case $\mathcal{W}_\varepsilon(u) = \int_\Omega W\left(x, \frac{x_\alpha}{\varepsilon}, \nabla_\alpha u \Big| \frac{1}{\varepsilon}\nabla_3 u\right) dx$

The motivation to study this case comes from the works of Braides, Fonseca and Francfort [20], of Babadjian and Francfort [11] and from Theorem 4.1.1.

*Starting point:* In [20] Braides, Fonseca and Francfort derived a homogenization result for energies $W_\varepsilon(x;\xi) = W(x_3, x_\alpha/\varepsilon, \xi)$. Namely, under the hypotheses that $W$ is a Carathéodory function satisfying standard coerciveness and growth conditions, they proved that if

$$\mathcal{W}_\varepsilon(u) = \int_\Omega W\left(x_3, \frac{x_\alpha}{\varepsilon}, \nabla_\alpha u \Big| \frac{1}{\varepsilon}\nabla_3 u\right) dx$$

then

$$\Gamma(L^p(\Omega))\text{-} \lim_{\varepsilon\to 0} \mathcal{W}_\varepsilon(u) = 2\int_\omega W_{\text{hom}}(\nabla_\alpha u)\, dx_\alpha$$

for all $u \in W^{1,p}(\omega;\mathbb{R}^3) \equiv \left\{u \in W^{1,p}(\Omega;\mathbb{R}^3) :\ D_3 u(x) = 0 \text{ for a.e.} x \in \Omega\right\}$, where

$$\begin{aligned} W_{\text{hom}}(\overline{\xi}) := \lim_{T\to+\infty} \inf_\phi \Bigg\{ \frac{1}{2T^2} \int_{(0,T)^2\times I} & W\big(y_3, y_\alpha, \overline{\xi} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y)\big)\, dy \\ & \phi \in W^{1,p}((0,T)^2 \times I;\mathbb{R}^3),\ \phi = 0 \text{ on } \partial(0,T)^2 \times I \Bigg\}. \end{aligned} \tag{3.10}$$

Their analysis is based on a compactness property for a family of energies of the form $W_\varepsilon(x,\xi)$. We state here this result since it is important for our applications in Chapter 5.[2]

**Theorem 3.2.1.** *(Theorem 2.5 in [20]) Let $\omega$ be an open bounded subset of $\mathbb{R}^2$ and let $\Omega := \omega \times I$. Let $W_\varepsilon : \Omega \times \mathbb{R}^{3\times 3} \to [0,\infty)$ be a family of Carathéodory functions such that for a.e. $x \in \Omega$ and any $\xi \in \mathbb{R}^{3\times 3}$*

$$\beta|\xi|^p \leqslant W_\varepsilon(x,\xi) \leqslant C(1+|\xi|^p),\ 0 < \beta \leqslant C,\ 1 < p < \infty.$$

*For each $\varepsilon > 0$ define $\mathcal{W}_\varepsilon : L^p(\Omega;\mathbb{R}^3) \times \mathcal{A}(\omega) \to [0,\infty)$ by*

$$\mathcal{W}_\varepsilon(u;A) := \begin{cases} \displaystyle\int_{A\times I} W_\varepsilon\left(x, D_\alpha u(x) \Big| \frac{1}{\varepsilon} D_3 u(x)\right) dx & \textit{if } u \in W^{1,p}(A\times I;\mathbb{R}^3), \\ \infty & \textit{otherwise.} \end{cases}$$

[2] In [20] the authors prove a more general result for thin films with varying profiles. For our purposes it is enough to present this simpler case.

*Let*

$$\mathcal{W}_{\{\varepsilon\}}(u;A) := \inf_{\{u_\varepsilon\}} \left\{ \liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon(u_\varepsilon;A) : u_\varepsilon \to u \text{ in } L^p(A\times I;\mathbb{R}^3) \right\}.$$

*Then every sequence $\varepsilon_n \downarrow 0$ admits a subsequence $\{\varepsilon_{n_j}\}_j \equiv \{\varepsilon_j\}_j$ such that $\mathcal{W}_{\{\varepsilon_j\}}(\,\cdot\,;A)$ is the $\Gamma(L^p(A\times I))$-limit of $\{\mathcal{W}_{\varepsilon_j}(\,\cdot\,;A)\}_j$ for all $A\in\mathcal{A}(\omega)$. Further there exists a Carathéodory function $W_{\{\varepsilon_j\}} : \omega\times\mathbb{R}^{3\times 2}\to\mathbb{R}$ such that*

$$\mathcal{W}_{\{\varepsilon_j\}}(u;A) = 2\int_A W_{\{\varepsilon_j\}}(x_\alpha, D_\alpha u(x_\alpha))\,dx_\alpha, \tag{3.11}$$

*for all $A\in\mathcal{A}(\omega)$ and all $u\in W^{1,p}(A;\mathbb{R}^3)$.*

The proof of Theorem 3.2.1 is based on the Direct Method of $\Gamma$-convergence. In a first step the authors derive a useful version of Proposition 2.5.13 that will allow us the matching of recovering sequences in the lateral boundary of open sets $A\times I$, with $A\subset\omega$. Its proof is based on De Giorgi's slicing argument and the possibility of considering cut-off functions independent on the transverse direction of the thin film.

**Lemma 3.2.2.** *(Lemma 2.6 in [20]) Under the hypotheses of Theorem 3.2.1 on $W_\varepsilon$, let $A\in\mathcal{A}(\omega)$, $u\in W^{1,p}(A;\mathbb{R}^3)$, and let $\varepsilon_j\downarrow 0$ be a sequence for which $\Gamma(L^p(A\times I))\text{-}\lim_{j\to\infty}\mathcal{W}_{\varepsilon_j}(u;A)$ exists. Then there exists $\{w_j\}_j\subset W^{1,p}(A\times I;\mathbb{R}^3)$ and a sequence of compact sets of $A$, $\{K_j\}_j$, such that*

$$\Gamma(L^p(A\times I))\text{-}\lim_{j\to\infty}\mathcal{W}_{\varepsilon_j}(u;A) = \lim_{j\to\infty}\mathcal{W}_{\varepsilon_j}(w_j,A)$$

*and $w_j = u$ in $(A\setminus K_j)\times I$.*

In Chapter 5 we study the asymptotic behavior of a heterogeneous $\varepsilon$-thin film whose microstructure oscillates on a scale that is comparable to that of the thickness of the domain (see Figure 3.4). We propose in Theorem 5.1.1 to establish a dimensional reduction and homogenization result, where both scales are identical, by adding in the stored energy density an explicit dependence on the in-plane variable $x_\alpha$. Namely, we assume that

$$W_\varepsilon(x;\xi) = W\left(x,\frac{x_\alpha}{\varepsilon},\xi\right).$$

We seek to find the $\Gamma(L^p(\Omega))$-limit of the following family of energies

$$\mathcal{W}_\varepsilon(u) := \int_\Omega W\left(x,\frac{x_\alpha}{\varepsilon},\nabla_\alpha u(x)\Big|\frac{1}{\varepsilon}\nabla_3 u(x)\right)dx$$

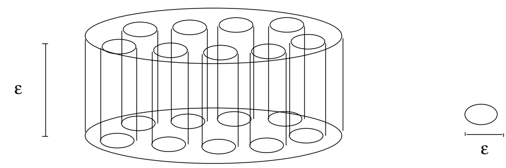



*Fig. 3.4:* Thin domain with periodic structure in the in-plane direction

for $u \in W^{1,p}(\Omega;\mathbb{R}^3)$, and some function $W : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ whose hypotheses will be introduced later $\big($in this case in the reference configuration $\Omega_\varepsilon$ we have $\mathcal{W}(\varepsilon)(x,\xi) = W(x_\alpha, \frac{x}{\varepsilon}, \xi)\big)$.

Two features differentiate our approach from what is available in most of the literature in the subject (see Shu [75] and Braides, Fonseca and Francfort [20]). The first one is the use of two-scale techniques as in Theorem 4.1.1. The second one is based on a decoupling argument used by Babadjian and Francfort [11] to derive a nonlinear membrane model for stored energy densities of the form $W(x,\xi)$ generalizing the case where $W_\varepsilon(x,\xi) \equiv W(x_3,\xi)$ that was studied in Braides, Fonseca and Francfort [20]. This decoupling procedure is necessary to take into account the different nature of the two variables $y_\alpha$ and $x_\alpha$ that appear in the structure of the limit functional (see (5.24) below).

### 3.2.2 The case $\mathcal{W}_\varepsilon(u) = \int_\Omega W\left(x, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}, \nabla_\alpha u \middle| \frac{1}{\varepsilon}\nabla_3 u\right) dx$

In Theorem 5.2.1 we want to study the asymptotic analysis of $\varepsilon$-thin elastic bodies whose microstructure is periodic of period $\varepsilon$ in the in-plane direction and periodic of period $\varepsilon^2$ in all directions. To take into account these heterogeneities, our goal is to study the sequence of energies

$$W_\varepsilon(x;\xi) = W\left(x, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}, \xi\right)$$

$\big($in this case in the reference configuration $\Omega_\varepsilon$ we have $\mathcal{W}(\varepsilon)(x,\xi) = W(x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha}{\varepsilon}, \frac{x}{\varepsilon^2}, \xi)\big)$.

We seek to find the $\Gamma(L^p(\Omega))$-limit of the following family of energies

$$\mathcal{W}_\varepsilon(u) := \int_\Omega W\left(x, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}, \nabla_\alpha u(x) \middle| \frac{1}{\varepsilon}\nabla_3 u(x)\right) dx,$$

for $u \in W^{1,p}(\Omega;\mathbb{R}^3)$, and for some function $W : \Omega \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ whose hypotheses will be introduced later. As a corollary, we derive a homogenization result for heterogeneous

$\varepsilon$-thin films of periodic structure of period $\varepsilon$ in the in-plane variable $x_\alpha$ and of period $\varepsilon^2$ in the transverse direction $x_3$. That is, we derive an homogenization formula for a family of energies

$$\mathcal{W}_\varepsilon(u) := \int_\Omega W\left(x, \frac{x}{\varepsilon}, \nabla_\alpha u(x)\Big|\frac{1}{\varepsilon}\nabla_3 u(x)\right)\,dx$$

for $u \in W^{1,p}(\Omega;\mathbb{R}^3)$. Integral functionals of the form

$$\int_\Omega W\left(\frac{x_\alpha}{\varepsilon^2}, \nabla_\alpha v\Big|\frac{1}{\varepsilon}\nabla_3 v\right)dx,$$

have been studied in Shu [75] (Theorem 5) under different length scales for the film thickness and the material microstructure. As far as we know there have no been previous results that allow for homogenization in the transverse direction of the film.

# Part II

# MAIN RESULTS

# 4. Γ-CONVERGENCE OF FUNCTIONALS WITH PERIODIC INTEGRANDS

The main goal of this chapter is to characterize the asymptotic behavior of a family of multiple scale integral functionals whose integrands have periodicity properties.

From now on, unless otherwise specified, $C$ will denote a generic constant, and for every $a \in \mathbb{R}^N$ and $\delta > 0$ we write $Q(a,\delta) := a + \delta Q \equiv (a,\delta)^N$, where $Q := (0,1)^N$. Throughout this chapter $\Omega$ stands for an open bounded set in $\mathbb{R}^N$ with $N \geqslant 1$.

## 4.1 An approach by 2-scale convergence

This section is devoted to proving the following result.

**Theorem 4.1.1.** *Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ be a function such that*

$(H_1)$ *$f(x,\cdot,\cdot)$ is continuous a.e. $x \in \Omega$;*

$(H_2)$ *$f(\cdot,y,\xi)$ is $\mathcal{L}^N$-measurable for all $y \in \mathbb{R}^N$ and all $\xi \in \mathbb{R}^{d\times N}$;*

$(H_3)$ *$f(x,\cdot,\xi)$ is $Q$-periodic for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^{d\times N}$;*

$(H_4)$ *there exist a real number $p > 1$ and a constant $\beta > 0$ such that*

$$\frac{|\xi|^p}{\beta} - \beta \leqslant f(x,y,\xi) \leqslant \beta(1 + |\xi|^p),$$

*for a.e. $x \in \Omega$, for all $y \in \mathbb{R}^N$ and all $\xi \in \mathbb{R}^{d\times N}$.*

*For each $\varepsilon > 0$ define the functional $\mathcal{I}_\varepsilon : L^p(\Omega;\mathbb{R}^d) \to [0,\infty]$ by*

$$\mathcal{I}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\left(x, \frac{x}{\varepsilon}, \nabla u(x)\right)\, dx & \textit{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\ \\ \infty & \textit{otherwise.} \end{cases} \tag{4.1}$$

*If* $u \in L^p(\Omega;\mathbb{R}^d)$ *then*

$$\mathcal{I}_{\mathrm{hom}}(u) := \Gamma(L^p(\Omega))\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega f_{\mathrm{hom}}(x, \nabla u(x))\, dx & \textit{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\ \\ \infty & \textit{otherwise,} \end{cases} \tag{4.2}$$

*where the integrand* $f_{\mathrm{hom}}$ *is given by*

$$f_{\mathrm{hom}}(x,\xi) := \lim_{T\to\infty} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(x,y,\xi + \nabla\phi(y))\, dy,\ \phi \in W_0^{1,p}((0,T)^N;\mathbb{R}^d) \right\} \tag{4.3}$$

*for a.e.* $x \in \Omega$ *and for all* $\xi \in \mathbb{R}^{d\times N}$*. It turns out that* $f_{\mathrm{hom}}$ *is (equivalent to) a Carathéodory function and satisfies p-coercivity and p-growth conditions similar to those of* $f$*. Moreover* $f_{\mathrm{hom}}(x,\,\cdot\,)$ *is quasiconvex for a.e.* $x \in \Omega$*.*

Theorem 4.1.1 was obtained in collaboration with I. Fonseca [12], and the main idea of its proof is to use the Direct Method of the Calculus of Variations (see Section 2.5.2) combined with the integral representation theorem of Buttazzo and Dal Maso (Theorem 2.4.1) to derive the existence of Γ-converging (sub)sequences to an abstract integral functional. Then the idea is to use arguments of two-scale convergence to derive an upper bound for the integrand of this functional. To get the other bound we use the fact that, under hypotheses $(H_1)$-$(H_4)$, the integrand $f$ is "uniformly continuous up to a small error". Indeed, since $f$ is a Carathéodory integrand, Scorza-Dragoni's Theorem (Theorem 2.3.26) implies that the restriction of $f$ to $K \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$ is continuous, for some compact set $K \subset \Omega$ whose complement has arbitrarily small Lebesgue measure. Then the periodicity of $f$ with respect to its second variable leads $f$ to be uniformly continuous on $K \times \mathbb{R}^N \times \overline{B}$, for some closed ball $\overline{B}$ of $\mathbb{R}^{d\times N}$ of sufficiently large radius. Finally, to ensure that the energy remains arbitrarily small on the complement of $K$ and on the set of $x$'s such that the gradient of the deformation does not belong to $\overline{B}$, we use the Decomposition Lemma (Theorem 2.2.16) which allows us to select minimizing sequences with $p^{th}$-equi-integrable gradients. Thus, in view of the $p$-growth character of the integrand, the energy over sets of arbitrarily small Lebesgue measure tends to zero.

Like for quasiconvex envelopes, there are very few explicit examples of homogenized densities in the literature. A classical explicit derivation of the function $f_{\text{hom}}$ for elliptic operators in the homogeneous case, that is, for integrands $f$ that do not depend on the variable $x$, can be found in De Giorgi and Spagnolo [41]. We present a classical example that can be found in the book of Dal Maso [35] (see references therein for more examples).

*Example.* Let $N = 1$ and let $f(y, \xi) := a(y)|\xi|^p$ for $(y, \xi) \in \mathbb{R}^2$ and $1 < p < \infty$, where $a$ is a measurable and $Q$-periodic function, such that for all $y \in \mathbb{R}$

$$\beta \leqslant a(y) \leqslant C, \;\; 0 < \beta \leqslant C.$$

Then $f_{\text{hom}}(\xi) = a_{\text{hom}}|\xi|^p$ where

$$a_{\text{hom}} := \left( \int_0^1 \left( \frac{1}{a} \right)^{p/p-1} \right)^{1-p}.$$

*Remark* 4.1.2. By hypothesis $(H_1)$ and $(H_2)$ the integrand $f$ is of Carathéodory-type and this ensures that $f(x, \cdot, \cdot)$ is a Borel function for a.e. $x \in \Omega$ (see Proposition 2.3.27; in particular the integral in (4.1) is well defined). Moreover, by hypothesis $(H_4)$ replacing $f$ by $f+\beta$ we may assume that $f$ is nonnegative almost everywhere. As a consequence of these two remarks, in the sequel, without loss of generality, we may assume that $f$ is a positive Borel function such that hypotheses $(H_1)$, $(H_3)$ and $(H_4)$ hold for every $(x, y, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{d \times N}$.

Two more remarks are worthy of note (see Müller [66] and Braides and Defranceschi [19]; see also Lemma 4.1.5 and Lemma 4.1.10 below). First, it can be seen that for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^{d \times N}$

$$f_{\text{hom}}(x, \xi) = \inf_{T \in \mathbb{N}} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(x, y, \xi + \nabla \phi(y))\, dy, \;\; \phi \in W_0^{1,p}((0,T)^N; \mathbb{R}^d) \right\} \tag{4.4}$$

and

$$f_{\text{hom}}(x, \xi) = \inf_{T \in \mathbb{N}} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(x, y, \xi + \nabla \phi(y))\, dy, \;\; \phi \in W_{\text{per}}^{1,p}((0,T)^N; \mathbb{R}^d) \right\}, \tag{4.5}$$

Secondly, we observe that under the additional hypothesis that $f(x, y, \cdot)$ is convex for a.e. $x$ and for all $y$ (in which case $(H_1)$ is equivalent to requiring that $f(x, \cdot, \xi)$ is continuous for a.e. $x$ and for all $\xi$), equalities (4.4) and (4.5) simplify to read, respectively,

$$f_{\text{hom}}(x, \xi) = \inf_{\phi} \left\{ \int_Q f(x, y, \xi + \nabla \phi(y))\, dy, \;\; \phi \in W_0^{1,p}((0,1)^N; \mathbb{R}^d) \right\} \tag{4.6}$$

and

$$f_{\mathrm{hom}}(x,\xi) = \inf_{\phi} \left\{ \int_Q f(x,y,\xi + \nabla\phi(y))\,dy,\ \phi \in W^{1,p}_{\mathrm{per}}((0,1)^N;\mathbb{R}^d) \right\}. \tag{4.7}$$

Moreover, by hypothesis $(H_1)$ and by Lemma 2.3.28, equalities (4.3)-(4.7) hold if the admissible test functions are taken in any smooth dense subset of $W^{1,p}_0((0,T)^N;\mathbb{R}^d)$ and $W^{1,p}_{\mathrm{per}}((0,T)^N;\mathbb{R}^d)$, respectively.

As a consequence of Theorem 4.1.1, Proposition 2.5.13 and Theorem 2.5.11 we get the convergence of (almost) minimizers of $\{\mathcal{I}_\varepsilon\}_\varepsilon$.

**Corollary 4.1.3.** *Under hypotheses $(H_1)$-$(H_4)$ the functional $\mathcal{I}_{\mathrm{hom}}$ defined in (4.2) has a minimum on $V_\varphi := \{u \in W^{1,p}(\Omega) : u - \varphi \in W^{1,p}_0(\Omega),\ \varphi \in W^{1,p}(\Omega)\}$ and*

$$\min_{u \in V_\varphi} \mathcal{I}_{\mathrm{hom}}(u) = \lim_{\varepsilon \to 0} \inf_{u \in V_\varphi} \mathcal{I}_\varepsilon(u).$$

*Moreover, given two sequences $\varepsilon_n \downarrow 0$ and $\{u_n\}_n \subset V_\varphi)$ such that*

$$\lim_{n\to\infty} \mathcal{I}_{\varepsilon_n}(u_n) = \lim_{n\to\infty} \inf_{u \in V_\varphi} \mathcal{I}_{\varepsilon_n}(u),$$

*then (up to subsequence) $\{u_n\}_n$ is $W^{1,p}$-weakly convergent to a minimum for the functional $\mathcal{I}_{\mathrm{hom}}$ on $V_\varphi$.*

### 4.1.1 Properties of the homogenized density

In this section we turn our attention to the main properties of the function $f_{\mathrm{hom}}$ for later use in the proof of Theorem 4.1.1. Most of these properties can be deduced from previous works (see references below), however we present alternative proofs for the sake of completeness.

By Remark 4.1.2 we restrict our analysis to the case where $f$ is a positive Borel function such that hypotheses $(H_1)$, $(H_3)$ and $(H_4)$ hold for every $(x,y,\xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$.

We start by showing that the limit in (4.3) is well defined. The argument is analogous to that used in Bouchitté, Fonseca and Mascarenhas [23] and relies on Lemma A.1 in the Appendix.

**Lemma 4.1.4.** *For all $(x,\xi) \in \Omega \times \mathbb{R}^{d\times N}$ there exists*

$$\lim_{T\to\infty} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(x,y,\xi + \nabla\phi(y))\,dy,\ \phi \in W^{1,p}_0((0,T)^N;\mathbb{R}^d) \right\}. \tag{4.8}$$

*Proof.* (see also Braides and Defranceschi [19]) Let $(x,\xi) \in \Omega \times \mathbb{R}^{d\times N}$ and let

$$S(A) := \inf_{\phi} \left\{ \int_A f(x,y,\xi+\nabla\phi(y))\,dy : \ \phi \in W_0^{1,p}(A;\mathbb{R}^d) \right\}$$

for $A \in \mathcal{A}(\mathbb{R}^N)$. Under the above assumptions on $f$ the function $S : \mathcal{A}(\mathbb{R}^N) \to [0,\infty)$ is well defined, and it satisfies the hypotheses of Lemma A.1 with $T = \mathbb{Z}^N$ and $M = 1$. Hence we conclude that the limit

$$\lim_{T\to\infty} \frac{S((0,T)^N)}{T^N},$$

or equivalently (4.8), exists. ■

We now want to show that $f_{\mathrm{hom}}$ is a Carathéodory integrand (see Definition 2.3.25) so that the functional $\mathcal{I}_{\mathrm{hom}}$ in (4.2) is well defined. We start by noting the following result.

**Lemma 4.1.5.** *Let $\overline{f}_{\mathrm{hom}}, \hat{f}_{\mathrm{hom}} : \Omega \times \mathbb{R}^{d\times N} \to [0,\infty)$ be defined, respectively, by*

$$\overline{f}_{\mathrm{hom}}(x,\xi) := \inf_{T\in\mathbb{N}} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(x,y,\xi+\nabla\phi(y))\,dy, \ \phi \in W_0^{1,p}((0,T)^N;\mathbb{R}^d) \right\}$$

*and*

$$\hat{f}_{\mathrm{hom}}(x,\xi) := \inf_{T\in\mathbb{N}} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(x,y,\xi+\nabla\phi(y))\,dy, \ \phi \in W_{\mathrm{per}}^{1,p}((0,T)^N;\mathbb{R}^d) \right\},$$

*for all $(x,\xi) \in \Omega \times \mathbb{R}^{d\times N}$. Then the relations $f_{\mathrm{hom}} = \overline{f}_{\mathrm{hom}} = \hat{f}_{\mathrm{hom}}$ hold.*

*Proof.* Let $(x,\xi) \in \Omega \times \mathbb{R}^{d\times N}$. We first show that $f_{\mathrm{hom}}(x,\xi) = \overline{f}_{\mathrm{hom}}(x,\xi)$. It is clear that $f_{\mathrm{hom}}(x,\xi) \geqslant \overline{f}_{\mathrm{hom}}(x,\xi)$. To prove the other inequality, fixed $\delta > 0$ let $S \in \mathbb{N}$ and $\varphi \in W_0^{1,p}((0,S)^N;\mathbb{R}^d)$ be such that

$$\overline{f}_{\mathrm{hom}}(x,\xi) + \delta \geqslant \frac{1}{S^N} \int_{(0,S)^N} f(x,y,\xi+\nabla\varphi(y))\,dy. \tag{4.9}$$

Extend $\varphi$ periodically to $\mathbb{R}^N$ with period $S$. Using Riemann-Lebesgue's Lemma (Lemma 2.6.2)

$$\begin{aligned} \frac{1}{S^N} \int_{(0,S)^N} f(x,y,\xi+\nabla\varphi(y))\,dy &= \lim_{\varepsilon\to 0} \frac{1}{S^N} \int_{(0,S)^N} f\left(x,\frac{y}{\varepsilon},\xi+\nabla\varphi\left(\frac{y}{\varepsilon}\right)\right)\,dy \\ &= \lim_{\varepsilon\to 0} \frac{\varepsilon^N}{S^N} \int_{\left(0,\frac{S}{\varepsilon}\right)^N} f(x,z,\xi+\nabla\theta_\varepsilon(z))\,dz, \end{aligned} \tag{4.10}$$

where $\theta_\varepsilon(z) := \frac{1}{\varepsilon}\varphi(\varepsilon z) \in W_0^{1,p}\Big(\big(0,\frac{S}{\varepsilon}\big)^N;\mathbb{R}^d\Big)$. Therefore from (4.9) and (4.10)

$$\begin{aligned}\overline{f}_{\text{hom}}(x,\xi)+\delta \quad &\geqslant \lim_{\varepsilon\to 0}\ \inf_{\theta}\left\{\frac{\varepsilon^N}{S^N}\int_{(0,\frac{S}{\varepsilon})^N} f(x,z,\xi+\nabla\theta(z))\,dz,\ \theta\in W_0^{1,p}\Big(\Big(0,\frac{S}{\varepsilon}\Big)^N;\mathbb{R}^d\Big)\right\}\\ &= f_{\text{hom}}(x,\xi).\end{aligned}$$

Letting $\delta\to 0$ we conclude that

$$\overline{f}_{\text{hom}}(x,\xi)\geqslant f_{\text{hom}}(x,\xi).$$

Finally we show that $\overline{f}_{\text{hom}}(x,\xi)=\hat{f}_{\text{hom}}(x,\xi)$ (see also Braides [19] and Müller [66] for an alternative justification). It is clear that $\overline{f}_{\text{hom}}(x,\xi)\geqslant\hat{f}_{\text{hom}}(x,\xi)$. To verify the opposite inequality, fix $\delta>0$ and take $S\in\mathbb{N}$ and a smooth function $\varphi\in W^{1,p}_{\text{per}}\big((0,S)^N;\mathbb{R}^d\big)$ such that

$$\hat{f}_{\text{hom}}(x,\xi)+\delta\geqslant\frac{1}{S^N}\int_{(0,S)^N} f(x,y,\xi+\nabla\varphi(y))\,dy. \tag{4.11}$$

By hypothesis $(H_3)$ the function $f(x,\cdot,\xi+\nabla\varphi(\cdot))$ is $(0,S)^N$-periodic, and thus

$$\begin{aligned}\frac{1}{S^N}\int_{(0,S)^N} f(x,y,\xi+\nabla\varphi(y))\,dy \quad &= \quad \lim_{\varepsilon\to 0}\int_Q f\Big(x,\frac{y}{\varepsilon},\xi+\nabla\varphi\Big(\frac{y}{\varepsilon}\Big)\Big)\,dy\\ &= \quad \lim_{\varepsilon\to 0}\int_Q f\Big(x,\frac{y}{\varepsilon},\xi+\nabla\psi_\varepsilon(y)\Big)\,dy,\end{aligned} \tag{4.12}$$

where $\psi_\varepsilon(y):=\varepsilon\varphi\Big(\frac{y}{\varepsilon}\Big)$. For each $\varepsilon>0$ define

$$Q_\varepsilon := \Big\{y\in Q:\ \operatorname{dist}(y,\partial Q)\geqslant\varepsilon\Big\}.$$

Take $\theta_\varepsilon\in C_c^\infty(Q)$ with $\theta_\varepsilon\in[0,1]$ such that $\theta_\varepsilon=1$ on $Q_\varepsilon$ and $||\nabla\theta_\varepsilon||_{L^\infty}\leqslant C\varepsilon^{-1}$. Then

$$\lim_{\varepsilon\to 0}\int_Q f\Big(x,\frac{y}{\varepsilon},\xi+\nabla(\theta_\varepsilon\psi_\varepsilon)(y)\Big)\,dy=\lim_{\varepsilon\to 0}\int_{Q_\varepsilon} f\Big(x,\frac{y}{\varepsilon},\xi+\nabla\psi_\varepsilon(y)\Big)\,dy, \tag{4.13}$$

since by the $p$-growth condition in $(H_4)$ we have

$$\begin{aligned}&\int_{Q\setminus Q_\varepsilon} f\Big(x,\frac{y}{\varepsilon},\xi+\nabla(\theta_\varepsilon\psi_\varepsilon)(y)\Big)\,dy\\ \leqslant\ & C\int_{Q\setminus Q_\varepsilon}\big(1+|\xi|^p+|\nabla\psi_\varepsilon(y)|^p+\varepsilon^{-p}|\psi_\varepsilon(y)|^p\big)\ dy\\ =\ & C\Big[\,|Q\setminus Q_\varepsilon|+\int_{Q\setminus Q_\varepsilon}\Big|\nabla\varphi\Big(\frac{y}{\varepsilon}\Big)\Big|^p dy+\int_{Q\setminus Q_\varepsilon}\Big|\varphi\Big(\frac{y}{\varepsilon}\Big)\Big|^p dy\Big]\to 0.\end{aligned}$$

Hence by (4.11)-(4.13), defining $\phi_\varepsilon(y) := \frac{1}{\varepsilon}(\theta_\varepsilon\psi_\varepsilon)(\varepsilon y) \in W_0^{1,p}\Big(\Big(0,\frac{1}{\varepsilon}\Big)^N;\mathbb{R}^N\Big)$, we obtain

$$\begin{aligned}\hat{f}_{\mathrm{hom}}(x,\xi)+\delta &\geqslant \lim_{\varepsilon\to 0}\int_Q f\Big(x,\frac{y}{\varepsilon},\xi+\nabla(\theta_\varepsilon\psi_\varepsilon)(y)\Big)\,dy\\ &\geqslant \lim_{\varepsilon\to 0}\varepsilon^N\int_{\left(0,\frac{1}{\varepsilon}\right)^N} f\Big(x,y,\xi+\nabla\phi_\varepsilon(y)\Big)\,dy\\ &\geqslant \overline{f}_{\mathrm{hom}}(x,\xi).\end{aligned}$$

Letting $\delta\to 0$ we get $f_{\mathrm{hom}}(x,\xi)\geqslant \overline{f}_{\mathrm{hom}}(x,\xi)$.

■

The measurability of $f_{\mathrm{hom}}$ follows as a consequence.

**Lemma 4.1.6.** *The function $f_{\mathrm{hom}}(\cdot,\xi)$ is measurable for all $\xi\in\mathbb{R}^{d\times N}$.*

*Proof.* Let $\xi\in\mathbb{R}^{d\times N}$. By Lemma 4.1.5 we can write

$$f_{\mathrm{hom}}(x,\xi)=\inf_{T\in\mathbb{N}}\ \inf_{\phi\in\mathcal{S}_T} f_{T,\phi}(x)$$

where

$$f_{T,\phi}(x):=\int_T f(x,y,\xi+\nabla\phi(z))\,dz$$

for $x\in\Omega$, and $\mathcal{S}_T$ is a countable subset of $C_c^\infty((0,T)^N;\mathbb{R}^d)$ dense in $W_0^{1,p}((0,T)^N;\mathbb{R}^d)$. By Tonelli's Theorem the functions $f_{T,\phi}$ are measurable, and so is $f_{\mathrm{hom}}(\cdot,\xi)$ as the infimum of a countable family of measurable functions.

■

Our next objective is to show that $f_{\mathrm{hom}}(x,\cdot)$ is continuous for all $x\in\Omega$. We are not able to prove this directly unless $f$ satisfy a $p$-Lipschitz condition as in (2.9). As quasiconvex functions satisfy inequality (2.9) (see Proposition 2.3.15), the first step will be to show that $f_{\mathrm{hom}}=(\mathcal{Q}f)_{\mathrm{hom}}$ where $\mathcal{Q}f:\Omega\times\mathbb{R}^N\times\mathbb{R}^{d\times N}\to\mathbb{R}$ denotes the usual quasiconvexification of $f$ with respect to the last variable $\xi$, that we know to be quasiconvex in $\xi$ (see Theorem 2.3.19). We remark that

$$\mathcal{Q}f(x,y,\xi)=\inf_\phi\Big\{\int_Q f(x,y,\xi+\nabla\phi(z))\,dz:\ \phi\in W_0^{1,p}(Q;\mathbb{R}^d)\Big\} \tag{4.14}$$

for all $(x,y,\xi)\in\Omega\times\mathbb{R}^N\times\mathbb{R}^{d\times N}$ (see Lemma 2.3.23) and that consequently $\mathcal{Q}f$ satisfies conditions $(H_3)$ and $(H_4)$. The following properties of $\mathcal{Q}f$ are of interest for the argument that follows.

**Lemma 4.1.7.** *We have that*

i) $\mathcal{Q}f(x,\cdot,\cdot)$ *is continuous for all* $x \in \Omega$*;*

ii) $\mathcal{Q}f(\cdot,y,\xi)$ *is measurable for all* $(y,\xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$*;*

iii) $$(\mathcal{Q}f)_{\mathrm{hom}}(x,\xi) := \liminf_{T\to\infty} \inf\left\{ \tfrac{1}{T^N}\int_{(0,T)^N} \mathcal{Q}f(x,y,\xi+\nabla\phi(y))\,dy,\ \phi \in W_0^{1,p}((0,T)^N;\mathbb{R}^d)\right\}$$
$$= \lim_{T\to\infty} \inf\left\{ \tfrac{1}{T^N}\int_{(0,T)^N} \mathcal{Q}f(x,y,\xi+\nabla\phi(y))\,dy,\ \phi \in W_0^{1,p}((0,T)^N;\mathbb{R}^d)\right\}$$
*for all* $(x,\xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$.

iv) $(\mathcal{Q}f)_{\mathrm{hom}}(x,\xi) = f_{\mathrm{hom}}(x,\xi)$ *for all* $(x,\xi) \in \Omega \times \mathbb{R}^{d\times N}$.

*Proof.* i) The upper semicontinuity of $\mathcal{Q}f(x,\cdot,\cdot)$ for $x \in \mathbb{R}^N$ follows from equality (4.14) and hypothesis $(H_1)$, while its lower semicontinuity can be obtained by an argument analogous to that of Lemma 2.3.24.

ii) The proof is identical to that of Lemma 4.1.6 above.

iii) Is a consequence of identities i) and ii), the coerciveness, growth and periodicity properties of $\mathcal{Q}f$, and of Lemma 4.1.4.

iv) Let $(x,\xi) \in \Omega \times \mathbb{R}^{d\times N}$. Obviously $f_{\mathrm{hom}}(x,\xi) \geqslant (\mathcal{Q}f)_{\mathrm{hom}}(x,\xi)$. Let us prove the converse inequality. Let $n \in \mathbb{N}$ and let $T_n \in \mathbb{N}$ and $\phi_n \in W_0^{1,p}((0,T_n)^N;\mathbb{R}^d)$ be such that

$$(\mathcal{Q}f)_{\mathrm{hom}}(x,\xi) + \frac{1}{n} \geqslant \frac{1}{T_n^N}\int_{(0,T_n)^N} \mathcal{Q}f(x,y;\xi+\nabla\phi_n(y))\,dy.$$

Thus
$$(\mathcal{Q}f)_{\mathrm{hom}}(x,\xi) \geqslant \limsup_{n\to\infty} \frac{1}{T_n^N}\int_{(0,T_n)^N} \mathcal{Q}f(x,y;\xi+\nabla\phi_n(y))\,dy. \tag{4.15}$$

To compare (4.15) with $f_{\mathrm{hom}}(x,\xi)$ we apply the Acerbi and Fusco Relaxation Theorem (Theorem 2.3.30) and the Decomposition Lemma (Lemma 2.2.16). As a consequence of Theorem 2.3.30, for every $n$ fixed there exists a sequence $\{\phi_{n,k}\}_k \subset W^{1,p}((0,T_n)^N;\mathbb{R}^d)$ such that $\phi_{n,k} \underset{k}{\rightharpoonup} \phi_n$ in $W^{1,p}((0,T_n)^N;\mathbb{R}^d)$ and

$$\frac{1}{T_n^N}\int_{(0,T_n)^N} \mathcal{Q}f(x,y;\xi+\nabla\phi_n(y))\,dy = \lim_{k\to\infty}\frac{1}{T_n^N}\int_{(0,T_n)^N} f(x,y;\xi+\nabla\phi_{n,k}(y))\,dy. \tag{4.16}$$

By Lemma 2.2.16 we can now find a subsequence (still denoted by $\{\phi_{n,k}\}_k$) and a sequence $\{\psi_{n,k}\}_k \subset W_0^{1,\infty}(\mathbb{R}^N;\mathbb{R}^d)$ such that $\psi_{n,k} \rightharpoonup \phi_n$ in $W^{1,p}((0,T_n)^N;\mathbb{R}^d)$ with

$$\{|\nabla\psi_{n,k}|^p\} \quad \text{equi-integrable} \tag{4.17}$$

and

$$\mathcal{L}^N\{y \in (0,T_n)^N :\ \psi_{n,k}(y) \neq \phi_{n,k}(y)\} \underset{k\to\infty}{\longrightarrow} 0. \tag{4.18}$$

As $f$ is nonnegative, by (4.17) and (4.18)

$$\begin{aligned}
&\lim_{k\to\infty} \frac{1}{T_n^N}\int_{(0,T_n)^N} f(x,y;\xi+\nabla\phi_{n,k}(y))\,dy\\
&\geqslant \limsup_{k\to\infty} \frac{1}{T_n^N}\int_{\{y\in(0,T_n)^N:\ \psi_{n,k}(y)=\phi_{n,k}(y)\}} f(x,y;\xi+\nabla\psi_{n,k}(y))\,dy\\
&= \limsup_{k\to\infty} \frac{1}{T_n^N}\int_{(0,T_n)^N} f(x,y;\xi+\nabla\psi_{n,k}(y))\,dy.
\end{aligned} \tag{4.19}$$

Thus from (4.15), (4.16) and (4.19)

$$(\mathcal{Q}f)_{\hom}(x,\xi) \geqslant \limsup_{n\to\infty}\ \limsup_{k\to\infty} \frac{1}{T_n^N}\int_{(0,T_n)^N} f(x,y,\xi+\nabla\psi_{n,k}(y))\,dy \geqslant f_{\hom}(x,\xi).$$

■

We are now in a position to prove the continuity property of $f_{\hom}$.

**Lemma 4.1.8.** *The function $f_{\hom}(x,\cdot)$ (or equivalently $(\mathcal{Q}f)_{\hom}(x,\cdot)$) is continuous for all $x \in \Omega$.*

*Proof.* (see also Braides [19]) Fix $x \in \Omega$. Let $\xi \in \mathbb{R}^{d\times N}$ and $\xi_n \to \xi$ in $\mathbb{R}^{d\times N}$. We wish to show that

$$f_{\hom}(x,\xi) = \lim_{n\to\infty} f_{\hom}(x,\xi_n).$$

We first establish that (upper semicontinuity)

$$f_{\hom}(x,\xi) \geqslant \limsup_{n\to\infty} f_{\hom}(x,\xi_n). \tag{4.20}$$

Fixed $\delta > 0$, choose $S \in \mathbb{N}$ and by density a function $\varphi \in C_0^\infty((0,S)^N;\mathbb{R}^d)$ such that

$$\begin{aligned}
f_{\hom}(x,\xi)+\delta &\geqslant \frac{1}{S^N}\int_{(0,S)^N} f(x,y,\xi+\nabla\varphi(y))\,dy\\
&= \frac{1}{S^N}\lim_{n\to\infty}\int_{(0,S)^N} f(x,y,\xi_n+\nabla\varphi(y))\,dy\\
&\geqslant \limsup_{n\to\infty} f_{\hom}(x,\xi_n),
\end{aligned}$$

as a consequence of Lemmas 2.1.15 and 4.1.5. Letting $\delta \to 0$ we get (4.20).

We show now the converse inequality (lower semicontinuity), i.e.

$$f_{\text{hom}}(x,\xi) \leqslant \liminf_{n\to\infty} f_{\text{hom}}(x,\xi_n). \tag{4.21}$$

We start by remarking that $f_{\text{hom}}(x,\xi_n) = (\mathcal{Q}f)_{\text{hom}}(x,\xi_n)$ for all $n \in \mathbb{N}$ (property iv) in Lemma 4.1.7). Given $n \in \mathbb{N}$, consider $T_n \in \mathbb{N}$ ($T_n \nearrow \infty$) and $\phi_n \in W_0^{1,p}((0,T_n)^N;\mathbb{R}^d)$ such that

$$\begin{aligned} f_{\text{hom}}(x,\xi_n) + \frac{1}{n} \quad &\geqslant \quad \frac{1}{{T_n}^N}\int_{(0,T_n)^N} \mathcal{Q}f(x,y,\xi_n+\nabla\phi_n(y))\,dy \\ &= \quad \int_{(0,1)^N} \mathcal{Q}f(x,T_n y,\xi_n+\nabla\phi_n(T_n y))\,dy \\ &= \quad \int_{(0,1)^N} \mathcal{Q}f(x,T_n y,\xi_n+\nabla\psi_n(y))\,dy, \end{aligned} \tag{4.22}$$

where $\psi_n(y) := \frac{1}{T_n}\phi_n(T_n y)$, $\psi_n \in W_0^{1,p}((0,1)^N;\mathbb{R}^d)$. We note that by the $p$-coervivity condition of $(\mathcal{Q}f)$ the sequence $\{||\nabla\psi_n||_{L^p((0,1)^N;\mathbb{R}^d)}\}$ is bounded. We write

$$\begin{aligned} &\int_{(0,1)^N} \mathcal{Q}f(x,T_n y,\xi_n+\nabla\psi_n(y))\,dy \\ = \ &\int_{(0,1)^N} \mathcal{Q}f(x,T_n y,\xi_n+\nabla\psi_n(y)) - \mathcal{Q}f(x,T_n y,\xi+\nabla\psi_n(y))\,dy \qquad (4.23) \\ + \ &\int_{(0,1)^N} \mathcal{Q}f(x,T_n y,\xi+\nabla\psi_n(y))\,dy. \end{aligned}$$

Our task now is to show that the term (4.23) goes to zero as $n$ goes to infinity. As $\xi_n \to \xi$ and the sequence $\{||\nabla\psi_n||_{L^p((0,1)^N;\mathbb{R}^d)}\}$ is bounded, using the $p$-Lipschitz condition (2.9) and Hölder Inequality we have

$$\begin{aligned} &\limsup_{n\to\infty}\int_{(0,1)^N} |\mathcal{Q}f(x,T_n y,\xi_n+\nabla\psi_n(y)) - \mathcal{Q}f(x,T_n y,\xi+\nabla\psi_n(y))|\ dy \\ &\leqslant C\ \limsup_{n\to\infty}\int_{(0,1)^N} \left(1+|\xi_n+\nabla\psi_n(y)|^{p-1}+|\xi+\nabla\psi_n(y)|^{p-1}\right)|\xi_n-\xi|\,dy \\ &\leqslant C\ \limsup_{n\to\infty}\int_{(0,1)^N} \left(1+|\nabla\psi_n(y)|^{p-1}\right)|\xi_n-\xi|\,dy \\ &\leqslant C\ \lim_{n\to\infty}|\xi_n-\xi| = 0, \end{aligned}$$

which, together with (4.22), leads to

$$f_{\text{hom}}(x,\xi_n) + \frac{1}{n} \geqslant \limsup_{n\to\infty}\int_{(0,1)^N} \mathcal{Q}f(x,T_n y,\xi+\nabla\psi_n(y))\,dy \geqslant (\mathcal{Q}f)_{\text{hom}}(x,\xi) = f_{\text{hom}}(x,\xi).$$

Inequality (4.21) holds letting $n \to \infty$.

■

By Lemmas 4.1.6 and 4.1.8, we conclude that $f_{\text{hom}}$ is a Carathéodory integrand and thus $\mathcal{I}_{\text{hom}}$ is well defined.

*Remark* 4.1.9. We note that $f_{\text{hom}}$ satisfies analogous growth and coercivity conditions to the ones of $f$, which, together with the continuity properties of $f_{\text{hom}}$, imply by standard arguments (approximation of $W^{1,p}$ by piecewise affine functions together with the invariance of the domain of $f_{\text{hom}}$) that this function is quasiconvex with respect to the last variable.

We will show next that in the convex case it is enough to consider one cell period for the definition of $f_{\text{hom}}$ (4.3) (see also Braides [19] or Müller [66]). We define for all $(x,\xi) \in \Omega \times \mathbb{R}^{d\times N}$

$$f^{\star}_{\text{hom}}(x,\xi) = \inf_{\phi} \left\{ \int_Q f(x,y,\xi+\nabla\phi(y))\,dy, \quad \phi \in W_0^{1,p}((0,1)^N;\mathbb{R}^d) \right\}$$

and

$$f^{\star\star}_{\text{hom}}(x,\xi) = \inf_{\phi} \left\{ \int_Q f(x,y,\xi+\nabla\phi(y))\,dy, \quad \phi \in W_{\text{per}}^{1,p}((0,1)^N;\mathbb{R}^d) \right\}.$$

**Lemma 4.1.10.** *Assume, in addition to the hypotheses on $f$, that $f(x,y,\cdot)$ is convex for all $(x,y) \in \Omega \times \mathbb{R}^N$. Then*

$$f^{\star}_{\text{hom}} = f^{\star\star}_{\text{hom}} = f_{\text{hom}}.$$

*Proof.* (see Braides [19] and Müller [66] for an alternative proof). Equality $f^{\star}_{\text{hom}} = f^{\star\star}_{\text{hom}}$ is proven by an argument analog to that of the proof of Lemma 4.1.5.

We show that $f^{\star}_{\text{hom}} = f_{\text{hom}}$. Let $(x,\xi) \in \Omega \times \mathbb{R}^{d\times N}$. By definition $f^{\star}_{\text{hom}}(x,\xi) \geqslant f_{\text{hom}}(x,\xi)$. To prove the opposite inequality, for each $n \in \mathbb{N}$ take $T_n \in \mathbb{N}$ and a function $\phi_n \in W_0^{1,p}((0,T_n)^N;\mathbb{R}^d)$ such that

$$\begin{aligned} f_{\text{hom}}(x,\xi) + \tfrac{1}{n} \;&\geqslant \tfrac{1}{T_n^N} \int_{(0,T_n)^N} f(x,y,\xi+\nabla\phi_n(y))\,dy \\ &= \int_Q f(x,T_n y,\xi+\nabla\psi_n(y))\,dy, \end{aligned} \tag{4.24}$$

where $\psi_n(y) := \frac{1}{T_n}\phi_n(T_n y)$, $\psi_n \in W_0^{1,p}(Q;\mathbb{R}^d)$. By the $p$-growth condition in $(H_4)$ the sequence $\{||\nabla\psi_n||_{L^p(Q;\mathbb{R}^d)}\}_n$ is bounded, and so is $\{||\psi_n||_{W^{1,p}(Q;\mathbb{R}^d)}\}_n$ by Poincaré Inequality.

Hence, there exists a subsequence (still denoted by $\{\psi_n\}_n$) such that

$$\psi_n \overset{W^{1,p}}{\rightharpoonup} \psi$$

for some $\psi = \psi(y) \in W_0^{1,p}(Q;\mathbb{R}^d)$. As a consequence, by Theorem 2.6.7 and up to a subsequence

$$\psi_n \overset{2s}{\rightharpoonup} \psi$$

and

$$\nabla \psi_n \overset{2s}{\rightharpoonup} \nabla \psi + \nabla_z \overline{\psi}$$

for some $\overline{\psi} = \overline{\psi}(y,z) \in L^p\big(Q; W^{1,p}_{\mathrm{per}}(Q;\mathbb{R}^d)\big)$. We divide the rest of the proof in two steps.

*Step 1.* We follow an argument of Allaire [4] assuming in addition that

$(H_5)$ $\frac{\partial f}{\partial \eta}(x,y,\eta)$ exists for all $(x,y,\eta) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$.

By inequalities (2.5) and (2.6), and by the $p$-growth condition in $(H_4)$ there exists a constant $C > 0$ such that for all $(x,y,\eta) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$

$$\left|\frac{\partial f}{\partial \eta}(x,y,\eta)\right| \leqslant C\big[1 + |\eta|^{p-1}\big]. \tag{4.25}$$

Let $\varphi = \varphi(y,z) \in C_c^\infty\big(Q; C^\infty_{\mathrm{per}}(Q;\mathbb{R}^{d\times N})\big)$. Since $f$ is convex in the last variable then by inequality (2.5)

$$\begin{aligned}\int_Q f\big(x, T_n y, \xi + \nabla\psi_n(y)\big)\,dy \geqslant \int_Q f\big(x, T_n y, \xi + \varphi(y, T_n y)\big)\,dy \\ + \int_Q \frac{\partial f}{\partial \eta}\big(x, T_n y, \varphi(y, T_n y)\big)\cdot(\nabla\psi_n(y) - \varphi(y, T_n y))\,dy,\end{aligned} \tag{4.26}$$

for each $n \in \mathbb{N}$. By Lemma 2.6.3, and as $\nabla\psi_n \overset{2s}{\rightharpoonup} \nabla\psi + \nabla_z\overline{\psi}$ (see Definition 2.6.5), we have

$$\lim_{n\to\infty} \int_Q f\big(x, T_n y, \xi + \varphi(y, T_n y)\big)\,dy = \int_Q \Big[\int_Q f\big(x, z, \xi + \varphi(y,z)\big)\,dz\Big]\,dy \tag{4.27}$$

and

$$\begin{aligned}&\lim_{n\to\infty} \int_Q \frac{\partial f}{\partial \eta}\Big(x, T_n y, \varphi(y, T_n y)\Big)\cdot(\nabla\psi_n(y) - \varphi(y, T_n y))\,dy \\ &= \int_Q \Big[\int_Q \frac{\partial f}{\partial \eta}\big(x, z, \varphi(y,z)\big)\cdot(\nabla\psi(y) + \nabla_z\overline{\psi}(y,z) - \varphi(y,z))\,dz\Big]\,dy.\end{aligned} \tag{4.28}$$

Therefore, from (4.26)-(4.28) we get

$$\liminf_{n\to\infty}\int_Q f\big(x,T_n y,\xi+\nabla\psi_n(y)\big)\,dy \geqslant \int_Q\left[\int_Q f\big(x,z,\xi+\varphi(y,z)\big)\,dz\right]dy$$
$$+\int_Q\left[\int_Q \frac{\partial f}{\partial\eta}\big(x,z,\varphi(y,z)\big)\cdot\big(\nabla\psi(y)+\nabla_z\overline{\psi}(y,z)-\varphi(y,z)\big)\,dz\right]dy, \tag{4.29}$$

for all $\varphi=\varphi(y,z)\in C_c^\infty\big(Q;C^\infty_{\mathrm{per}}(Q;\mathbb{R}^{d\times N})\big)$. Let now $\{\varphi_k\}_k\subset C_c^\infty\big(Q;C^\infty_c(Q;\mathbb{R}^{d\times N})\big)$ be a convergent sequence to $\nabla\psi+\nabla_z\overline{\psi}$ in $L^p\big(Q\times Q;\mathbb{R}^{d\times N}\big)$. In particular from (4.29) we get

$$\liminf_{n\to\infty}\int_Q f\big(x,T_n y,\xi+\nabla\psi_n(y)\big)\,dy \geqslant \int_Q\left[\int_Q f\big(x,z,\xi+\varphi_k(y,z)\big)\,dz\right]dy$$
$$+\int_Q\left[\int_Q \frac{\partial f}{\partial\eta}\big(x,z,\varphi_k(y,z)\big)\cdot\big(\nabla\psi(y)+\nabla_z\overline{\psi}(y,z)-\varphi_k(y,z)\big)\,dz\right]dy, \tag{4.30}$$

for every $k\in\mathbb{N}$. Inequality (4.25) and the growth conditions on $f$ and $\frac{\partial f}{\partial\eta}$ imply that we can pass to the limit in (4.30) and get

$$\liminf_{n\to\infty}\int_Q f\big(x,T_n y,\xi+\nabla\psi_n(y)\big)\,dy \geqslant \int_Q\left[\int_Q f\big(x,z,\xi+\nabla\psi(y)+\nabla_z\overline{\psi}(y,z)\big)\,dz\right]dy. \tag{4.31}$$

By Jensen's Inequality and Fubini's Theorem, for each $z\in Q$

$$\begin{aligned}\int_Q f\big(x,z,\xi+\nabla\psi(y)+\nabla_z\overline{\psi}(y,z)\big)\,dy &\geqslant f\Big(x,z,\int_Q\big[\,\xi+\nabla\psi(y)+\nabla_z\overline{\psi}(y,z)\,\big]\,dy\Big)\\ &= f\Big(x,z,\xi+\nabla_z\Big(\int_Q\overline{\psi}(y,z)\,dy\Big)\Big).\end{aligned} \tag{4.32}$$

Thus, by (4.24), (4.31), (4.32), and once more Fubini's Theorem

$$f_{\mathrm{hom}}(\xi)\geqslant\int_Q f\Big(z,\xi+\nabla_z\Big(\int_Q\overline{\psi}(y,z)\,dy\Big)\Big)\,dz\geqslant f^{\star\star}_{\mathrm{hom}}(\xi)=f^{\star}_{\mathrm{hom}}(\xi).$$

*Step 2.* We address now the general case. For each $\varepsilon>0$ set $\zeta_\varepsilon(\eta):=\frac{1}{\varepsilon^N}\zeta\Big(\frac{\eta}{\varepsilon}\Big)$ where $\zeta\in C^\infty(\mathbb{R}^{d\times N})$ denotes the standard mollifier, that is,

$$\zeta(\eta):=\begin{cases} C\exp\Big(\frac{1}{|\eta|^2-1}\Big) & \text{if } |\eta|<1,\\[2ex] 0 & \text{if } |\eta|\geqslant 1,\end{cases}$$

and the constant $C$ is selected so that $\int_{\mathbb{R}^{d\times N}} \zeta(\eta)\, d\eta = 1$. Let

$$f_\varepsilon(x, y, \xi) := \int_{B(0,\varepsilon)} \zeta_\varepsilon(\eta) f(x, y, \xi - \eta)\, d\eta$$

for all $\varepsilon > 0$ and all $(x, y, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$. It is straightforward to show that $f_\varepsilon$ satisfies conditions $(H_1)$-$(H_5)$. Fix $\delta > 0$ and, by a density argument, let $S \in \mathbb{N}$ and $\psi \in W^{1,\infty}((0, S)^N; \mathbb{R}^d)$ be such that

$$f_{\text{hom}}(x, \xi) + \delta \geqslant \frac{1}{S^N} \int_{(0,S)^N} f(x, y, \xi + \nabla\psi(y))\, dy.$$

Then

$$f_{\text{hom}}(x, \xi) + \delta \quad \geqslant \quad \lim_{\varepsilon \to 0} \frac{1}{S^N} \int_{(0,S)^N} f_\varepsilon(x, y, \xi + \nabla\psi(y))\, dy,$$

and thus

$$\begin{aligned} f_{\text{hom}}(x, \xi) + \delta \quad &\geqslant \quad \limsup_{\varepsilon \to 0}\, (f_\varepsilon)_{\text{hom}}(x, \xi) \\ &= \quad \limsup_{\varepsilon \to 0}\, (f_\varepsilon)^{\star\star}_{\text{hom}}(x, \xi) \\ &\geqslant \quad f^{\star\star}_{\text{hom}}(x, \xi) \end{aligned}$$

since $f_\varepsilon$ is convex and $f_\varepsilon \geqslant f$ for all $\varepsilon > 0$. Indeed, it is easy to show that $f_\varepsilon$ is convex and the last assertion is a consequence of the convexity of $f$ and Jensen's Inequality.

■

### 4.1.2 Proof of the main result

As in the previous subsection, by Remark 4.1.2 we may suppose that $f$ is a positive Borel function satisfying hypotheses $(H_1)$, $(H_3)$ and $(H_4)$ for every $(x, y, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$. Due to the $p$-coercivity condition in $(H_4)$, to prove Theorem 4.1.1 it suffices to show that

$$\Gamma(L^p(\Omega))\text{-}\lim_{\varepsilon \to 0} \mathcal{I}_\varepsilon(u) = \int_\Omega f_{\text{hom}}(x, u(x), \nabla u(x))\, dx, \tag{4.33}$$

for all $u \in W^{1,p}(\Omega; \mathbb{R}^d)$, since

$$\Gamma(L^p(\Omega))\text{-}\lim_{\varepsilon \to 0} \mathcal{I}_\varepsilon(u) = \infty$$

for all $u \in L^p(\Omega;\mathbb{R}^N) \setminus W^{1,p}(\Omega;\mathbb{R}^d)$. To prove identity (4.33) we use the Direct Method of Γ- convergence. Accordingly, we start by localizing the functionals $\mathcal{I}_\varepsilon$ in order to highlight their dependence on the domain of integration, that is, we consider a family of functionals $\mathcal{I}_\varepsilon : L^p(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,\infty]$ defined by

$$\mathcal{I}_\varepsilon(u;A) := \begin{cases} \displaystyle\int_A f\left(x, \frac{x}{\varepsilon}, \nabla u(x)\right) dx & \text{if } u \in W^{1,p}(A;\mathbb{R}^d), \\ \\ \infty & \text{otherwise.} \end{cases}$$

Our goal is to show that

$$\Gamma(L^p(\Omega))\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u,A) = \int_A f_{\text{hom}}(x,\nabla u(x))\,dx, \tag{4.34}$$

for all $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$. In particular (4.33) will follow by taking $A = \Omega$.

The next step toward the proof of (4.34) is to establish a compactness property that ensures the existence of Γ-converging subsequences of $\{\mathcal{I}_\varepsilon\}_\varepsilon$.

**Proposition 4.1.11.** *For every sequence $\{\varepsilon_n\}_n$ of positive real numbers converging to zero there exists a further subsequence $\{\varepsilon_{n_j}\}_j \equiv \{\varepsilon_j\}_j$ such that*

$$\Gamma(L^p(A))\text{-}\lim_{j\to\infty} \mathcal{I}_{\varepsilon_j}(\cdot;A)(u) =: \mathcal{I}_{\{\varepsilon_j\}}(u;A) \tag{4.35}$$

*exists for all $u \in W^{1,p}(\Omega,\mathbb{R}^d)$ and all $A \in \mathcal{A}(\Omega)$.*

The proof of this proposition follows an argument analog to the one used in Braides, Fonseca and Francfort [20], but for completeness we present it here.

Let $\mathcal{R}$ be a countable collection of subsets of $\Omega$ such that, for any $\delta > 0$ and any $A \in \mathcal{A}(\Omega)$, there exists a finite union $C_A$ of disjoint elements of $\mathcal{R}$ satisfying

$$\begin{cases} \overline{C_A} \subset A, \\ \\ \mathcal{L}^N(A) \leqslant \mathcal{L}^N(C_A) + \delta. \end{cases}$$

We may take $\mathcal{R}$ as the set of open squares with faces parallel to the axes, centered at points $x \in \Omega \cap \mathbb{Q}^N$ and with rational edge lengths. We denote by $\mathfrak{R}$ the countable collection of all finite unions of elements of $\mathcal{R}$, i.e.

$$\mathfrak{R} = \left\{ \bigcup_{i=1}^k C_i : \ k \in \mathbb{N},\, C_i \in \mathcal{R} \right\}.$$

The next lemma is the starting point for the proof of Proposition 4.1.11.

**Lemma 4.1.12.** *For every sequence $\{\varepsilon_n\}_n$ of positive real numbers converging to zero there exists a further subsequence $\{\varepsilon_{n_j}\}_j$ (depending on $\mathfrak{R}$) such that the $\Gamma$-limit*

$$\Gamma(L^p(C))\text{-}\lim_{j\to\infty} \mathcal{I}_{\varepsilon_{n_j}}(\cdot;C)(u) =: \mathcal{I}_{\{\varepsilon_{n_j}\}}(u;C) \tag{4.36}$$

*exists for all $u \in W^{1,p}(\Omega,\mathbb{R}^d)$ and all $C \in \mathfrak{R}$.*

*Proof.* Let $C \in \mathfrak{R}$. From Proposition 2.5.8 and as $L^p(\Omega;\mathbb{R}^d)$ is separable there exist a subsequence $\{\varepsilon_{n_j}\}_j$ (depending on $C$) such that the $\Gamma(L^p(C))$-limit of $\mathcal{I}_{\varepsilon_{n_j}}(\cdot;C)$ exists for all $u \in W^{1,p}(\Omega,\mathbb{R}^d)$. But then by a diagonalization procedure we can find a subsequence $\{\varepsilon_{n_j}\}_j$ (depending on $\mathfrak{R}$) such that (4.36) holds.

■

Let now $\{\varepsilon_n\}_n$ be a fixed sequence of positive real numbers converging to zero and $\{\varepsilon_j\}_j$ a subsequence for which (4.36) holds.

*Proof of Proposition 4.1.11.* We wish to show that for all $A \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(\Omega,\mathbb{R}^d)$

$$\inf\left\{ \liminf_{j\to\infty} \int_A f\Big(x,\frac{x}{\varepsilon_j},\nabla u_j(x)\Big)\,dx:\ u_j \in W^{1,p}(A,\mathbb{R}^d), u_j \overset{L^p(A;\mathbb{R}^d)}{\longrightarrow} u\right\} \tag{4.37}$$

$$= \inf\left\{ \limsup_{j\to\infty} \int_A f\Big(x,\frac{x}{\varepsilon_j},\nabla u_j(x)\Big)\,dx:\ u_j \in W^{1,p}(A,\mathbb{R}^d), u_j \overset{L^p(A;\mathbb{R}^d)}{\longrightarrow} u\right\}.$$

Let $A \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(\Omega,\mathbb{R}^d)$. To prove (4.37) it suffices to find a sequence $\{v_j\}_j \subset W^{1,p}(A,\mathbb{R}^d)$ with $v_j \overset{L^p(A;\mathbb{R}^d)}{\longrightarrow} u$ and such that

$$\begin{aligned}&\inf\left\{ \liminf_{j\to\infty} \int_A f\left(x,\frac{x}{\varepsilon_j},\nabla u_j(x)\right)\,dx:\ u_j \in W^{1,p}(A,\mathbb{R}^d), u_j \overset{L^p(A;\mathbb{R}^d)}{\longrightarrow} u\right\}\\ &\geqslant \limsup_{j\to\infty} \int_A f\left(x,\frac{x}{\varepsilon_j},\nabla v_j(x)\right)\,dx.\end{aligned} \tag{4.38}$$

Fix $\delta > 0$ and choose $C_\delta \in \mathfrak{R}$ with $\overline{C_\delta} \subset A$ and $\mathcal{L}^N(A \setminus C_\delta) << 1$, so that

$$\int_{A\setminus C_\delta} (1 + |\nabla u|^p)\,dx \leqslant \frac{\delta}{\beta}, \tag{4.39}$$

where $\beta$ is the constant in $(H_4)$. By Proposition 2.5.13 consider a sequence $\{w_j^\delta\} \in W^{1,p}(C_\delta, \mathbb{R}^d)$ such that

$$w_j^\delta \overset{L^p(C_\delta;\mathbb{R}^d)}{\longrightarrow} u,$$

$$\Gamma(L^p(C_\delta))\text{-}\lim_{j\to\infty} \mathcal{I}_{\varepsilon_j}(\cdot\,; C_\delta)(u) = \lim_{j\to\infty} \int_{C_\delta} f\left(x, \frac{x}{\varepsilon_j}, Dw_j^\delta(x)\right) dx, \tag{4.40}$$

and $w_j^\delta = u$ on $\partial C_\delta$. Extending $w_j^\delta$ by $u$ outside $C_\delta$ (still denoted by $w_j^\delta$) it follows that $w_j^\delta \overset{L^p(A;\mathbb{R}^d)}{\longrightarrow} u$, and in view of $(H_3)$, (4.39) and (4.40) we obtain that

$$\begin{aligned}
& \limsup_{\delta\to 0}\ \limsup_{j\to\infty} \int_A f\left(x, \frac{x}{\varepsilon_j}, \nabla w_j^\delta(x)\right) dx \qquad\qquad (4.41)\\
\leqslant\ & \limsup_{\delta\to 0}\ \lim_{j\to\infty} \int_{C_\delta} f\left(x, \frac{x}{\varepsilon_j}, \nabla w_j^\delta(x)\right) dx + \beta\ \limsup_{\delta\to 0} \int_{A\setminus C_\delta} C(1+|\nabla u|^p)\, dx\\
=\ & \limsup_{\delta\to 0}\ \Gamma(L^p(C_\delta))\text{-}\lim_{j\to\infty} \mathcal{I}_{\varepsilon_j}(\cdot; C_\delta)(u)\\
=\ & \limsup_{\delta\to 0}\ \inf\left\{ \liminf_{j\to\infty} \int_{C_\delta} f\left(x, \frac{x}{\varepsilon_j}, \nabla u_j(x)\right) dx :\ u_j \in W^{1,p}(C_\delta, \mathbb{R}^d),\ u_j \overset{L^p(C_\delta;\mathbb{R}^d)}{\longrightarrow} u\right\}\\
=\ & \inf\left\{ \liminf_{j\to\infty} \int_A f\left(x, \frac{x}{\varepsilon_j}, \nabla u_j(x)\right) dx :\ u_j \in W^{1,p}(A, \mathbb{R}^d),\ u_j \overset{L^p(A;\mathbb{R}^d)}{\longrightarrow} u\right\} \text{ (by 4.41)}\\
\leqslant\ & \liminf_{\delta\to 0}\ \liminf_{j\to\infty} \int_A f\left(x, \frac{x}{\varepsilon_j}, \nabla w_j^\delta(x)\right) dx.
\end{aligned}$$

By Lemma A.2 in the Appendix there exists a decreasing sequence $\delta(\varepsilon_j) \downarrow 0$ such that

$$\begin{aligned}
& \lim_{j\to\infty} \int_A f\left(x, \frac{x}{\varepsilon_j}, \nabla w_j^{\delta(\varepsilon_j)}(x)\right) dx\\
& = \inf\left\{ \liminf_{j\to\infty} \int_A f\left(x, \frac{x}{\varepsilon_j}, \nabla u_j(x)\right) dx :\ u_j \in W^{1,p}(A, \mathbb{R}^d),\ u_j \overset{L^p(A;\mathbb{R}^d)}{\longrightarrow} u\right\},
\end{aligned}$$

for $v_j := w_j^{\delta(\varepsilon_j)}$, and this implies (4.38).

■

We now seek to ensure that $\mathcal{I}_{\{\varepsilon_j\}}$, regarded both as a functional on $W^{1,p}(\Omega, \mathbb{R}^d)$ and as a set function, admits an integral representation of the form

$$\mathcal{I}_{\{\varepsilon_j\}}(u; A) = \int_A f_{\{\varepsilon_j\}}(x, \nabla u(x))\, dx.$$

We will verify that the hypotheses of Theorem 2.4.1 hold. Using Lemma 2.1.20 and the conditions imposed on $f$, it is possible to show that $\mathcal{I}_{\{\varepsilon_j\}}(u;.)$ is a measure, more precisely we prove the following result.

**Lemma 4.1.13.** *For each $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, $\mathcal{I}_{\{\varepsilon_j\}}(u;.)$ is the trace of a finite, positive Radon measure restricted to $\mathcal{A}(\Omega)$.*

*Proof.* Let $u \in W^{1,p}(\Omega;\mathbb{R}^d)$. In view of Lemma 4.1.11, let $\{u_j\} \subset W^{1,p}(\Omega;\mathbb{R}^d)$ be a sequence such that

$$\mathcal{I}_{\{\varepsilon_j\}}(u;\Omega) = \lim_{j\to\infty} \int_\Omega f\left(x, \frac{x}{\varepsilon_j}, \nabla u_j(x)\right) dx,$$

and consider $\mu_j := f(\cdot, \frac{\cdot}{\varepsilon_j}, \nabla u_j)\chi_\Omega(\cdot)\mathcal{L}^N$. By $(H_4)$, and up to a subsequence (still denoted by $\mu_j$), there exists a finite positive Radon measure on $\mathbb{R}^N$ such that

$$\mu_j \overset{\star}{\rightharpoonup} \mu$$

(see Proposition 2.1.31). We claim that $\mathcal{I}_{\{\varepsilon_j\}}(u;.)\lfloor\mathcal{A}(\Omega) = \mu$, i.e. $\mathcal{I}_{\{\varepsilon_j\}}(u;A) = \mu(A)$ for all $A \in \mathcal{A}(\Omega)$. We apply Lemma 2.1.20 with $\Pi(\cdot) = \mathcal{I}_{\{\varepsilon_j\}}(u;.)$

We start by proving condition i) in Lemma 2.1.20, i.e. $\mathcal{I}_{\{\varepsilon_j\}}(u;.)$ is nested-subadditive. Given $A$, $B$, $C \in \mathcal{A}(\Omega)$ with $C \subset\subset B \subset A$ we have to show that

$$\mathcal{I}_{\{\varepsilon_j\}}(u;A) \leqslant \mathcal{I}_{\{\varepsilon_j\}}(u;B) + \mathcal{I}_{\{\varepsilon_j\}}(u;A \setminus \overline{C}).$$

Choose $C_\delta$, $D^\delta \in \mathfrak{R}$ with $C_\delta \subset C$ and $D^\delta \subset A \setminus \overline{C}$ such that

$$\int_{C\setminus C_\delta} (1 + |\nabla u|^p)\, dx < \delta \quad \text{and} \quad \int_{(A\setminus\overline{C})\setminus D^\delta} (1 + |\nabla u|^p)\, dx < \delta.$$

By Proposition 2.5.13 there exist two sequences $\{v^{C_\delta}\} \subset W^{1,p}(C_\delta;\mathbb{R}^d)$ and $\{v_j^{D^\delta}\} \subset W^{1,p}(D^\delta;\mathbb{R}^d)$ satisfying

$$||v_j^{D^\delta} - u||_{L^p(D^\delta;\mathbb{R}^d)} \to 0, \quad ||v_j^{C_\delta} - u||_{L^p(C_\delta;\mathbb{R}^d)} \to 0,$$

$$\mathcal{I}_{\{\varepsilon_j\}}(u;D^\delta) = \lim \mathcal{I}_{\varepsilon_j}(v_j^{D^\delta};D^\delta), \quad \mathcal{I}_{\{\varepsilon_j\}}(u;C_\delta) = \lim \mathcal{I}_{\varepsilon_j}(v_j^{C_\delta};C_\delta),$$

$$v_j^{D^\delta} = u \quad \text{on} \quad \partial D^\delta \quad \text{and} \quad v_j^{C_\delta} = u \quad \text{on} \quad \partial C_\delta.$$

Extend $v_j^{C_\delta}$ and $v_j^{D^\delta}$ by $u$ to all $A$ and set

$$w_j^\delta := \begin{cases} v_j^{D^\delta} & \text{if} \quad x \in A \setminus \overline{C} \\ v_j^{C_\delta} & \text{if} \quad x \in C. \end{cases}$$

Clearly $||w_j^\delta - u||_{L^p(A;\mathbb{R}^d)} \to 0$ and we have

$$\begin{aligned} \mathcal{I}_{\{\varepsilon_j\}}(u;A) &\leqslant \liminf_{\delta\to 0} \ \liminf_{j\to\infty} I_{\varepsilon_j}(w_j^\delta;A) \\ &\leqslant \mathcal{I}_{\{\varepsilon_j\}}(u;D^\delta) + \mathcal{I}_{\{\varepsilon_j\}}(u;C_\delta) + \lim_{\delta\to 0}\int_{(A\setminus\overline{C})\setminus D^\delta \cup (C\setminus C_\delta)} (1+|\nabla u|^p)\, dx \\ &= \mathcal{I}_{\{\varepsilon_j\}}(u;D^\delta) + \mathcal{I}_{\{\varepsilon_j\}}(u;C_\delta). \end{aligned}$$

To establish condition ii) in Lemma 2.1.20: Given $A \in \mathcal{A}(\Omega)$ and $\varepsilon > 0$, consider $A_\varepsilon \in \mathcal{A}(\Omega)$ such that $\bar{A}_\varepsilon \subset A$ and

$$\beta(1+|\nabla u(\cdot)|^p)\chi_\Omega(\cdot)\mathcal{L}^N\,(A\setminus\bar{A}_\varepsilon) < \varepsilon.$$

Due to the growth conditions $(H_4)$

$$\begin{aligned} \mathcal{I}_{\{\varepsilon_j\}}(u;A\setminus\bar{A}_\varepsilon) &\leqslant \liminf_{j\to\infty}\int_{A\setminus\bar{A}_\varepsilon} f\left(x,\frac{x}{\varepsilon_j},\nabla u(x)\right)\,dx \\ &\leqslant \beta\int_{A\setminus\bar{A}_\varepsilon}(1+|\nabla u(x)|^p)\,dx \\ &\leqslant \varepsilon. \end{aligned}$$

To show iv) fix $A \in \mathcal{A}(\Omega)$. By Proposition 2.1.32

$$\begin{aligned} \mathcal{I}_{\{\varepsilon_j\}}(u;A) &\leqslant \liminf_{j\to\infty}\int_A f\left(x,\frac{x}{\varepsilon_j},\nabla u_j(x)\right)\,dx \\ &= \liminf_{j\to\infty}\mu_j(A) \\ &\leqslant \mu(\overline{A}). \end{aligned}$$

Finally, to establish iii) take $\Omega' \subset \mathbb{R}^N$ such that $\Omega \subset\subset \Omega'$. As $\mu_j \overset{\star}{\rightharpoonup} \mu$

$$\mu(\Omega') \leqslant \lim_{j\to\infty} \mu_j(\Omega') = \lim_{j\to\infty} \int_\Omega f\Big(x, \frac{x}{\varepsilon_j}, \nabla u_j(x)\Big)\, dx = \mathcal{I}_{\{\varepsilon_j\}}(u;\Omega).$$

Therefore $\mu(\Omega') \leqslant \mathcal{I}_{\{\varepsilon_j\}}(u;\Omega)$ for all such $\Omega'$. Hence $\mathcal{I}_{\{\varepsilon_j\}}(u;\Omega) \geqslant \mu(\mathbb{R}^N)$, and as a consequence of Lemma 2.1.20 we conclude that $\mathcal{I}_{\{\varepsilon_j\}}(u;A) = \mu(A)$ for all $A \in \mathcal{A}(\Omega)$.

■

By Proposition 2.5.5 the functional $\mathcal{I}_{\{\varepsilon_j\}}(.,A)$ is lower semicontinuous with respect to the $L^p$- topology for all $A \in \mathcal{A}(\Omega)$, hence it is sequentially lower semicontinuous with respect to the weak topology in $W^{1,p}$. As a consequence of the integral representation Theorem 2.4.1 and Remark 2.4.2 we derive the following result.

**Lemma 4.1.14.** *There exist a Carathéodory function*

$$f_{\{\varepsilon_j\}} : \Omega \times \mathbb{R}^{d\times N} \to [0,\infty)$$

*quasiconvex with respect to its second variable for a.e. $x \in \Omega$ satisfying the same growth conditions than $f$ does, and such that*

$$\mathcal{I}_{\{\varepsilon_j\}}(u,A) = \int_A f_{\{\varepsilon_j\}}(x, \nabla u(x))\, dx$$

*for all $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$.*

To conclude that (4.34) holds, by Remark 2.5.9 it suffices to prove that the function $f_{\{\varepsilon_j\}}$ is independent on this particular (sub)sequence, so that each Γ-convergent (sub)sequence has the same limit. The remaining of this section is devoted to showing that

$$f_{\{\varepsilon_j\}}(x,\xi) = f_{\hom}(x,\xi) \tag{4.42}$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^{d\times N}$. To start, let $T \in \mathbb{N}$ and let $\mathcal{S}_T$ denote a countable set of $C_c^\infty((0,T)^N;\mathbb{R}^d)$-functions dense in $W_0^{1,p}((0,T)^N;\mathbb{R}^d)$. Let $L$ be the set of Lebesgue points $x_0$ for all functions

$$f_{\{\varepsilon_j\}}(\cdot,\eta) \tag{4.43}$$

and

$$x \to \int_Q f(x, Ty, \eta + \nabla\phi(Ty))\, dy, \tag{4.44}$$

with $\eta \in \mathbb{Q}^{d\times N}$, $\phi \in \mathcal{S}_T$ and $T \in \mathbb{N}$. Since the function $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,\infty)$ is of Carathéodory-type, by Scorza-Dragoni Theorem there exists a non decreasing sequence of compact subsets $\{K_m\}_{m\in\mathbb{N}} \subset \Omega$ with $|\Omega\backslash K_m| \leqslant \frac{1}{m}$ such that $f : K_m \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,\infty)$ is continuous for all $m \in \mathbb{N}$. Let $K_m^\star$ be the set of Lebesgue points for $\chi_{K_m}$ with $m \in \mathbb{N}$, and define

$$W := \bigcup_{m=0}^{\infty} (K_m \cap K_m^\star) \quad \text{and} \quad E := L \cap W.$$

We have $|\Omega \setminus L| = 0$ and $|\Omega\backslash W| \leqslant |\Omega\backslash K_m| \leqslant \frac{1}{m}$ for each $m \in \mathbb{N}$. Consequently $|\Omega \setminus W| = 0$ and $|\Omega \setminus E| = 0$. In a first step to prove identity (4.42) we derive the following equality.

**Proposition 4.1.15.** *$f_{\{\varepsilon_j\}}(x_0, \xi) = f_{\hom}(x_0, \xi)$ for all $x_0 \in E$ and $\xi \in \mathbb{Q}^{d\times N}$.*

*Proof.* Consider $x_0 \in E$ and $\xi \in \mathbb{Q}^{d\times N}$. By (4.43) we have

$$\begin{aligned} f_{\{\varepsilon_j\}}(x_0, \xi) &= \lim_{\delta\to 0} \tfrac{1}{\delta^N} \int_{Q(x_0,\delta)} f_{\{\varepsilon_j\}}(x, \xi)\, dx \\ &= \lim_{\delta\to 0} \tfrac{\mathcal{I}_{\{\varepsilon_j\}}(\xi\,\cdot\,; Q(x_0,\delta))}{\delta^N}. \end{aligned} \tag{4.45}$$

*Step 1.* We first establish the upper bound inequality for the Γ-limit of $\{\mathcal{I}_{\varepsilon_j}\}_j$, i.e.

$$f_{\{\varepsilon_j\}}(x_0, \xi) \leqslant f_{\hom}(x_0, \xi).$$

Given $n \in \mathbb{N}$, let $T_n \in \mathbb{N}$ and $\overline{\phi}_n \in W_0^{1,p}((0,T_n)^N; \mathbb{R}^d)$ such that

$$f_{\hom}(x_0, \xi) + \frac{1}{2n} \geqslant \frac{1}{{T_n}^N} \int_{(0,T_n)^N} f(x_0, y, \xi + \nabla\overline{\phi}_n(y))\, dy.$$

By conditions $(H_1)$ and $(H_4)$, and by the density of $\mathcal{S}_{T_n}$ in $W_0^{1,p}((0,T_n)^N; \mathbb{R}^d)$ we may take $\phi_n \in \mathcal{S}_{T_n}$ with

$$f_{\hom}(x_0, \xi) + \frac{1}{n} \geqslant \frac{1}{{T_n}^N} \int_{(0,T_n)^N} f(x_0, y, \xi + \nabla\phi_n(y))\, dy.$$

Extend $\phi_n$ periodically with period $T_n$ to $\mathbb{R}^N$ (still denoted by $\phi_n$). For $x \in \mathbb{R}^N$ define

$$u_j^n(x) := \xi \cdot x + \varepsilon_j \phi_n\Big(\frac{x}{\varepsilon_j}\Big)$$

and let $\delta > 0$ be small enough so that $Q(x_0, \delta) \in \mathcal{A}(\Omega)$. As $\{\phi_n\}_n$ is bounded in $W^{1,p}$, for fixed $n$ we have $\lim_{j \to \infty} u_j^n = v$ in $L^p(Q(x_0;\delta);\mathbb{R}^d)$ where $v(x) = \xi \cdot x$. Hence by equality (4.45)

$$f_{\{\varepsilon_j\}}(x_0,\xi) \leqslant \liminf_{\delta \to 0} \liminf_{j \to \infty} \frac{1}{\delta^N} \int_{Q(x_0;\delta)} f\left(x, \frac{x}{\varepsilon_j}, \xi + \nabla\phi_n\left(\frac{x}{\varepsilon_j}\right)\right) dx. \tag{4.46}$$

Define now $h_n(x,y) := f(x, T_n y, \xi + \nabla\phi_n(T_n y))$ for all $x \in \Omega$ and $y \in \mathbb{R}^N$, and for all $n \in \mathbb{N}$. Clearly $h_n \in L^1(Q(x_0;\delta); C_{\text{per}}(Q;\mathbb{R}^d))$ for $n \in \mathbb{N}$ - recall Definition 2.6.1 and Lemma 2.6.4 - and then by Lemma 2.6.3

$$\begin{aligned}
&\lim_{j\to\infty} \int_{Q(x_0;\delta)} f\left(x, \frac{x}{\varepsilon_j}, \xi + \nabla\phi_n\left(\frac{x}{\varepsilon_j}\right)\right) dx \\
&= \lim_{j\to\infty} \int_{Q(x_0;\delta)} f\left(x, \frac{T_n x}{T_n\varepsilon_j}, \xi + \nabla\phi_n\left(\frac{T_n x}{T_n\varepsilon_j}\right)\right) dx \\
&= \lim_{j\to\infty} \int_{Q(x_0;\delta)} h_n\left(x, \frac{x}{T_n\varepsilon_j}\right) dx \\
&= \int_{Q(x_0;\delta)} \int_Q h_n(x,y)\, dy\, dx \\
&= \int_{Q(x_0;\delta)} \int_Q f(x, T_n y, \xi + \nabla\phi_n(T_n y))\, dy\, dx
\end{aligned} \tag{4.47}$$

(Allaire [3] shows with counterexamples that this convergence does not hold if the continuity in one of the variables of $f$ is not assumed). Therefore by identities (4.44), (4.46) and (4.47)

$$\begin{aligned}
f_{\{\varepsilon_j\}}(x_0,\xi) &\leqslant \liminf_{\delta\to 0} \frac{1}{\delta^N} \int_{Q(x_0;\delta)} \int_Q f(x, T_n y, \xi + \nabla\phi_n(T_n y))\, dy\, dx \\
&= \int_Q f(x_0, T_n y, \xi + \nabla\phi_n(T_n y))\, dy \\
&= \frac{1}{T_n{}^N} \int_{(0,T_n)^N} f(x_0, y, \xi + \nabla\phi_n(y))\, dy \\
&\leqslant f_{\text{hom}}(x_0,\xi) + \frac{1}{n}.
\end{aligned}$$

Letting $n \to \infty$ we get

$$f_{\{\varepsilon_j\}}(x_0,\xi) \leqslant f_{\text{hom}}(x_0,\xi).$$

*Step 2.* We now show that the converse inequality holds, a.e.

$$f_{\{\varepsilon_j\}}(x_0,\xi) \geqslant f_{\text{hom}}(x_0,\xi).$$

Fix $\delta > 0$ small enough so that $Q(x_0;\delta) \in \mathcal{A}(\Omega)$, and consider $\{u_j^\delta\}_j \subset W^{1,p}(Q(x_0;\delta);\mathbb{R}^d)$ with $\lim_{j\to\infty} u_j^\delta = 0$ in $L^p(Q(x_0;\delta);\mathbb{R}^d)$ and

$$\mathcal{I}_{\{\varepsilon_j\}}(\xi\,\cdot\,; Q(x_0;\delta)) = \lim_{j\to\infty} \int_{Q(x_0;\delta)} f\left(x, \frac{x}{\varepsilon_j}, \xi + \nabla u_j^\delta(x)\right) dx.$$

By identity (4.45)

$$\begin{aligned} f_{\{\varepsilon_j\}}(x_0,\xi) &= \lim_{\delta\to 0}\ \lim_{j\to\infty} \tfrac{1}{\delta^N} \int_{Q(x_0;\delta)} f\left(x, \frac{x}{\varepsilon_j}, \xi + \nabla u_j^\delta(x)\right) dx \\ &= \lim_{\delta\to 0} \tfrac{1}{\delta^N} \int_{Q(x_0;\delta)} f\left(x, \frac{x}{\varepsilon_{j(\delta)}}, \xi + \nabla u_{j(\delta)}^\delta(x)\right) dx \\ &= \lim_{\delta\to 0} \int_Q f\left(x_0+\delta y, \frac{x_0+\delta y}{\varepsilon_{j(\delta)}}, \xi + \nabla v_{j(\delta)}^\delta(y)\right) dy, \end{aligned}$$

where $v_{j(\delta)}^\delta(y) := \frac{1}{\delta} u_{j(\delta)}^\delta(x_0+\delta y) \in W^{1,p}(Q;\mathbb{R}^d)$. Using a diagonalization argument, we choose the sequence $\{j(\delta)\}$ in such a way that

$$\frac{\delta}{\varepsilon_{j(\delta)}} > \frac{1}{\delta} \quad \text{and} \quad \lim_{\delta\to 0} ||v_{j(\delta)}^\delta||_{L^p(Q;\mathbb{R}^d)} = 0. \tag{4.48}$$

For simplicity denote $j(\delta) \equiv \delta$, $v_{j(\delta)}^\delta \equiv v_\delta$ and $u_{j(\delta)}^\delta \equiv u_\delta$. By Theorem 2.2.16 there exists a subsequence of $\{v_\delta\}$ (still denoted by $\{v_\delta\}$) and a sequence $\{w_\delta\} \subset W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^d)$ such that

$$w_\delta \rightharpoonup 0 \text{ in } W^{1,p}, \quad w_\delta = 0 \text{ in a neighborhood of } \partial Q,$$

$$\{|\ \nabla w_\delta\ |^p\} \text{ is equi-integrable} \tag{4.49}$$

and

$$|\ \{y \in Q:\ v_\delta(y) \neq w_\delta(y)\}\ | \to 0. \tag{4.50}$$

As, by assumption, $f$ is nonnegative

$$\begin{aligned} f_{\{\varepsilon_j\}}(x_0,\xi) &\geqslant \liminf_{\delta\to 0} \int_{\{y\in Q:\ v_\delta(y)=w_\delta(y)\}} f\left(x_0+\delta y, \frac{x_0+\delta y}{\varepsilon_\delta}, \xi + \nabla w_\delta(y)\right) dy \\ &= \liminf_{\delta\to 0} \int_Q f\left(x_0+\delta y, \frac{x_0+\delta y}{\varepsilon_\delta}, \xi + \nabla w_\delta(y)\right) dy \end{aligned}$$

from (4.49), (4.50) and $(H_4)$. Since $x_0 \in W$, there exists $m_0 \in \mathbb{N}$ such that $x_0 \in K_{m_0} \cap K^\star_{m_0}$ and then we can write

$$f_{\{\varepsilon_j\}}(x_0, \xi) \geqslant \liminf_{m\to\infty} \liminf_{\delta\to\delta} \int_{Q_{m,\delta}} f\left(x_0 + \delta y, \frac{x_0 + \delta y}{\varepsilon_\delta}, \xi + \nabla w_\delta(y)\right) dy, \tag{4.51}$$

where the set

$$Q_{m,\delta} := \{y \in Q : \quad x_0 + \delta y \in K_{m_0} \text{ and } |\nabla w_\delta(y)| \leqslant m\}$$

is such that

$$\lim_{m\to\infty} \lim_{\delta\to 0} |Q \setminus Q_{m,\delta}| = 0. \tag{4.52}$$

Indeed, this set has measure zero because on the one hand, as

$$|\{y \in Q : \ x_0 + \delta y \notin K_{m_0}\}| \quad \leqslant \quad 1 - \tfrac{1}{\delta^N} \int_{Q(x_0;\delta)} \chi_{K_{m_0}}(y)\, dy$$

and as $x_0 \in K^\star_{m_0}$, we have

$$\lim_{m\to\infty} \lim_{\delta\to 0} |\{y \in Q : \ x_0 + \delta y \notin K_{m_0}\}| = 0,$$

and on the other hand, by Chebyshev Inequality (Lemma 2.1.10) and the fact that $\{\nabla w_\delta\}_\delta$ is bounded in $L^p(Q; \mathbb{R}^d)$, we have

$$\limsup_{m\to\infty} \limsup_{\delta\to 0} |\{y \in Q : \ |\nabla w_\delta(y)| > m\}| = 0.$$

Write

$$\frac{x_0}{\varepsilon_\delta} = m_\delta + s_\delta$$

with $m_\delta \in \mathbb{Z}^N$ and $s_\delta \in [0,1)^N$, and define

$$x_\delta := \frac{-\varepsilon_\delta}{\delta} s_\delta.$$

Note that by (4.48) $x_\delta \to 0$ as $\delta \to 0$. After changing variables once more,

$$\begin{aligned}
&\int_{Q_{m,\delta}} f\left(x_0 + \delta y, \frac{x_0 + \delta y}{\varepsilon_\delta}, \xi + \nabla w_\delta(y)\right) dy \\
&= \int_{Q_{m,\delta} - x_\delta} f\left(x_0 + \delta(y + x_\delta), \frac{x_0 + \delta(y + x_\delta)}{\varepsilon_\delta}, \xi + \nabla w_\delta(y + x_\delta)\right) dy \\
&= \int_{Q_{m,\delta} - x_\delta} f\left(x_0 + \delta(y + x_\delta), \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y + x_\delta)\right) dy,
\end{aligned}$$

by the periodicity hypothesis $(H_3)$ and the fact that $\frac{x_0+\delta x_\delta}{\varepsilon_\delta} = m_\delta \in \mathbb{Z}^N$. Then by inequality (4.51) we have

$$f_{\{\varepsilon_j\}}(x_0,\xi) \geqslant \liminf_{m\to 0}\ \liminf_{\delta\to 0} \int_{Q_{m,\delta}-x_\delta} f\left(x_0+\delta(y+x_\delta), \frac{\delta}{\varepsilon_\delta}y, \xi+\nabla w_\delta(y+x_\delta)\right)\, dy. \tag{4.53}$$

We now write

$$\begin{aligned}
&\int_{Q_{m,\delta}-x_\delta} f\left(x_0+\delta(y+x_\delta), \frac{\delta}{\varepsilon_\delta}y, \xi+\nabla w_\delta(y+x_\delta)\right)\, dy \\
&= \int_{Q_{m,\delta}-x_\delta} \left[f\left(x_0+\delta(y+x_\delta), \frac{\delta}{\varepsilon_\delta}y, \xi+\nabla w_\delta(y+x_\delta)\right) - f\left(x_0, \frac{\delta}{\varepsilon_\delta}y, \xi+\nabla w_\delta(y+x_\delta)\right)\right] dy \\
&\qquad + \int_{Q_{m,\delta}-x_\delta} f\left(x_0, \frac{\delta}{\varepsilon_\delta}y, \xi+\nabla w_\delta(y+x_\delta)\right)\, dy.
\end{aligned} \tag{4.54}$$

By hypothesis $(H_3)$ and the continuity of $f$ with respect to $y$, given $m \in \mathbb{N}$ the restriction of $f$ to the set $K_{m_0} \times \mathbb{R}^N \times \overline{B(\xi,m)}$, where $B(\xi,m) := \{x \in \mathbb{R}^N : |\xi - x| < m\}$, is uniformly continuous - recall that if $g : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function $kQ$-periodic with respect to the first variable, then for each compact $K \subset \mathbb{R}^N$, $g : \mathbb{R}^N \times K \to \mathbb{R}$ is uniformly continuous. Hence there exist $\rho_m \in (0,1)$ such that

$$|f(x,y,\phi) - f(\overline{x},\overline{y},\overline{\phi})| < \frac{1}{m}$$

for all $x, \overline{x} \in K_{m_0}$, $y, \overline{y} \in \mathbb{R}^N$, and $\phi, \overline{\phi} \in \overline{B(\xi,m)}$ satisfying $|x-\overline{x}| + |y-\overline{y}| + |\phi-\overline{\phi}| < \rho_m$. As a result

$$\lim_{m\to\infty}\lim_{\delta\to 0}\int_{Q_{m,\delta}-x_\delta}\left|f\left(x_0+\delta(y+x_\delta), \frac{\delta}{\varepsilon_\delta}y, \xi+\nabla w_\delta(y+x_\delta)\right) - f\left(x_0, \frac{\delta}{\varepsilon_\delta}y, \xi+\nabla w_\delta(y+x_\delta)\right)\right| dy = 0,$$

and then by inequality (4.53) and identity (4.54) we get

$$f_{\{\varepsilon_j\}}(x_0,\xi) \quad \geqslant \liminf_{m\to\infty}\ \liminf_{\delta\to 0} \int_{Q_{m,\delta}-x_\delta} f\left(x_0, \frac{\delta}{\varepsilon_\delta}y, \xi+\nabla w_\delta(y+x_\delta)\right)\, dy.$$

Consequently,

$$f_{\{\varepsilon_j\}}(x_0,\xi) \geqslant \liminf_{\delta\to 0} \int_{Q-x_\delta} f\left(x_0, \frac{\delta}{\varepsilon_\delta}y, \xi+\nabla w_\delta(y+x_\delta)\right)\, dy. \tag{4.55}$$

Indeed, first we note that

$$\begin{aligned}
&\int_{(Q-x_\delta)\setminus(Q_{m,\delta}-x_\delta)} f\left(x_0, \frac{\delta}{\varepsilon_\delta}y, \xi+\nabla w_\delta(y+x_\delta)\right)\, dy \\
&\leqslant \beta \int_{(Q-x_\delta)\setminus(Q_{m,\delta}-x_\delta)} (1+|\xi+\nabla w_\delta(y+x_\delta)|^p)\, dy
\end{aligned}$$

$$= \beta \int_{Q\setminus Q_{m,\delta}} (1 + |\xi + \nabla w_\delta(y)|^p)\, dy. \tag{4.56}$$

As $\{|\nabla w_\delta|^p\}$ is equi-integrable, by inequality (4.56) and condition (4.52) we have

$$\limsup_{m\to\infty} \limsup_{\delta\to 0} \int_{(Q-x_\delta)\setminus(Q_{m,\delta}-x_\delta)} f\left(x_0, \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y + x_\delta)\right) dy = 0,$$

which, in turn, implies inequality (4.55). In addition, since

$$\begin{aligned}\int_{Q\setminus Q-x_\delta} f\left(x_0, \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y + x_\delta)\right) dy \quad &\leqslant \beta \int_{Q\setminus Q-x_\delta} (1 + |\xi + \nabla w_\delta(y + x_\delta)|^p) dy \\ &= \beta \int_{Q+x_\delta\setminus Q} (1 + |\xi + \nabla w_\delta(z)|^p) dz \to 0\end{aligned}$$

as $\delta \to 0$ (once again because $\{|\nabla w_\delta|^p\}$ is equi-integrable and $|(Q + x_\delta) \setminus Q| \to 0$ as $\delta \to 0$), we get from inequality (4.55)

$$f_{\{\varepsilon_j\}}(x_0, \xi) \quad \geqslant \liminf_{\delta\to 0} \int_Q f\left(x_0, \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y + x_\delta)\right) dy. \tag{4.57}$$

In order to compare $f_{\{\varepsilon_j\}}(x_0, \xi)$ with $f_{\text{hom}}(x_0, \xi)$ we need to modify $w_\delta(\cdot + x_\delta)$ close to the boundary of $Q$ so that it become admissible for $f_{\text{hom}}$. For this purpose, define the sets

$$L_\delta := \{y \in Q : \ \mathrm{dist(y, \partial Q)} \leqslant |x_\delta|\}, \quad M_\delta = \{y \in Q : \ \mathrm{dist(y, \partial Q)} \geqslant 2|x_\delta|\},$$

and

$$S_\delta = \{y \in Q : \ \mathrm{dist(y, \partial Q)} \in (|x_\delta|, 2|x_\delta|)\}.$$

Consider a function $\phi_\delta \in C_c^\infty(Q; \mathbb{R})$ with $||\nabla \phi_\delta||_{L^\infty} \leqslant \frac{C}{|x_\delta|}$ such that

$$\phi_\delta(y) = \begin{cases} 1 & \text{if } y \in \ M_\delta, \\ \\ 0 & \text{if } y \in \ L_\delta \end{cases}$$

and finally set $v_\delta(y) := \phi_\delta(y) w_\delta(y + x_\delta) + (1 - \phi_\delta(y)) w_\delta(y) \in W_0^{1,\infty}(Q;\mathbb{R}^d)$. We claim that

$$f_{\{\varepsilon_j\}}(x_0,\xi) \quad \geqslant \quad \liminf_{\delta\to 0} \int_Q f\Big(x_0, \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla v_\delta(y)\Big)\, dy, \tag{4.58}$$

which implies

$$f_{\{\varepsilon_j\}}(x_0,\xi) \quad \geqslant \quad \liminf_{\delta\to 0} \Big(\frac{\varepsilon_\delta}{\delta}\Big)^N \int_{\frac{\delta}{\varepsilon_\delta}Q} f\Big(x_0, y, \xi + \nabla \phi_\delta(y)\Big)\, dy$$

with $\phi_\delta(y) := \frac{\delta}{\varepsilon_\delta} v_\delta(\frac{\varepsilon_\delta}{\delta} y) \in W_0^{1,\infty}(\frac{\delta}{\varepsilon_\delta}Q;\mathbb{R}^d)$, and consequently,

$$f_{\{\varepsilon_j\}}(x_0,\xi) \geqslant f_{\mathrm{hom}}(x_0,\xi).$$

To prove inequality (4.58) we note that

$$\begin{aligned}
&\liminf_{\delta\to 0} \int_Q f\left(x_0, \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla v_\delta(y)\right)\, dy \\
&\leqslant \liminf_{\delta\to 0} \int_Q f\left(x_0, \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y + x_\delta)\right)\, dy \\
&+\ \limsup_{\delta\to 0} \beta \int_{L_\delta} (1 + |\nabla w_\delta(x)|^p)\, dx \\
&+\ \limsup_{\delta\to 0} \beta \int_{S_\delta} (|\nabla w_\delta(x + x_\delta)|^p + |\nabla w_\delta(x)|^p)\, dx \\
&+\ \limsup_{\delta\to 0} \beta ||\nabla \phi_\delta||_{L^\infty}^p \int_{S_\delta} |w_\delta(x + x_\delta) - w_\delta(x)|^p\, dx.
\end{aligned} \tag{4.59}$$

Due to the integrability property of $\{|\nabla w_\delta|^p\}_\delta$

$$\limsup_{\delta\to 0} \int_{L_\delta} (1 + |\nabla w_\delta(x)|^p)\, dx = 0 = \limsup_{\delta\to 0} \int_{S_\delta} (|\nabla w_\delta(x + x_\delta)|^p + |\nabla w_\delta(x)|^p)\, dx. \tag{4.60}$$

This property also implies that

$$\limsup_{\delta\to 0} ||\nabla \phi_\delta||_{L^\infty}^p \int_{S_\delta} |w_\delta(x + x_\delta) - w_\delta(x)|^p\, dx = 0, \tag{4.61}$$

because

$$
\begin{aligned}
||\nabla\phi_\delta||^p_{L^\infty}\int_{S_\delta}|w_\delta(x+x_\delta)-w_\delta(x)|^p\,dx &\leqslant \frac{C}{|x_\delta|^p}\int_{S_\delta}\left|\int_0^1\frac{dw_\delta(x+tx_\delta)}{dt}\,dt\right|^p\,dx\\
&\leqslant \frac{C}{|x_\delta|^p}\int_{S_\delta}\int_0^1|\nabla w_\delta(x+tx_\delta)|^p\,.\,|x_\delta|^p\,dt\,dx\\
&\leqslant C\int_{S_\delta}\int_0^1|\nabla w_\delta(x+tx_\delta)|^p\,dt\,dx\\
&= C\int_0^1\int_{S_\delta-tx_\delta}|\nabla w_\delta(y)|^p\,dy\,dt\\
&\leqslant C\int_{N_\delta}|\nabla w_\delta(y)|^p\,dy,
\end{aligned}
$$

where

$$N_\delta=\{x\in Q:\ \mathrm{dist(x,\partial Q)}\leqslant 3|x_\delta|\}.$$

Hence, inequality (4.58) holds by (4.57), (4.59), (4.60) and (4.61).

■

As a consequence of this proposition equality (4.42) holds.

**Corollary 4.1.16.** *$f_{\{\varepsilon_j\}}(x,\xi)=f_{\mathrm{hom}}(x,\xi)$ for a.e. $x\in\Omega$ and for all $\xi\in\mathbb{R}^{d\times N}$.*

*Proof.* By Proposition 4.1.15 we have that for a.e. $x\in\Omega$ and for all $\xi\in\mathbb{Q}^{d\times N}$, $f_{\{\varepsilon_j\}}(x;\xi)=f_{\mathrm{hom}}(x;\xi)$. By the continuity properties of $f_{\{\varepsilon_j\}}$ and $f_{\mathrm{hom}}$ with respect to their second variable, the equality $f_{\{\varepsilon_j\}}(x;\xi)=f_{\mathrm{hom}}(x;\xi)$ holds true for a.e. $x\in\Omega$ and for all $\xi\in\mathbb{R}^{d\times N}$.

■

The proof of Theorem 4.1.1 is now straightforward.

*Proof of Theorem 4.1.1* As a consequence of Corollary 4.1.16 the $\Gamma(L^p(A))$-limit of $\mathcal{I}_{\varepsilon_j}(\,\cdot\,;A)$ is equal to $\mathcal{I}_{\mathrm{hom}}(\,\cdot\,,A)$. In particular, since it does not depend upon the extracted subsequence, in view of Remark 2.5.9, the whole sequence $\mathcal{I}_\varepsilon(\,\cdot\,;A)$ $\Gamma(L^p(A))$-converges to $\mathcal{I}_{\mathrm{hom}}(\,\cdot\,;A)$. Taking $A=\Omega$ we conclude the proof of Theorem 4.1.1. ■

## 4.2 Multiple scale functionals

In this section we allow our functionals to depend on one more scale of periodicity, namely our goal is to prove the following result.

**Theorem 4.2.1.** *Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ be a function satisfying*

$(H_1)$ *$f(x, \cdot, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$;*

$(H_2)$ *$f(\cdot, y, z, \xi)$ is measurable for all $(y, z, \xi) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$;*

$(H_3)$ *$f(x, \cdot, z, \xi)$ is $Q$-periodic for all $(z, \xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and for a.e. $x \in \Omega$; $f(x, y, \cdot, \xi)$ is $Q$-periodic for all $(y, \xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and for a.e. $x \in \Omega$;*

$(H_4)$ *there exists $\beta > 0$ and a real number $p > 1$ such that for all $(y, z, \xi) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and for a.e. $x \in \Omega$*
$$\frac{1}{\beta}|\xi|^p - \beta \leqslant f(x, y, z, \xi) \leqslant \beta(1 + |\xi|^p).$$

*For each $\varepsilon > 0$ define $\mathcal{I}_\varepsilon : L^p(\Omega, \mathbb{R}^d) \to \overline{\mathbb{R}}$ by*
$$\mathcal{I}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \nabla u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^d), \\ \infty & \text{otherwise.} \end{cases}$$

*Then the $\Gamma(L^p(\Omega))$-limit of the family $\{\mathcal{I}_\varepsilon\}_\varepsilon$ is given by the functional*
$$\mathcal{I}_{\mathrm{hom}}(u) := \begin{cases} \displaystyle\int_\Omega \overline{f}_{\mathrm{hom}}(x, \nabla u(x))\, dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^d), \\ \infty & \text{otherwise,} \end{cases}$$

*where $\overline{f}_{\mathrm{hom}}$ is defined for all $\xi \in \mathbb{R}^{d\times N}$ and a.e. $x \in \Omega$ by*
$$\overline{f}_{\mathrm{hom}}(x, \xi) := \lim_{T\to\infty} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f_{\mathrm{hom}}(x, y, \xi + \nabla\phi(y))\, dy : \ \phi \in W_0^{1,p}((0,T)^N; \mathbb{R}^d) \right\}, \tag{4.62}$$

*and*
$$f_{\mathrm{hom}}(x, y, \xi) := \lim_{T\to\infty} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(x, y, z, \xi + \nabla\phi(z))\, dz : \ \phi \in W_0^{1,p}((0,T)^N; \mathbb{R}^d) \right\} \tag{4.63}$$

*for a.e.* $x \in \Omega$ *and all* $(y, \xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$.

Theorem 4.2.1 was obtained in collaboration with J.F. Babadjian [10]. As previously remarked, formula (4.62) is obtained by homogenizing first with respect to $z$, considering $y$ as a parameter, and then homogenizing with respect to $y$. That is, $\overline{f}_{\text{hom}}$ is the density obtained by iterating twice the homogenization formula (4.3). The generalization of this result to any number of scales $k \geqslant 2$ follows by an iterated argument similar to the one used in Braides and Defranceschi (see Remark 22.8 in [19]).

As in Section 4.1 (see Remark 4.1.2), without loss of generality we may assume that $f$ is a positive Borel function such that hypothesis $(H_1)$, $(H_3)$ and $(H_4)$ hold for every $(x, y, z, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$.

We note that most of the proofs presented here follow along the lines of the ones in Braides and Defranceschi [19] (Theorem 22.1 and Remark 22.8), and that our main contribution is to weaken the strong uniform continuity hypothesis (1.6). Let us briefly describe how we proceed: The idea consists in proving the result for integrands which do not depend explicitly on $x$ (see Theorem 4.2.2), and then to treat the general case by freezing this macroscopic variable and proceeding as in the proof of Theorem 4.1.1 by using the "uniformly continuous structure" of the integrand $f$ up to sets of measure zero.

We divide this section as follows. In Subsubsection 4.2.1 we state the main properties of $f_{\text{hom}}$ and $\overline{f}_{\text{hom}}$ that are basic for our analysis. In Subsubsection 4.2.2 we present some auxiliary results for the proof of the homogeneous counterpart of Theorem 4.2.1, Theorem 4.2.2, in which we assume that $f$ does not depend explicitly on $x$. The proof of Theorem 4.2.1 in its full generality is presented in Subsection 4.2.3. Finally, in Subsection 4.2.4 we remark an alternative proof for convex integrands relying on arguments of multiscale convergence.

### 4.2.1 Properties of the homogenized density

Repeating the argument used in Section 4.1, we can see that the function $f_{\text{hom}}$ given in (4.63) is well defined and is a Carathéodory function:

$$f_{\text{hom}}(\cdot, \cdot, \xi) \text{ is } \mathcal{L}^N \otimes \mathcal{L}^N\text{-measurable for all } \xi \in \mathbb{R}^{d\times N}, \tag{4.64}$$

$$f_{\text{hom}}(x, y, \cdot) \text{ is continuous for all } (x, y) \in \Omega \times \mathbb{R}^N. \tag{4.65}$$

By condition $(H_3)$ it follows that

$$f_{\text{hom}}(x, \cdot, \xi) \text{ is } Q\text{-periodic for all } x \in \Omega \text{ and all } \xi \in \mathbb{R}^{d\times N}. \tag{4.66}$$

Moreover, $f_{\text{hom}}$ is quasiconvex in the variable $\xi$ and satisfies the same $p$-growth and $p$-coercivity condition as $f$:

$$\frac{1}{\beta}|\xi|^p - \beta \leqslant f_{\text{hom}}(x, y, \xi) \leqslant \beta(1 + |\xi|^p) \tag{4.67}$$

for all $x \in \Omega$ and all $(y, \xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$, where $\beta$ is the constant in $(H_4)$. As a consequence of (4.64)-(4.67), the function $\overline{f}_{\text{hom}}$ given in (4.62) is also well defined, and is a Carathéodory function, which implies the functional $\mathcal{I}_{\text{hom}}$ is well defined on $W^{1,p}(\Omega;\mathbb{R}^d)$. Finally, $\overline{f}_{\text{hom}}$ is also quasiconvex in the variable $\xi$ and satisfies the same $p$-growth and $p$-coercivity condition as $f$ and $f_{\text{hom}}$:

$$\frac{1}{\beta}|\xi|^p - \beta \leqslant \overline{f}_{\text{hom}}(x, \xi) \leqslant \beta(1 + |\xi|^p) \tag{4.68}$$

for all $x \in \Omega$ and all $\xi \in \mathbb{R}^{d\times N}$, where, as before, $\beta$ is the constant in $(H_4)$.

In what follows $\lim\limits_{k,m,n} := \lim\limits_{k} \lim\limits_{m} \lim\limits_{n}$ with obvious generalizations.

### 4.2.2 Main result when the integrands do not depend on the macroscopic variable

We assume that $f$ does not depend explicitly on $x$, namely $f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0, \infty)$, and that it satisfies hypotheses $(H_3)$-$(H_4)$. In addition, according to $(H_1)$-$(H_2)$, and unless we specify the contrary, we assume $f$ to be continuous.

For each $\varepsilon > 0$ consider the functional $\mathcal{I}_\varepsilon : L^p(\Omega;\mathbb{R}^d) \to [0, \infty]$ defined by

$$\mathcal{I}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \nabla u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\ \\ \infty & \text{otherwise.} \end{cases} \tag{4.69}$$

Our objective is to prove the following result.

**Theorem 4.2.2.** *Under the above assumptions on $f$ the $\Gamma(L^p(\Omega))$-limit of the family $\{\mathcal{I}_\varepsilon\}_\varepsilon$ is given by*

$$\mathcal{I}_{\mathrm{hom}}(u) = \begin{cases} \displaystyle\int_\Omega \overline{f}_{\mathrm{hom}}(\nabla u(x))\,dx & \textit{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\ \\ \infty & \textit{otherwise}, \end{cases}$$

*where* $\overline{f}_{\mathrm{hom}}$ *is defined by*

$$\overline{f}_{\mathrm{hom}}(\xi) := \lim_{T\to\infty} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f_{\mathrm{hom}}(y, \xi + \nabla\phi(y))\,dy : \ \phi \in W_0^{1,p}\big((0,T)^N;\mathbb{R}^d\big) \right\} \tag{4.70}$$

*for all* $\xi \in \mathbb{R}^{d\times N}$*, and where*

$$f_{\mathrm{hom}}(y,\xi) := \lim_{T\to\infty} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(y, z, \xi + \nabla\phi(z))\,dz : \ \phi \in W_0^{1,p}\big((0,T)^N;\mathbb{R}^d\big) \right\} \tag{4.71}$$

*for all* $(y,\xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$*.*

This result can be seen as a generalization of Theorem 1.9 in Fonseca and Zappale [51] (for first derivatives), in which, as it is usual for the convex case, it is enough to consider variations that are periodic in the cell $Q$. Their multiscale argument (see Subsection 4.2.4 below) does not apply here since, as it is expected in the non convex case, the variations should be considered to be periodic over an infinite ensemble of cells, as it is seen from (4.70) and (4.71).

We divide the proof of Theorem 4.2.2 in four steps.

*STEP 1.* Localization of our functionals (4.69).

We highlight their dependence on the class of bounded, open subsets of $\mathbb{R}^N$, denoted by $\mathcal{A}_0(\mathbb{R}^N)$. As it will be clear from the proofs of Lemmas 4.2.7 and 4.2.8 below, it would not be sufficient to localize, as in Section 4.1, on any open subset of $\Omega$. Indeed, formulas (4.70) and (4.71) suggest working in cubes of the type $(0,T)^N$, with $T$ arbitrarily large, not necessarily contained in $\Omega$.

For each $\varepsilon > 0$ consider $\mathcal{I}_\varepsilon : L^p(\mathbb{R}^N;\mathbb{R}^d) \times \mathcal{A}_0(\mathbb{R}^N) \to [0,\infty]$ defined by

$$\mathcal{I}_\varepsilon(u;A) := \begin{cases} \displaystyle\int_A f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \nabla u(x)\right)\,dx & \text{if } u \in W^{1,p}(A;\mathbb{R}^d), \\ \\ \infty & \text{otherwise.} \end{cases} \tag{4.72}$$

We will prove that the family of functionals $\{\mathcal{I}_\varepsilon(\cdot\,;A)\}_\varepsilon$, with $A \in \mathcal{A}_0(\Omega)$, Γ-converges with respect to the strong $L^p(A;\mathbb{R}^d)$-topology to the functional $\mathcal{I}_{\text{hom}}(\cdot\,;A)$, where $\mathcal{I}_{\text{hom}} : L^p(\mathbb{R}^N;\mathbb{R}^d) \times \mathcal{A}_0(\mathbb{R}^N) \to [0,\infty]$ is given by

$$\mathcal{I}_{\text{hom}}(u;A) = \begin{cases} \displaystyle\int_A \overline{f}_{\text{hom}}(\nabla u(x))\,dx & \text{if } u \in W^{1,p}(A;\mathbb{R}^d), \\ \\ \infty & \text{otherwise.} \end{cases}$$

As a consequence, taking $A = \Omega$ yields Theorem 4.2.2.

*STEP 2.* Existence of Γ-convergent subsequences.

Let $\varepsilon_n \downarrow 0$. For every $A \in \mathcal{A}_0(\mathbb{R}^N)$ consider the Γ-lower limit of $\{\mathcal{I}_{\varepsilon_n}(\cdot\,;A)\}_n$ for the $L^p(A;\mathbb{R}^d)$-topology defined for $u \in L^p(\mathbb{R}^N;\mathbb{R}^d)$ by

$$\mathcal{I}_{\{\varepsilon_n\}}(u;A) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} \mathcal{I}_{\varepsilon_n}(u_n;A) : \; u_n \to u \text{ in } L^p(A;\mathbb{R}^d) \right\}.$$

In view of the $p$-coercivity condition $(H_4)$ it follows that $\mathcal{I}_{\{\varepsilon_n\}}(u;A)$ is infinite whenever $u \in L^p(\mathbb{R}^N;\mathbb{R}^d) \setminus W^{1,p}(A;\mathbb{R}^d)$ for each $A \in \mathcal{A}_0(\mathbb{R}^N)$, so it suffices to study the case where $u \in W^{1,p}(A;\mathbb{R}^d)$.

By a similar argument to the one used in Section 4.1 it can be seen that there exists a subsequence $\{\varepsilon_{n_j}\}_j \equiv \{\varepsilon_j\}_j$ such that $\mathcal{I}_{\{\varepsilon_j\}}(\cdot\,;A)$ is the Γ-limit of $\{\mathcal{I}_{\varepsilon_j}(\cdot\,;A)\}_j$ for each $A \in \mathcal{A}_0(\mathbb{R}^N)$.

*STEP 3.* Integral representation of the Γ-limit.

Our goal is to study the behavior of $\mathcal{I}_{\{\varepsilon_j\}}(u;\cdot)$ in $\mathcal{A}(A)$ for each $u \in W^{1,p}(A;\mathbb{R}^d)$ and $A \in \mathcal{A}_0(\mathbb{R}^N)$. Following the proof of Lemma 4.1.13, it is possible to show that $\mathcal{I}_{\{\varepsilon_j\}}(u;\cdot)$ is a measure on $\mathcal{A}(A)$ for all $A \in \mathcal{A}_0(\mathbb{R}^N)$. Namely, the following result holds.

**Lemma 4.2.3.** *For each $A \in \mathcal{A}_0(\mathbb{R}^N)$ and all $u \in W^{1,p}(A;\mathbb{R}^d)$, the restriction of $\mathcal{I}_{\{\varepsilon_j\}}(u;\cdot)$ to $\mathcal{A}(A)$ is a Radon measure, absolutely continuous with respect to the $N$-dimensional Lebesgue measure.*

For the moment, we are not in position to apply Buttazzo-Dal Maso Integral Representation Theorem (Theorem 2.4.1) because, a priori, the integrand would depend on the open set

$A \in \mathcal{A}_0(\mathbb{R}^N)$. The following result prevents this dependence from holding since it leads to an homogeneous integrand as it will be seen in Lemma 4.2.5 below.

**Lemma 4.2.4.** *For all $\xi \in \mathbb{R}^{d\times N}$, $y_0$ and $z_0 \in \mathbb{R}^N$, and $\delta > 0$*

$$\mathcal{I}_{\{\varepsilon_j\}}(\xi\,\cdot\,; Q(y_0,\delta)) = \mathcal{I}_{\{\varepsilon_j\}}(\xi\,\cdot\,; Q(z_0,\delta)).$$

*Proof.* Clearly, it suffices to establish the inequality

$$\mathcal{I}_{\{\varepsilon_j\}}(\xi\,\cdot\,; Q(y_0,\delta)) \geqslant \mathcal{I}_{\{\varepsilon_j\}}(\xi\,\cdot\,; Q(z_0,\delta)).$$

Let $\{w_j\}_j \subset W_0^{1,p}(Q(y_0,\delta);\mathbb{R}^d)$, with $w_j \to 0$ in $L^p(Q(y_0,\delta);\mathbb{R}^d)$, be such that

$$\mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot; Q(y_0,\delta)) = \lim_{j\to\infty} \int_{Q(y_0,\delta)} f\Big(\frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla w_j(x)\Big)\, dx$$

(see Proposition 2.5.13). By hypothesis $(H_4)$ and the Poincaré Inequality, we can suppose that the sequence $\{w_j\}_j$ is uniformly bounded in $W^{1,p}(Q(y_0,\delta);\mathbb{R}^d)$. Thus by the Decomposition Lemma, there exists a subsequence of $\{w_j\}_j$ (still denoted by $\{w_j\}_j$) and a sequence $\{u_j\}_j \subset W_0^{1,\infty}(Q(y_0,\delta);\mathbb{R}^d)$ such that $u_j \rightharpoonup 0$ in $W^{1,p}(Q(y_0,\delta);\mathbb{R}^d)$,

$$\{|\nabla u_j|^p\} \text{ is equi-integrable} \tag{4.73}$$

and

$$\mathcal{L}^N(\{y \in Q(y_0,\delta): \quad u_j(y) \neq w_j(y)\}) \to 0. \tag{4.74}$$

Then, in view of (4.73), (4.74) and the $p$-growth condition $(H_4)$,

$$\begin{aligned}\mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot; Q(y_0,\delta)) &\geqslant \limsup_{j\to\infty} \int_{Q(y_0,\delta)\cap\{u_j=w_j\}} f\Big(\frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j(x)\Big)\, dx \\ &\geqslant \limsup_{j\to\infty} \int_{Q(y_0,\delta)} f\Big(\frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j(x)\Big)\, dx \end{aligned}\tag{4.75}$$

For all $j \in \mathbb{N}$ we write

$$\frac{y_0 - z_0}{\varepsilon_j} = m_{\varepsilon_j} + s_{\varepsilon_j}$$

with $m_{\varepsilon_j} \in \mathbb{Z}^N$ and $s_{\varepsilon_j} \in [0,1)^N$,

$$\frac{m_{\varepsilon_j}}{\varepsilon_j} = \theta_{\varepsilon_j} + l_{\varepsilon_j} \tag{4.76}$$

with $\theta_{\varepsilon_j} \in \mathbb{Z}^N$ and $l_{\varepsilon_j} \in [0,1)^N$, and we define

$$x_{\varepsilon_j} := m_{\varepsilon_j}\varepsilon_j - \varepsilon_j^2 l_{\varepsilon_j}. \tag{4.77}$$

Note that $x_{\varepsilon_j} = y_0 - z_0 - \varepsilon_j s_{\varepsilon_j} - \varepsilon_j^2 l_{\varepsilon_j} \to y_0 - z_0$ as $j \to \infty$. For all $j \in \mathbb{N}$, extend $u_j$ by zero to the whole $\mathbb{R}^N$ and set $v_j(x) = u_j(x + x_{\varepsilon_j})$ for $x \in Q(z_0, \delta)$. Then $\{v_j\}_j \subset W^{1,p}(Q(z_0,\delta);\mathbb{R}^d)$ and $v_j \to 0$ in $L^p(Q(z_0,\delta);\mathbb{R}^d)$ because

$$\begin{aligned}
\int_{Q(z_0,\delta)} |v_j(x)|^p dx &= \int_{Q(z_0,\delta)} |u_j(x + x_{\varepsilon_j})|^p dx \\
&= \int_{Q(z_0 + x_{\varepsilon_j},\delta)} |u_j(x)|^p dx \\
&\leqslant \int_{Q(y_0,\delta)} |u_j(x)|^p dx,
\end{aligned} \tag{4.78}$$

since $u_j \equiv 0$ outside $Q(y_0,\delta)$. We also remark that by the translation invariance of the Lebesgue measure, the sequence $\{|\nabla v_j|^p\}_j$ is equi-integrable. In view of (4.75), (4.76), (4.77) and $(H_3)$,

$$\begin{aligned}
\mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot\,; Q(y_0,\delta)) &\geqslant \limsup_{j\to\infty} \int_{Q(y_0 - x_{\varepsilon_j},\delta)} f\Big(\frac{x + x_{\varepsilon_j}}{\varepsilon_j}, \frac{x + x_{\varepsilon_j}}{\varepsilon_j^2}, \xi + \nabla u_j(x + x_{\varepsilon_j})\Big)\, dx \\
&= \limsup_{j\to\infty} \int_{Q(y_0 - x_{\varepsilon_j},\delta)} f\Big(\frac{x}{\varepsilon_j} - \varepsilon_j l_{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla v_j(x)\Big)\, dx \\
&\geqslant \limsup_{j\to\infty} \int_{Q(z_0,\delta)} f\Big(\frac{x}{\varepsilon_j} - \varepsilon_j l_{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla v_j(x)\Big)\, dx \\
&\qquad - \limsup_{j\to\infty} \int_{Q(z_0,\delta)\setminus Q(y_0 - x_{\varepsilon_j},\delta)} f\Big(\frac{x}{\varepsilon_j} - \varepsilon_j l_{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla v_j(x)\Big)\, dx.
\end{aligned}$$

Since $v_j \equiv 0$ outside $Q(y_0 - x_{\varepsilon_j},\delta)$, the $p$-growth condition $(H_4)$ and the fact that $\mathcal{L}^N(Q(z_0,\delta)\setminus Q(y_0 - x_{\varepsilon_j},\delta)) \to 0$ yield

$$\begin{aligned}
&\limsup_{j\to\infty} \int_{Q(z_0,\delta)\setminus Q(y_0 - x_{\varepsilon_j},\delta)} f\Big(\frac{x}{\varepsilon_j} - \varepsilon_j l_{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla v_j(x)\Big)\, dx \\
&\qquad\qquad \leqslant \limsup_{j\to\infty} \beta(1 + |\xi|^p)\mathcal{L}^N(Q(z_0,\delta)\setminus Q(y_0 - x_{\varepsilon_j},\delta)) = 0,
\end{aligned}$$

and therefore

$$\mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot\,; Q(y_0,\delta)) \geqslant \limsup_{j\to\infty} \int_{Q(z_0,\delta)} f\Big(\frac{x}{\varepsilon_j} - \varepsilon_j l_{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla v_j(x)\Big)\, dx. \tag{4.79}$$

To eliminate the term $\varepsilon_j\, l_{\varepsilon_j}$ in (4.79), and thus to recover $\mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot\,; Q(z_0,\delta))$, we would like to apply a uniform continuity argument. Since $f$ is continuous on $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and separately $Q$-periodic with respect to its two first variables, by hypothesis $(H_3)$, then $f$ is uniformly continuous on $\mathbb{R}^N \times \mathbb{R}^N \times \overline{B}(0,\lambda)$ for any $\lambda > 0$. We define

$$R_j^\lambda := \{x \in Q(z_0,\delta) : |\xi + \nabla v_j(x)| \leqslant \lambda\},$$

and we note that by Chebyshev's inequality

$$\mathcal{L}^N(Q(z_0,\delta) \setminus R_j^\lambda) \leqslant C/\lambda^p, \tag{4.80}$$

for some constant $C > 0$ independent of $\lambda$ or $j$. Thus, in view of (4.79) and the fact that $f$ is nonnegative,

$$\mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot\,; Q(y_0,\delta)) \geqslant \limsup_{\lambda\to\infty}\limsup_{j\to\infty}\int_{R_j^\lambda} f\Big(\frac{x}{\varepsilon_j} - \varepsilon_j l_{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla v_j(x)\Big)\,dx.$$

Denoting by $\omega_\lambda : [0,\infty) \to [0,\infty)$ the modulus of continuity of $f$ on $\mathbb{R}^N \times \mathbb{R}^N \times \overline{B}(0,\lambda)$, we get that for any $x \in R_j^\lambda$

$$\left| f\Big(\frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla v_j(x)\Big) - f\Big(\frac{x}{\varepsilon_j} - \varepsilon_j l_{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla v_j(x)\Big) \right| \leqslant \omega_\lambda(\varepsilon_j l_{\varepsilon_j}).$$

Then, the continuity of $\omega_\lambda$ and the fact that $\omega_\lambda(0) = 0$ yield

$$\begin{aligned}\mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot\,; Q(y_0,\delta)) &\geqslant \limsup_{\lambda\to\infty}\limsup_{j\to\infty}\left\{\int_{R_j^\lambda} f\Big(\frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla v_j(x)\Big)\,dx - \delta^N \omega_\lambda(\varepsilon_j l_{\varepsilon_j})\right\}\\ &= \limsup_{\lambda\to\infty}\limsup_{j\to\infty}\int_{R_j^\lambda} f\Big(\frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla v_j(x)\Big)\,dx.\end{aligned}$$

The equi-integrability of $\{|\nabla v_j|^p\}$, the $p$-growth condition $(H_4)$ and (4.80), imply that

$$\begin{aligned}&\limsup_{\lambda\to\infty}\limsup_{j\to\infty}\int_{Q(z_0,\delta)\setminus R_j^\lambda} f\Big(\frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla v_j(x)\Big)\,dx\\ &\qquad\leqslant \beta\limsup_{\lambda\to\infty}\sup_{j\in\mathbb{N}}\int_{Q(z_0,\delta)\setminus R_j^\lambda}(1 + |\nabla v_j(x)|^p)dx = 0,\end{aligned}$$

and since $v_j \to 0$ in $L^p(Q(z_0,\delta);\mathbb{R}^d)$,

$$\begin{aligned}\mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot\,; Q(y_0,\delta)) &\geqslant \limsup_{j\to\infty}\int_{Q(z_0,\delta)} f\Big(\frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla v_j(x)\Big)\,dx\\ &\geqslant \mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot\,; Q(z_0,\delta)).\end{aligned}$$

■

As a consequence of this lemma, we derive the following result.

**Lemma 4.2.5.** *There exists a continuous function* $f_{\{\varepsilon_j\}} : \mathbb{R}^{d\times N} \to [0,\infty)$ *such that for all* $A \in \mathcal{A}_0(\mathbb{R}^N)$ *and all* $u \in W^{1,p}(A;\mathbb{R}^d)$,

$$\mathcal{I}_{\{\varepsilon_j\}}(u;A) = \int_A f_{\{\varepsilon_j\}}(\nabla u(x))\,dx.$$

*Proof.* Let $A \in \mathcal{A}_0(\mathbb{R}^N)$. By Theorem 2.4.1, there exists a Carathéodory function $f^A_{\{\varepsilon_j\}} : A \times \mathbb{R}^{d\times N} \to [0,\infty)$ satisfying

$$\mathcal{I}_{\{\varepsilon_j\}}(u;U) = \int_U f^A_{\{\varepsilon_j\}}(x, \nabla u(x))\,dx$$

for all $U \in \mathcal{A}(A)$ and all $u \in W^{1,p}(U;\mathbb{R}^d)$. Furthermore, for a.e. $x \in A$ and all $\xi \in \mathbb{R}^{d\times N}$

$$f^A_{\{\varepsilon_j\}}(x,\xi) = \lim_{\delta\to 0} \frac{\mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot\,; Q(x,\delta))}{\delta^N}.$$

Define $f_{\{\varepsilon_j\}} : \mathbb{R}^{d\times N} \to [0,\infty)$ by

$$f_{\{\varepsilon_j\}}(\xi) = \lim_{\delta\to 0} \frac{\mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot\,; Q(0,\delta))}{\delta^N}.$$

As a consequence of Lemma 4.2.4, $f^A_{\{\varepsilon_j\}}(x,\xi) = f_{\{\varepsilon_j\}}(\xi)$ for a.e. $x \in A$ and for all $\xi \in \mathbb{R}^{d\times N}$, and we conclude that

$$\mathcal{I}_{\{\varepsilon_j\}}(u;A) = \int_A f_{\{\varepsilon_j\}}(\nabla u(x))\,dx$$

holds for all $u \in W^{1,p}(A;\mathbb{R}^d)$. ■

*STEP 4.* Characterization of the Γ-limit.

Our next objective is to show that $\mathcal{I}_{\{\varepsilon_j\}}(u;A) = \mathcal{I}_{\text{hom}}(u;A)$ for any $A \in \mathcal{A}_0(\mathbb{R}^N)$ and all $u \in W^{1,p}(A;\mathbb{R}^d)$. In view of Lemma 4.2.5, we only need to prove that $\overline{f}_{\text{hom}}(\xi) = f_{\{\varepsilon_j\}}(\xi)$ for all $\xi \in \mathbb{R}^{d\times N}$, thus it suffices to work with affine functions instead of general Sobolev functions. In order to estimate $f_{\{\varepsilon_j\}}$ from below in terms of $\overline{f}_{\text{hom}}$, we will need the following result, close in spirit to Proposition 22.4 in Braides and Defranceschi [19].

**Proposition 4.2.6.** *Let $f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,\infty)$ be a (not necessarily continuous) function such that $f(x,\cdot,\cdot)$ is continuous, $f(\cdot, y;\xi)$ is measurable, and $(H_3)$ and $(H_4)$ hold. Let $A$ be an open, bounded, connected and Lipschitz subset of $\mathbb{R}^N$. Given $M$ and $\eta$ two positive numbers, and $\varphi : [0,\infty) \to [0,\infty]$ a continuous and increasing function satisfying $\varphi(t)/t \to \infty$ as $t \to \infty$, there exists $\varepsilon_0 \equiv \varepsilon_0(\varphi, M, \eta) > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, every $a \in \mathbb{R}^N$ and every $u \in W^{1,p}(a + A;\mathbb{R}^d)$ with*

$$\int_{a+A} \varphi(|\nabla u|^p)\, dx \leqslant M,$$

*there exists $v \in W_0^{1,p}(a + A;\mathbb{R}^d)$ with $\|v\|_{L^p(a+A;\mathbb{R}^d)} \leqslant \eta$ satisfying*

$$\int_{a+A} f\Big(x, \frac{x}{\varepsilon}, \nabla u\Big)\, dx \geqslant \int_{a+A} f_{\text{hom}}(x, \nabla u + \nabla v)\, dx - \eta.$$

*Proof.* The proof is divided into two steps. First, we prove this proposition under the additional hypothesis that $a$ belongs to a compact set of $\mathbb{R}^N$. Then, we conclude the result in its full generality replacing $a$ by its fractional part $a - [[a]]$ and using the periodicity of the integrands $f$ and $f_{\text{hom}}$.

*Step 1.* For $a \in [-1, 1]^N$, the claim of Proposition 4.2.6 holds. Indeed, if not then we may find $\varphi$, $M$ and $\eta$ as above, and sequences $\varepsilon_j \downarrow 0$, $\{a_j\}_j \subset [-1,1]^N$ and $\{u_j\}_j \subset W^{1,p}(a_j + A;\mathbb{R}^d)$ with

$$\int_{a_j+A} \varphi(|\nabla u_j|^p)\, dx \leqslant M \tag{4.81}$$

such that, for every $j \in \mathbb{N}$

$$\begin{aligned}&\int_{a_j+A} f\left(x, \frac{x}{\varepsilon_j}, \nabla u_j\right) dx\\ &< \inf_{v\in W_0^{1,p}(a_j+A;\mathbb{R}^d)} \left\{\int_{a_j+A} f_{\text{hom}}(x, \nabla u_j + \nabla v)\, dx : \|v\|_{L^p(a_j+A;\mathbb{R}^d)} \leqslant \eta\right\} - \eta.\end{aligned} \tag{4.82}$$

From (4.81) and the Poincaré Inequality, up to a translation argument, we can suppose that the sequence $\{\|u_j\|_{W^{1,p}(a_j+A;\mathbb{R}^d)}\}$ is uniformly bounded. From this fact and since the set $a_j + A$ is an extension domain (see Theorem 2.2.2), there is no loss of generality in assuming that $\{u_j\}_j$ is bounded in $W^{1,p}(\mathbb{R}^N;\mathbb{R}^d)$ and that, due to (4.81),

$$\sup_{j\in\mathbb{N}} \int_{\mathbb{R}^N} \varphi(|\nabla u_j|^p)\, dx \leqslant M_1 \tag{4.83}$$

for some constant $M_1 > 0$ depending only on $M$ (see the proof of the Extension Theorem for Sobolev functions, Theorem 1, Section 4.4 in Evans and Gariepy [46]). Passing to a subsequence, we can also assume that $u_j \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N;\mathbb{R}^d)$. Let $B$ a ball of sufficiently large radius so that $a_j + A \subset B$ for all $j \in \mathbb{N}$. De La Vallée Poussin criterion (see Proposition 2.2.10) and (4.83) guarantee that the sequence $\{|\nabla u_j|^p\}_j$ is equi-integrable on $B$. This implies that there exists $\delta = \delta(\eta)$ such that

$$\sup_{j\in\mathbb{N}} \beta \int_E (1 + |\nabla u|^p + |\nabla u_j|^p)\, dx \leqslant \frac{\eta}{2} \tag{4.84}$$

whenever $E$ is a measurable subset of $B$ satisfying $\mathcal{L}^N(E) \leqslant \delta$, and where $\beta$ is the constant given in $(H_4)$. As $\{a_j\} \subset [-1,1]^N$ we may suppose, without loss of generality, that this sequence $a_j \to a \in [-1,1]^N$, and that for fixed $0 < \rho < 1$, with $\rho^N \lll \delta$, the following holds for $j$ large enough:

$$\begin{cases} a + (1-\rho)A \subset a_j + A \subset a + (1+\rho)A, \\ \\ \mathcal{L}^N(S_j) \leqslant \delta, \text{ where } S_j := [a_j + A] \setminus [a + (1-\rho)A] \subset B, \\ \\ \text{and} \quad \|u_j - u\|_{L^p(a+(1+\rho)A;\mathbb{R}^d)} \leqslant \eta. \end{cases} \tag{4.85}$$

Take now a sequence of cut-off functions $\varphi_j \in C_c^\infty(\mathbb{R}^N;[0,1])$ such that

$$\varphi_j = \begin{cases} 1 & \text{on } a + (1-\rho)A, \\ \\ 0 & \text{outside } a_j + A, \end{cases}$$

and $\|\nabla \varphi_j\|_{L^\infty(\mathbb{R}^N)} \leqslant C/\rho$ for some constant $C > 0$. Let $w_j = \varphi_j u + (1-\varphi_j)u_j$. Then $w_j - u_j \in W_0^{1,p}(a_j + A;\mathbb{R}^d)$ and

$$\int_{a_j+A} |w_j - u_j|^p\, dx \leqslant \int_{a_j+A} \varphi_j |u - u_j|^p\, dx \leqslant \int_{a+(1+\rho)A} |u - u_j|^p\, dx \leqslant \eta^p.$$

Then, taking $v := w_j - u_j$ as test function in (4.82), it follows from (4.67), (4.84), and (4.85) that

$$\begin{aligned}
\int_{a_j+A} f\left(x, \frac{x}{\varepsilon_j}, \nabla u_j\right) dx &< \int_{a_j+A} f_{\text{hom}}(x, \nabla w_j)\, dx - \eta \\
&\leqslant \int_{a+(1-\rho)A} f_{\text{hom}}(x, \nabla u)\, dx \\
&\qquad + \beta \int_{S_j} \left(1 + |\nabla u|^p + |\nabla u_j|^p + \frac{C}{\rho}|u-u_j|^p\right) dx - \eta \\
&\leqslant \int_{a+(1-\rho)A} f_{\text{hom}}(x, \nabla u)\, dx - \frac{\eta}{2} + \frac{\beta\, C}{\rho}\int_{S_j} |u-u_j|^p\, dx. \quad (4.86)
\end{aligned}$$

Since $u_j \to u$ in $L^p(\mathbb{R}^N;\mathbb{R}^d)$, by (4.85) and (4.86) we have

$$\limsup_{j\to\infty} \int_{a_j+A} f\left(x, \frac{x}{\varepsilon_j}, \nabla u_j\right) dx \leqslant \int_{a+(1-\rho)A} f_{\text{hom}}(x, \nabla u)\, dx - \frac{\eta}{2}, \tag{4.87}$$

and as $u_j \rightharpoonup u$ in $W^{1,p}(a+(1-\rho)A;\mathbb{R}^d)$, by Theorem 1.1 and (4.87) we get

$$\begin{aligned}
\int_{a+(1-\rho)A} f_{\text{hom}}(x, \nabla u)\, dx &\leqslant \liminf_{j\to\infty} \int_{a+(1-\rho)A} f\left(x, \frac{x}{\varepsilon_j}, \nabla u_j\right) dx \\
&\leqslant \liminf_{j\to\infty} \int_{a_j+A} f\left(x, \frac{x}{\varepsilon_j}, \nabla u_j\right) dx \\
&\leqslant \int_{a+(1-\rho)A} f_{\text{hom}}(x, \nabla u)\, dx - \frac{\eta}{2}
\end{aligned}$$

which is a contradiction.

*Step 2. (General case)* Let $a \in \mathbb{R}^N$. Then $a - [[a]] \in [-1,1]^N$. Given $u \in W^{1,p}(a+A;\mathbb{R}^d)$, set $\tilde{u}(x) := u(x+[[a]])$ and thus $\tilde{u} \in W^{1,p}(a-[[a]]+A;\mathbb{R}^d)$. Applying Step 1 with $\eta/3$, we get the existence of $0 < \varepsilon_0' \equiv \varepsilon_0'(M,\varphi,\eta)$ such that, for all $0 < \varepsilon < \varepsilon_0'$, there exist $\tilde{v} \in W_0^{1,p}(a-[[a]]+A;\mathbb{R}^d)$ satisfying $\|\tilde{v}\|_{L^p(a-[[a]]+A;\mathbb{R}^d)} \leqslant \eta/3$ and

$$\int_{a-[[a]]+A} f\left(x, \frac{x}{\varepsilon}, \nabla\tilde{u}(x)\right) dx \geqslant \int_{a-[[a]]+A} f_{\text{hom}}(x, \nabla\tilde{u}(x) + \nabla\tilde{v}(x))\, dx - \frac{\eta}{3}.$$

Setting $v(x) := \tilde{v}(x-[[a]])$, then $v \in W_0^{1,p}(a+A;\mathbb{R}^d)$ and $\|v\|_{L^p(a+A;\mathbb{R}^d)} \leqslant \frac{\eta}{3} \leqslant \eta$. Therefore, by a change of variables

$$\int_{a+A} f\left(x, \frac{x-[[a]]}{\varepsilon}, \nabla u(x)\right) dx \geqslant \int_{a+A} f_{\text{hom}}(x, \nabla u(x) + \nabla v(x))\, dx - \frac{\eta}{3}, \tag{4.88}$$

where we have used condition $(H_3)$ and (4.66). Writing

$$\frac{[[a]]}{\varepsilon} =: m_\varepsilon + r_\varepsilon, \qquad \text{with } m_\varepsilon \in \mathbb{Z}^N \text{ and } |r_\varepsilon| < \sqrt{N}\varepsilon,$$

by $(H_3)$ the inequality (4.88) reduces to

$$\int_{a+A} f\left(x, \frac{x}{\varepsilon} - r_\varepsilon, \nabla u(x)\right)\, dx \geqslant \int_{a+A} f_{\hom}(x, \nabla u(x) + \nabla v(x))\, dx - \frac{\eta}{3}. \tag{4.89}$$

Choose $\lambda > 0$ large enough (depending on $\eta$) so that

$$\beta \int_{\{|\nabla u| > \lambda\} \cap [a+A]} (1 + |\nabla u|^p)\, dx \leqslant \frac{\eta}{6}. \tag{4.90}$$

Fixed $\rho > 0$, by Scorza-Dragoni's Theorem there exists a compact set $K_\rho \subset a + A$ with $\mathcal{L}^N([a + A] \setminus K_\rho) \leqslant \rho$ such that $f : K_\rho \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ is continuous. Take $\rho \equiv \rho(\eta)$ small enough so that

$$\beta \int_{[a+A]\setminus K_\rho} (1 + |\nabla u|^p)\, dx \leqslant \frac{\eta}{6}. \tag{4.91}$$

Then, from (4.90), (4.91) and the $p$-growth condition $(H_4)$

$$\int_{a+A} f\left(x, \frac{x}{\varepsilon} - r_\varepsilon, \nabla u(x)\right)\, dx \leqslant \int_{\{|\nabla u| \leqslant \lambda\} \cap K_\rho} f\left(x, \frac{x}{\varepsilon} - r_\varepsilon, \nabla u(x)\right)\, dx + \frac{\eta}{3}. \tag{4.92}$$

But since $f$ is $Q$-periodic in its second variable, then $f$ is uniformly continuous on $K_\rho \times \mathbb{R}^N \times \overline{B}(0, \lambda)$. Thus, as $r_\varepsilon \to 0$, for any $\eta > 0$ there exists $\varepsilon_0'' \equiv \varepsilon_0''(\eta) > 0$ such that for all $\varepsilon < \varepsilon_0''$ and all $x \in \{|\nabla u| \leqslant \lambda\} \cap K_\rho$,

$$\left| f\left(x, \frac{x}{\varepsilon} - r_\varepsilon, \nabla u(x)\right) - f\left(x, \frac{x}{\varepsilon}, \nabla u(x)\right)\right| < \frac{\eta}{3\mathcal{L}^N(A)}.$$

Hence

$$\int_{\{|\nabla u| \leqslant \lambda\} \cap K_\rho} f\left(x, \frac{x}{\varepsilon} - r_\varepsilon, \nabla u(x)\right)\, dx \leqslant \int_{\{|\nabla u| \leqslant \lambda\} \cap K_\rho} f(x, \frac{x}{\varepsilon}, \nabla u)\, dx + \frac{\eta}{3} \tag{4.93}$$

and, consequently, by (4.92) and (4.93) we have

$$\int_{a+A} f\left(x, \frac{x}{\varepsilon} - r_\varepsilon, \nabla u(x)\right)\, dx \leqslant \int_{\{|\nabla u| \leqslant \lambda\} \cap K_\rho} f\left(x, \frac{x}{\varepsilon}, \nabla u(x)\right)\, dx + \frac{2\eta}{3}. \tag{4.94}$$

Thus, for all $\varepsilon < \varepsilon_0 := \min\{\varepsilon_0', \varepsilon_0''\}$, and as a result of (4.89), (4.94) and the fact that $f$ is nonnegative,

$$\int_{a+A} f\left(x, \frac{x}{\varepsilon}, \nabla u(x)\right)\, dx \geqslant \int_{a+A} f_{\hom}(x, \nabla u(x) + \nabla v(x))\, dx - \eta.$$

■

We are now in position to prove that $f_{\{\varepsilon_j\}} = \overline{f}_{\hom}$.

**Lemma 4.2.7.** *For all* $\xi \in \mathbb{R}^{d\times N}$, $\overline{f}_{\hom}(\xi) \leqslant f_{\{\varepsilon_j\}}(\xi)$.

*Proof.* By Lemma 4.2.5, given $\xi \in \mathbb{R}^{d\times N}$

$$f_{\{\varepsilon_j\}}(\xi) = \int_Q f_{\{\varepsilon_j\}}(\xi)\,dx = \mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot\,;Q). \tag{4.95}$$

Let $\{w_j\} \subset W_0^{1,p}(Q;\mathbb{R}^d)$ be a sequence such that $w_j \to 0$ in $L^p(Q;\mathbb{R}^d)$ and

$$\mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot\,;Q) = \lim_{j\to\infty}\int_Q f\left(\frac{x}{\varepsilon_j},\frac{x}{{\varepsilon_j}^2};\xi+\nabla w_j(x)\right)\,dx$$

(Proposition 2.5.13). Following the same argument as in Lemma 4.2.4, by the Decomposition Lemma, there is no loss of generality in assuming that $\{|\nabla w_j|^p\}$ is equi-integrable. Thus, from De La Vallée Poussin criterion (see Proposition 2.2.10) there exists an increasing continuous function $\varphi : [0,\infty) \to [0,\infty]$ satisfying $\varphi(t)/t \to \infty$ as $t \to \infty$ and such that

$$\sup_{j\in\mathbb{N}}\int_Q \varphi(|\nabla w_j|^p)\,dx \leqslant 1.$$

Changing variables

$$f_{\{\varepsilon_j\}}(\xi) = \lim_{j\to\infty}\frac{1}{T_j^N}\int_{(0,T_j)^N} f\left(x,\frac{x}{\varepsilon_j},\xi+\nabla z_j(x)\right)\,dx$$

and

$$\sup_{j\in\mathbb{N}}\frac{1}{T_j^N}\int_{(0,T_j)^N}\varphi(|\nabla z_j|^p)\,dx \leqslant 1, \tag{4.96}$$

where we set $T_j := 1/\varepsilon_j$ and $z_j(x) := T_j w_j(x/T_j)$ with $z_j \in W_0^{1,p}((0,T_j)^N;\mathbb{R}^d)$. For any $j \in \mathbb{N}$ define $I_j := \big\{1,...,[[T_j]]^N\big\}$, and for $i \in I_j$ take $a_i^j \in \mathbb{Z}^N$ such that

$$\bigcup_{i\in I_j}(a_i^j + Q) \subseteq (0,T_j)^N. \tag{4.97}$$

Thus

$$f_{\{\varepsilon_j\}}(\xi) \geqslant \limsup_{j\to\infty}\frac{1}{T_j^N}\sum_{i\in I_j}\int_{a_i^j+Q} f\left(x,\frac{x}{\varepsilon_j},\xi+\nabla z_j(x)\right)\,dx. \tag{4.98}$$

Let $M > 2$ and $\eta > 0$. For $j \in \mathbb{N}$ define

$$I_j^M := \left\{ i \in I_j : \int_{a_i^j+Q}\varphi(|\nabla z_j|^p)\,dx \leqslant M \right\}.$$

We note that for any $M > 2$, there exists $j(M) \in \mathbb{N}$ such that for all $j \geqslant n(M)$ sufficiently large so that $T_j > M$, $I_j^M \neq \emptyset$. In fact, otherwise we could find $M > 2$ and a subsequence $j_k \in \mathbb{N}$ satisfying

$$\int_{a_i^{j_k}+Q} \varphi(|\nabla z_{j_k}|^p) dx > M,$$

for all $i \in I_{j_k}$. Summation in $i$ and (4.97) would yield

$$\int_{(0,T_{j_k})^N} \varphi(|\nabla z_{j_k}|^p) dx > M[[T_{j_k}]]^N$$

which is in contradiction with (4.96). We also note that in view of (4.96)

$$\mathrm{Card}(I_j \setminus I_j^M)M \leqslant \sum_{i\in I_j\setminus I_j^M} \int_{a_i^j+Q} \varphi(|\nabla z_j|^p)\, dx \leqslant \int_{(0,T_j)^N} \varphi(|\nabla z_j|^p)\, dx \leqslant T_j^N,$$

and so

$$\mathrm{Card}(I_j \setminus I_j^M) \leqslant \frac{T_j^N}{M}. \tag{4.99}$$

By Lemma 4.2.6 there exists $\varepsilon_0 \equiv \varepsilon_0(M,\eta)$ such that, for any $j$ large enough satisfying $0 \leqslant \varepsilon_j < \varepsilon_0$ and for any $i \in I_j^M$, we can find $v_i^{j,M,\eta} \in W_0^{1,p}(a_i^j+Q;\mathbb{R}^d)$ with $\|v_i^{j,M,\eta}\|_{L^p(a_i^j+Q;\mathbb{R}^d)} \leqslant \eta$ and

$$\int_{a_i^j+Q} f\left(x, \frac{x}{\varepsilon_j}, \xi + \nabla z_j(x)\right)\, dx \geqslant \int_{a_i^j+Q} f_{\mathrm{hom}}\left(x; \xi + \nabla z_j + \nabla v_i^{j,M,\eta}\right)\, dx - \eta.$$

Consequently, for $j$ large enough

$$\begin{aligned}\sum_{i\in I_j} \int_{a_i^j+Q} f\left(x, \frac{x}{\varepsilon_j}, \xi + \nabla z_j(x)\right)\, dx \quad &\geqslant \quad \sum_{i\in I_j^M} \int_{a_i^j+Q} f_{\mathrm{hom}}(x, \xi + \nabla z_j + \nabla v_i^{j,M,\eta})\, dx \\ &\qquad -\eta\, \mathrm{card}(I_j^M).\end{aligned}$$

As $\mathrm{Card}(I_j^M) \leqslant [[T_j]]^N$, dividing by $T_j^N$ and passing to the limit when $j \to \infty$ we obtain

$$\begin{aligned}&\limsup_{j\to\infty} \frac{1}{T_j^N} \sum_{i\in I_j} \int_{a_i^j+Q} f\left(x, \frac{x}{\varepsilon_j}, \xi + \nabla z_j(x)\right)\, dx \\ &\qquad \geqslant \limsup_{j\to\infty} \frac{1}{T_j^N} \sum_{i\in I_j^M} \int_{a_i^j+Q} f_{\mathrm{hom}}(x, \xi + \nabla z_j + \nabla v_i^{j,M,\eta})\, dx - \eta.\end{aligned} \tag{4.100}$$

Hence, from (4.98) and (4.100),

$$f_{\{\varepsilon_j\}}(\xi) \quad \geqslant \quad \limsup_{M,\eta,j} \frac{1}{T_j^N} \sum_{i \in I_j^M} \int_{a_i^j + Q} f_{\text{hom}}(x, \xi + \nabla \phi^{j,M,\eta})\, dx \tag{4.101}$$

where $\phi^{j,M,\eta} \in W_0^{1,p}((0;T_j)^N;\mathbb{R}^d)$ is defined by

$$\phi^{j,M,\eta}(x) := \begin{cases} z_j(x) + v_i^{j,M,\eta}(x) & \text{if } x \in a_i^j + Q \text{ and } i \in I_j^M, \\ \\ z_j(x) & \text{otherwise.} \end{cases}$$

Now, in view of the definition of $\phi^{j,M,\eta}$, the $p$-growth condition (4.67) and (4.99),

$$\begin{aligned} &\frac{1}{T_j^N} \sum_{i \in I_j \setminus I_j^M} \int_{a_i^j + Q} f_{\text{hom}}(x; \xi + \nabla \phi^{j,M,\eta})\, dx \\ &\qquad = \frac{1}{T_j^N} \sum_{i \in I_j \setminus I_j^M} \int_{a_i^j + Q} f_{\text{hom}}(x; \xi + \nabla z_j)\, dx \\ &\qquad \leqslant \frac{\beta}{T_j^N} \sum_{i \in I_j \setminus I_j^M} \int_{a_i^j + Q} (1 + |\nabla z_j|^p)\, dx \\ &\qquad \leqslant \frac{\beta}{M} + \frac{\beta}{T_j^N} \sum_{i \in I_j \setminus I_j^M} \int_{a_i^j + Q} |\nabla z_j|^p\, dx \\ &\qquad \leqslant \frac{\beta}{M} + \beta \int_{\bigcup_{i \in I_j \setminus I_j^M} \frac{1}{T_j}(a_i^j + Q)} |\nabla w_j|^p\, dx. \end{aligned} \tag{4.102}$$

By (4.99)

$$\mathcal{L}^N \left( \bigcup_{i \in I_j \setminus I_j^M} \frac{1}{T_j}(a_i^j + Q) \right) \leqslant \frac{1}{M}.$$

Consequently, in view of the equi-integrability of $\{|\nabla w_j|^p\}$ and (4.102), we get

$$\limsup_{M,\eta,j} \frac{1}{T_j^N} \sum_{i \in I_j \setminus I_j^M} \int_{a_i^j + Q} f_{\text{hom}}(x; \xi + \nabla \phi^{j,M,\eta})\, dx = 0. \tag{4.103}$$

Therefore, (4.101) and (4.103) imply

$$
\begin{aligned}
f_{\{\varepsilon_j\}}(\xi) &\geqslant \limsup_{M,\eta,j} \frac{1}{T_j^N} \sum_{i\in I_j} \int_{a_i^j+Q} f_{\mathrm{hom}}(x,\xi+\nabla\phi^{j,M,\eta})\,dx \\
&= \limsup_{M,\eta,j} \frac{1}{T_j^N} \int_{(0,T_j)^N} f_{\mathrm{hom}}(x,\xi+\nabla\phi^{j,M,\eta})\,dx,
\end{aligned} \tag{4.104}
$$

because by definition of $\phi^{j,M,\eta}$ and the $p$-growth property of $f_{\mathrm{hom}}$, (4.67),

$$
\begin{aligned}
&\frac{1}{T_j^N} \int_{\left(0,T_j^N\right)\setminus\left[\bigcup_{i\in I_j}(a_i^j+Q)\right]} f_{\mathrm{hom}}(x,\xi+\nabla\phi^{j,M,\eta})\,dx \\
&\qquad= \frac{1}{T_j^N} \int_{\left(0,T_j^N\right)\setminus\left[\bigcup_{i\in I_j}(a_i^j+Q)\right]} f_{\mathrm{hom}}(x,\xi+\nabla z_j)\,dx \\
&\qquad\leqslant \frac{\beta}{T_j^N} \int_{\left(0,T_j^N\right)\setminus\left[\bigcup_{i\in I_j}(a_i^j+Q)\right]} (1+|\nabla z_j|^p)\,dx \\
&\qquad= \beta \int_{Q\setminus\left[\bigcup_{i\in I_j}\frac{1}{T_j}(a_i^j+Q)\right]} (1+|\nabla w_j|^p)\,dx.
\end{aligned}
$$

and consequently, the equi-integrability of $\{|\nabla w_j|^p\}$ and the fact that

$$
\mathcal{L}^N\left(Q\setminus\left[\bigcup_{i\in I_j}\frac{1}{T_j}(a_i^j+Q)\right]\right)\to 0
$$

as $j\to\infty$ yield

$$
\limsup_{M,\eta,j} \frac{1}{T_j^N} \int_{\left(0,T_j^N\right)\setminus\left[\bigcup_{i\in I_j}(a_i^j+Q)\right]} f_{\mathrm{hom}}(x,\xi+\nabla\phi^{j,M,\eta})\,dx = 0.
$$

Hence by (4.104) and (4.70) we get that

$$
f_{\{\varepsilon_j\}}(\xi) \geqslant \overline{f}_{\mathrm{hom}}(\xi).
$$

■

Let us now prove the converse inequality.

**Lemma 4.2.8.** *For all $\xi\in\mathbb{R}^{d\times N}$, $\overline{f}_{\mathrm{hom}}(\xi)\geqslant f_{\{\varepsilon_j\}}(\xi)$.*

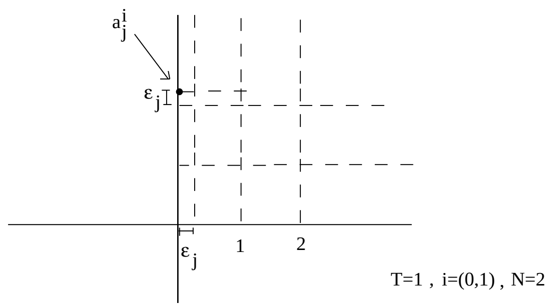
*Proof.* In view of (4.70), for $\delta > 0$ fixed take $T \equiv T_\delta \in \mathbb{N}$, with $T_\delta \to \infty$ as $\delta \to 0$, and let $\phi \equiv \phi_\delta \in W_0^{1,p}((0,T)^N; \mathbb{R}^d)$ be such that

$$\overline{f}_{\text{hom}}(\xi) + \delta \geqslant \frac{1}{T^N} \int_{(0,T)^N} f_{\text{hom}}(x; \xi + \nabla\phi(x))\, dx. \tag{4.105}$$

By Theorem 1.1 and Proposition 2.5.13, there exists a sequence $\{\phi_j\} \subset W_0^{1,p}((0,T)^N; \mathbb{R}^d)$ with $\phi_j \to \phi$ in $L^p((0,T)^N; \mathbb{R}^d)$ such that

$$\int_{(0,T)^N} f_{\text{hom}}(x; \xi + \nabla\phi(x))\, dx = \lim_{j\to\infty} \int_{(0,T)^N} f\left(x, \frac{x}{\varepsilon_j}, \xi + \nabla\phi_j(x)\right) dx. \tag{4.106}$$

Further, in view of the Decomposition Lemma, we can assume – upon extracting a subsequence still denoted by $\{\phi_j\}$ – $\{|\nabla\phi_j|^p\}$ to be equi-integrable. Fix $j \in \mathbb{N}$ such that $\varepsilon_j \ll 1$. For all $i \in \mathbb{Z}^N$ let $a_i^j \in \varepsilon_j\mathbb{Z}^N \cap [i(T+1), \varepsilon_j)^N$ (uniquely defined).




In particular, the cubes $Q(a_i^j, T)$ are not overlapping because if $i, k \in \mathbb{Z}^N$ with $i \neq k$, then $|i - k| \geqslant 1$ and thus $|a_i^j - a_j^k| > T$. Set

$$\tilde{\phi}_j(x) := \begin{cases} \phi_j(x - a_i^j) & \text{if } x \in Q(a_i^j, T) \text{ and } i \in \mathbb{Z}^N, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Then $\tilde{\phi}_j \in W^{1,p}(\mathbb{R}^N; \mathbb{R}^d)$. Let $I_j := \{i \in \mathbb{Z}^N : (0, T/\varepsilon_j)^N \cap Q(a_i^j, T) \neq \emptyset\}$. Note that

$$\text{Card}(I_j) \leqslant \left( \left[\left[ \frac{1}{\varepsilon_j} \right]\right] + 1 \right)^N. \tag{4.107}$$

If $\psi_j(x) := \varepsilon_j \tilde{\phi}_j(x/\varepsilon_j)$, then $\psi_j \to 0$ in $L^p((0,T)^N;\mathbb{R}^d)$, as $j \to \infty$, because

$$\begin{aligned}
\int_{(0,T)^N} |\psi_j(x)|^p dx &= \varepsilon_j^p \int_{(0,T)^N} \left|\tilde{\phi}_j\left(\frac{x}{\varepsilon_j}\right)\right|^p dx \\
&= \varepsilon_j^{p+N} \int_{(0,T/\varepsilon_j)^N} |\tilde{\phi}_j(x)|^p dx \\
&\leqslant \varepsilon_j^{p+N} \sum_{i\in I_j} \int_{Q(a_i^j,T)} |\phi_j(x-a_i^j)|^p dx \\
&= \varepsilon_j^{p+N} \mathrm{Card}(I_j) \int_{(0,T)^N} |\phi_j(x)|^p dx \\
&\leqslant \varepsilon_j^{p+N} \left(\left[\left[\frac{1}{\varepsilon_j}\right]\right]+1\right)^N \int_{(0,T)^N} |\phi_j(x)|^p dx \to 0,
\end{aligned}$$

where we have used the fact that $\tilde{\phi}_j \equiv 0$ on $(0,T/\varepsilon_j)^N \setminus \bigcup_{i\in I_j} Q(a_i^j,T)$ and that

$$\sup_{j\in\mathbb{N}} \|\phi_j\|_{L^p((0,T)^N;\mathbb{R}^d)} < \infty$$

by the Poincaré Inequality and (4.106). Consequently,

$$\begin{aligned}
\mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot\,;(0,T)^N) &\leqslant \liminf_{j\to\infty} \int_{(0,T)^N} f\left(\frac{x}{\varepsilon_j},\frac{x}{\varepsilon_j^2},\xi+\nabla\psi_j(x)\right) dx \\
&= \liminf_{j\to\infty} \int_{(0,T)^N} f\left(\frac{x}{\varepsilon_j},\frac{x}{\varepsilon_j^2},\xi+\nabla\tilde{\phi}_j\left(\frac{x}{\varepsilon_j}\right)\right) dx \\
&= \liminf_{j\to\infty} \varepsilon_j^N \int_{(0,T/\varepsilon_j)^N} f\left(x,\frac{x}{\varepsilon_j};\xi+\nabla\tilde{\phi}_j(x)\right) dx. \qquad (4.108)
\end{aligned}$$

Note that

$$\varepsilon_j^N \mathcal{L}^N\left(\left(0,\frac{T}{\varepsilon_j}\right)^N \setminus \bigcup_{i\in I_j} Q(a_i^j,T)\right) = \mathcal{L}^N\left((0,T)^N \setminus \bigcup_{i\in I_j} Q(a_i^j,\varepsilon_j T)\right)$$

Thus

$$\begin{aligned}
&\varepsilon_j^N \mathcal{L}^N \left( \left(0, \frac{T}{\varepsilon_j}\right)^N \setminus \bigcup_{i \in I_j} Q(a_i^j, T) \right) \\
\leqslant\ & \mathcal{L}^N \left( (0,T)^N \setminus \bigcup_{i \in I_j : Q(a_i^j, \varepsilon_j T) \subset (0,T)^N} Q(a_i^j, \varepsilon_j T) \right) \\
=\ & T^N - \mathrm{Card}\left( \left\{ i \in I_j : Q(a_i^j, \varepsilon_j T) \subset (0,T)^N \right\} \right) \varepsilon_j^N T^N \\
\leqslant\ & T^N \left( 1 - \varepsilon_j^N \left[\left[ \frac{T}{\varepsilon_j (T+1)} \right]\right]^N \right).
\end{aligned}$$

Hence, from inequality (4.108) and the $p$-growth condition $(H_4)$ it follows that

$$\begin{aligned}
\mathcal{I}_{\{\varepsilon_j\}}(\xi \cdot \,; (0,T)^N) \ \leqslant\ & \liminf_{j \to \infty} \varepsilon_j^N \left[ \sum_{i \in I_j} \int_{Q(a_i^j, T)} f\left(x, \frac{x}{\varepsilon_j}, \xi + \nabla \tilde{\phi}_j(x)\right) dx \right. \\
& \left. + \beta(1 + |\xi|^p) \mathcal{L}^N \left( \left(0, \frac{T}{\varepsilon_j}\right)^N \setminus \bigcup_{i \in I_j} Q(a_i^j, T) \right) \right] \\
\leqslant\ & \liminf_{j \to \infty} \varepsilon_j^N \sum_{i \in I_j} \int_{Q(a_i^j, T)} f\left(x, \frac{x}{\varepsilon_j}, \xi + \nabla \tilde{\phi}_j(x)\right) dx \\
& + \beta(1 + |\xi|^p) T^N \left( 1 - \left( \frac{T}{T+1} \right)^N \right). \qquad (4.109)
\end{aligned}$$

By a change of variables, for all $i \in I_j$

$$\begin{aligned}
&\int_{Q(a_i^j, T)} f\left(x, \frac{x}{\varepsilon_j}, \xi + \nabla \tilde{\phi}_j(x)\right) dx \\
&\qquad = \int_{(0,T)^N} f\left(x + a_i^j, \frac{x + a_i^j}{\varepsilon_j}, \xi + \nabla \phi_j(x)\right) dx \\
&\qquad = \int_{(0,T)^N} f\left(x + a_i^j - i(T+1), \frac{x}{\varepsilon_j}, \xi + \nabla \phi_j(x)\right) dx, \quad (4.110)
\end{aligned}$$

where we have used $(H_3)$, the fact that $T \in \mathbb{N}$ and $a_i^j / \varepsilon_j \in \mathbb{Z}^N$. In order to apply a uniform continuity argument and recover (4.106) we define

$$R_j^\lambda := \{ x \in (0,T)^N : |\xi + \nabla \phi_j(x)| \leqslant \lambda \}$$

and we observe that, according to Chebyshev's inequality,

$$\mathcal{L}^N((0,T)^N \setminus R_j^\lambda) \leqslant C/\lambda^p,$$

for some constant $C > 0$ which does not depend on $j$ and $\lambda$. Then by (4.109) and (4.110)

$$\begin{aligned}
&\mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot\,;(0,T)^N)\\
&\qquad\leqslant \liminf_{\lambda\to\infty}\liminf_{j\to\infty}\sum_{i\in I_j}\varepsilon_j^N\int_{R_j^\lambda} f\left(x+a_i^j-i(T+1),\frac{x}{\varepsilon_j},\xi+\nabla\phi_j(x)\right)dx\\
&\qquad\quad+\beta(1+|\xi|^p)T^N\left(1-\left(\frac{T}{T+1}\right)^N\right).
\end{aligned}\tag{4.111}$$

Indeed, the $p$-growth condition $(H_4)$ and the equi-integrability of $\{|\nabla\phi_j|^p\}_j$ imply that

$$\begin{aligned}
&\limsup_{\lambda\to\infty}\limsup_{j\to\infty}\sum_{i\in I_j}\varepsilon_j^N\int_{(0,T)^N\setminus R_j^\lambda} f\left(x+a_i^j-i(T+1),\frac{x}{\varepsilon_j},\xi+\nabla\phi_j(x)\right)dx\\
&\qquad\leqslant \limsup_{\lambda\to\infty}\limsup_{j\to\infty}\beta\,\varepsilon_j^N\,\mathrm{Card}(I_j)\int_{(0,T)^N\setminus R_j^\lambda}(1+|\xi+\nabla\phi_j(x)|^p)dx\\
&\qquad\leqslant \limsup_{\lambda\to\infty}\left\{C_1\mathcal{L}^N((0,T)^N\setminus R_j^\lambda)+C_2\sup_{j\in\mathbb{N}}\int_{(0,T)^N\setminus R_j^\lambda}|\nabla\phi_j(x)|^p dx\right\}=0.
\end{aligned}$$

As $f$ is continuous and separately periodic in its two first variables, $f$ is uniformly continuous on $\mathbb{R}^N\times\mathbb{R}^N\times\overline{B}(0,\lambda)$ for any $\lambda>0$. Let $\omega_\lambda:[0,\infty)\to[0,\infty)$ the modulus of continuity of $f$ on $\mathbb{R}^N\times\mathbb{R}^N\times\overline{B}(0,\lambda)$. Then, for all $x\in R_j^\lambda$,

$$\begin{aligned}
&\left|f\left(x,\frac{x}{\varepsilon_j},\xi+\nabla\phi_j(x)\right)-f\left(x+a_i^j-i(T+1),\frac{x}{\varepsilon_j},\xi+\nabla\phi_j(x)\right)\right|\\
&\qquad\qquad\leqslant\omega_\lambda(|a_i^j-i(T+1)|)\leqslant\omega_\lambda(\varepsilon_j).
\end{aligned}\tag{4.112}$$

In view of (4.112), (4.107), and since $\omega_\lambda$ is continuous and satisfies $\omega_\lambda(0)=0$, we get

$$\begin{aligned}
&\liminf_{\lambda\to\infty}\liminf_{j\to\infty}\varepsilon_j^N\,\mathrm{Card}(I_j)\left\{\int_{R_j^\lambda} f\left(x,\frac{x}{\varepsilon_j},\xi+\nabla\phi_j(x)\right)dx+T^N\omega_\lambda(\varepsilon_j)\right\}\\
&\qquad\leqslant\liminf_{\lambda\to\infty}\liminf_{j\to\infty}(1+\varepsilon_j^N)\left\{\int_{R_j^\lambda} f\left(x,\frac{x}{\varepsilon_j},\xi+\nabla\phi_j(x)\right)dx+T^N\omega_\lambda(\varepsilon_j)\right\}\\
&\qquad\leqslant\liminf_{\lambda\to\infty}\liminf_{j\to\infty}\int_{R_j^\lambda} f\left(x,\frac{x}{\varepsilon_j},\xi+\nabla\phi_j(x)\right)dx\\
&\qquad\leqslant\liminf_{j\to\infty}\int_{(0,T)^N} f\left(x,\frac{x}{\varepsilon_j},\xi+\nabla\phi_j(x)\right)dx.
\end{aligned}$$

Consequently by (4.105), (4.106), (4.111) and Lemma 4.2.5,

$$f_{\{\varepsilon_j\}}(\xi) \leqslant \overline{f}_{\mathrm{hom}}(\xi) + \delta + \beta(1+|\xi|^p)\left(1 - \left(\frac{T}{T+1}\right)^N\right).$$

The result follows by letting $\delta$ tend to zero. ■

*Proof of Theorem 4.2.2.* From Lemma 4.2.7 and Lemma 4.2.8, we conclude that $\overline{f}_{\mathrm{hom}}(\xi) = f_{\{\varepsilon_j\}}(\xi)$ for all $\xi \in \mathbb{R}^{d\times N}$. As a consequence, $\mathcal{I}_{\{\varepsilon_j\}}(u;A) = \mathcal{I}_{\mathrm{hom}}(u;A)$ for all $A \in \mathcal{A}_0(\mathbb{R}^N)$ and all $u \in W^{1,p}(A;\mathbb{R}^d)$. Since the Γ-limit does not depend upon the extracted subsequence, Remark 2.5.9 implies that the whole sequence $\mathcal{I}_\varepsilon(\cdot\,;A)$ $\Gamma(L^p(A))$-converges to $\mathcal{I}_{\mathrm{hom}}(\cdot\,;A)$. In particular,

$$\Gamma(L^p(A))\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u;A) = \mathcal{I}_{\mathrm{hom}}(u;A) := \int_A \overline{f}_{\mathrm{hom}}(\nabla u)\,dx.$$

### 4.2.3 The general case

Our objective now is to prove Theorem 4.2.1.

*STEP 1.* Existence and integral representation of the Γ-limit.

The idea in this case is to freeze the macroscopic variable and to use Theorem 4.2.2 through a blow up argument. This leads us to work on small cubes centered at convenient Lebesgue points of $\Omega$ which, contrary to the homogeneous case, allow us to localize our functionals on $\mathcal{A}(\Omega)$.

We define $\mathcal{I}_\varepsilon : L^p(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,\infty]$ by

$$\mathcal{I}_\varepsilon(u;A) := \begin{cases} \displaystyle\int_A f\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \nabla u(x)\right) dx & \text{if } u \in W^{1,p}(A;\mathbb{R}^d), \\ \infty & \text{otherwise,} \end{cases}$$

and we introduce the functional $\mathcal{I}_{\mathrm{hom}} : L^p(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,\infty]$

$$\mathcal{I}_{\mathrm{hom}}(u;A) := \begin{cases} \displaystyle\int_A \overline{f}_{\mathrm{hom}}(x, \nabla u(x))\,dx & \text{if } u \in W^{1,p}(A;\mathbb{R}^d), \\ \infty & \text{otherwise.} \end{cases}$$

Given $\varepsilon_n \downarrow 0$ and $A \in \mathcal{A}(\Omega)$, consider the Γ-lower limit of $\{\mathcal{I}_{\varepsilon_n}(\cdot\,; A)\}_n$ for the $L^p(A;\mathbb{R}^d)$-topology defined, for $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, by

$$\mathcal{I}_{\{\varepsilon_n\}}(u;A) := \inf\left\{\liminf_{j\to\infty} \mathcal{I}_{\varepsilon_n}(u_j;A) : \quad u_n \to u \text{ in } L^p(A;\mathbb{R}^d)\right\}.$$

Due to the $p$-coercivity condition in $(H_4)$, to prove Theorem 4.2.1 it suffices to show that for all $u \in W^{1,p}(\Omega;\mathbb{R}^d)$

$$\Gamma(L^p(\Omega))\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u) = \int_\Omega \overline{f}_{\mathrm{hom}}(x, \nabla u(x))\, dx.$$

As in Section 4.1 there exists a subsequence $\{\varepsilon_j\}_j \equiv \{\varepsilon_{n_j}\}_j$ such that for any $A \in \mathcal{A}(\Omega)$, $\mathcal{I}_{\{\varepsilon_j\}}(\cdot\,;A)$ is the $\Gamma(L^p(A))$-limit of $\mathcal{I}_{\varepsilon_j}(\cdot\,;A)$ and, for all $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, the set function $\mathcal{I}_{\{\varepsilon_j\}}(u;\cdot)$ is the restriction of a Radon measure to $\mathcal{A}(\Omega)$. Furthermore, from the integral representation Theorem 2.4.1, we have

**Lemma 4.2.9.** *There exists a Carathéodory function $f_{\{\varepsilon_j\}} : \Omega \times \mathbb{R}^{d\times N} \to \mathbb{R}$, quasiconvex in its second variable, satisfying the same coercivity and growth conditions as $f$, such that*

$$\mathcal{I}_{\{\varepsilon_j\}}(u;A) = \int_A f_{\{\varepsilon_j\}}(x, \nabla u(x))\, dx$$

*for every $A \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(\Omega;\mathbb{R}^d)$. Moreover, for all $\xi \in \mathbb{R}^{d\times N}$ and a.e. $x \in \Omega$,*

$$f_{\{\varepsilon_j\}}(x,\xi) = \lim_{\delta\to 0} \frac{\mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot\,; Q(x,\delta))}{\delta^N}.$$

*STEP 2.* Characterization of the Γ-limit.

As before we only need to prove that $f_{\{\varepsilon_j\}}(x,\xi) = \overline{f}_{\mathrm{hom}}(x,\xi)$ for a.e. $x$ and all $\xi$. For this purpose let $L$ be the set of Lebesgue points $x_0$ for all functions $f_{\{\varepsilon_j\}}(\cdot\,,\xi)$ and $\overline{f}_{\mathrm{hom}}(\cdot\,,\xi)$, for all $\xi \in \mathbb{Q}^{d\times N}$. We have $\mathcal{L}^N(\Omega \setminus L) = 0$ and we will first show in Lemma 4.2.10 and 4.2.11 below that the equality $f_{\{\varepsilon_j\}}(x,\xi) = \overline{f}_{\mathrm{hom}}(x,\xi)$ holds for all $x \in L$ and all $\xi \in \mathbb{Q}^{d\times N}$. By definition of the set $L$ it is enough to show that

$$\int_{Q(x_0,\delta)} f_{\{\varepsilon_j\}}(x,\xi)\, dx = \int_{Q(x_0,\delta)} \overline{f}_{\mathrm{hom}}(x,\xi)\, dx, \tag{4.113}$$

for every $x_0 \in L$ and each $\delta > 0$ small enough so that $Q(x_0,\delta) \in \mathcal{A}(\Omega)$.

**Lemma 4.2.10.** *For all $\xi \in \mathbb{Q}^{d\times N}$ and all $x_0 \in L$,*

$$\int_{Q(x_0,\delta)} f_{\{\varepsilon_j\}}(x,\xi)\,dx \geqslant \int_{Q(x_0,\delta)} \overline{f}_{\mathrm{hom}}(x,\xi)\,dx.$$

*Proof.* Let $\xi \in \mathbb{Q}^{d\times N}$ and $x_0 \in L$. From Lemma 4.2.9, we have

$$\int_{Q(x_0,\delta)} f_{\{\varepsilon_j\}}(x,\xi)\,dx = \mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot\,; Q(x_0;\delta)).$$

Let $\{u_j\} \subset W^{1,p}(Q(x_0,\delta);\mathbb{R}^d)$ be a recovery sequence for $\mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot\,;Q(x_0;\delta))$, that is, a sequence $\{u_j\}$ such that $u_j \to 0$ in $L^p(Q(x_0,\delta);\mathbb{R}^d)$ and

$$\mathcal{I}_{\{\varepsilon_j\}}(\xi\cdot\,;Q(x_0;\delta)) = \lim_{j\to\infty} \int_{Q(x_0,\delta)} f\left(x, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j(x)\right) dx.$$

As before, the Decomposition Lemma lets us assume that $\{|\nabla u_j|^p\}$ is equi-integrable. We split $Q(x_0,\delta)$ into $h^N$ small disjoint cubes $Q_{i,h}$ such that

$$Q(x_0,\delta) = \bigcup_{i=1}^{h^N} Q_{i,h} \quad \text{and} \quad \mathcal{L}^N(Q_{i,h}) = (\delta/h)^N. \tag{4.114}$$

Then

$$\int_{Q(x_0,\delta)} f_{\{\varepsilon_j\}}(x,\xi)\,dx = \lim_{h\to\infty}\lim_{j\to\infty}\sum_{i=1}^{h^N}\int_{Q_{i,h}} f\left(x, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j(x)\right) dx.$$

Let $\eta > 0$. By Scorza-Dragoni's Theorem, there exists a compact set $K_\eta \subset \Omega$ such that

$$\mathcal{L}^N(\Omega \setminus K_\eta) < \eta, \tag{4.115}$$

and the restriction of $f$ to $K_\eta \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$ is a continuous function. Given $\lambda > 0$, we introduce

$$R_j^\lambda := \{x \in \Omega : |\xi + \nabla u_j(x)| \leqslant \lambda\},$$

for all $j \in \mathbb{N}$ and we note that due to Chebyshev's inequality, we have

$$\mathcal{L}^N(\Omega \setminus R_j^\lambda) \leqslant \frac{C}{\lambda^p}, \tag{4.116}$$

for some constant $C > 0$ independent of $j$ and $\lambda$. Then

$$\int_{Q(x_0,\delta)} f_{\{\varepsilon_j\}}(x,\xi)\,dx \geqslant \limsup_{\lambda,\eta,h,j}\sum_{i=1}^{h^N}\int_{Q_{i,h}\cap K_\eta\cap R_j^\lambda} f\left(x, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j(x)\right) dx.$$

In view of condition $(H_3)$, $f$ is uniformly continuous on $K_\eta \times \mathbb{R}^N \times \mathbb{R}^N \times \overline{B}(0,\lambda)$. Denoting by $\omega_{\eta,\lambda} : [0,\infty) \to [0,\infty)$ the modulus of continuity of $f$ on $K_\eta \times \mathbb{R}^N \times \mathbb{R}^N \times \overline{B}(0,\lambda)$, for every $(x,x') \in [Q_{i,h} \cap K_\eta \cap R_j^\lambda] \times [Q_{i,h} \cap K_\eta]$,

$$\begin{aligned}\left| f\left(x, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j(x)\right) - f\left(x', \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j(x)\right)\right| &\leqslant \omega_{\eta,\lambda}(|x-x'|) \\ &\leqslant \omega_{\eta,\lambda}\left(\frac{\sqrt{N}\delta}{h}\right). \end{aligned} \tag{4.117}$$

From (4.114) and (4.117), after integrating in $(x,x')$ over $[Q_{i,h} \cap K_\eta \cap R_j^\lambda] \times [Q_{i,h} \cap K_\eta]$, we get, since $\omega_{\eta,\lambda}$ is continuous and satisfies $\omega_{\eta,\lambda}(0) = 0$,

$$\begin{aligned}\sum_{i=1}^{h^N} \frac{h^N}{\delta^N} \int_{Q_{i,h}\cap K_\eta} \Bigg\{ \int_{Q_{i,h}\cap K_\eta \cap R_j^\lambda} \Bigg| f\left(x, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j(x)\right) & \\ -f\left(x', \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j(x)\right)\Bigg| \, dx \Bigg\} \, dx' & \\ \leqslant \delta^N \omega_{\eta,\lambda}\left(\frac{\sqrt{N}\delta}{h}\right) \xrightarrow[h\to\infty]{} 0, &\end{aligned}$$

uniformly in $j \in \mathbb{N}$, for all $\eta > 0$ and $\lambda > 0$. Hence, by Fubini's Theorem

$$\begin{aligned}&\int_{Q(x_0,\delta)} f_{\{\varepsilon_j\}}(x,\xi)\, dx \\ &\geqslant \limsup_{\lambda,\eta,h,j} \frac{h^N}{\delta^N} \sum_{i=1}^{h^N} \int_{Q_{i,h}\cap K_\eta \cap R_j^\lambda} \left\{ \int_{Q_{i,h}\cap K_\eta} f\left(x', \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j(x)\right) dx' \right\} dx. \end{aligned} \tag{4.118}$$

However, as a consequence of $(H_4)$ and (4.115) we have that for all $\lambda > 0$,

$$\begin{aligned}&\frac{h^N}{\delta^N} \sum_{i=1}^{h^N} \int_{Q_{i,h}\cap K_\eta \cap R_j^\lambda} \left\{ \int_{Q_{i,h}\setminus K_\eta} f\left(x', \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j(x)\right) dx' \right\} dx \\ &\qquad \leqslant \beta \frac{h^N}{\delta^N} \sum_{i=1}^{h^N} \frac{\delta^N}{h^N} (1+\lambda^p) \mathcal{L}^N(Q_{i,h}\setminus K_\eta) \\ &\qquad \leqslant \beta(1+\lambda^p)\mathcal{L}^N(\Omega \setminus K_\eta) \\ &\qquad \leqslant \beta(1+\lambda^p)\eta \xrightarrow[\eta\to 0]{} 0 \end{aligned} \tag{4.119}$$

uniformly in $j \in \mathbb{N}$ and $h \in \mathbb{N}$, and similarly

$$\frac{h^N}{\delta^N} \sum_{i=1}^{h^N} \int_{Q_{i,h}\cap R_j^\lambda \setminus K_\eta} \left\{ \int_{Q_{i,h}} f\left(x', \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j(x)\right) dx' \right\} dx \leqslant \beta(1+\lambda^p)\eta \xrightarrow[\eta\to 0]{} 0, \tag{4.120}$$

uniformly in $j \in \mathbb{N}$ and $h \in \mathbb{N}$. Moreover, (4.114) and (4.116), together with the equi-integrability of $\{|\nabla u_j|^p\}$, yield

$$\begin{aligned}\sup_{j,h,\eta} \frac{h^N}{\delta^N} \sum_{i=1}^{h^N} \int_{Q_{i,h}\setminus R_j^\lambda} \left\{ \int_{Q_{i,h}} f\left(x', \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j(x)\right) dx' \right\} dx \\ \leqslant \sup_{j\in\mathbb{N}} \beta \int_{Q(x_0,\delta)\setminus R_j^\lambda} (1 + |\nabla u_j(x)|^p)\, dx \xrightarrow[\lambda\to\infty]{} 0.\end{aligned} \tag{4.121}$$

Finally, (4.118)-(4.121) and Fubini's Theorem lead to

$$\begin{aligned}&\int_{Q(x_0,\delta)} f_{\{\varepsilon_j\}}(x,\xi)\, dx \\ &\geqslant \limsup_{h\to\infty} \limsup_{j\to\infty} \frac{h^N}{\delta^N} \sum_{i=1}^{h^N} \int_{Q_{i,h}} \left\{ \int_{Q_{i,h}} f\left(x', \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j(x)\right) dx \right\} dx' \\ &\geqslant \limsup_{h\to\infty} \frac{h^N}{\delta^N} \sum_{i=1}^{h^N} \int_{Q_{i,h}} \left\{ \liminf_{j\to\infty} \int_{Q_{i,h}} f\left(x', \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j(x)\right) dx \right\} dx',\end{aligned} \tag{4.122}$$

where we have used Fatou's Lemma. Fix $x' \in Q_{i,h}$ such that $\overline{f}_{\mathrm{hom}}(x',\xi)$ is well defined and apply Theorem 4.2.2 to the continuous function $(y,z,\xi) \mapsto f(x',y,z,\xi)$. Since $u_j \to 0$ in $L^p(Q(x_0,\delta);\mathbb{R}^d)$, we can use the Γ-lim inf inequality to get

$$\liminf_{j\to\infty} \int_{Q_{i,h}} f\left(x', \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j(x)\right) dx \geqslant \frac{\delta^N}{h^N} \overline{f}_{\mathrm{hom}}(x',\xi).$$

Then, in view of (4.122) we conclude (4.113), that is,

$$\int_{Q(x_0,\delta)} f_{\{\varepsilon_j\}}(x,\xi)\, dx \geqslant \int_{Q(x_0,\delta)} \overline{f}_{\mathrm{hom}}(x,\xi)\, dx.$$

■

**Lemma 4.2.11.** *For all $\xi \in \mathbb{Q}^{d\times N}$ and all $x_0 \in L$,*

$$\int_{Q(x_0,\delta)} f_{\{\varepsilon_j\}}(x,\xi)\, dx \leqslant \int_{Q(x_0,\delta)} \overline{f}_{\mathrm{hom}}(x,\xi)\, dx.$$

*Proof.* As in Lemma 4.2.10, we decompose $Q(x_0,\delta)$ into $h^N$ small disjoints cubes $Q_{i,h}$ such that

$$Q(x_0,\delta) = \bigcup_{i=1}^{h^N} Q_{i,h} \quad \text{and} \quad \mathcal{L}^N(Q_{i,h}) = (\delta/h)^N.$$

Since $f$ and $\overline{f}_{\text{hom}}$ are Carathéodory functions, by Scorza-Dragoni's Theorem for each $\eta > 0$, we can find a compact set $K_\eta \subset Q(x_0, \delta)$ such that

$$\mathcal{L}^N(Q(x_0,\delta) \setminus K_\eta) < \eta, \tag{4.123}$$

$f$ is continuous on $K_\eta \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and $\overline{f}_{\text{hom}}$ is continuous on $K_\eta \times \mathbb{R}^{d\times N}$. Let

$$I_{h,\eta} := \left\{ i \in \{1, \cdots, h^N\} : \quad K_\eta \cap Q_{i,h} \neq \emptyset \right\}.$$

For $i \in I_{h,\eta}$, choose $x_i^{h,\eta} \in K_\eta \cap Q_{i,h}$. Theorem 4.2.2, together with e.g. Theorem 21.1 in Dal Maso [19], implies the existence of a sequence $\{u_i^{j,h,\eta}\} \subset W_0^{1,p}(Q_{i,h};\mathbb{R}^d)$ such that $u_i^{j,h,\eta} \to 0$ in $L^p(Q_{i,h};\mathbb{R}^d)$ as $j \to \infty$ and

$$\int_{Q_{i,h}} \overline{f}_{\text{hom}}(x_i^{h,\eta}, \xi)\, dx = \left(\frac{\delta}{h}\right)^N \overline{f}_{\text{hom}}(x_i^{h,\eta}, \xi) = \lim_{j\to\infty} \int_{Q_{i,h}} f\left(x_i^{h,\eta}, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_i^{j,h,\eta}\right) dx.$$

Set

$$u_j^\eta(x) := \begin{cases} u_i^{j,h,\eta}(x) & \text{if } x \in Q_{i,h} \text{ and } i \in I_{h,\eta}, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{4.124}$$

Then $\{u_j^\eta\} \subset W_0^{1,p}(Q(x_0,\delta);\mathbb{R}^d)$, $u_j^\eta \to 0$ in $L^p(Q(x_0,\delta);\mathbb{R}^d)$ as $j \to \infty$ and

$$\begin{aligned} &\liminf_{h\to\infty} \sum_{i\in I_{h,\eta}} \int_{Q_{i,h}} \overline{f}_{\text{hom}}(x_i^{h,\eta}, \xi)\, dx \\ &\qquad = \liminf_{h\to\infty} \lim_{j\to\infty} \sum_{i\in I_{h,\eta}} \int_{Q_{i,h}} f\left(x_i^{h,\eta}, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j^\eta\right) dx. \end{aligned} \tag{4.125}$$

In view of (4.68) and (4.123) we have

$$\sup_{h\in\mathbb{N}} \sum_{i\in I_{h,\eta}} \int_{Q_{i,h}\setminus K_\eta} \overline{f}_{\text{hom}}(x_i^{h,\eta}, \xi)\, dx \leqslant \beta(1+|\xi|^p)\mathcal{L}^N(Q(x_0,\delta)\setminus K_\eta) \xrightarrow[\eta\to 0]{} 0, \tag{4.126}$$

thus from (4.125) and (4.126),

$$\begin{aligned} &\liminf_{\eta\to 0} \liminf_{h\to\infty} \sum_{i\in I_{h,\eta}} \int_{Q_{i,h}\cap K_\eta} \overline{f}_{\text{hom}}(x_i^{h,\eta}, \xi)\, dx \\ &\qquad \geqslant \liminf_{\eta,h,j} \sum_{i\in I_{h,\eta}} \int_{Q_{i,h}\cap K_\eta} f\left(x_i^{h,\eta}, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j^\eta\right) dx. \end{aligned} \tag{4.127}$$

Since $\overline{f}_{\hom}(\cdot\,,\xi)$ is continuous on $K_\eta$, it is uniformly continuous. Thus, denoting by $\omega_\eta$ its modulus of continuity on $K_\eta$, we have for all $x \in Q_{i,h} \cap K_\eta$,

$$|\overline{f}_{\hom}(x,\xi) - \overline{f}_{\hom}(x_i^{h,\eta},\xi)| \leqslant \omega_\eta(|x - x_i^{h,\eta}|) \leqslant \omega_\eta\left(\frac{\sqrt{N}\delta}{h}\right) \xrightarrow[h\to\infty]{} 0. \tag{4.128}$$

Since $Q_{i,h} \cap K_\eta = \emptyset$ for $i \notin I_{h,\eta}$, then by (4.123), (4.128) and (4.127) we get

$$\begin{aligned}
&\int_{Q(x_0,\delta)} \overline{f}_{\hom}(x,\xi)\,dx \\
&\quad= \lim_{\eta\to 0} \int_{K_\eta} \overline{f}_{\hom}(x,\xi)\,dx \\
&\quad= \liminf_{\eta\to 0}\, \liminf_{h\to\infty} \sum_{i\in I_{h,\eta}} \int_{Q_{i,h}\cap K_\eta} \overline{f}_{\hom}(x,\xi)\,dx \\
&\quad= \liminf_{\eta\to 0}\, \liminf_{h\to\infty} \sum_{i\in I_{h,\eta}} \int_{Q_{i,h}\cap K_\eta} \overline{f}_{\hom}(x_i^{h,\eta},\xi)\,dx \\
&\quad\geqslant \liminf_{\lambda,\eta,h,j} \sum_{i\in I_{h,\eta}} \int_{Q_{i,h}\cap K_\eta \cap R^\lambda_{j,\eta}} f\left(x_i^{h,\eta}, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j^\eta\right) dx,
\end{aligned} \tag{4.129}$$

where $R^\lambda_{j,\eta} := \left\{x \in Q(x_0,\delta) : |\xi + \nabla u_j^\eta(x)| \leqslant \lambda\right\}$. From (4.125) and the fact that $u_j^\eta \equiv 0$ on $Q(x_0,\delta) \setminus \bigcup_{i\in I_{h,\eta}} Q_{i,h}$, we get

$$\sup_{j\in\mathbb{N},\eta>0} \int_{Q(x_0,\delta)} |\nabla u_j^\eta|^p dx < \infty. \tag{4.130}$$

In particular, according to Chebyshev's inequality, we have

$$\mathcal{L}^N(Q(x_0,\delta) \setminus R^\lambda_{j,\eta}) \leqslant \frac{C}{\lambda^p}, \tag{4.131}$$

for some constant $C > 0$ independent of $j$, $\eta$ and $\lambda$. Since $f$ is continuous on $K_\eta \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and separately $Q$-periodic in its second and third variable (see assumption $(H_3)$), it is uniformly continuous on $K_\eta \times \mathbb{R}^N \times \mathbb{R}^N \times \overline{B}(0,\lambda)$. Thus, denoting by $\omega_{\eta,\lambda}$ its modulus of continuity on $K_\eta \times \mathbb{R}^N \times \mathbb{R}^N \times \overline{B}(0,\lambda)$, we have for all $x \in Q_{i,h} \cap K_\eta \cap R^\lambda_{j,\eta}$,

$$\begin{aligned}
&\left| f\left(x, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j^\eta(x)\right) - f\left(x_i^{h,\eta}, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j^\eta(x)\right)\right| \\
&\qquad\leqslant \omega_{\eta,\lambda}(|x - x_i^{h,\eta}|) \\
&\qquad\leqslant \omega_{\eta,\lambda}\left(\frac{\sqrt{N}\delta}{h}\right) \xrightarrow[h\to\infty]{} 0.
\end{aligned}$$

Then, according to (4.129) and the fact that $Q_{i,h} \cap K_\eta = \emptyset$ for $i \notin I_{h,\eta}$,

$$\begin{aligned}
&\int_{Q(x_0,\delta)} \overline{f}_{\text{hom}}(x,\xi)\,dx \\
&\quad \geqslant \liminf_{\lambda,\eta,h,j} \sum_{i\in I_{h,\eta}} \int_{Q_{i,h}\cap K_\eta \cap R^\lambda_{j,\eta}} f\left(x, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j^\eta\right) dx, \\
&\quad = \liminf_{\lambda,\eta,j} \int_{K_\eta \cap R^\lambda_{j,\eta}} f\left(x, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j^\eta\right) dx.
\end{aligned}$$

In view of the $p$-growth condition $(H_4)$, (4.123) and the definition of $R^\lambda_{j,\eta}$,

$$\sup_{j\in\mathbb{N}} \int_{R^\lambda_{j,\eta}\setminus K_\eta} f\left(x, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j^\eta\right) dx \leqslant \beta(1+\lambda^p)\eta \xrightarrow[\eta\to 0]{} 0,$$

so

$$\int_{Q(x_0,\delta)} \overline{f}_{\text{hom}}(x,\xi)\,dx \geqslant \liminf_{\lambda,\eta,j} \int_{R^\lambda_{j,\eta}} f\left(x, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla u_j^\eta\right) dx.$$

Let $\lambda_k \nearrow \infty$ and $\eta_k \downarrow 0$. By a diagonalization procedure, it is possible to find a subsequence $\{j_k\}$ of $\{j\}$ such that, upon setting $v_k := u^{\eta_k}_{j_k}$ and $R_k := R^{\lambda_k}_{j_k,\eta_k}$, then $v_k \in W^{1,p}_0(Q(x_0,\delta);\mathbb{R}^d)$, $v_k \to 0$ in $L^p(Q(x_0,\delta);\mathbb{R}^d)$ and

$$\int_{Q(x_0,\delta)} \overline{f}_{\text{hom}}(x,\xi)\,dx \geqslant \liminf_{k\to\infty} \int_{R_k} f\left(x, \frac{x}{\varepsilon_{j_k}}, \frac{x}{\varepsilon_{j_k}^2}, \xi + \nabla v_k\right) dx.$$

By (4.130) and the Poincaré Inequality, the sequence $\{v_k\}$ is bounded in $W^{1,p}(Q(x_0,\delta);\mathbb{R}^d)$ uniformly with respect to $k \in \mathbb{N}$ so that, according to the Decomposition Lemma, it is no loss of generality to assume that $\{|\nabla v_k|^p\}$ is equi-integrable. It turns out, in view of the $p$-growth condition $(H_4)$ and (4.131) that

$$\int_{Q(x_0,\delta)\setminus R_k} f\left(x, \frac{x}{\varepsilon_{j_k}}, \frac{x}{\varepsilon_{j_k}^2}, \xi + \nabla v_k\right) dx \leqslant \beta \sup_{l\in\mathbb{N}} \int_{Q(x_0,\delta)\setminus R_k} (1+|\nabla v_l|^p)\,dx \xrightarrow[k\to\infty]{} 0.$$

Thus, using the Γ-lim inf inequality,

$$\begin{aligned}
\int_{Q(x_0,\delta)} \overline{f}_{\text{hom}}(x,\xi)\,dx &\geqslant \liminf_{k\to\infty} \int_{Q(x_0,\delta)} f\left(x, \frac{x}{\varepsilon_{j_k}}, \frac{x}{\varepsilon_{j_k}^2}, \xi + \nabla v_k\right) dx \\
&\geqslant \int_{Q(x_0,\delta)} f_{\{\varepsilon_j\}}(x,\xi)\,dx.
\end{aligned}$$

■

*Proof of Theorem 4.2.1.* As a consequence of Lemma 4.2.10 and 4.2.11, we have $\overline{f}_{\text{hom}}(x,\xi) = f_{\{\varepsilon_j\}}(x,\xi)$ for all $x \in L$ and all $\xi \in \mathbb{Q}^{d\times N}$. By Lemma 4.2.9 and the fact that $\overline{f}_{\text{hom}}$ is a Carathéodory function we obtain $\overline{f}_{\text{hom}}(x,\xi) = f_{\{\varepsilon_j\}}(x,\xi)$ for all a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^{d\times N}$. Since the result does not depend upon the specific choice of the subsequence, we get by Remark 2.5.9 that the whole sequence $\mathcal{I}_\varepsilon(\cdot\,;A)$ $\Gamma(L^p(A))$-converges to $\mathcal{I}_{\text{hom}}(\cdot\,;A)$. Taking $A = \Omega$ we conclude the proof of Theorem 4.2.1. ∎

### 4.2.4 Some remarks in the convex case

As in Section 4.1, we note that under the additional hypothesis that $f(x,y,z,\cdot)$ is convex for all $x$ and all $(y,z)$ equality (4.62) and (4.63) simplify to read

$$\overline{f}_{\text{hom}}(x,\xi) = \inf_{\phi}\left\{\int_Q f_{\text{hom}}(x,y,\xi+\nabla\phi(y))\,dy, \quad \phi \in W_0^{1,p}(Q;\mathbb{R}^d)\right\}$$

for all $\xi \in \mathbb{R}^{d\times N}$ and all $x \in \Omega$, and

$$f_{\text{hom}}(x,y,\xi) = \inf_{\phi}\left\{\int_Q f(x,y,z,\xi+\nabla\phi(z))\,dz, \quad \phi \in W_0^{1,p}(Q;\mathbb{R}^d)\right\}$$

for all $x \in \Omega$ and all $(y,\xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$ (see Müller [66] and Braides and Defranceschi [19]).

Our objective here is to present an alternative proof of Lemma 4.2.11 in the convex case. Namely, we would like to show that $f_{\{\varepsilon_j\}}(x_0,\xi) \leqslant \overline{f}_{\text{hom}}(x_0,\xi)$ a.e. $x_0 \in \Omega$ and all $\xi \in \mathbb{R}^{d\times N}$, without appealing to Theorem 4.2.2. For this purpose, let us denote by $\mathcal{S}$ (resp. $\mathfrak{S}$) a countable set of functions in $C_c^\infty(Q;\mathbb{R}^d)$ (resp. $C_c^\infty(Q\times Q;\mathbb{R}^d)$) dense in $W_0^{1,p}(Q;\mathbb{R}^d)$ (resp. $L^p(Q;W_0^{1,p}(Q;\mathbb{R}^d))$). Define $L$ to be the set of Lebesgue points $x_0$ for all functions

$$f_{\{\varepsilon_j\}}(\cdot\,,\xi), \quad \overline{f}_{\text{hom}}(\cdot\,,\xi)$$

and

$$x \to \int_Q\int_Q f(x,y,z,\xi+\nabla_y\phi(y)+\nabla_z\psi(y,z))\,dy\,dz,$$

with $\phi \in \mathcal{S}$, $\psi \in \mathfrak{S}$. Note that $\mathcal{L}^N(\Omega \setminus L) = 0$. If $x_0 \in L$ and $\xi \in \mathbb{Q}^{d\times N}$, then

$$\begin{aligned} f_{\{\varepsilon_j\}}(x_0,\xi) &= \lim_{\delta\to 0} \frac{1}{\delta^N} \int_{Q(x_0,\delta)} f_{\{\varepsilon_j\}}(x,\xi)\,dx \\ &= \lim_{\delta\to 0} \frac{\mathcal{I}_{\{\varepsilon_j\}}(\xi\,\cdot\,; Q(x_0,\delta))}{\delta^N}. \end{aligned} \tag{4.132}$$

Given $m \in \mathbb{N}$ consider $\phi_m \in \mathcal{S}$ such that

$$\overline{f}_{\text{hom}}(x_0,\xi) + \frac{1}{m} \geqslant \int_Q f_{\text{hom}}(x_0,y,\xi+\nabla\phi_m(y))\,dy. \tag{4.133}$$

Then by Theorem 2.1.9, and following a similar argument as in Lemma 4.6 in Fonseca and Zappale [51], there exist $\Phi_m \in L^p\big(Q; W_0^{1,p}(Q;\mathbb{R}^d)\big)$ such that

$$f_{\text{hom}}(x_0,y,\xi+\nabla\phi_m(y)) + \frac{1}{m} \geqslant \int_Q f(x_0,y,z,\xi+\nabla_y\phi_m(y)+\nabla_z\Phi_m(y,z))\,dz.$$

We now choose $\Phi_{m,k} \in \mathfrak{S}$ such that

$$\|\Phi_{m,k}-\Phi_m\|_{L^p(Q;W_0^{1,p}(Q;\mathbb{R}^d))} \xrightarrow[k\to\infty]{} 0, \tag{4.134}$$

and we extend $\phi_m$ and $\Phi_{m,k}$ periodically to $\mathbb{R}^N$ and $\mathbb{R}^N\times\mathbb{R}^N$, respectively. For each $x \in \mathbb{R}^N$ define

$$u^j_{m,k}(x) := \xi\cdot x + \varepsilon_j\phi_m\Big(\frac{x}{\varepsilon_j}\Big) + \varepsilon_j^2\Phi_{m,k}\Big(\frac{x}{\varepsilon_j},\frac{x}{\varepsilon_j^2}\Big)$$

and consider $\delta > 0$ small enough so that $Q(x_0,\delta) \in \mathcal{A}(\Omega)$. For fixed $m$ and $k$ we have $u^j_{m,k} \to v$ in $L^p(Q(x_0;\delta);\mathbb{R}^d)$ as $j\to\infty$, where $v(x) = \xi\cdot x$. Hence by (4.132) and the $p$-Lipschitz property of $f(x,y,\cdot)$ (see 2.9)

$$\begin{aligned} f_{\{\varepsilon_j\}}(x_0,\xi) &\leqslant \liminf_{k,\delta,j} \frac{1}{\delta^N}\int_{Q(x_0;\delta)} f\left(x,\frac{x}{\varepsilon_j},\frac{x}{\varepsilon_j^2},\xi+\nabla_y\phi_m\left(\frac{x}{\varepsilon_j}\right)\right. \\ &\qquad \left. +\varepsilon_j\nabla_y\Phi_{m,k}\Big(\frac{x}{\varepsilon_j},\frac{x}{\varepsilon_j^2}\Big)+\nabla_z\Phi_{m,k}\Big(\frac{x}{\varepsilon_j},\frac{x}{\varepsilon_j^2}\Big)\right)\,dx \\ &\leqslant \liminf_{k,\delta,j} \frac{1}{\delta^N}\int_{Q(x_0;\delta)} f\left(x,\frac{x}{\varepsilon_j},\frac{x}{\varepsilon_j^2},\xi+\nabla_y\phi_m\left(\frac{x}{\varepsilon_j}\right)\right. \\ &\qquad \left. +\nabla_z\Phi_{m,k}\Big(\frac{x}{\varepsilon_j},\frac{x}{\varepsilon_j^2}\Big)\right)\,dx. \end{aligned} \tag{4.135}$$

Arguing as in Proposition 4.1.15 we define

$$h_{m,k}(x,y,z) := f\big(x,y,z;\xi + \nabla_y \phi_m(y) + \nabla_z \Phi_{m,k}(y,z)\big).$$

Then by Lemma 2.6.3 since $h_{m,k} \in L^p(Q(x_0,\delta); C_{\rm per}(Q\times Q))$, we get

$$\begin{aligned}
&\liminf_{j\to\infty} \int_{Q(x_0;\delta)} f\left(x, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla_y \phi_m\left(\frac{x}{\varepsilon_j}\right) + \nabla_z \Phi_{m,k}\Big(\frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}\Big)\right) dx \\
&= \lim_{j\to\infty} \int_{Q(x_0;\delta)} h_{m,k}\left(x, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}\right) dx \\
&= \int_{Q(x_0;\delta)} \int_Q \int_Q h_{m,k}(x,y,z)\, dz\, dy\, dx \\
&= \int_{Q(x_0;\delta)} \int_Q \int_Q f\big(x,y,z;\xi + \nabla_y \phi_m\,(y) + \nabla_z \Phi_{m,k}(y,z)\big) dz\, dy\, dx.
\end{aligned}$$

Therefore, by (4.44)

$$\liminf_{\delta\to 0}\ \liminf_{j\to\infty} \frac{1}{\delta^N} \int_{Q(x_0;\delta)} f\left(x, \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}, \xi + \nabla_y \phi_m\left(\frac{x}{\varepsilon_j}\right) + \nabla_z \Phi_{m,k}\Big(\frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j^2}\Big)\right) dx$$

$$= \int_Q \int_Q f(x_0,y,z;\xi + \nabla_y \phi_m(y) + \nabla_z \Phi_{m,k}(y,z))\, dz\, dy,$$

and thus, by (4.133)-(4.135), $(H_4)$, and Fubini's Theorem, we obtain

$$\begin{aligned}
f_{\{\varepsilon_j\}}(x_0,\xi) &\leqslant \int_Q \int_Q f(x_0,y,z,\xi + \nabla_y \phi_m(y) + \nabla_z \Phi_m(y,z))\, dz\, dy \\
&= \int_Q \left[\int_Q f(x_0,y,z,\xi + \nabla_y \phi_m(y) + \nabla_z \Phi_m(y,z))\, dz\right] dy \\
&\leqslant \int_Q f_{\rm hom}(x_0,y,\xi + \nabla \phi_m(y))\, dy + \frac{1}{m} \\
&\leqslant \overline{f}_{\rm hom}(x_0,\xi) + \frac{2}{m}.
\end{aligned}$$

Letting $m \to \infty$ we deduce that

$$f_{\{\varepsilon_j\}}(x_0,\xi) \leqslant \overline{f}_{\rm hom}(x_0,\xi).$$

■

# 5. APPLICATION TO THIN FILMS

This part is devoted to studying a reiterated homogenization problem in thin domains of elastic type with multiple scale and periodic microstructure. The results presented here were obtained in collaboration with J. F. Babadjian [10, 9].

Throughout this chapter $\omega$ stands for an open bounded set in $\mathbb{R}^2$ and $\Omega := \omega \times I$, with $I := (-1, 1)$. We will identify $W^{1,p}(\omega; \mathbb{R}^3)$ with the set of functions $u \in W^{1,p}(\Omega; \mathbb{R}^3)$ such that $D_3 u(x) = 0$ for a.e. $x \in \Omega$, and we set $Q'(a, \delta) := a + \delta Q'$ for $a \in \mathbb{R}^2$ and $\delta > 0$, where $Q' = (0, 1)^2$.

## 5.1 Thin films with periodic microstructure in the in-plane direction

We start with the case where heterogeneities are allowed in the in-plane direction of the film and they scale as the thickness of this body.

For each $\varepsilon > 0$ we define $\mathcal{W}_\varepsilon : L^p(\Omega; \mathbb{R}^3) \to \overline{\mathbb{R}}$ by

$$\mathcal{W}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega W\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon}, \nabla_\alpha u(x) \Big| \frac{1}{\varepsilon} \nabla_3 u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^3), \\ \infty & \text{otherwise,} \end{cases} \tag{5.1}$$

with $1 < p < \infty$, where we assume that $W : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfies the following hypotheses:

$(A_1)$ $W(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$;

$(A_2)$ $W(\cdot, \cdot, \xi)$ is $\mathcal{L}^3 \otimes \mathcal{L}^2$-measurable for all $\xi \in \mathbb{R}^{3\times 3}$;

$(A_3)$ there exists $0 < \beta < \infty$ such that

$$\frac{1}{\beta}|\xi|^p - \beta \leqslant W(x, y_\alpha, \xi) \leqslant \beta(1+|\xi|^p), \quad \text{for a.e. } x \in \Omega \text{ and for all } (y_\alpha, \xi) \in \mathbb{R}^2 \times \mathbb{R}^{3\times 3};$$

$(A_4)$ $W(x, \cdot, \xi)$ is $Q'$-periodic for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^{3\times 3}$, where we denote by $Q' = (0,1)^2$ the unit cube of $\mathbb{R}^2$.

The goal of this section is to prove the following result.

**Theorem 5.1.1.** *If $W$ satisfies $(A_1)$-$(A_4)$, then the family $\{\mathcal{W}_\varepsilon\}_\varepsilon$ $\Gamma(L^p(\Omega))$-converges to the functional $\mathcal{W}_{\text{hom}} : L^p(\Omega; \mathbb{R}^3) \to \overline{\mathbb{R}}$ defined by*

$$\mathcal{W}_{\text{hom}}(u) := \begin{cases} 2\displaystyle\int_\omega W_{\text{hom}}(x_\alpha, \nabla_\alpha u(x_\alpha))\, dx_\alpha & \text{if } u \in W^{1,p}(\omega; \mathbb{R}^3), \\ \\ \infty & \text{otherwise}, \end{cases} \tag{5.2}$$

*where $W_{\text{hom}}$ is given by*

$$\begin{aligned} W_{\text{hom}}(x_\alpha, \overline{\xi}) \quad := \quad & \lim_{T\to\infty} \inf_\phi \Big\{ \frac{1}{2T^2} \int_{(0,T)^2\times I} W\big(x_\alpha, y_3, y_\alpha, \overline{\xi} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y)\big)\, dy : \\ & \phi \in W^{1,p}((0,T)^2 \times I; \mathbb{R}^3),\ \phi = 0 \text{ on } \partial(0,T)^2 \times I \Big\} \end{aligned} \tag{5.3}$$

*for a.e. $x_\alpha \in \omega$ and all $\overline{\xi} \in \mathbb{R}^{3\times 2}$.*

*Remark* 5.1.2. We remark that, due to hypotheses $(A_1)$ and $(A_2)$, the function $W$ is a Carathéodory integrand as $W(x, \cdot; \cdot)$ is continuous a.e. $x \in \Omega$ and $W(\cdot, y_\alpha; \xi)$ is measurable for all $y_\alpha \in \mathbb{R}^2$ and $\xi \in \mathbb{R}^{3\times 3}$. This implies (see Proposition 2.3.27) that $W$ is equivalent to a Borel function, that is there exist a Borel function $\tilde{W}$ such that $W(x, \cdot; \cdot) = \tilde{W}(x, \cdot; \cdot)$ for a.e. $x \in \Omega$. As a consequence the integral in (5.1) is well defined. As in the results of Chapter 4 to prove Theorem 5.1.1 we may assume, without loss of generality, that $W$ is non negative. Indeed, in view of $(A_3)$ it suffices to replace $W$ by $W + \beta$.

As a consequence of Theorem 5.1.1, we deduce the usual convergence of (almost) minimizers.

**Corollary 5.1.3.** *Let $f \in L^{p'}(\Omega; \mathbb{R}^3)$ and $g \in L^{p'}(\Sigma; \mathbb{R}^3)$, where $\Sigma := \omega \times \{-1, 1\}$ $(1/p + 1/p' = 1)$. Then every sequence $\{u_\varepsilon\}_\varepsilon \subset \mathcal{V}_\varepsilon := \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : u(x) = (x_\alpha, \varepsilon x_3) \text{ on } \partial\omega \times I\}$ of (almost) minimizers of*

$$\mathcal{W}_\varepsilon^{f,g}(u) = \int_\Omega W\left(x, \frac{x_\alpha}{\varepsilon}, \nabla_\alpha u(x)\Big|\frac{1}{\varepsilon}\nabla_3 u(x)\right) dx - \int_\Omega f \cdot u\, dx - \int_\Sigma g \cdot u\, d\mathcal{S}$$

*is weakly relatively compact in* $W^{1,p}(\Omega;\mathbb{R}^3)$. *Furthermore, any limit point* $u$ *of this sequence is a solution of the minimization problem*

$$\min_{v-(x_\alpha,0)\in W_0^{1,p}(\omega;\mathbb{R}^3)} \left\{2\int_\omega W_{\text{hom}}(x_\alpha, \nabla_\alpha v(x_\alpha))\, dx_\alpha - \int_\omega (\overline{f} + g^+ + g^-)(x_\alpha)\cdot v(x_\alpha)\, dx_\alpha\right\},$$

*where* $\overline{f} := \frac{1}{2}\int_I f(\cdot, x_3)\, dx_3$ *and* $g^\pm := g(\cdot, \pm 1)$.

Corollary 5.1.3 departs from Corollary 4.1.3 on the type of boundary condition that has been considered. This difficulty is overcome thanks to Remark 5.1.11, which says that we can prescribe the lateral boundary condition of the recovery sequence. We do not include the proof of this corollary because it is similar to that of Corollary 1.3 in Bouchitté, Fonseca and Mascarenhas [25].

The plan of this section is as follows. In Subsection 5.1.1 we will discuss some properties of $W_{\text{hom}}$, namely that it is well defined and that $W_{\text{hom}}(x_\alpha; \cdot\,)$ is continuous for a.e. $x_\alpha \in \omega$. Section 5.1.2 is devoted to the proof of Theorem 5.1.1. The starting point of our analysis is the $\Gamma$-limit integral representation result Theorem 3.2.1. Our objective is to identify the integrand, showing that it coincides (almost everywhere) with $W_{\text{hom}}$. As in Proposition 4.1.15, we will use a two-scale convergence argument to derive an upper bound for the limit integral (Lemma 5.1.12). However, we cannot use the same argument as in Proposition 4.1.15 to derive a lower bound. The argument in this case (see Lemma 5.1.13) is more technical and the difficulty comes from the fact that the problem, at fixed $\varepsilon$, and the asymptotic problem, when $\varepsilon \to 0$, are of different nature (one is a full three-dimensional problem, the other a two-dimensional one). We will need to use a decoupling argument to take into account the different nature of the two variables $x_\alpha$ and $y_\alpha$ that appear in the structure of the limit functional. For this purpose it will be convenient to extend $W$ to a function which is (separately) continuous everywhere. This is the aim of Lemma B.1 which provides conditions under which a Carathéodory function such as $W$ can be extended to a separately continuous function in the macroscopic variable $x_\alpha$ and the microscopic variable $x_\alpha/\varepsilon$.[1]

[1] We could also have used the Scorza Dragoni Theorem and the Tietze Extension Theorem (see Theorem B.3 in the Appendix). This argument will be used in Subsection 5.2.

### 5.1.1 Properties of the homogenized density

In this section we follow very closely the arguments of Section 4.1 to derive some properties of the stored energy $W_{\rm hom}$ that will be of use in the proof of Theorem 5.1.1.

We begin by showing that in the definition (5.3) of $W_{\rm hom}$ the limit as $T \to \infty$ exists. We introduce the following new condition :

$(A_1')$ $W(x, y_\alpha; \cdot)$ is continuous for a.e. $x \in \Omega$ and all $y_\alpha \in \mathbb{R}^2$.

*Remark* 5.1.4. Note that $(A_1)$ implies $(A_1')$. Furthermore, if $W$ satisfies $(A_1')$ and $(A_2)$, then $W$ is a Carathéodory function in the following sense : $W(\cdot,\cdot;\xi)$ is $\mathcal{L}^3 \otimes \mathcal{L}^2$-measurable for all $\xi \in \mathbb{R}^{3\times 3}$ and $W(x, y_\alpha; \cdot)$ is continuous for $\mathcal{L}^3 \otimes \mathcal{L}^2$-a.e. $(x, y_\alpha) \in \Omega \times \mathbb{R}^2$. As a consequence, there exists a Borel function $W'$ on $\Omega \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$ such that $W(x, y_\alpha; \cdot) = W'(x, y_\alpha; \cdot)$ for $\mathcal{L}^3 \otimes \mathcal{L}^2$-a.e. $(x, y_\alpha) \in \Omega \times \mathbb{R}^2$. Thus the integral in (5.3) is well defined. We insist on the fact that, in principle, $W'$ and $\tilde{W}$ (see Remark 5.1.2) need not to be equal.

**Lemma 5.1.5.** *If $W$ satisfies $(A_1')$, $(A_2)$-$(A_4)$, then*

$$\begin{aligned} W_{\rm hom}(x_\alpha, \overline{\xi}) \;\; = \;\; & \lim_{T\to\infty} \inf_{\phi} \Big\{ \frac{1}{2T^2} \int_{(0,T)^2 \times I} W\big(x_\alpha, y_3, y_\alpha, \overline{\xi} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y)\big)\, dy : \\ & \phi \in W^{1,p}((0,T)^2 \times I; \mathbb{R}^3),\ \phi = 0 \ \text{on}\ \partial(0,T)^2 \times I \Big\} \end{aligned}$$

*exists for a.e. $x_\alpha \in \omega$ and all $\overline{\xi} \in \mathbb{R}^{3\times 2}$.*

*Proof.* Let $x_\alpha \in \omega$ be such that $(A_1')$, $(A_3)$ and $(A_4)$ hold and let $\overline{\xi} \in \mathbb{R}^{3\times 2}$. Define $\mu : \mathcal{A}(\mathbb{R}^2) \to \mathbb{R}^+$ by

$$\begin{aligned} \mu(A) \;\; := \;\; & \inf_{\phi} \Big\{ \frac{1}{2} \int_{A\times I} W(x_\alpha, y_3, y_\alpha, \overline{\xi} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y))\, dy : \\ & \phi \in W^{1,p}(A \times I; \mathbb{R}^3),\ \phi = 0 \text{ on } \partial A \times I \Big\}. \end{aligned}$$

In view of Remark 5.1.4, $\mu$ is well defined and, thanks to $(A_3)$, it is a finite function. Moreover this set function satisfies the assumptions of Lemma A.1. Indeed firstly, by $(A_3)$, $\mu(A) \leqslant \beta(1 + |\overline{\xi}|^p)\, \mathcal{L}^2(A)$ for all $A \in \mathcal{A}(\mathbb{R}^2)$. Secondly, $\mu$ is subadditive, that is

$\mu(C) \leqslant \mu(A) + \mu(B)$ for all $A$, $B$, $C \in \mathcal{A}(\mathbb{R}^2)$ with $A \cap B \neq \emptyset$ and $\overline{C} = \overline{A} \cup \overline{B}$. Finally, by $(A_4)$, for any $\mathbf{i} \in \mathbb{Z}^2$, $\mu(A + \mathbf{i}) = \mu(A)$ for all $A \in \mathcal{A}(\mathbb{R}^2)$. As a consequence the limit

$$\lim_{T\to\infty} \frac{\mu((0,T)^2)}{T^2} = W_{\mathrm{hom}}(x_\alpha, \overline{\xi})$$

exists. ■

*Remark* 5.1.6. As in Section 4.1, the limit as $T \to \infty$ in (5.3) can be replaced by an infimum taken for every $T > 0$.

Now that $W_{\mathrm{hom}}$ is well defined, we will show that $W_{\mathrm{hom}}(x_\alpha; \cdot)$ is continuous for a.e. $x_\alpha \in \omega$, for later use in Theorem 5.1.1. To prove this property directly it seems that we would need a little bit more than only the continuity condition imposed on $W(x, y_\alpha; \cdot)$ (e.g. a $p$-Lipschitz condition). We remark that if $W(x, y_\alpha; \cdot)$ were quasiconvex, then by the $p$-growth condition $(A_3)$, $W(x, y_\alpha; \cdot)$ would satisfy a $p$-Lipschitz condition (see Lemma 5.1.9 below). Since we do not want to a priori restrict too much the stored energy density, in order to compensate for this lack of regularity we prove first in Lemma 5.1.8 that the value of $W_{\mathrm{hom}}$ does not change if we replace $W$ by its quasiconvexification $\mathcal{Q}W$ (see Remark 5.1.7 below).

*Remark* 5.1.7. For a.e. $x \in \Omega$, all $y_\alpha \in \mathbb{R}^2$ and all $\xi \in \mathbb{R}^{3\times3}$ the functions $\mathcal{Q}W(x, y_\alpha;\, \cdot)$ (usual quasiconvexification with respect to the last variable $\xi$) are quasiconvex. By Remark 5.1.4 if $W$ satisfies $(A_1)$-$(A_4)$, then so does $\mathcal{Q}W$, except that $\mathcal{Q}W(x, \cdot; \xi)$ may only be upper semicontinuous (as the infimum of continuous functions) for a.e. $x \in \Omega$, and all $\xi \in \mathbb{R}^{3\times3}$. In particular, since $\mathcal{Q}W$ satisfies $(A_1')$, $(A_2)$-$(A_4)$, by Lemma 5.1.5 it follows that

$$\begin{aligned}(\mathcal{Q}W)_{\mathrm{hom}}(x_\alpha, \overline{\xi}) \quad = \quad & \lim_{T\to\infty} \inf_{\phi} \Big\{ \frac{1}{2T^2} \int_{(0,T)^2\times I} \mathcal{Q}W\big(x_\alpha, y_3, y_\alpha, \overline{\xi} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y)\big)\, dy : \\ & \phi \in W^{1,p}((0,T)^2 \times I; \mathbb{R}^3),\ \phi = 0 \text{ on } \partial(0,T)^2 \times I \Big\}\end{aligned}$$

exists for a.e. $x_\alpha \in \omega$ and all $\overline{\xi} \in \mathbb{R}^{3\times2}$.

**Lemma 5.1.8.** *If $W$ satisfies $(A_1)$-$(A_4)$, then $(\mathcal{Q}W)_{\mathrm{hom}}(x_\alpha, \overline{\xi}) = W_{\mathrm{hom}}(x_\alpha, \overline{\xi})$ for a.e. $x_\alpha \in \omega$ and all $\overline{\xi} \in \mathbb{R}^{3\times2}$.*

*Proof.* Let $x_\alpha \in \omega$ be such that both $(\mathcal{Q}W)_{\mathrm{hom}}(x_\alpha;\, \cdot\,)$ and $W_{\mathrm{hom}}(x_\alpha;\, \cdot\,)$ are well defined. Since $W \geqslant \mathcal{Q}W$, we have $W_{\mathrm{hom}}(x_\alpha, \overline{\xi}) \geqslant (\mathcal{Q}W)_{\mathrm{hom}}(x_\alpha, \overline{\xi})$ for all $\overline{\xi} \in \mathbb{R}^{3\times2}$. Let us prove now the opposite inequality. Let $\overline{\xi} \in \mathbb{R}^{3\times2}$. For each $n > 0$, let $T_n \in \mathbb{N}$ and $\phi_n \in W^{1,\infty}((0,T_n)^2 \times$

$I;\mathbb{R}^3)$, satisfying $\phi_n = 0$ on $\partial(0,T_n)^2 \times I$, be such that

$$(\mathcal{Q}W)_{\mathrm{hom}}(x_\alpha,\overline{\xi}) + \frac{1}{n} \geqslant \frac{1}{2T_n^2}\int_{(0,T_n)^2\times I} \mathcal{Q}W(x_\alpha,y_3,y_\alpha,\overline{\xi}+\nabla_\alpha\phi_n(y)|\nabla_3\phi_n(y))\,dy.$$

The Lipschitz regularity of $\phi_n$ may be ensured due to the density of $W^{1,\infty}((0,T_n)^2\times I;\mathbb{R}^3)$ in $W^{1,p}((0,T_n)^2\times I;\mathbb{R}^3)$ together with the $p$-growth condition $(A_3)$. Thus

$$(\mathcal{Q}W)_{\mathrm{hom}}(x_\alpha,\overline{\xi}) \geqslant \limsup_{n\to\infty} \frac{1}{2T_n^2}\int_{(0,T_n)^2\times I} \mathcal{Q}W(x_\alpha,y_3,y_\alpha,\overline{\xi}+\nabla_\alpha\phi_n(y)|\nabla_3\phi_n(y))\,dy. \quad (5.4)$$

For each $n\in\mathbb{N}$ fixed, by the Acerbi-Fusco Relaxation Theorem 2.3.30 and Remark 5.1.2, there exists a sequence $\{\phi_{n,k}\}_k \subset W^{1,\infty}((0,T_n)^2\times I;\mathbb{R}^3)$ satisfying $\phi_{n,k}=\phi_n$ on $\partial[(0,T_n)^2\times I]$ with $\phi_{n,k} \underset{k\to\infty}{\rightharpoonup} \phi_n$ and such that

$$\begin{aligned}
&\frac{1}{2T_n^2}\int_{(0,T_n)^2\times I} \mathcal{Q}W(x_\alpha,y_3,y_\alpha,\overline{\xi}+\nabla_\alpha\phi_n(y)|\nabla_3\phi_n(y))\,dy\\
&= \lim_{k\to\infty}\frac{1}{2T_n^2}\int_{(0,T_n)^2\times I} W(x_\alpha,y_3,y_\alpha,\overline{\xi}+\nabla_\alpha\phi_{n,k}(y)|\nabla_3\phi_{n,k}(y))\,dy.
\end{aligned}$$

From (5.4) we have

$$\begin{aligned}
(\mathcal{Q}W)_{\mathrm{hom}}(x_\alpha,\overline{\xi}) &\geqslant \limsup_{n\to\infty}\limsup_{k\to\infty}\frac{1}{2T_n^2}\int_{(0,T_n)^2\times I} W(x_\alpha,y_3,y_\alpha,\overline{\xi}+\nabla_\alpha\phi_{n,k}(y)|\nabla_3\phi_{n,k}(y))\,dy\\
&\geqslant \limsup_{n\to\infty}\inf_{\phi}\Big\{\frac{1}{2T_n^2}\int_{(0,T_n)^2\times I} W(x_\alpha,y_3,y_\alpha,\overline{\xi}+\nabla_\alpha\phi(y)|\nabla_3\phi(y))\,dy :\\
&\qquad\qquad \phi\in W^{1,p}((0,T_n)^2\times I;\mathbb{R}^3),\ \phi=0 \text{ on } \partial(0,T_n)^2\times I\Big\}\\
&= W_{\mathrm{hom}}(x_\alpha,\overline{\xi}).
\end{aligned}$$

■

We are now in position to prove the continuity of $W_{\mathrm{hom}}$ in its second variable :

**Lemma 5.1.9.** *Let $W$ satisfying $(A_1)$-$(A_4)$, then $W_{\mathrm{hom}}(x_\alpha;\cdot)$ is continuous on $\mathbb{R}^{3\times 2}$ for a.e. $x_\alpha\in\omega$.*

*Proof.* We observe that by the $p$-growth condition in $(A_3)$ and Remark 5.1.7, $\mathcal{Q}W$ satisfies a $p$-Lipschitz condition: There exists $\beta>0$ such that for all $y_\alpha\in\mathbb{R}^2$ and a.e. $x\in\Omega$,

$$|\mathcal{Q}W(x,y_\alpha;\xi_1)-\mathcal{Q}W(x,y_\alpha;\xi_2)| \leqslant \beta(1+|\xi_1|^{p-1}+|\xi_2|^{p-1})|\xi_1-\xi_2|, \quad \xi_1,\,\xi_2\in\mathbb{R}^{3\times 3}. \quad (5.5)$$

Take $x_\alpha \in \omega$ such that both $(\mathcal{Q}W)_{\hom}(x_\alpha;\,\cdot\,)$ and $W_{\hom}(x_\alpha;\,\cdot\,)$ are well defined. By Lemma 5.1.8 we have $(\mathcal{Q}W)_{\hom}(x_\alpha;\,\cdot\,) = W_{\hom}(x_\alpha;\,\cdot\,)$. Given $\overline{\xi} \in \mathbb{R}^{3\times 2}$ let $\overline{\xi}_n \to \overline{\xi}$ in $\mathbb{R}^{3\times 2}$. From the definition of $W_{\hom}(x_\alpha, \overline{\xi})$, for fixed $\delta > 0$ choose $T \in \mathbb{N}$ and $\phi \in W^{1,p}((0,T)^2 \times I; \mathbb{R}^3)$, $\phi = 0$ on $\partial(0,T)^2 \times I$, such that

$$W_{\hom}(x_\alpha, \overline{\xi}) + \delta \geqslant \frac{1}{2T^2}\int_{(0,T)^2\times I} W(x_\alpha, y_3, y_\alpha, \overline{\xi} + \nabla_\alpha \phi(y)|\nabla_3\phi(y))\,dy. \tag{5.6}$$

Therefore, Remark 5.1.6 yields

$$\begin{aligned}\limsup_{n\to\infty} W_{\hom}(x_\alpha, \overline{\xi}_n) &\leqslant \limsup_{n\to\infty} \frac{1}{2T^2}\int_{(0,T)^2\times I} W(x_\alpha, y_3, y_\alpha; \overline{\xi}_n + \nabla_\alpha \phi(y)|\nabla_3\phi(y))\,dy \\ &= \frac{1}{2T^2}\int_{(0,T)^2\times I} W(x_\alpha, y_3, y_\alpha, \overline{\xi} + \nabla_\alpha \phi(y)|\nabla_3\phi(y))\,dy\end{aligned}$$

due to hypothesis $(A_1)$, the $p$-growth condition in $(A_3)$ and Lebesgue's Dominated Convergence Theorem. So by (5.6) and letting $\delta \to 0$ we conclude that

$$\limsup_{n\to\infty} W_{\hom}(x_\alpha, \overline{\xi_n}) \leqslant W_{\hom}(x_\alpha, \overline{\xi}). \tag{5.7}$$

Similarly, for each $n \in \mathbb{N}$ consider $T_n \in \mathbb{N}$ ($T_n \nearrow \infty$) and $\phi_n \in W^{1,p}((0,T_n)^2 \times I; \mathbb{R}^3)$, $\phi_n = 0$ on $\partial(0,T_n)^2 \times I$, such that

$$\begin{aligned}W_{\hom}(x_\alpha, \overline{\xi}_n) + \frac{1}{n} &\geqslant \frac{1}{2T_n^2}\int_{(0,T_n)^2\times I} \mathcal{Q}W(x_\alpha, y_3, y_\alpha, \overline{\xi}_n + \nabla_\alpha\phi_n(y)|\nabla_3\phi_n(y))\,dy \\ &= \frac{1}{2}\int_{Q'\times I} \mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha, \overline{\xi}_n + \nabla_\alpha\phi_n(T_n y_\alpha, y_3)|\nabla_3\phi_n(T_n y_\alpha, y_3))\,dy \\ &= \frac{1}{2}\int_{Q'\times I} \mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha, \overline{\xi}_n + \nabla_\alpha\psi_n(y)|T_n\nabla_3\psi_n(y))\,dy,\end{aligned}$$

after a change of variables and where $\psi_n(y) := \frac{1}{T_n}\phi_n(T_n y_\alpha, y_3)$. Clearly the function $\psi_n$ belongs to $W^{1,p}(Q' \times I; \mathbb{R}^3)$ and $\psi_n = 0$ on $\partial Q' \times I$. By the $p$-coercivity hypothesis in $(A_3)$ and (5.7), the sequence $\{(\nabla_\alpha\psi_n|T_n\nabla_3\psi_n)\}$ is bounded in $L^p(Q' \times I; \mathbb{R}^{3\times 3})$ uniformly in $n$. We can write that

$$\begin{aligned}&\liminf_{n\to\infty}\int_{Q'\times I} \mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha, \overline{\xi}_n + \nabla_\alpha\psi_n(y)|T_n\nabla_3\psi_n(y))\,dy \\ &\geqslant \liminf_{n\to\infty}\int_{Q'\times I}\big[\,\mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha, \overline{\xi}_n + \nabla_\alpha\psi_n(y)|T_n\nabla_3\psi_n(y)) \\ &\qquad\qquad - \mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha, \overline{\xi} + \nabla_\alpha\psi_n(y)|T_n\nabla_3\psi_n(y))\,\big]\;dy \\ &\quad + \liminf_{n\to\infty}\int_{Q'\times I} \mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha, \overline{\xi} + \nabla_\alpha\psi_n(y)|T_n\nabla_3\psi_n(y))\,dy.\end{aligned}$$

Using (5.5), Hölder inequality, the fact that $\{\|(\nabla_\alpha \psi_n | T_n \nabla_3 \psi_n)\|_{L^p(Q' \times I;\mathbb{R}^{3\times 3})}\}$ is bounded and $\overline{\xi}_n \to \overline{\xi}$, we obtain

$$\liminf_{n\to\infty} \int_{Q'\times I} \Big[\, \mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha, \overline{\xi}_n + \nabla_\alpha \psi_n(y) | T_n \nabla_3 \psi_n(y)) \\ - \mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha, \overline{\xi} + \nabla_\alpha \psi_n(y) | T_n \nabla_3 \psi_n(y)) \Big]\, dy = 0,$$

and consequently

$$\begin{aligned} \liminf_{n\to\infty} W_{\text{hom}}(x_\alpha, \overline{\xi}_n) &\geqslant \liminf_{n\to\infty} \frac{1}{2} \int_{Q'\times I} \mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha, \overline{\xi} + \nabla_\alpha \psi_n(y) | T_n \nabla_3 \psi_n(y))\, dy \\ &= \liminf_{n\to\infty} \frac{1}{2T_n^2} \int_{(0,T_n)^2\times I} \mathcal{Q}W(x_\alpha, y_3, y_\alpha, \overline{\xi} + \nabla_\alpha \phi_n(y) | \nabla_3 \phi_n(y))\, dy \\ &\geqslant (\mathcal{Q}W)_{\text{hom}}(x_\alpha, \overline{\xi}) \\ &= W_{\text{hom}}(x_\alpha, \overline{\xi}). \end{aligned} \tag{5.8}$$

From (5.7) and (5.8), we conclude that $W_{\text{hom}}(x_\alpha; \cdot)$ is continuous at $\overline{\xi}$. ■

### 5.1.2 Main result

We start by localizing our functionals. Define $\mathcal{W}_\varepsilon : L^p(\Omega;\mathbb{R}^3) \times \mathcal{A}(\omega) \to \overline{\mathbb{R}}$ by

$$\mathcal{W}_\varepsilon(u; A) := \begin{cases} \displaystyle\int_{A\times I} W\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon}, \nabla_\alpha u(x) \Big| \frac{1}{\varepsilon} \nabla_3 u(x)\right) dx & \text{if } u \in W^{1,p}(A\times I;\mathbb{R}^3), \\ \infty & \text{otherwise.} \end{cases}$$

We will prove that the family of functionals $\{\mathcal{W}_\varepsilon(\cdot; A)\}_\varepsilon$ $\Gamma$-converges with respect to the $L^p(A\times I;\mathbb{R}^3)$-topology to the functional $\mathcal{W}_{\text{hom}}(\cdot; A) : L^p(\Omega;\mathbb{R}^3) \to \overline{\mathbb{R}}$,

$$\mathcal{W}_{\text{hom}}(u; A) := \begin{cases} \displaystyle 2\int_A W_{\text{hom}}(x_\alpha, \nabla_\alpha u(x_\alpha))\, dx_\alpha & \text{if } u \in W^{1,p}(A;\mathbb{R}^3), \\ \infty & \text{otherwise,} \end{cases} \tag{5.9}$$

for all $A \in \mathcal{A}(\omega)$. As a consequence, taking $A = \omega$ yields Theorem 5.1.1.

For any $A \in \mathcal{A}(\omega)$ and any sequence $\varepsilon_n \downarrow 0$, consider $\mathcal{W}_{\{\varepsilon_n\}}(\cdot; A) : L^p(\Omega;\mathbb{R}^3) \to \overline{\mathbb{R}}$ the $\Gamma$-lower limit of $\{\mathcal{W}_{\varepsilon_n}(\cdot; A)\}_n$,

$$\mathcal{W}_{\{\varepsilon_n\}}(u; A) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} \mathcal{W}_{\varepsilon_j}(u_n; A) : u_n \to u \text{ in } L^p(A\times I;\mathbb{R}^3) \right\}. \tag{5.10}$$

*Remark* 5.1.10. In view of the coercivity condition $(A_4)$, for all $A \in \mathcal{A}(\omega)$ we have that $\mathcal{W}_{\{\varepsilon_n\}}(u; A) = \infty$ whenever $u \in L^p(\Omega; \mathbb{R}^3) \setminus W^{1,p}(A; \mathbb{R}^3)$, hence our objective is to characterize $\mathcal{W}_{\{\varepsilon_n\}}(u; A)$ for $u \in W^{1,p}(A; \mathbb{R}^3)$.

By virtue of Remark 5.1.10, together with Theorem 3.2.1, it follows that every sequence $\{\varepsilon_n\}_n$ admits a subsequence $\{\varepsilon_{n_j}\}_j \equiv \{\varepsilon_j\}_j$ such that $\mathcal{W}_{\{\varepsilon_j\}}(\,\cdot\,; A)$, defined in (5.10), is the $\Gamma(L^p(A \times I))$-limit of $\{\mathcal{W}_{\varepsilon_n}(\,\cdot\,; A)\}_j$ for all $A \in \mathcal{A}(\omega)$. Further, there exists a Carathéodory function $W_{\{\varepsilon_j\}} : \omega \times \mathbb{R}^{3\times 2} \to \mathbb{R}$ such that

$$\mathcal{W}_{\{\varepsilon_j\}}(u; A) = 2 \int_A W_{\{\varepsilon_j\}}(x_\alpha, \nabla_\alpha u(x_\alpha))\, dx_\alpha, \tag{5.11}$$

for all $A \in \mathcal{A}(\omega)$ and all $u \in W^{1,p}(A; \mathbb{R}^3)$.

Our aim is to show that $\mathcal{W}_{\{\varepsilon_j\}}(\cdot; A) = \mathcal{W}_{\mathrm{hom}}(\cdot; A)$ on $W^{1,p}(A; \mathbb{R}^3)$ for all $A \in \mathcal{A}(\omega)$. Given $A \in \mathcal{A}(\omega)$, in view of the integral representation (5.11) and (5.9), it is enough to show that $W_{\{\varepsilon_j\}}(x_\alpha, \overline{\xi}) = W_{\mathrm{hom}}(x_\alpha, \overline{\xi})$ for a.e. $x_\alpha \in A$ and all $\overline{\xi} \in \mathbb{R}^{3\times 2}$. We will prove that $W_{\{\varepsilon_j\}}(x_\alpha, \overline{\xi}) = W_{\mathrm{hom}}(x_\alpha, \overline{\xi})$ for a.e. $x_\alpha \in \omega$ and all $\overline{\xi} \in \mathbb{R}^{3\times 2}$.

*Remark* 5.1.11. Lemma 3.2.2 implies that $\mathcal{W}_{\{\varepsilon_j\}}(u; A)$ is unchanged if the approximating sequences $\{u_j\}$ are constrained to match the lateral boundary condition of their target, i.e. $u_j \equiv u$ on $\partial A \times I$.

From now on, $\{\varepsilon_j\}_j$ will denote a subsequence of $\{\varepsilon_j\}$ for which the $\Gamma(L^p(A \times I))$-limit of $\{\mathcal{W}_{\varepsilon_j}(\cdot; A)\}_{j\in\mathbb{N}}$ exists and coincides with $\mathcal{W}_{\{\varepsilon_j\}}(\cdot; A)$ for all $A \in \mathcal{A}(\omega)$.

For each $T > 0$ consider $\mathcal{S}_T$ a countable set of functions in $\mathcal{C}^\infty([0, T]^2 \times [-1, 1]; \mathbb{R}^3)$ that is dense in

$$\mathcal{W}_T = \{\phi \in W^{1,p}((0, T)^2 \times I; \mathbb{R}^3) : \ \phi = 0 \ \text{ on } \ \partial(0, T)^2 \times I\}.$$

Let $L$ be the set of Lebesgue points $x^0_\alpha$ for all functions

$$W_{\{\varepsilon_j\}}(\cdot, \overline{\xi}), \quad W_{\mathrm{hom}}(\cdot, \overline{\xi}) \tag{5.12}$$

and

$$x_\alpha \mapsto \int_{Q'\times I} W(x_\alpha, y_3, Ty_\alpha, \overline{\xi} + \nabla_\alpha \phi(Ty_\alpha, y_3) | \nabla_3 \phi(Ty_\alpha, y_3))\, dy_\alpha\, dy_3, \tag{5.13}$$

with $T \in \mathbb{N}$, $\phi \in \mathcal{S}_T$ and $\overline{\xi} \in \mathbb{Q}^{3\times 2}$, and for which $W_{\mathrm{hom}}(x^0_\alpha; \,\cdot\,)$ is well defined.

To prove that $W_{\{\varepsilon_j\}}(x_\alpha, \overline{\xi}) = W_{\rm hom}(x_\alpha, \overline{\xi})$ for a.e. $x_\alpha \in \omega$ and all $\overline{\xi} \in \mathbb{R}^{3\times 2}$ we first show in Lemmas 5.1.12 and 5.1.13 below that both functions coincide on $L \times \mathbb{Q}^{3\times 2}$. The general case will only be treated at the end of that section using the Carathéodory property of both integrands.

Fix $\overline{\xi} \in \mathbb{Q}^{3\times 2}$ and set $v(x) := \overline{\xi} \cdot x_\alpha$. By (5.11) and (5.12)

$$\begin{aligned} W_{\{\varepsilon_j\}}(x^0_\alpha, \overline{\xi}) &= \lim_{\delta\to 0} \frac{1}{\delta^2} \int_{Q'(x^0_\alpha,\delta)} W_{\{\varepsilon_j\}}(x_\alpha; \overline{\xi})\, dx_\alpha \\ &= \lim_{\delta\to 0} \frac{\mathcal{I}_{\{\varepsilon_j\}}(v; Q'(x^0_\alpha,\delta))}{2\delta^2}. \end{aligned} \tag{5.14}$$

**Lemma 5.1.12.** *$W_{\{\varepsilon_j\}}(x^0_\alpha, \overline{\xi}) \leqslant W_{\rm hom}(x^0_\alpha, \overline{\xi})$ for all $x^0_\alpha \in L$ and all $\overline{\xi} \in \mathbb{Q}^{3\times 2}$.*

*Proof.* Given $k \in \mathbb{N}$, let $T_k \in \mathbb{N}$ and $\phi_k \in \mathcal{S}_{T_k}$ with $\phi_k = 0$ on $\partial(0,T_k)^2 \times I$, be such that

$$W_{\rm hom}(x^0_\alpha, \overline{\xi}) + \frac{1}{k} \geqslant \frac{1}{2{T_k}^2} \int_{(0,T_k)^2\times I} W(x^0_\alpha, y_3, y_\alpha; \overline{\xi} + \nabla_\alpha \phi_k(y) | \nabla_3 \phi_k(y))\, dy.$$

This is possible because of the continuity properties $(A_1)$ of $W$, the growth conditions $(A_3)$ and the density of $\mathcal{S}_{T_k}$ in $\mathcal{W}_{T_k}$. Extend $\phi_k$ periodically with period $T_k$ to $\mathbb{R}^2 \times I$. For $x \in \mathbb{R}^2 \times I$, define $u^k_j(x) := \overline{\xi} \cdot x_\alpha + \varepsilon_j \phi_k(\frac{x_\alpha}{\varepsilon_j}, x_3)$. Let $\delta$ small enough so that $Q'(x^0_\alpha, \delta) \in \mathcal{A}(\omega)$. For fixed $k$, $u^k_j \to v$ in $L^p(Q'(x^0_\alpha,\delta) \times I; \mathbb{R}^3)$ as $j \to \infty$, hence, by (5.14)

$$\begin{aligned} W_{\{\varepsilon_j\}}(x^0_\alpha, \overline{\xi}) &\leqslant \liminf_{\delta\to 0} \liminf_{j\to\infty} \frac{1}{2\delta^2} \int_{Q'(x^0_\alpha,\delta)\times I} W\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \nabla_\alpha u^k_j \Big| \frac{1}{\varepsilon_j} \nabla_3 u^k_j\right) dx \\ &= \liminf_{\delta\to 0} \liminf_{j\to\infty} \frac{1}{2\delta^2} \int_{Q'(x^0_\alpha,\delta)\times I} W\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \overline{\xi} + \nabla_\alpha \phi_k\left(\frac{x_\alpha}{\varepsilon_j}, x_3\right) \Big| \nabla_3 \phi_k\left(\frac{x_\alpha}{\varepsilon_j}, x_3\right)\right) dx. \end{aligned}$$

Define

$$h_k(x_\alpha, y_\alpha) := \int_{-1}^{1} W(x_\alpha, x_3, T_k y_\alpha, \overline{\xi} + \nabla_\alpha \phi_k(T_k y_\alpha, x_3) | \nabla_3 \phi_k(T_k y_\alpha, x_3)) dx_3,$$

for $x_\alpha \in \omega$ and $y_\alpha \in \mathbb{R}^2$.

The continuity of $W$ with respect to $y_\alpha$, its measurability and periodicity properties, and the fact that $T_k \in \mathbb{N}$ lead us to conclude that the function $h_k \in L^1(Q'(x^0_\alpha,\delta); \mathcal{C}_{\rm per}(Q'))$ for

fixed $\delta > 0$. Lemma 2.6.3 together with Fubini's Theorem yields

$$\begin{aligned}
&\lim_{j\to\infty}\int_{Q'(x_\alpha^0,\delta)\times I} W\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \overline{\xi} + \nabla_\alpha \phi_k\left(\frac{x_\alpha}{\varepsilon_j}, x_3\right)\Big|\nabla_3\phi_k\left(\frac{x_\alpha}{\varepsilon_j}, x_3\right)\right)\,dx\\
=\;& \lim_{j\to\infty}\int_{Q'(x_\alpha^0,\delta)} h_k\left(x_\alpha, \frac{x_\alpha}{T_k\varepsilon_j}\right)\,dx_\alpha\\
=\;& \int_{Q'(x_\alpha^0,\delta)}\int_{Q'} h_k(x_\alpha, y_\alpha)\,dy_\alpha\,dx_\alpha\\
=\;& \int_{Q'(x_\alpha^0,\delta)}\int_{Q'\times I} W(x_\alpha, x_3, T_k y_\alpha; \overline{\xi} + \nabla_\alpha\phi_k(T_k y_\alpha, x_3)|\nabla_3\phi_k(T_k y_\alpha, x_3))dy_\alpha\,dx_3\,dx_\alpha.
\end{aligned}$$

Using (5.13) we have

$$\begin{aligned}
&W_{\{\varepsilon_j\}}(x_\alpha^0, \overline{\xi})\\
&\leqslant \liminf_{\delta\to 0}\frac{1}{2\delta^2}\int_{Q'(x_\alpha^0,\delta)}\int_{Q'\times I} W(x_\alpha, x_3, T_k y_\alpha; \overline{\xi} + \nabla_\alpha\phi_k(T_k y_\alpha, x_3)|\nabla_3\phi_k(T_k y_\alpha, x_3))dy_\alpha\,dx_3 dx_\alpha\\
&= \frac{1}{2}\int_{Q'\times I} W(x_\alpha^0, x_3, T_k y_\alpha; \overline{\xi} + \nabla_\alpha\phi_k(T_k y_\alpha, x_3)|\nabla_3\phi_k(T_k y_\alpha, x_3))dy_\alpha\,dx_3\\
&\leqslant W_{\mathrm{hom}}(x_\alpha^0, \overline{\xi}) + \frac{1}{k}.
\end{aligned}$$

Letting $k \to \infty$, we assert the claim.

■

Note that the same argument could be used to prove Lemma 2.5 in Babadjian and Francfort [11].

**Lemma 5.1.13.** $W_{\{\varepsilon_j\}}(x_\alpha^0, \overline{\xi}) \geqslant W_{\mathrm{hom}}(x_\alpha^0, \overline{\xi})$ *for all* $x_\alpha^0 \in L$ *and all* $\overline{\xi} \in \mathbb{Q}^{3\times 2}$.

*Proof.* Let $\{v_j\} \subset W^{1,p}(Q'(x_\alpha^0, \delta) \times I; \mathbb{R}^3)$ be a recovery sequence for the $\Gamma$-limit, i.e.

$$v_j \to 0 \text{ in } L^p(Q'(x_\alpha^0, \delta) \times I; \mathbb{R}^3)$$

and

$$\mathcal{W}_{\{\varepsilon_j\}}(v; Q'(x_\alpha^0, \delta)) = \lim_{j\to\infty}\int_{Q'(x_\alpha^0,\delta)\times I} W\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \overline{\xi} + \nabla_\alpha v_j\Big|\frac{1}{\varepsilon_j}\nabla_3 v_j\right)dx.$$

According to the Decomposition Lemma result for a sequence of scaled gradients, Theorem 2.2.17, there exists a subsequence of $\{\varepsilon_j\}$ (not relabelled) and a sequence $\{u_j\} \subset$

$W^{1,p}(Q'(x_\alpha^0,\delta)\times I;\mathbb{R}^3)$ such that, upon setting $E_j := \{x \in Q'(x_\alpha^0,\delta)\times I : u_j(x) = v_j(x)\}$, then

$$\begin{cases} u_j \to 0 \text{ in } L^p(Q'(x_\alpha^0,\delta)\times I;\mathbb{R}^3), \\ \\ \left\{\left|\left(\nabla_\alpha u_j\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j\right)\right|^p\right\} \text{ is equi-integrable}, \\ \\ \lim\limits_{j\to\infty} \mathcal{L}^3([Q'(x_\alpha^0,\delta)\times I]\setminus E_j) = 0. \end{cases} \tag{5.15}$$

Thus, in view of the $p$-growth condition $(A_3)$ together with (5.15) and Remark 5.1.2 it follows that

$$\begin{aligned} \mathcal{W}_{\{\varepsilon_j\}}(v;Q'(x_\alpha^0,\delta)) &\geqslant \limsup_{j\to\infty}\int_{E_j} W\left(x_\alpha,x_3,\frac{x_\alpha}{\varepsilon_j},\overline{\xi}+\nabla_\alpha u_j\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j\right)dx \\ &= \limsup_{j\to\infty}\int_{Q'(x_\alpha^0,\delta)\times I} W\left(x_\alpha,x_3,\frac{x_\alpha}{\varepsilon_j},\overline{\xi}+\nabla_\alpha u_j\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j\right)dx \\ &\quad -\limsup_{j\to\infty}\int_{[Q'(x_\alpha^0,\delta)\times I]\setminus E_j} W\left(x_\alpha,x_3,\frac{x_\alpha}{\varepsilon_j},\overline{\xi}+\nabla_\alpha u_j\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j\right)dx \\ &\geqslant \limsup_{j\to\infty}\int_{Q'(x_\alpha^0,\delta)\times I} W\left(x_\alpha,x_3,\frac{x_\alpha}{\varepsilon_j},\overline{\xi}+\nabla_\alpha u_j\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j\right)dx. \end{aligned}$$

For any $h\in\mathbb{N}$, we split $Q'(x_\alpha^0,\delta)$ into $h^2$ disjoints cubes $Q'_{i,h}$ of side length $\delta/h$ so that $Q'(x_\alpha^0,\delta) = \bigcup_{i=1}^{h^2} Q'_{i,h}$ and

$$\mathcal{W}_{\{\varepsilon_j\}}(v;Q'(x_\alpha^0,\delta)) \geqslant \limsup_{h\to\infty}\limsup_{j\to\infty}\sum_{i=1}^{h^2}\int_{Q'_{i,h}\times I} W\left(x_\alpha,x_3,\frac{x_\alpha}{\varepsilon_j},\overline{\xi}+\nabla_\alpha u_j\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j\right)dx. \tag{5.16}$$

For every $\eta>0$ and $\lambda>0$, let $K_\eta\subset\Omega$ and $W^{\eta,\lambda}$ be given by Lemma B.1 below (with $N=d=3$, $m=2$ and $f=W$). Then

$$\mathcal{L}^3(\Omega\setminus K_\eta) < \eta. \tag{5.17}$$

On the other hand, define

$$R_j^\lambda := \left\{x\in Q'(x_\alpha^0,\delta)\times I : \left|\left(\overline{\xi}+\nabla_\alpha u_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j(x)\right)\right| \leqslant \lambda\right\}.$$

Chebyshev's inequality implies that there exists a constant $C>0$ – which does not depend on $j$ or $\lambda$ – such that

$$\mathcal{L}^3([Q'(x_\alpha^0,\delta)\times I]\setminus R_j^\lambda) < \frac{C}{\lambda^p}. \tag{5.18}$$

Since $W$ and $W^{\eta,\lambda}$ coincide on $K_\eta \times \mathbb{R}^2 \times \overline{B}(0,\lambda)$, we get in view of (5.16)

$$\mathcal{W}_{\{\varepsilon_j\}}(v; Q'(x_\alpha^0,\delta)) \geqslant \limsup_{\lambda,\eta,h,j} \sum_{i=1}^{h^2} \int_{[Q'_{i,h}\times I]\cap R_j^\lambda \cap K_\eta} W^{\eta,\lambda}\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \overline{\xi} + \nabla_\alpha u_j \Big| \frac{1}{\varepsilon_j}\nabla_3 u_j\right) dx.$$

By virtue of inequality (0.1) below (Appendix) and (5.17),

$$\sum_{i=1}^{h^2} \int_{([Q'_{i,h}\times I]\cap R_j^\lambda)\setminus K_\eta} W^{\eta,\lambda}\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \overline{\xi} + \nabla_\alpha u_j \Big| \frac{1}{\varepsilon_j}\nabla_3 u_j\right) dx \leqslant \beta(1+\lambda^p)\eta \xrightarrow[\eta\to 0]{} 0,$$

uniformly in $(j,h)$, so that

$$\mathcal{W}_{\{\varepsilon_j\}}(v; Q'(x_\alpha^0,\delta)) \geqslant \limsup_{\lambda,\eta,h,j} \sum_{i=1}^{h^2} \int_{[Q'_{i,h}\times I]\cap R_j^\lambda} W^{\eta,\lambda}\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \overline{\xi} + \nabla_\alpha u_j \Big| \frac{1}{\varepsilon_j}\nabla_3 u_j\right) dx.$$

Fix $y_\alpha \in Q'$. Since $W^{\eta,\lambda}(\cdot, y_\alpha; \cdot)$ is continuous, it is uniformly continuous on $\overline{\Omega} \times \overline{B}(0,\lambda)$, and we define the modulus of continuity $\omega_{\eta,\lambda} : Q' \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\omega_{\eta,\lambda}(y_\alpha, t) := \sup_{(x,\xi),\,(x',\xi')\in\overline{\Omega}\times\overline{B}(0,\lambda)} \{|W^{\eta,\lambda}(x, y_\alpha;\xi) - W^{\eta,\lambda}(x', y_\alpha;\xi')| : |(x;\xi)-(x';\xi')| \leqslant t\}.$$

Then

$$\begin{cases} \omega_{\eta,\lambda}(\cdot,t) \text{ is lower semicontinuous for all } t\in\mathbb{R}^+, \\ \\ \omega_{\eta,\lambda}(y_\alpha,\cdot) \text{ is continuous and increasing for all } y_\alpha \in Q', \\ \\ \omega_{\eta,\lambda}(y_\alpha,0) = 0 \text{ for all } y_\alpha\in Q', \end{cases}$$

and

$$|W^{\eta,\lambda}(x, y_\alpha;\xi) - W^{\eta,\lambda}(x', y_\alpha;\xi')| \leqslant \omega_{\eta,\lambda}(y_\alpha, |x-x'|+|\xi-\xi'|) \text{ for all } (x,\xi),\ (x',\xi') \in \overline{\Omega}\times\overline{B}(0,\lambda). \tag{5.19}$$

The first property is a consequence of the fact that the supremum of continuous functions is lower semicontinuous, while the other ones are classical properties of moduli of continuity.

For all $t\in\mathbb{R}^+$, we extend $\omega_{\eta,\lambda}(\cdot,t)$ to $\mathbb{R}^2$ by $Q'$-periodicity. Since $W^{\eta,\lambda}(x,\cdot\,;\xi)$ is $Q'$-periodic, inequality (5.19) holds for all $y_\alpha \in \mathbb{R}^2$. Consequently, for every $(x_\alpha, x_3) \in [Q'_{i,h}\times I]\cap R_j^\lambda$

and every $x'_\alpha \in Q'_{i,h}$,

$$\left| W^{\eta,\lambda}\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \overline{\xi} + \nabla_\alpha u_j(x_\alpha, x_3) \Big| \frac{1}{\varepsilon_j}\nabla_3 u_j(x_\alpha, x_3)\right) \right.$$
$$\left. - W^{\eta,\lambda}\left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \overline{\xi} + \nabla_\alpha u_j(x_\alpha, x_3) \Big| \frac{1}{\varepsilon_j}\nabla_3 u_j(x_\alpha, x_3)\right)\right|$$
$$\leqslant \omega_{\eta,\lambda}\left(\frac{x_\alpha}{\varepsilon_j}, |x_\alpha - x'_\alpha|\right) \leqslant \omega_{\eta,\lambda}\left(\frac{x_\alpha}{\varepsilon_j}, \frac{\sqrt{2}\delta}{h}\right).$$

We get, after integration in $(x_\alpha, x_3, x'_\alpha)$ and summation,

$$\sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \int_{Q'_{i,h}} \left\{ \int_{R_j^\lambda \cap [Q'_{i,h} \times I]} \left| W^{\eta,\lambda}\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \overline{\xi} + \nabla_\alpha u_j(x_\alpha, x_3) \Big| \frac{1}{\varepsilon_j}\nabla_3 u_j(x_\alpha, x_3)\right) \right.\right.$$
$$\left.\left. - W^{\eta,\lambda}\left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \overline{\xi} + \nabla_\alpha u_j(x_\alpha, x_3) \Big| \frac{1}{\varepsilon_j}\nabla_3 u_j(x_\alpha, x_3)\right)\right| dx \right\} dx'_\alpha$$
$$\leqslant 2 \int_{Q'(x^0_\alpha, \delta)} \omega_{\eta,\lambda}\left(\frac{x_\alpha}{\varepsilon_j}, \frac{\sqrt{2}\delta}{h}\right) dx_\alpha.$$

Riemann-Lebesgue's Lemma applied to the $Q'$-periodic function $\omega_{\eta,\lambda}(\,\cdot\,, \sqrt{2}\delta/h)$ yields,

$$\lim_{j\to\infty} 2 \int_{Q'(x^0_\alpha, \delta)} \omega_{\eta,\lambda}\left(\frac{x_\alpha}{\varepsilon_j}, \frac{\sqrt{2}\delta}{h}\right) dx_\alpha = 2\delta^2 \int_{Q'} \omega_{\eta,\lambda}\left(x_\alpha, \frac{\sqrt{2}\delta}{h}\right) dx_\alpha,$$

and by Levi's Monotone Convergence Theorem

$$\lim_{h\to\infty} 2\delta^2 \int_{Q'} \omega_{\eta,\lambda}\left(x_\alpha, \frac{\sqrt{2}\delta}{h}\right) dx_\alpha = 0.$$

Hence

$$\mathcal{W}_{\{\varepsilon_j\}}(v; Q'(x^0_\alpha, \delta)) \geqslant$$
$$\limsup_{\lambda,\eta,h,j} \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \int_{Q'_{i,h}} \left\{ \int_{[Q'_{i,h} \times I] \cap R_j^\lambda} W^{\eta,\lambda}\left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}; \overline{\xi} + \nabla_\alpha u_j(x_\alpha, x_3) \Big| \frac{1}{\varepsilon_j}\nabla_3 u_j(x_\alpha, x_3)\right) dx \right\} dx'_\alpha.$$

We define the following sets which depend on all parameters $(\eta, \lambda, i, h, n)$

$$\begin{cases} T := \{(x'_\alpha, x_\alpha, x_3) \in Q'_{i,h} \times Q'_{i,h} \times I : (x'_\alpha, x_3) \in K_\eta \text{ and } (x_\alpha, x_3) \in R_j^\lambda\}, \\[2ex] T_1 := \{(x'_\alpha, x_\alpha, x_3) \in Q'_{i,h} \times Q'_{i,h} \times I : (x'_\alpha, x_3) \notin K_\eta \text{ and } (x_\alpha, x_3) \in R_j^\lambda\}, \\[2ex] T_2 := \{(x'_\alpha, x_\alpha, x_3) \in Q'_{i,h} \times Q'_{i,h} \times I : (x_\alpha, x_3) \notin R_j^\lambda\}, \end{cases}$$

and note that $Q'_{i,h} \times Q'_{i,h} \times I = T \cup T_1 \cup T_2$. Since $W(\cdot, y_\alpha; \cdot)$ and $W^{\eta,\lambda}(\cdot, y_\alpha; \cdot)$ coincide on $K_\eta \times \overline{B}(0,\lambda)$, we have

$$\begin{aligned}
&\mathcal{W}_{\{\varepsilon_j\}}(v; Q'(x^0_\alpha, \delta)) \\
&\quad \geqslant \limsup_{\lambda,\eta,h,j} \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \int_T W^{\eta,\lambda}\left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \overline{\xi} + \nabla_\alpha u_j(x_\alpha, x_3) \Big| \frac{1}{\varepsilon_j} \nabla_3 u_j(x_\alpha, x_3)\right) dx\, dx'_\alpha \\
&\quad = \limsup_{\lambda,\eta,h,j} \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \int_T W\left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \overline{\xi} + \nabla_\alpha u_j(x_\alpha, x_3) \Big| \frac{1}{\varepsilon_j} \nabla_3 u_j(x_\alpha, x_3)\right) dx\, dx'_\alpha.
\end{aligned} \tag{5.20}$$

We will prove that the corresponding integrals over $T_1$ and $T_2$ are zero. Indeed, in view of (5.17) and the $p$-growth condition $(A_3)$,

$$\begin{aligned}
&\sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \int_{T_1} W\left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \overline{\xi} + \nabla_\alpha u_j(x_\alpha, x_3) \Big| \frac{1}{\varepsilon_j} \nabla_3 u_j(x_\alpha, x_3)\right) dx\, dx'_\alpha \\
&\qquad\qquad \leqslant \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \mathcal{L}^2(Q'_{i,h})\, \mathcal{L}^3([Q'_{i,h} \times I] \setminus K_\eta)\, \beta(1+\lambda^p) \\
&\qquad\qquad < \beta(1+\lambda^p)\eta \xrightarrow[\eta \to 0]{} 0,
\end{aligned} \tag{5.21}$$

uniformly in $(j,h)$. The bound from above in $(A_3)$, the equi-integrability of the sequence $\left\{\left|\left(\nabla_\alpha u_j \big| \frac{1}{\varepsilon_j} \nabla_3 u_j\right)\right|^p\right\}$ and (5.18) imply that

$$\begin{aligned}
&\sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \int_{T_2} W\left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \overline{\xi} + \nabla_\alpha u_j(x_\alpha, x_3) \Big| \frac{1}{\varepsilon_j} \nabla_3 u_j(x_\alpha, x_3)\right) dx\, dx'_\alpha \\
&\qquad \leqslant \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \mathcal{L}^2(Q'_{i,h})\, \beta \int_{[Q'_{i,h} \times I] \setminus R^\lambda_j} \left(1 + \left|\left(\nabla_\alpha u_j \Big| \frac{1}{\varepsilon_j} \nabla_3 u_j\right)\right|^p\right) dx \\
&\qquad = \beta \int_{[Q'(x^0_\alpha,\delta) \times I] \setminus R^\lambda_j} \left(1 + \left|\left(\nabla_\alpha u_j \Big| \frac{1}{\varepsilon_j} \nabla_3 u_j\right)\right|^p\right) dx \xrightarrow[\lambda \to \infty]{} 0,
\end{aligned} \tag{5.22}$$

uniformly in $(\eta, n, h)$. Thus, in view of (5.20), (5.21), (5.22), Fatou's Lemma yields

$$\begin{aligned}
&\mathcal{W}_{\{\varepsilon_j\}}(v; Q'(x^0_\alpha, \delta)) \\
&\geqslant \limsup_{h\to\infty} \limsup_{j\to\infty} \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \int_{Q'_{i,h}} \int_{Q'_{i,h} \times I} W\left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \overline{\xi} + \nabla_\alpha u_j(x_\alpha, x_3) \Big| \frac{1}{\varepsilon_j} \nabla_3 u_j(x_\alpha, x_3)\right) dx\, dx'_\alpha \\
&\geqslant \limsup_{h\to\infty} \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \int_{Q'_{i,h}} \liminf_{j\to\infty} \int_{Q'_{i,h} \times I} W\left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \overline{\xi} + \nabla_\alpha u_j(x_\alpha, x_3) \Big| \frac{1}{\varepsilon_j} \nabla_3 u_j(x_\alpha, x_3)\right) dx\, dx'_\alpha.
\end{aligned}$$

Fix $x'_\alpha \in Q'_{i,h}$ such that $W_{\rm hom}(x'_\alpha, \overline{\xi})$ is well defined and set $Z(x;\xi) := W(x'_\alpha, x_3, x_\alpha; \xi)$. It is easy to check that $Z$ is a Carathéodory integrand. Hence, applying (3.10), we get since $u_j \to 0$ in $L^p(Q'(x^0_\alpha, \delta) \times I; \mathbb{R}^3)$,

$$2\frac{\delta^2}{h^2}\overline{Z}(\overline{\xi}) \leqslant \liminf_{j\to\infty} \int_{Q'(x^0_\alpha,\delta)\times I} Z\left(\frac{x_\alpha}{\varepsilon_j}, x_3, \overline{\xi} + \nabla_\alpha u_j(x) \Big| \frac{1}{\varepsilon}\nabla_3 u_j(x)\right)\,dx,$$

where

$$\begin{aligned}\overline{Z}(\overline{\xi}) \quad := \quad & \inf_{T>0,\,\phi}\Big\{\int_{(0,T)^2\times I} Z(x, \overline{\xi} + \nabla_\alpha\phi(x)|\nabla_3\phi(x))\,dx : \\ & \qquad \phi \in W^{1,p}((0,T)^2\times I;\mathbb{R}^3), \quad \phi = 0 \text{ on } \partial(0,T)^2\times I\Big\}.\end{aligned}$$

In view of the previous formula together with (5.3) and Remark 5.1.6, we have that $\overline{Z}(\overline{\xi}) = W_{\rm hom}(x'_\alpha, \overline{\xi})$. Then

$$\liminf_{j\to\infty}\int_{Q'_{i,h}\times I} W\left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_j}, \overline{\xi} + \nabla_\alpha u_j(x_\alpha, x_3)\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j(x_\alpha, x_3)\right)dx \geqslant \frac{2\delta^2}{h^2}\, W_{\rm hom}(x'_\alpha, \overline{\xi}),$$

and so

$$\mathcal{W}_{\{\varepsilon_j\}}(v; Q'(x^0_\alpha,\delta)) \geqslant \limsup_{h\to\infty}\sum_{i=1}^{h^2}\frac{h^2}{\delta^2}\int_{Q'_{i,h}}\frac{2\delta^2}{h^2}\, W_{\rm hom}(x'_\alpha,\overline{\xi})dx'_\alpha = 2\int_{Q'(x^0_\alpha,\delta)} W_{\rm hom}(x'_\alpha, \overline{\xi})dx'_\alpha.$$

Dividing both sides of the previous inequality by $\delta^2$ and passing to the limit when $\delta \downarrow 0$, we obtain by (5.12) and (5.14)

$$W_{\{\varepsilon_j\}}(x^0_\alpha, \overline{\xi}) \geqslant W_{\rm hom}(x^0_\alpha, \overline{\xi}).$$

■

**Proposition 5.1.14.** *$W_{\{\varepsilon_j\}}(x_\alpha, \overline{\xi}) = W_{\rm hom}(x_\alpha, \overline{\xi})$ a.e. $x_\alpha \in \omega$ and all $\overline{\xi} \in \mathbb{R}^{3\times 2}$.*

*Proof.* Let $E$ be the intersection of the set $L$ with the subset of points $x^0_\alpha \in \omega$ where $W_{\{\varepsilon_j\}}(x^0_\alpha; \cdot)$ and $W_{\rm hom}(x^0_\alpha; \cdot)$ are continuous (see Lemma 5.1.9). Then $\mathcal{L}^2(\omega \setminus E) = 0$ and in view of Lemma 5.1.12 and 5.1.13, we have that for all $x^0_\alpha \in E$ and for all $\overline{\xi} \in \mathbb{Q}^{3\times 2}$, $W_{\{\varepsilon_j\}}(x^0_\alpha, \overline{\xi}) = W_{\rm hom}(x^0_\alpha, \overline{\xi})$. Since $W_{\{\varepsilon_j\}}(x^0_\alpha; \cdot)$ and $W_{\rm hom}(x^0_\alpha; \cdot)$ are continuous for all $x^0_\alpha \in E$, the equality $W_{\{\varepsilon_j\}}(x^0_\alpha, \overline{\xi}) = W_{\rm hom}(x^0_\alpha, \overline{\xi})$ holds true for $x^0_\alpha \in E$ and all $\overline{\xi} \in \mathbb{R}^{3\times 2}$.

■

**Corollary 5.1.15.** $\Gamma(L^p(A \times I))$- $\lim_{\varepsilon} \mathcal{W}_\varepsilon(\,\cdot\,; A) = \mathcal{W}_{\text{hom}}(\,\cdot\,; A)$ *for all* $A \in \mathcal{A}(\omega)$, *where* $\mathcal{W}_{\text{hom}}(\cdot; A)$ *is the functional defined in (5.9).*

*Proof.* From Proposition 5.1.14 we can conclude that $\mathcal{W}_{\text{hom}}(\cdot; A)$ is well defined and

$$\Gamma(L^p(A \times I))\text{-} \lim_{j} \mathcal{W}_{\varepsilon_j}(\,\cdot\,; A) = \mathcal{W}_{\text{hom}}(\,\cdot\,; A)$$

for all $A \in \mathcal{A}(\omega)$ (see Remark 5.1.10). Since this limit does not depend upon the extracted subsequence, in view of Remark 2.5.9, the whole sequence $\{\mathcal{W}_\varepsilon(\,\cdot\,; A)\}_\varepsilon$ $\Gamma(L^p(A\times I))$-converges to $\mathcal{W}_{\text{hom}}(\,\cdot\,; A)$ for each $A \in \mathcal{A}(\omega)$.

■

The proof of Theorem 5.1.1 is a consequence of Corollary 5.1.15 taking $A = \omega$.

## 5.2 When heterogeneities are allowed also in the transverse direction

Following the lines of the previous section and those of Babadjian and Francfort [11] we assume that

$(A_1)$ $W(x, \cdot\,, \cdot\,; \cdot\,)$ is continuous for a.e. $x \in \Omega$;

$(A_2)$ $W(\,\cdot\,, \cdot\,, \cdot\,; \xi)$ is $\mathcal{L}^3 \otimes \mathcal{L}^3 \otimes \mathcal{L}^2$-measurable for all $\xi \in \mathbb{R}^{3\times3}$;

$$(A_3)\quad \begin{cases} y_\alpha \mapsto W(x, y_\alpha, y_3, z_\alpha; \xi) \text{ is } Q'\text{-periodic for all } (z_\alpha, y_3, \xi) \in \mathbb{R}^3 \times \mathbb{R}^{3\times3} \text{ and a.e. } x \in \Omega, \\[2ex] (z_\alpha, y_3) \mapsto W(x, y_\alpha, y_3, z_\alpha; \xi) \text{ is } Q \text{ -periodic for all } (y_\alpha, \xi) \in \mathbb{R}^2 \times \mathbb{R}^{3\times3} \text{ and a.e. } x \in \Omega. \end{cases}$$

$(A_4)$ there exists $\beta > 0$ such that

$$\frac{1}{\beta}|\xi|^p - \beta \leqslant W(x, y, z_\alpha; \xi) \leqslant \beta(1+|\xi|^p) \quad \text{for all } (y, z_\alpha, \xi) \in \mathbb{R}^3\times\mathbb{R}^2\times\mathbb{R}^{3\times3} \text{ and a.e } x \in \Omega.$$

We prove the following theorem.

**Theorem 5.2.1.** *Let* $W : \Omega \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times3} \to \mathbb{R}$ *be a function satisfying* $(A_1)$-$(A_4)$. *For each* $\varepsilon > 0$, *consider the functional* $\mathcal{W}_\varepsilon : L^p(\Omega; \mathbb{R}^3) \to \overline{\mathbb{R}}$ *defined by*

$$\mathcal{W}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega W\left(x, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}; \nabla_\alpha u(x)\Big|\frac{1}{\varepsilon}\nabla_3 u(x)\right) dx & \textit{if } u \in W^{1,p}(\Omega; \mathbb{R}^3), \\[2ex] \infty & \textit{otherwise.} \end{cases} \tag{5.23}$$

*Then the $\Gamma(L^p(\Omega))$-limit of the family $\{\mathcal{W}_\varepsilon\}_\varepsilon$ is given by the functional*

$$\mathcal{W}_{\mathrm{hom}}(u) := \begin{cases} 2\displaystyle\int_\omega \overline{W}_{\mathrm{hom}}(x_\alpha; \nabla_\alpha u(x_\alpha))\,dx_\alpha & \text{if } u \in W^{1,p}(\omega;\mathbb{R}^3), \\ \\ \infty & \text{otherwise,} \end{cases} \tag{5.24}$$

*where $\overline{W}_{\mathrm{hom}}$ is defined, for all $\overline{\xi} \in \mathbb{R}^{3\times 2}$ and a.e. $x_\alpha \in \omega$, by*

$$\begin{aligned} \overline{W}_{\mathrm{hom}}(x_\alpha;\overline{\xi}) \quad := \quad & \lim_{T\to\infty} \inf_\phi \Big\{ \frac{1}{2T^2} \int_{(0,T)^2\times I} W_{\mathrm{hom}}(x_\alpha, y_3, y_\alpha; \overline{\xi} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y))\,dy : \\ & \phi \in W^{1,p}((0,T)^2 \times I;\mathbb{R}^3),\ \phi = 0 \text{ on } \partial(0,T)^2 \times I \Big\} \end{aligned} \tag{5.25}$$

*and*

$$\begin{aligned} W_{\mathrm{hom}}(x, y_\alpha; \xi) \quad := \quad & \lim_{T\to\infty} \inf_\phi \Big\{ \frac{1}{T^3} \int_{(0,T)^3} W(x, y_\alpha, z_3, z_\alpha; \xi + \nabla\phi(z))\,dz : \\ & \phi \in W_0^{1,p}((0,T)^3;\mathbb{R}^3) \Big\}, \end{aligned} \tag{5.26}$$

*for a.e $x \in \Omega$ and all $(y_\alpha, \xi) \in \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$.*

*Remark* 5.2.2. As before, the limits as $T \to \infty$ in (5.25) and (5.26) can be replaced by an infimum taken over $T > 0$.

Let us formally justify the periodicity assumptions $(A_3)$: Since the volume of $\Omega_\varepsilon$ is of order $\varepsilon$ and $\varepsilon^2 \ll \varepsilon$, in a first step, we can think of $\varepsilon$ as being a fixed parameter and let $\varepsilon^2$ tend to zero. At this point dimension reduction is not occurring and (5.23) can be seen as a single one-scale homogenization problem as in (4.1), in which it is natural to assume $(z_\alpha, y_3) \mapsto W(x, y_\alpha, y_3, z_\alpha; \xi)$ to be $Q$-periodic. The homogenization formula for this case gives us an homogenized stored energy density $W_{\mathrm{hom}}(x, y_\alpha; \xi)$ that, in a second step, is used as the integrand of a problem similar to the one treated in Section 5.1. In particular, the required $Q'$- periodicity of $W_{\mathrm{hom}}(x, \cdot\,; \xi)$ can be obtained from the $Q'$-periodicity of $y_\alpha \mapsto W(x, y_\alpha, y_3, z_\alpha; \xi)$.

We would also like to remark the difference between assumptions $(H_2)$ in Section 4.2 and $(A_2)$: In the 3D-2D case, if we assume only $W(\cdot, y, z_\alpha; \xi)$ to be $\mathcal{L}^3$-measurable for all $(y, z_\alpha, \xi) \in \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$, then the functional (5.23) is well defined (at least for $u \in W^{1,p}(\Omega;\mathbb{R}^3)$) but this would not be the case for the integrals in (5.25) and (5.26) because their integrands would be only separately measurable.

We will first study the case where $W$ does not depend on the macroscopic variable $x$ (Theorem 5.2.4). We observe that the proof of Theorem 5.2.4 is very close to its $N$-dimensional analogue Theorem 4.2.2, the main difference being the use of the Scaled Gradients Decomposition Lemma, Lemma 2.2.17, in place of Lemma 2.2.16. As before (compare Theorems 4.1.1 and 5.1.1), Theorem 5.2.1 and Theorem 4.2.1 cannot be treated similarly. We will need an argument along the lines of what is done in Section 5.1 where we had to consider a suitable continuous extension of $W$; in this case we will use a corollary of the Scorza-Dragoni Theorem and Tietze Extension Theorem (see Lemma B.3 in the Appendix).

We organize this section as follows. In Subsection 5.2.1 we discuss the main properties of $W_{\text{hom}}$ and $\overline{W}_{\text{hom}}$. Then, in Subsection 5.2.2 we address the case where $W$ is independent of the macroscopic in-plane variable $x_\alpha$ (Theorem 5.2.4). Finally, Theorem 5.2.1 is proved in Subsection 5.2.3.

*Remark* 5.2.3. As before, without loss of generality we assume that $W$ is non negative upon replacing $W$ by $W + \beta$ which is non negative in view of $(A_4)$.

### 5.2.1 Properties of the homogenized density

As in Section 4.2.1 we can see that the function $W_{\text{hom}}$ given in (5.26) is well defined and is (equivalent to) a Carathéodory function:

$$W_{\text{hom}}(\cdot, \cdot; \xi) \text{ is } \mathcal{L}^3 \otimes \mathcal{L}^2\text{-measurable for all } \xi \in \mathbb{R}^{3\times 3}, \tag{5.27}$$

$$W_{\text{hom}}(x, y_\alpha; \cdot) \text{ is continuous for } \mathcal{L}^3 \otimes \mathcal{L}^2\text{-a.e } (x, y_\alpha) \in \Omega \times \mathbb{R}^2. \tag{5.28}$$

By condition $(A_3)$ it follows that

$$W_{\text{hom}}(x, \cdot; \xi) \text{ is } Q'\text{-periodic for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^{3\times 3}. \tag{5.29}$$

Moreover, $W_{\text{hom}}$ is quasiconvex in the $\xi$ variable and satisfies the same $p$-growth and $p$-coercivity condition as $W$:

$$\frac{1}{\beta}|\xi|^p - \beta \leqslant W_{\text{hom}}(x, y_\alpha; \xi) \leqslant \beta(1+|\xi|^p) \quad \text{for a.e. } x \in \Omega \text{ and all } (y_\alpha, \xi) \in \mathbb{R}^2 \times \mathbb{R}^{3\times 3}, \tag{5.30}$$

where $\beta$ is the constant in $(A_4)$. Just as before, (5.27), (5.28) and (5.30) imply that the function $\overline{W}_{\text{hom}}$ given in (5.25) is also well defined, and is (equivalent to) a Carathéodory function, which implies that the definition of $\mathcal{W}_{\text{hom}}$ makes sense on $W^{1,p}(\Omega; \mathbb{R}^3)$. Finally,

$\overline{W}_{\text{hom}}$ is also quasiconvex in the $\overline{\xi}$ variable and satisfies the same $p$-growth and $p$-coercivity condition as $W$ and $W_{\text{hom}}$:

$$\frac{1}{\beta}|\overline{\xi}|^p - \beta \leqslant \overline{W}_{\text{hom}}(x_\alpha;\overline{\xi}) \leqslant \beta(1+|\overline{\xi}|^p) \quad \text{for a.e. } x_\alpha \in \omega \text{ and all } \overline{\xi} \in \mathbb{R}^{3\times 2}, \tag{5.31}$$

where, as before, $\beta$ is the constant in $(A_4)$.

### 5.2.2 Main result when the integrands do not depend on the macroscopic variable

In this section, we assume that $W$ does not depend explicitly on $x_\alpha$, namely $W : I \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3} \to [0,\infty)$. For each $\varepsilon > 0$, consider the functional $\mathcal{W}_\varepsilon : L^p(\Omega;\mathbb{R}^3) \to [0,\infty]$ defined by

$$\mathcal{W}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega W\left(x_3, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}; \nabla_\alpha u(x)\Big|\frac{1}{\varepsilon}\nabla_3 u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^3), \\ \\ \infty & \text{otherwise.} \end{cases} \tag{5.32}$$

Our objective is to prove the following result.

**Theorem 5.2.4.** *Under assumptions $(A_1)$-$(A_4)$ the $\Gamma(L^p(\Omega))$-limit of the family $\{\mathcal{W}_\varepsilon\}_\varepsilon$ is given by*

$$\mathcal{W}_{\text{hom}}(u) = \begin{cases} 2\displaystyle\int_\omega \overline{W}_{\text{hom}}(\nabla_\alpha u(x_\alpha))\, dx_\alpha & \textit{if } u \in W^{1,p}(\omega;\mathbb{R}^3), \\ \\ \infty & \textit{otherwise,} \end{cases}$$

*where $\overline{W}_{\text{hom}}$ is defined, for all $\overline{\xi} \in \mathbb{R}^{3\times 2}$, by*

$$\begin{aligned} \overline{W}_{\text{hom}}(\overline{\xi}) \quad := \quad \lim_{T\to\infty} \inf_\phi \Big\{ \frac{1}{2T^2} \int_{(0,T)^2\times I} & W_{\text{hom}}(y_3, y_\alpha; \overline{\xi} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y))\, dy : \\ & \phi \in W^{1,p}((0,T)^2 \times I;\mathbb{R}^3) \textit{ and } \phi = 0 \textit{ on } \partial(0,T)^2 \times I \Big\} \end{aligned} \tag{5.33}$$

*and*

$$\begin{aligned} W_{\text{hom}}(y_3, y_\alpha; \xi) \quad := \quad \lim_{T\to\infty} \inf_\phi \Big\{ \frac{1}{T^3} \int_{(0,T)^3} & W(y_3, y_\alpha, z_3, z_\alpha; \xi + \nabla\phi(z))\, dz : \\ & \phi \in W_0^{1,p}((0,T)^3;\mathbb{R}^3) \Big\}, \end{aligned} \tag{5.34}$$

*for all $(y,\xi) \in \mathbb{R}^3 \times \mathbb{R}^{3\times 3}$.*

Since the proofs are very similar to those of Section 4.2.2, we just sketch them highlighting the main differences.

*STEP 1.* Localization and existence of $\Gamma$-convergent subsequences.

For the same reason than in the proof of Theorem 4.2.2 in Section 4.2.2, we localize the functionals given in (5.32) on the class of bounded open subsets of $\mathbb{R}^2$, denoted by $\mathcal{A}_0(\omega)$. For each $\varepsilon > 0$, consider $\mathcal{W}_\varepsilon : L^p(\mathbb{R}^2 \times I; \mathbb{R}^3) \times \mathcal{A}_0(\omega) \to [0, \infty]$ defined by

$$\mathcal{W}_\varepsilon(u; A) := \begin{cases} \displaystyle\int_{A\times I} W\left(x_3, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}; \nabla_\alpha u(x) \Big| \frac{1}{\varepsilon}\nabla_3 u(x)\right) dx & \text{if } u \in W^{1,p}(A \times I; \mathbb{R}^3), \\ \\ \infty & \text{otherwise.} \end{cases} \tag{5.35}$$

Given $\{\varepsilon_n\}_n \downarrow 0$ and $A \in \mathcal{A}_0(\mathbb{R}^2)$, consider the $\Gamma$-lower limit of $\{\mathcal{W}_{\varepsilon_n}(\cdot\,; A)\}_n$ for the $L^p(A \times I; \mathbb{R}^3)$-topology, defined for $u \in L^p(\mathbb{R}^2 \times I; \mathbb{R}^3)$, by

$$\mathcal{W}_{\{\varepsilon_n\}}(u; A) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} \mathcal{W}_{\varepsilon_n}(u_n; A) : \quad u_n \to u \text{ in } L^p(A \times I; \mathbb{R}^3) \right\}.$$

In view of the $p$-coercivity condition $(A_4)$, for each $A \in \mathcal{A}_0(\mathbb{R}^2)$ it follows that $\mathcal{W}_{\{\varepsilon_n\}}(u; A)$ is infinite whenever $u \in L^p(\mathbb{R}^2 \times I; \mathbb{R}^3) \setminus W^{1,p}(A; \mathbb{R}^3)$, so it suffices to study the case where $u \in W^{1,p}(A; \mathbb{R}^3)$. Arguing exactly as in Section 5.1, we can prove the existence of a subsequence $\{\varepsilon_{n_j}\}_j \equiv \{\varepsilon_j\}_j$ such that $\mathcal{W}_{\{\varepsilon_j\}}(\cdot\,; A)$ is the $\Gamma(L^p(A\times I)$-limit of $\{\mathcal{W}_{\varepsilon_j}(\cdot\,; A)\}_{n\in\mathbb{N}}$ for each $A \in \mathcal{A}_0(\mathbb{R}^2)$.

Our next ojective is to show that for every $A \in \mathcal{A}_0(\mathbb{R}^2)$ and every $u \in W^{1,p}(A; \mathbb{R}^3)$, then $\mathcal{W}_{\{\varepsilon_j\}}(u; A) = \mathcal{W}_{\mathrm{hom}}(u; A)$, where $\mathcal{W}_{\mathrm{hom}} : L^p(\mathbb{R}^2 \times I; \mathbb{R}^3) \times \mathcal{A}_0(\mathbb{R}^2) \to [0, \infty]$ is given by

$$\mathcal{W}_{\mathrm{hom}}(u; A) = \begin{cases} 2\displaystyle\int_A \overline{W}_{\mathrm{hom}}(\nabla_\alpha u(x_\alpha))\, dx_\alpha & \text{if } u \in W^{1,p}(A; \mathbb{R}^3), \\ \\ \infty & \text{otherwise.} \end{cases}$$

*STEP 2.* Integral representation of the $\Gamma$-limit.

Following the proof of Lemma 3.2.2, it is possible to show that for each $A \in \mathcal{A}_0(\mathbb{R}^2)$ and all $u \in W^{1,p}(A; \mathbb{R}^3)$, the restriction of $\mathcal{W}_{\{\varepsilon_j\}}(u; \cdot)$ to $\mathcal{A}(A)$ is a Radon measure, absolutely

continuous with respect to the two-dimensional Lebesgue measure. But as before, one has to ensure that the integral representation given by Theorem 2.4.1 is independent of the open set $A \in \mathcal{A}_0(\mathbb{R}^2)$. The following result, prevents this dependence from holding since it leads to an homogeneous integrand, as will be seen in Lemma 5.2.6 below.

**Lemma 5.2.5.** *For all $\overline{\xi} \in \mathbb{R}^{3\times 2}$, $y_\alpha^0$ and $z_\alpha^0 \in \mathbb{R}^2$, and $\delta > 0$*

$$\mathcal{W}_{\{\varepsilon_j\}}(\overline{\xi}\,\cdot\,; Q'(y_\alpha^0,\delta)) = \mathcal{W}_{\{\varepsilon_j\}}(\overline{\xi}\,\cdot\,; Q'(z_\alpha^0,\delta)).$$

*Proof.* It is obviously enough to show that

$$\mathcal{W}_{\{\varepsilon_j\}}(\overline{\xi}\,\cdot\,; Q'(y_\alpha^0,\delta)) \geqslant \mathcal{W}_{\{\varepsilon_j\}}(\overline{\xi}\,\cdot\,; Q'(z_\alpha^0,\delta)).$$

According to Theorem 2.2.17 and Lemma 3.2.2 there exists a sequence $\{u_j\} \subset W^{1,p}(Q'(y_\alpha^0,\delta)\times I;\mathbb{R}^3)$ such that $\left\{\left|\left(\nabla_\alpha u_j\middle|\frac{1}{\varepsilon_j}\nabla_3 u_j\right)\right|^p\right\}$ is equi-integrable, $u_j = 0$ on $\partial Q'(y_\alpha^0,\delta)\times I$, $u_j \to 0$ in $L^p(Q'(y_\alpha^0,\delta)\times I;\mathbb{R}^3)$ and

$$\mathcal{W}_{\{\varepsilon_j\}}(\overline{\xi}\cdot; Q'(y_\alpha^0,\delta)) = \lim_{j\to\infty}\int_{Q'(y_\alpha^0,\delta)\times I} W\left(x_3, \frac{x}{\varepsilon_j}, \frac{x_\alpha}{\varepsilon_j^2}; \overline{\xi} + \nabla_\alpha u_j(x)\middle|\frac{1}{\varepsilon_j}\nabla_3 u_j(x)\right) dx.$$

We argue exactly as in the proof of Lemma 4.2.4 with $y_\alpha^0$ and $z_\alpha^0$ in place of $y_0$ and $z_0$. For all $j \in \mathbb{N}$, extend $u_j$ by zero to the whole $\mathbb{R}^2 \times I$ and set $v_j(x_\alpha, x_3) = u_j(x_\alpha + x_\alpha^{\varepsilon_j}, x_3)$ for $(x_\alpha, x_3) \in Q'(z_\alpha^0,\delta)\times I$, where $x_\alpha^{\varepsilon_j} := m_{\varepsilon_j}\varepsilon_j - \varepsilon_j^2 l_{\varepsilon_j}$. Then $\{v_j\} \subset W^{1,p}(Q'(z_\alpha^0,\delta)\times I;\mathbb{R}^3)$, $v_j \to 0$ in $L^p(Q'(z_\alpha^0,\delta)\times I;\mathbb{R}^3)$, the sequence $\left\{\left|\left(\nabla_\alpha v_j\middle|\frac{1}{\varepsilon_j}\nabla_3 v_j\right)\right|^p\right\}$ is equi-integrable and

$$\begin{aligned}&\mathcal{W}_{\{\varepsilon_j\}}(\overline{\xi}\cdot; Q'(y_\alpha^0,\delta))\\ &= \limsup_{j\to\infty}\int_{Q'(z_\alpha^0,\delta)\times I} W\left(x_3, \frac{x_\alpha}{\varepsilon_j} - \varepsilon_j l_{\varepsilon_j}, \frac{x_3}{\varepsilon_j}, \frac{x_\alpha}{\varepsilon_j^2}; \overline{\xi} + \nabla_\alpha v_j(x)\middle|\frac{1}{\varepsilon_j}\nabla_3 v_j(x)\right) dx,\end{aligned} \tag{5.36}$$

where we have used the $p$-growth condition $(A_4)$ and the fact that $\mathcal{L}^2(Q'(z_\alpha^0,\delta) \setminus Q'(y_\alpha^0 - x_\alpha^{\varepsilon_j},\delta)) \to 0$. To eliminate the term $\varepsilon_j\, l_{\varepsilon_j}$ in (5.36), we would like to apply a uniform continuity argument. Since for a.e. $x_3 \in I$ the function $W(x_3,\cdot,\cdot;\cdot)$ is continuous on $\mathbb{R}^3\times\mathbb{R}^2\times\mathbb{R}^{3\times 3}$, then $(A_3)$ implies that it is uniformly continuous on $\mathbb{R}^3\times\mathbb{R}^2\times\overline{B}(0,\lambda)$ for any $\lambda > 0$. We define

$$R_j^\lambda := \left\{x \in Q'(z_\alpha^0,\delta)\times I : \left|\left(\overline{\xi} + \nabla_\alpha v_j(x)\middle|\frac{1}{\varepsilon_j}\nabla_3 v_j(x)\right)\right| \leqslant \lambda\right\},$$

and we note that by Chebyshev's inequality

$$\mathcal{L}^3([Q'(z_\alpha^0,\delta)\times I]\setminus R_j^\lambda)\leqslant C/\lambda^p, \tag{5.37}$$

for some constant $C>0$ independent of $\lambda$ or $j$. Thus, in view of (5.36) and the fact that $W$ is nonnegative,

$$\begin{aligned}&\mathcal{W}_{\{\varepsilon_j\}}(\overline{\xi}\cdot;Q'(y_\alpha^0,\delta))\\ &\qquad\geqslant \limsup_{\lambda\to\infty}\limsup_{j\to\infty}\int_{R_j^\lambda} W\left(x_3,\frac{x_\alpha}{\varepsilon_j}-\varepsilon_j l_{\varepsilon_j},\frac{x_3}{\varepsilon_j},\frac{x_\alpha}{\varepsilon_j^2};\overline{\xi}+\nabla_\alpha v_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 v_j(x)\right)dx.\end{aligned}$$

Denoting by $\omega_\lambda(x_3,\cdot):[0,\infty)\to[0,\infty)$ the modulus of continuity of $W(x_3,\cdot,\cdot;\cdot)$ on $\mathbb{R}^3\times\mathbb{R}^2\times\overline{B}(0,\lambda)$, we can check that for a.e. $x_3\in I$, the function $t\mapsto\omega_\lambda(x_3,t)$ is continuous, increasing and satisfies $\omega_\lambda(x_3,0)=0$ while, for all $t\in[0,\infty)$, the function $x_3\mapsto\omega_\lambda(x_3,t)$ is measurable (as the supremum of measurable functions). We get, for any $x\in R_j^\lambda$

$$\begin{aligned}&\Bigg|W\left(x_3,\frac{x}{\varepsilon_j},\frac{x_\alpha}{\varepsilon_j^2};\overline{\xi}+\nabla_\alpha v_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 v_j(x)\right)\\ &\qquad\qquad -W\left(x_3,\frac{x_\alpha}{\varepsilon_j}-\varepsilon_j l_{\varepsilon_j},\frac{x_3}{\varepsilon_j}\frac{x_\alpha}{\varepsilon_j^2};\overline{\xi}+\nabla_\alpha v_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 v_j(x)\right)\Bigg|\\ &\qquad\leqslant \omega_\lambda(x_3,\varepsilon_j l_{\varepsilon_j}).\end{aligned}$$

The properties of $\omega_\lambda$, Levi's Monotone Convergence Theorem and (5.36) yield

$$\begin{aligned}\mathcal{W}_{\{\varepsilon_j\}}(\overline{\xi}\cdot;Q'(y_\alpha^0,\delta)) &\geqslant \limsup_{\lambda\to\infty}\limsup_{j\to\infty}\Bigg\{\int_{R_j^\lambda} W\left(x_3,\frac{x}{\varepsilon_j},\frac{x_\alpha}{\varepsilon_j^2};\overline{\xi}+\nabla_\alpha v_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 v_j(x)\right)dx\\ &\qquad\qquad -\delta^2\int_{-1}^{1}\omega_\lambda(x_3,\varepsilon_j l_{\varepsilon_j})\,dx_3\Bigg\}\\ &= \liminf_{j\to\infty}\int_{Q'(z_\alpha^0,\delta)\times I} W\left(x_3,\frac{x}{\varepsilon_j},\frac{x_\alpha}{\varepsilon_j^2};\overline{\xi}+\nabla_\alpha v_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 v_j(x)\right)dx\\ &\geqslant \mathcal{W}_{\{\varepsilon_j\}}(\overline{\xi}\cdot;Q'(z_\alpha^0,\delta)),\end{aligned}$$

where we have used the equi-integrability of $\left\{\left|\left(\nabla_\alpha v_j\big|\frac{1}{\varepsilon_j}\nabla_3 v_j\right)\right|^p\right\}$, the $p$-growth condition $(A_4)$, (5.37) and the fact that $v_j\to 0$ in $L^p(Q'(z_\alpha^0,\delta)\times I;\mathbb{R}^3)$. ■

As a consequence of this lemma, and adapting the argument used in the proof of Lemma 4.2.5, we deduce the following integral representation result.

**Lemma 5.2.6.** *There exists a continuous function $W_{\{\varepsilon_j\}} : \mathbb{R}^{3\times 2} \to [0,\infty)$ such that for all $A \in \mathcal{A}_0(\mathbb{R}^2)$ and all $u \in W^{1,p}(A;\mathbb{R}^3)$,*

$$\mathcal{W}_{\{\varepsilon_j\}}(u;A) = 2\int_A W_{\{\varepsilon_j\}}(\nabla_\alpha u(x_\alpha))\,dx_\alpha.$$

*STEP 3.* Characterization of the $\Gamma$-limit.

In view of Lemma 5.2.6, we only need to prove that $\overline{W}_{\mathrm{hom}}(\overline{\xi}) = W_{\{\varepsilon_j\}}(\overline{\xi})$ for all $\overline{\xi} \in \mathbb{R}^{3\times 2}$, and thus it suffices to work with affine functions instead of with general Sobolev functions.

We state, without proof, a result equivalent to Proposition 4.2.6 for the dimension reduction case.

**Proposition 5.2.7.** *Given $M > 0$, $\eta > 0$, and $\varphi : [0,\infty) \to [0,\infty]$ a continuous and increasing function satisfying $\varphi(t)/t \to \infty$ as $t \to \infty$, there exists $\varepsilon_0 \equiv \varepsilon_0(M,\eta) > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, every $a \in \mathbb{R}^2$ and every $u \in W^{1,p}((a+Q')\times I;\mathbb{R}^3)$ with*

$$\int_{(a+Q')\times I} \varphi(|\nabla u|^p)\,dx \leqslant M, \tag{5.38}$$

*there exists $v \in W_0^{1,p}((a+Q')\times I;\mathbb{R}^3)$ with $\|v\|_{L^p((a+Q')\times I;\mathbb{R}^3)} \leqslant \eta$ satisfying*

$$\int_{(a+Q')\times I} W\Big(x_3, x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha}{\varepsilon}; \nabla u\Big)\,dx \geqslant \int_{(a+Q')\times I} W_{\mathrm{hom}}(x_3, x_\alpha; \nabla u + \nabla v)\,dx - \eta.$$

**Lemma 5.2.8.** *For all $\overline{\xi} \in \mathbb{R}^{3\times 2}$, $\overline{W}_{\mathrm{hom}}(\overline{\xi}) \leqslant W_{\{\varepsilon_j\}}(\overline{\xi})$.*

*Proof.* From Lemma 5.2.6, Theorem 2.2.17 and Lemma 3.2.2 we may find a sequence $\{w_j\} \subset W^{1,p}(Q'\times I;\mathbb{R}^3)$ such that $\big\{\big|\big(\nabla_\alpha w_j\big|\frac{1}{\varepsilon_j}\nabla_3 w_j\big)\big|^p\big\}_j$ is equi-integrable, $w_j = 0$ on $\partial Q' \times I$, $w_j \to 0$ in $L^p(Q'\times I;\mathbb{R}^3)$ and

$$2W_{\{\varepsilon_j\}}(\overline{\xi}) = \lim_{j\to\infty} \int_{Q'\times I} W\left(x_3, \frac{x}{\varepsilon_j}, \frac{x_\alpha}{{\varepsilon_j}^2}; \overline{\xi} + \nabla_\alpha w_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 w_j(x)\right)\,dx.$$

Thus, from De La Vallée Poussin criterion (Proposition 2.2.10) there exists an increasing continuous function $\varphi : [0,\infty) \to [0,\infty]$ satisfying $\varphi(t)/t \to \infty$ as $t \to \infty$ and such that

$$\sup_{j\in\mathbb{N}} \int_{Q'\times I} \varphi\left(\left|\left(\nabla_\alpha w_j\Big|\frac{1}{\varepsilon_j}\nabla_3 w_j\right)\right|^p\right)\,dx \leqslant 1.$$

Changing variables yields

$$W_{\{\varepsilon_j\}}(\overline{\xi}) = \lim_{j\to\infty} \frac{1}{2T_j^2} \int_{(0,T_j)^2\times I} W\left(x_3, x_\alpha, \frac{x_3}{\varepsilon_j}, \frac{x_3}{\varepsilon_j^2}; \overline{\xi} + \nabla_\alpha z_j(x)|\nabla_3 z_j(x)\right)\,dx$$

and

$$\sup_{j\in\mathbb{N}} \frac{1}{T_j^2} \int_{(0,T_j)^2\times I} \varphi(|\nabla z_j|^p)\,dx \leqslant 1,$$

where we set $T_j := 1/\varepsilon_j$ and $z_j(x) := T_j w_j(x_\alpha/T_j, x_3)$. Note that $z_j \in W^{1,p}((0,T_j)^2\times I;\mathbb{R}^3)$ and $z_j = 0$ on $\partial(0,T_j)^2 \times I$. For all $j \in \mathbb{N}$, define $I_j := \{1, \cdots, [[T_j]]^2\}$ and for any $i \in I_j$, take $a_i^j \in \mathbb{Z}^2$ such that

$$\bigcup_{i\in I_j} (a_i^j + Q') \subset (0,T_j)^2.$$

Moreover, for all $M > 0$, let

$$I_j^M := \left\{ i \in I_j : \int_{(a_i^j+Q')\times I} \varphi(|\nabla z_j|^p)\,dx \leqslant M \right\}.$$

Applying Proposition 5.2.7, we get for any $\eta > 0$ and any $i \in I_j^M$ the existence of $v_i^{j,M,\eta} \in W_0^{1,p}((a_i^j + Q') \times I;\mathbb{R}^3)$ with $\|v_i^{j,M,\eta}\|_{L^p((a_i^j+Q')\times I;\mathbb{R}^3)} \leqslant \eta$ and

$$\begin{aligned}\int_{(a_i^j+Q')\times I} & W\left(x_3, x_\alpha, \frac{x_3}{\varepsilon_j}, \frac{x_\alpha}{\varepsilon_j}; \overline{\xi} + \nabla_\alpha z_j | \nabla_3 z_j\right) dx \\ & \geqslant \frac{1}{T_j^2} \int_{(a_i^j+Q')\times I} W_{\mathrm{hom}}\left(x_3, x_\alpha; \overline{\xi} + \nabla_\alpha z_j + \nabla_\alpha v_i^{j,M,\eta} | \nabla_3 z_j + \nabla_3 v_i^{j,M,\eta}\right) dx - \eta.\end{aligned}$$

Hence,

$$W_{\{\varepsilon_j\}}(\overline{\xi}) \geqslant \limsup_{M,\eta,j} \frac{1}{2T_j^2} \sum_{i\in I_j^M} \int_{(a_i^j+Q')\times I} W_{\mathrm{hom}}\left(x_3, x_\alpha; (\overline{\xi}|0) + \nabla\phi^{j,M,\eta}\right) dx \tag{5.39}$$

where $\phi^{j,M,\eta} \in W^{1,p}((0;T_j)^2 \times I;\mathbb{R}^3)$ is defined by

$$\phi^{j,M,\eta}(x) := \begin{cases} z_j(x) + v_i^{j,M,\eta}(x) & \text{if } x \in (a_i^j + Q') \times I \text{ and } i \in I_j^M, \\ \\ z_j(x) & \text{otherwise} \end{cases}$$

and satisfies $\phi^{j,M,\eta} = 0$ on $\partial(0,T_j)^2 \times I$. In view of the definition of $\phi^{j,M,\eta}$, the $p$-growth condition (5.30) and the equi-integrability of $\left\{\left|\left(\nabla_\alpha w_j\middle|\frac{1}{\varepsilon_j}\nabla_3 w_j\right)\right|^p\right\}$, arguing exactly as in Lemma 4.2.7, we get

$$\begin{aligned} W_{\{\varepsilon_j\}}(\overline{\xi}) & \geqslant \limsup_{M,\eta,j} \frac{1}{2T_j^2} \int_{(0,T_j)^2\times I} W_{\mathrm{hom}}(x_3, x_\alpha; \overline{\xi} + \nabla_\alpha \phi^{j,M,\eta} | \nabla_3 \phi^{j,M,\eta})\,dx \\ & \geqslant \overline{W}_{\mathrm{hom}}(\overline{\xi}). \end{aligned}$$

■

Let us now prove the opposite inequality.

**Lemma 5.2.9.** *For all* $\xi \in \mathbb{R}^{3\times 2}$, $\overline{W}_{\text{hom}}(\overline{\xi}) \geqslant W_{\{\varepsilon_j\}}(\overline{\xi})$.

*Proof.* In view of (5.33), for $\delta > 0$ fixed take $T_\delta \equiv T \in \mathbb{N}$, with $T_\delta \to \infty$ as $\delta \to 0$, and let $\phi_\delta \equiv \phi \ \in W^{1,p}((0,T)^2 \times I; \mathbb{R}^3)$ be such that $\phi = 0$ on $\partial(0,T)^2 \times I$ and

$$\overline{W}_{\text{hom}}(\overline{\xi}) + \delta \geqslant \frac{1}{2T^2} \int_{(0,T)^2 \times I} W_{\text{hom}}(x_3, x_\alpha; \overline{\xi} + \nabla_\alpha \phi(x) | \nabla_\alpha \phi(x))\, dx. \tag{5.40}$$

From Theorem 4.1.1, Theorem 2.5.13 and the Decomposition Lemma 2.2.16 there exists $\{\phi_j\} \subset W_0^{1,p}((0,T)^2 \times I; \mathbb{R}^3)$ such that $\{|\nabla \phi_j|^p\}$ is equi-integrable, $\phi_j \to \phi$ in $L^p((0,T)^2 \times I; \mathbb{R}^3)$ and

$$\begin{aligned}\int_{(0,T)^2 \times I} & W_{\text{hom}}(x_3, x_\alpha; \overline{\xi} + \nabla_\alpha \phi(x) | \nabla_3 \phi(x))\, dx \\ &= \lim_{j \to \infty} \int_{(0,T)^2 \times I} W\left(x_3, x_\alpha, \frac{x_3}{\varepsilon_j}, \frac{x_3}{\varepsilon_j}; \overline{\xi} + \nabla_\alpha \phi_j(x) | \nabla_3 \phi_j(x)\right) dx. \end{aligned} \tag{5.41}$$

Fix $j \in \mathbb{N}$ such that $\varepsilon_j \ll 1$. For all $i \in \mathbb{Z}^2$ let $a_i^j \in \varepsilon_j \mathbb{Z}^2 \cap [i(T+1), \varepsilon_j)^2$ (uniquely defined). Set

$$\tilde{\phi}_j(x) := \begin{cases} \phi_j(x_\alpha - a_i^j, x_3) & \text{if } x \in Q'(a_i^j, T) \times I \text{ and } i \in \mathbb{Z}^2, \\ \\ 0 & \text{otherwise,} \end{cases}$$

then $\tilde{\phi}_j \in W^{1,p}(\mathbb{R}^2 \times I; \mathbb{R}^3)$. Let $I_j := \{i \in \mathbb{Z}^2 : (0, T/\varepsilon_j)^2 \cap Q'(a_i^j, T) \neq \emptyset\}$. If $\psi_j(x) := \varepsilon_j \tilde{\phi}_j(x_\alpha/\varepsilon_j, x_3)$ then $\psi_j \to 0$ in $L^p((0,T)^2 \times I; \mathbb{R}^3)$, as $j \to \infty$. Consequently, the $p$-growth

condition $(A_4)$ implies that

$$
\begin{aligned}
&\mathcal{W}_{\{\varepsilon_j\}}(\overline{\xi}\cdot\,;(0,T)^2)\\
&\quad\leqslant \liminf_{j\to\infty}\int_{(0,T)^2\times I} W\left(x_3,\frac{x}{\varepsilon_j},\frac{x_\alpha}{\varepsilon_j^2};\overline{\xi}+\nabla_\alpha\psi_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3\psi_j(x)\right)dx\\
&\quad\leqslant \liminf_{j\to\infty}\varepsilon_j^2\left\{\sum_{i\in I_j}\int_{Q'(a_i^j,T)\times I} W\left(x_3,x_\alpha,\frac{x_3}{\varepsilon_j},\frac{x_\alpha}{\varepsilon_j};\overline{\xi}+\nabla_\alpha\tilde{\phi}_j(x)|\nabla_3\tilde{\phi}_j(x)\right)dx\right.\\
&\qquad\qquad\left.+2\beta(1+|\overline{\xi}|^p)\mathcal{L}^N\left(\left(0,\frac{T}{\varepsilon_j}\right)^2\setminus\bigcup_{i\in I_j}Q'(a_i^j,T)\right)\right\}\\
&\quad= \liminf_{j\to\infty}\varepsilon_j^2\sum_{i\in I_j}\int_{(0,T)^2\times I} W\left(x_3,x_\alpha+a_i^j-i(T+1),\frac{x_3}{\varepsilon_j},\frac{x_\alpha}{\varepsilon_j};\overline{\xi}+\nabla_\alpha\phi_j(x)|\nabla_3\phi_j(x)\right)dx\\
&\qquad\qquad+2\beta(1+|\overline{\xi}|^p)T^2\left(1-\left(\frac{T}{T+1}\right)^2\right) \qquad (5.42)
\end{aligned}
$$

where we have used $(A_3)$, the fact that $T\in\mathbb{N}$ and $a_i^j/\varepsilon_j\in\mathbb{Z}^2$. We now use the same uniform continuity argument than in the proof of Lemmas 5.2.5 and 4.2.8. We get

$$
W_{\{\varepsilon_j\}}(\overline{\xi})\leqslant\overline{W}_{\mathrm{hom}}(\overline{\xi})+\delta+2\beta(1+|\overline{\xi}|^p)\left(1-\left(\frac{T}{T+1}\right)^2\right).
$$

The result follows by letting $\delta$ tend to zero. ■

*Proof of Theorem 5.2.4.* From Lemma 5.2.8 and Lemma 5.2.9, we conclude that $\overline{W}_{\mathrm{hom}}(\overline{\xi})=W_{\{\varepsilon_j\}}(\overline{\xi})$ for all $\overline{\xi}\in\mathbb{R}^{3\times 2}$. As a consequence, $\mathcal{W}_{\{\varepsilon_j\}}(u;A)=\mathcal{W}_{\mathrm{hom}}(u;A)$ for all $A\in\mathcal{A}_0(\mathbb{R}^2)$ and all $u\in W^{1,p}(A;\mathbb{R}^3)$. Since the $\Gamma$-limit does not depend upon the extracted subsequence, Proposition 2.5.8 implies that the whole sequence $\mathcal{W}_\varepsilon(\cdot\,;A)$ $\Gamma(L^p(A\times I))$-converges to $\mathcal{W}_{\mathrm{hom}}(\cdot\,;A)$. ■

### 5.2.3 The general case

Our aim here is to study the case where the function $W$ depends also on the in-plane variable.

*STEP 1.* Localization of the functionals.

As in Subsection 4.2.3, to prove Theorem 5.2.1 it is convenient to localize the functionals $\mathcal{W}_\varepsilon$ in (5.23) on the class of all bounded open subsets of $\omega$, denoted by $\mathcal{A}(\omega)$. For each $\varepsilon > 0$ we consider the family of functionals $\mathcal{W}_\varepsilon : L^p(\Omega;\mathbb{R}^3) \times \mathcal{A}(\omega) \to [0,\infty]$ defined by

$$\mathcal{W}_\varepsilon(u;A) := \begin{cases} \displaystyle\int_{A\times I} W\left(x, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}; \nabla_\alpha u(x) \Big| \frac{1}{\varepsilon}\nabla_3 u(x)\right) dx & \text{if } u \in W^{1,p}(A\times I;\mathbb{R}^3), \\ \\ \infty & \text{otherwise.} \end{cases} \tag{5.43}$$

Given $\varepsilon_n \downarrow 0$ and $A \in \mathcal{A}(\omega)$ we define the $\Gamma$-lower limit of $\{\mathcal{W}_{\varepsilon_n}(\cdot\,;A)\}_n$ with respect to the $L^p(A\times I;\mathbb{R}^3)$-topology by

$$\mathcal{W}_{\{\varepsilon_n\}}(u;A) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} \mathcal{W}_{\varepsilon_n}(u_n;A) : \quad u_n \to u \text{ in } L^p(A\times I;\mathbb{R}^3) \right\}$$

for all $u \in L^p(\Omega;\mathbb{R}^3)$. Our main objective is to show that

$$\mathcal{W}_{\{\varepsilon_n\}} = \mathcal{W}_{\text{hom}} \tag{5.44}$$

where $\mathcal{W}_{\text{hom}} : L^p(\Omega;\mathbb{R}^3) \times \mathcal{A}(\omega) \to [0,\infty]$ is given by

$$\mathcal{W}_{\text{hom}}(u;A) = \begin{cases} 2\displaystyle\int_A \overline{W}_{\text{hom}}(x_\alpha; \nabla_\alpha u(x_\alpha))\, dx_\alpha & \text{if } u \in W^{1,p}(A;\mathbb{R}^3), \\ \\ \infty & \text{otherwise.} \end{cases}$$

The conclusion of Theorem 5.2.1 would follow taking $A = \omega$.

By hypotheses $(A_4)$ it follows that $\mathcal{W}_{\{\varepsilon_n\}}(u;A) = \infty$ for each $A \in \mathcal{A}(\omega)$ whenever $u \in L^p(\Omega;\mathbb{R}^3) \setminus W^{1,p}(A;\mathbb{R}^3)$.

As a consequence of Theorem 3.2.1, given $\varepsilon_n \downarrow 0$ there exists a subsequence $\{\varepsilon_{n_j}\}_j \equiv \{\varepsilon_j\}_j$ for which the functional $\mathcal{W}_{\{\varepsilon_j\}}(\cdot\,;A)$ is the $\Gamma(L^p(A\times I))$-limit of $\{\mathcal{W}_{\varepsilon_j}(\cdot\,;A)\}_j$ for each $A \in \mathcal{A}(\omega)$. Moreover given $u \in W^{1,p}(A;\mathbb{R}^3)$

$$\mathcal{W}_{\{\varepsilon_j\}}(u;A) = 2\int_A W_{\{\varepsilon_j\}}(x_\alpha; \nabla_\alpha(x_\alpha))\, dx_\alpha,$$

for some Carathéodory function $W_{\{\varepsilon_j\}} : \omega \times \mathbb{R}^{3\times 2} \to \mathbb{R}$.

Accordingly, to prove equality (5.44) it suffices to show that $W_{\{\varepsilon_j\}}(x_\alpha;\overline{\xi}) = \overline{W}_{\text{hom}}(x_\alpha;\overline{\xi})$ for a.e. $x_\alpha \in \omega$ and all $\overline{\xi} \in \mathbb{R}^{3\times 2}$.

*STEP 2.* Characterization of the Γ-limit.

For each $T > 0$ consider $\mathcal{S}_T$ a countable set of functions in $C^\infty([0,T]^2 \times [-1,1];\mathbb{R}^3)$ that is dense in

$$\big\{\phi \in W^{1,p}((0,T)^2 \times I;\mathbb{R}^3):\ \phi = 0 \ \text{on} \ \partial(0,T)^2 \times I\big\}.$$

Let $L$ be the set of Lebesgue points $x_\alpha^0$ for all functions

$$W_{\{\varepsilon_j\}}(\cdot;\overline{\xi}), \quad \overline{W}_{\mathrm{hom}}(\cdot;\overline{\xi})$$

and

$$x_\alpha \mapsto \int_{(0,T)^2\times I} W_{\mathrm{hom}}(x_\alpha, y_3, y_\alpha; \overline{\xi} + \nabla_\alpha\phi(y)|\nabla_3\phi(y))\,dy,$$

with $T \in \mathbb{N}$, $\phi \in \mathcal{S}_T$ and $\overline{\xi} \in \mathbb{Q}^{3\times 2}$, and for which $\overline{W}_{\mathrm{hom}}(x_\alpha^0;\,\cdot)$ is well defined. Note that $\mathcal{L}^2(\omega \setminus L) = 0$.

We start by proving the following inequality.

**Lemma 5.2.10.** *For all $x_\alpha^0 \in L$ and all $\overline{\xi} \in \mathbb{Q}^{3\times 2}$, we have $W_{\{\varepsilon_j\}}(x_\alpha^0;\overline{\xi}) \geqslant \overline{W}_{\mathrm{hom}}(x_\alpha^0;\overline{\xi})$.*

*Proof.* Let $\delta > 0$ small enough so that $Q'(x_\alpha^0,\delta) \in \mathcal{A}(\omega)$. By Theorem 2.2.17 we can find a sequence $\{u_j\}_j \subset W^{1,p}(Q'(x_\alpha^0,\delta)\times I;\mathbb{R}^3)$ with $u_j \to 0$ in $L^p(Q'(x_\alpha^0,\delta)\times I;\mathbb{R}^3)$, such that the sequence of scaled gradients $\big\{\big(\nabla_\alpha u_j|\frac{1}{\varepsilon_j}\nabla_3 u_j\big)\big\}$ is $p$-equi-integrable and

$$\begin{aligned}
\mathcal{W}_{\{\varepsilon_j\}}(\overline{\xi}\cdot\,; Q'(x_\alpha^0,\delta)) &= 2\int_{Q'(x_\alpha^0,\delta)} W_{\{\varepsilon_j\}}(x_\alpha;\overline{\xi})\,dx_\alpha \\
&= \lim_{j\to\infty}\int_{Q'(x_\alpha^0,\delta)\times I} W\left(x, \frac{x}{\varepsilon_j}, \frac{x_\alpha}{\varepsilon_j^2}; \overline{\xi} + \nabla_\alpha u_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j(x)\right)dx.
\end{aligned}$$

Given $m \in \mathbb{N}$ let $\mathcal{C}_m$ and $W^m$ be given by Lemma B.3. Then, since $W \geqslant 0$ and $W = W^m$ on $\mathcal{C}_m \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$, we get

$$\mathcal{W}_{\{\varepsilon_j\}}(\overline{\xi}\cdot\,; Q'(x_\alpha^0,\delta)) \geqslant \limsup_{m\to\infty}\limsup_{j\to\infty}\int_{[Q'(x_\alpha^0,\delta)\times I]\cap\mathcal{C}_m} W^m\left(x, \frac{x}{\varepsilon_j}, \frac{x_\alpha}{\varepsilon_j^2}; \overline{\xi} + \nabla_\alpha u_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j(x)\right)dx.$$

By the $p$-growth condition (0.5), the equi-integrability of $\big\{\big|\big(\nabla_\alpha u_j|\frac{1}{\varepsilon_j}\nabla_3 u_j\big)\big|^p\big\}$ and relation (0.4), we obtain

$$\begin{aligned}
&\int_{[Q'(x_\alpha^0,\delta)\times I]\setminus\mathcal{C}_m} W^m\left(x, \frac{x}{\varepsilon_j}, \frac{x_\alpha}{\varepsilon_j^2}; \overline{\xi} + \nabla_\alpha u_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j(x)\right)dx \\
&\qquad \leqslant \beta\int_{[Q'(x_\alpha^0,\delta)\times I]\setminus\mathcal{C}_m}\left(1 + \left|\left(\overline{\xi} + \nabla_\alpha u_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j(x)\right)\right|^p\right)dx \xrightarrow[m\to\infty]{} 0,
\end{aligned}$$

uniformly with respect to $j \in \mathbb{N}$. Then, we get

$$\mathcal{W}_{\{\varepsilon_j\}}(\overline{\xi}\cdot;Q'(x_\alpha^0,\delta)) \geqslant \limsup_{m\to\infty}\limsup_{j\to\infty}\int_{Q'(x_\alpha^0,\delta)\times I} W^m\left(x,\frac{x}{\varepsilon_j},\frac{x_\alpha}{\varepsilon_j^2};\overline{\xi}+\nabla_\alpha u_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j(x)\right)dx.$$

For any $h \in \mathbb{N}$, we split $Q'(x_\alpha^0,\delta)$ into $h^2$ disjoint cubes $Q'_{i,h}$ of side length $\delta/h$ so that

$$Q'(x_\alpha^0,\delta) = \bigcup_{i=1}^{h^2} Q'_{i,h}$$

and

$$\begin{aligned}\mathcal{W}_{\{\varepsilon_j\}}(\overline{\xi}\cdot;Q'(x_\alpha^0,\delta)) &\geqslant \limsup_{m,h,j}\sum_{i=1}^{h^2}\int_{Q'_{i,h}\times I} W^m\left(x,\frac{x}{\varepsilon_j},\frac{x_\alpha}{\varepsilon_j^2};\overline{\xi}+\nabla_\alpha u_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j(x)\right)dx\\ &\geqslant \limsup_{m,\lambda,h,j}\sum_{i=1}^{h^2}\int_{[Q'_{i,h}\times I]\cap R_j^\lambda} W^m\left(x,\frac{x}{\varepsilon_j},\frac{x_\alpha}{\varepsilon_j^2};\overline{\xi}+\nabla_\alpha u_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j(x)\right)dx\end{aligned}$$

where, given $\lambda > 0$, we define

$$R_j^\lambda := \left\{x \in Q'(x_\alpha^0,\delta)\times I : \quad \left|\left(\overline{\xi}+\nabla_\alpha u_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j(x)\right)\right| \leqslant \lambda\right\}.$$

Since $W^m$ is continuous and separately periodic it is in particular uniformly continuous on $\overline{\Omega}\times\mathbb{R}^3\times\mathbb{R}^2\times\overline{B}(0,\lambda)$. With similar arguments to that used in the proof of Lemma 5.1.13 (with $W^m$ in place of $W^{m,\lambda}$), we obtain

$$\begin{aligned}&\mathcal{W}_{\{\varepsilon_j\}}(\overline{\xi}\cdot;Q'(x_\alpha^0,\delta))\\ &\geqslant \limsup_{h\to\infty}\limsup_{j\to\infty}\frac{h^2}{\delta^2}\sum_{i=1}^{h^2}\int_{Q'_{i,h}}\int_{Q'_{i,h}\times I} W\left(x'_\alpha,x_3,\frac{x}{\varepsilon_j},\frac{x_\alpha}{\varepsilon_j^2};\overline{\xi}+\nabla_\alpha u_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j(x)\right)dx\,dx'_\alpha\\ &\geqslant \limsup_{h\to\infty}\frac{h^2}{\delta^2}\sum_{i=1}^{h^2}\int_{Q'_{i,h}}\liminf_{j\to\infty}\int_{Q'_{i,h}\times I} W\left(x'_\alpha,x_3,\frac{x}{\varepsilon_j},\frac{x_\alpha}{\varepsilon_j^2};\overline{\xi}+\nabla_\alpha u_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j(x)\right)dx\,dx'_\alpha, \quad (5.45)\end{aligned}$$

where we have used Fatou's Lemma. We now fix $x'_\alpha \in Q'_{i,h}$ such that $\overline{W}_{\mathrm{hom}}(x'_\alpha,x_3,y,z_\alpha;\overline{\xi})$ is well defined. Then by Theorem 5.2.4 we get that

$$\liminf_{j\to\infty}\int_{Q'_{i,h}\times I} W\left(x'_\alpha,x_3,\frac{x}{\varepsilon_j},\frac{x_\alpha}{\varepsilon_j^2};\overline{\xi}+\nabla_\alpha u_j(x)\Big|\frac{1}{\varepsilon_j}\nabla_3 u_j(x)\right)dx \geqslant 2\frac{\delta^2}{h^2}\overline{W}_{\mathrm{hom}}(x'_\alpha;\overline{\xi}). \quad (5.46)$$

Gathering (5.45) and (5.46), it turns out that

$$\int_{Q'(x^0_\alpha,\delta)} W_{\{\varepsilon_j\}}(x_\alpha;\overline{\xi})\,dx_\alpha \geqslant \int_{Q'(x^0_\alpha,\delta)} \overline{W}_{\hom}(x'_\alpha;\overline{\xi})\,dx'_\alpha.$$

As a consequence the claim follows by the choice of $x^0_\alpha$, after dividing the previous inequality by $\delta^2$ and letting $\delta \to 0$. ■

We now prove the converse inequality.

**Lemma 5.2.11.** *For all $\xi \in \mathbb{Q}^{3\times 2}$ and all $x^0_\alpha \in L$, $W_{\{\varepsilon_j\}}(x^0_\alpha;\overline{\xi}) \leqslant \overline{W}_{\hom}(x_\alpha;\overline{\xi})$.*

*Proof.* For every $m \in \mathbb{N}$, consider the set $\mathcal{C}_m$ and the function $W^m$ given by Lemma B.3, and define $\overline{(W^m)}_{\hom}$ and $(W^m)_{\hom}$ as (5.25) and (5.26), with $W^m$ in place of $W$. For fixed $\eta > 0$ and any $m \in \mathbb{N}$ let $K^m_\eta$ be a compact subset of $\omega$ given by Scorza-Dragoni's Theorem with $\mathcal{L}^2(\omega \setminus K^m_\eta) \leqslant \eta$ and such that $\overline{(W^m)}_{\hom} : K^m_\eta \times \mathbb{R}^{3\times 2} \to \mathbb{R}$ is continuous .

*Step 1.* There exists a subsequence $\{m_k\} \nearrow \infty$ such that

$$2\liminf_{k\to\infty} \int_{Q'(x^0_\alpha,\delta)} \overline{(W^{m_k})}_{\hom}(x_\alpha;\overline{\xi})\,dx_\alpha \;\;\geqslant\;\; 2\int_{Q'(x^0_\alpha,\delta)} W_{\{\varepsilon_j\}}\left(x_\alpha,\overline{\xi}\right)\,dx_\alpha. \tag{5.47}$$

To show this inequality we follow an argument similar to that of Lemma 4.2.11. We first decompose $Q'(x^0_\alpha,\delta)$ into $h^2$ small disjoint cubes $Q'_{i,h}$ such that

$$Q'(x^0_\alpha,\delta) = \bigcup_{i=1}^{h^2} Q'_{i,h} \quad\text{ and }\quad \mathcal{L}^2(Q'_{i,h}) = (\delta/h)^2.$$

Let

$$I^m_{h,\eta} := \left\{ i \in \{1,\cdots,h^2\} : \quad K^m_\eta \cap Q'_{i,h} \neq \emptyset \right\}.$$

For $i \in I^m_{h,\eta}$ choose $x_i^{h,\eta,m} \in K^m_\eta \cap Q'_{i,h}$. By Theorem 5.2.4 and Lemma 3.2.2 there exists a sequence $\{u_i^{j,h,\eta,m}\} \subset W^{1,p}(Q'_{i,h}\times I;\mathbb{R}^3)$ with $u_i^{j,h,\eta,m} = 0$ on $\partial Q'_{i,h}\times I$, $u_i^{j,h,\eta,m} \xrightarrow[n\to\infty]{} 0$ in $L^p(Q'_{i,h}\times I;\mathbb{R}^3)$, and such that

$$\begin{aligned} &2\int_{Q'_{i,h}} \overline{(W^m)}_{\hom}(x_i^{h,\eta,m};\overline{\xi})\,dx_\alpha \\ &\qquad = \lim_{j\to\infty}\int_{Q'_{i,h}\times I} W^m\left(x_i^{h,\eta,m}, x_3, \frac{x}{\varepsilon_j}, \frac{x_\alpha}{\varepsilon_j^2}; \overline{\xi} + \nabla_\alpha u_i^{j,h,\eta,m} \Big| \frac{1}{\varepsilon_j}\nabla_3 u_i^{j,h,\eta,m}\right)dx. \end{aligned}$$

Setting

$$u_j^{\eta,m}(x) := \begin{cases} u_i^{j,h,\eta,m}(x) & \text{if } x_\alpha \in Q'_{i,h} \text{ and } i \in I^m_{h,\eta}, \\ \\ 0 & \text{otherwise,} \end{cases}$$

it follows that $\{u_j^{\eta,m}\} \subset W^{1,p}(Q'(x^0_\alpha,\delta)\times I;\mathbb{R}^3)$ and $u_j^{\eta,m} \xrightarrow[n\to\infty]{} 0$ in $L^p(Q'(x^0_\alpha,\delta)\times I;\mathbb{R}^3)$. Thus,

$$2\liminf_{\eta,h} \sum_{i\in I^m_{h,\eta}} \int_{Q'_{i,h}} \overline{(W^m)}_{\hom}(x_i^{h,\eta,m};\overline{\xi})\,dx_\alpha$$
$$\geqslant \liminf_{\eta,h,j} \sum_{i\in I^m_{h,\eta}} \int_{Q'_{i,h}\times I} W^m\left(x_i^{h,\eta,m}, x_3, \frac{x}{\varepsilon_j}, \frac{x_\alpha}{\varepsilon_j^2}; \overline{\xi} + \nabla_\alpha u_j^{\eta,m} \Big| \frac{1}{\varepsilon_j}\nabla_3 u_j^{\eta,m}\right) dx.$$

As in Lemma 4.2.11 we obtain

$$2\int_{Q'(x^0_\alpha,\delta)} \overline{(W^m)}_{\hom}(x_\alpha;\overline{\xi})\,dx_\alpha$$
$$\geqslant \liminf_{\lambda,\eta,h,j} \sum_{i\in I^m_{h,\eta}} \int_{(Q'_{i,h}\times I)\cap R^\lambda_{j,\eta,m}} W^m\left(x_i^{h,\eta,m}, x_3, \frac{x}{\varepsilon_j}, \frac{x_\alpha}{\varepsilon_j^2}; \overline{\xi} + \nabla_\alpha u_j^{\eta,m} \Big| \frac{1}{\varepsilon_j}\nabla_3 u_j^{\eta,m}\right) dx,$$

where

$$R^\lambda_{j,\eta,m} := \left\{ x \in Q'(x^0_\alpha,\delta)\times I : \left| \left( \overline{\xi} + \nabla_\alpha u_j^{\eta,m}(x) \Big| \frac{1}{\varepsilon_j}\nabla_3 u_j^{\eta,m}(x)\right)\right| \leqslant \lambda \right\}$$

with

$$\mathcal{L}^3\left(\left[Q'(x^0_\alpha,\delta)\times I\right] \setminus R^\lambda_{j,\eta,m}\right) \leqslant \frac{C}{\lambda^p}, \tag{5.48}$$

for some constant $C > 0$ independent of $j$, $\eta$, $m$ and $\lambda$. Taking into account that $W^m$ is continuous we get

$$2\int_{Q'(x^0_\alpha,\delta)} \overline{(W^m)}_{\hom}(x_\alpha;\overline{\xi})\,dx_\alpha \geqslant \liminf_{\lambda,\eta,j} \int_{R^\lambda_{j,\eta,m}} W^m\left(x, \frac{x}{\varepsilon_j}, \frac{x_\alpha}{\varepsilon_j^2}; \overline{\xi} + \nabla_\alpha u_j^{\eta,m} \Big| \frac{1}{\varepsilon_j}\nabla_3 u_j^{\eta,m}\right) dx.$$

By a diagonalization argument, given $m_k \nearrow \infty$, $\lambda_k \nearrow \infty$, and $\eta_k \downarrow 0$ there exists $j_k \nearrow \infty$ such that

$$2\liminf_{k\to\infty} \int_{Q'(x^0_\alpha,\delta)} \overline{(W^{m_k})}_{\hom}(x_\alpha;\overline{\xi})\,dx_\alpha \geqslant \liminf_{k\to\infty} \int_{R_k} W^{m_k}\left(x, \frac{x}{\varepsilon_{j_k}}, \frac{x_\alpha}{\varepsilon_{j_k}^2}; \overline{\xi} + \nabla_\alpha v_k \Big| \frac{1}{\varepsilon_{j_k}}\nabla_3 v_k\right) dx,$$

where $v_k := u_{j_k}^{\eta_k,m_k} \in W^{1,p}(Q'(x^0_\alpha,\delta)\times I;\mathbb{R}^3)$ with $v_k \to 0$ in $L^p(Q'(x^0_\alpha,\delta)\times I;\mathbb{R}^3)$, and where $R_k := R^{\lambda_k}_{j_k,\eta_k,m_k}$. Using Theorem 2.2.17 we can assume, without loss of generality,

that the sequence $\left\{\left|\left(\nabla_\alpha v_k\middle|\frac{1}{\varepsilon_{j_k}}\nabla_3 v_k\right)\right|^p\right\}$ is equi-integrable. Then, since $W^{m_k} = W$ on $\mathcal{C}_{m_k} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$,

$$\begin{aligned}
&2\liminf_{k\to\infty}\int_{Q'(x^0_\alpha,\delta)} \overline{(W^{m_k})}_{\text{hom}}(x_\alpha;\overline{\xi})\,dx_\alpha \\
&\qquad\geqslant \liminf_{k\to\infty}\int_{Q'(x^0_\alpha,\delta)\times I} W^{m_k}\left(x,\frac{x}{\varepsilon_{j_k}},\frac{x_\alpha}{\varepsilon^2_{j_k}};\overline{\xi}+\nabla_\alpha v_k\middle|\frac{1}{\varepsilon_{j_k}}\nabla_3 v_k\right)dx \\
&\qquad\geqslant \liminf_{k\to\infty}\int_{[Q'(x^0_\alpha,\delta)\times I]\cap\mathcal{C}_{m_k}} W\left(x,\frac{x}{\varepsilon_{j_k}},\frac{x_\alpha}{\varepsilon^2_{j_k}};\overline{\xi}+\nabla_\alpha v_k\middle|\frac{1}{\varepsilon_{j_k}}\nabla_3 v_k\right)dx \\
&\qquad= \liminf_{k\to\infty}\int_{Q'(x^0_\alpha,\delta)\times I} W\left(x,\frac{x}{\varepsilon_{j_k}},\frac{x_\alpha}{\varepsilon^2_{j_k}};\overline{\xi}+\nabla_\alpha v_k\middle|\frac{1}{\varepsilon_{j_k}}\nabla_3 v_k\right)dx
\end{aligned}$$

by the growth conditions on $W$, the $p$-equi-integrability of the above sequence of scaled gradients, (0.4) (see the Appendix) and (5.48). As a result we get inequality (5.47).

*Step 2.* Fixed $\rho > 0$, let $T \in \mathbb{N}$ and $\phi \in \mathcal{S}_T$ be such that

$$\overline{W}_{\text{hom}}(x^0_\alpha;\overline{\xi})+\rho \geqslant \frac{1}{2T^2}\int_{(0,T)^2\times I} W_{\text{hom}}(x^0_\alpha,y_3,y_\alpha;\overline{\xi}+\nabla_\alpha\phi(y)|\nabla_3\phi(y))\,dy. \tag{5.49}$$

Taking $(T,\phi)$ in the definition of $\overline{(W^{m_k})}_{\text{hom}}$ and recalling Remark 5.2.2 (with $W^{m_k}$ in place of $W$) it follows that

$$\begin{aligned}
&\int_{Q'(x^0_\alpha,\delta)} \overline{(W^{m_k})}_{\text{hom}}(x_\alpha;\overline{\xi})\,dx_\alpha \\
&\leqslant \frac{1}{2T^2}\int_{Q'(x^0_\alpha,\delta)}\int_{(0,T)^2\times I}(W^{m_k})_{\text{hom}}(x_\alpha,y_3,y_\alpha;\overline{\xi}+\nabla_\alpha\phi(y)|\nabla_3\phi(y))\,dy\,dx_\alpha.
\end{aligned} \tag{5.50}$$

Define

$$E_k := \{(x_\alpha,y_\alpha,y_3)\in Q'(x^0_\alpha,\delta)\times(0,T)^2\times I: \quad (x_\alpha,y_3)\in\mathcal{C}_{m_k}\}.$$

From (0.4) it follows that

$$\begin{aligned}
\mathcal{L}^2\otimes\mathcal{L}^3\big([Q'(x^0_\alpha,\delta)\times(0,T)^2\times I]\setminus E_k\big) &= \mathcal{L}^3\otimes\mathcal{L}^2\big(\big([Q'(x^0_\alpha,\delta)\times I]\setminus\mathcal{C}_{m_k}\big)\times(0,T)^2\big) \\
&= T^2\mathcal{L}^3\big([Q'(x^0_\alpha,\delta)\times I]\setminus\mathcal{C}_{m_k}\big) \\
&\leqslant T^2/m_k.
\end{aligned} \tag{5.51}$$

Since $(W^{m_k})_{\text{hom}} = W_{\text{hom}}$ on $\mathcal{C}_{m_k} \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$,

$$\begin{aligned}
\int_{Q'(x_\alpha^0,\delta)} \int_{(0,T)^2\times I} & (W^{m_k})_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{\xi} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y))\, dy\, dx_\alpha \\
&= \int_{E_k} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{\xi} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y))\, dy\, dx_\alpha \\
&\quad + \int_{[Q'(x_\alpha^0,\delta)\times(0,T)^2\times I]\setminus E_k} (W^{m_k})_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{\xi} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y))\, dy\, dx_\alpha \\
&\leqslant \int_{Q'(x_\alpha^0,\delta)\times(0,T)^2\times I} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{\xi} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y))\, dy\, dx_\alpha \\
&\quad + C \int_{[Q'(x_\alpha^0,\delta)\times(0,T)^2\times I]\setminus E_k} (1 + |\nabla \phi(y)|^p)\, dy\, dx_\alpha
\end{aligned} \tag{5.52}$$

by property (5.30) with $W^{m_k}$ in place of $W$. Passing to the limit as $k \to \infty$, relations (5.50), (5.51) and (5.52) yield

$$\begin{aligned}
\limsup_{k\to\infty} \int_{Q'(x_\alpha^0,\delta)} & \overline{(W^{m_k})}_{\text{hom}}(x_\alpha; \overline{\xi})\, dx_\alpha \\
&\leqslant \frac{1}{2T^2} \int_{Q'(x_\alpha^0,\delta)} \int_{(0,T)^2\times I} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{\xi} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y))\, dy\, dx_\alpha.
\end{aligned}$$

Hence by (5.47) we obtain

$$\int_{Q'(x_\alpha^0,\delta)} W_{\{\varepsilon_j\}}\left(x_\alpha, \overline{\xi}\right)\, dx_\alpha \;\leqslant\; \frac{1}{2T^2} \int_{Q'(x_\alpha^0,\delta)} \int_{(0,T)^2\times I} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{\xi} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y))\, dy\, dx_\alpha.$$

As a consequence, by the choice of $x_\alpha^0$ together with (5.49) we finally get, after dividing the previous inequality by $\delta^2$ and letting $\delta \to 0$, that

$$\begin{aligned}
W_{\{\varepsilon_j\}}\left(x_\alpha^0, \overline{\xi}\right) &\leqslant \frac{1}{2T^2} \int_{(0,T)^2\times I} W_{\text{hom}}(x_\alpha^0, y_3, y_\alpha; \overline{\xi} + \nabla_\alpha \phi(y) | \nabla_3 \phi(y))\, dy \\
&\leqslant \overline{W}_{\text{hom}}(x_\alpha^0; \overline{\xi}) + \rho
\end{aligned}$$

and the result follows by letting $\rho \to 0$. ∎

*Proof of Theorem 5.2.1.* As a consequence of Lemmas 5.2.11 and 5.2.10, we have $\overline{W}_{\text{hom}}(x_\alpha; \overline{\xi}) = W_{\{\varepsilon_j\}}(x_\alpha; \overline{\xi})$ for all $x_\alpha \in L$ and all $\overline{\xi} \in \mathbb{Q}^{3\times 2}$. Since $\overline{W}_{\text{hom}}$ and $W_{\{\varepsilon_j\}}$ are Carathéodory functions, denoting by $E$ the intersection between $L$ and the set of points $x_\alpha \in \omega$ such that both $\overline{W}_{\text{hom}}(x_\alpha; \cdot)$ and $W_{\{\varepsilon_j\}}(x_\alpha; \cdot)$ are well defined and continuous, we obtain $\overline{W}_{\text{hom}}(x_\alpha; \overline{\xi}) = W_{\{\varepsilon_j\}}(x_\alpha; \overline{\xi})$ for all $x_\alpha \in E$ and all $\overline{\xi} \in \mathbb{R}^{3\times 2}$. But since $\mathcal{L}^2(\omega \setminus E) = 0$, it follows that

the equality holds for a.e. $x_\alpha \in \omega$. Therefore, we have $\mathcal{W}_{\{\varepsilon_j\}}(u;A) = \mathcal{W}_{\rm hom}(u;A)$ for all $A \in \mathcal{A}(\omega)$ and all $u \in W^{1,p}(A;\mathbb{R}^3)$. Since the result does not depend upon the specific choice of the subsequence, we obtain by Proposition 8.3 in Dal Maso [35] that the whole sequence $\mathcal{W}_\varepsilon(\cdot\,;A)$ $\Gamma(L^p(A\times I))$-converges to $\mathcal{W}_{\rm hom}(\cdot\,;A)$. The proof of Theorem 5.2.1 follows by taking $A=\omega$. ■

To conclude we state an interesting consequence of Theorem 5.2.1 that could not be obtained from the analysis made in Subsection 5.1.

**Corollary 5.2.12.** *Let $W:\Omega\times\mathbb{R}^3\times\mathbb{R}^{3\times3}\to\mathbb{R}$ be a function satisfying $(A_4)$, and such that $W(x,\cdot\,;\xi)$ is $Q$-periodic for all $\xi\in\mathbb{R}^{3\times3}$ and a.e. $x\in\Omega$. Define the functional $\mathcal{W}_\varepsilon : L^p(\Omega;\mathbb{R}^3)\to\overline{\mathbb{R}}$ by*

$$\mathcal{W}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega W\left(x,\frac{x}{\varepsilon};\nabla_\alpha u(x)\Big|\frac{1}{\varepsilon}\nabla_3 u(x)\right)dx & \text{if } u\in W^{1,p}(\Omega;\mathbb{R}^3),\\ \infty & \text{otherwise.}\end{cases}$$

*Then the $\Gamma(L^p(\Omega))$-limit of the family $\{\mathcal{W}_\varepsilon\}_\varepsilon$ is given by $\mathcal{W}_{\rm hom}: L^p(\Omega;\mathbb{R}^3)\to\overline{\mathbb{R}}$ with*

$$\mathcal{W}_{\rm hom}(u) := \begin{cases} \displaystyle 2\int_\omega \overline{W}_{\rm hom}(x_\alpha,\nabla_\alpha u(x_\alpha))\,dx_\alpha & \text{if } u\in W^{1,p}(\omega;\mathbb{R}^3),\\ \infty & \text{otherwise,}\end{cases}$$

*where, for all $\overline{\xi}\in\mathbb{R}^{3\times2}$ and a.e. $x_\alpha\in\omega$*

$$\begin{aligned}\overline{W}_{\rm hom}(x_\alpha,\overline{\xi}) \;:=\; \lim_{T\to\infty}\inf_\phi\Bigg\{&\frac{1}{2T^2}\int_{(0,T)^2\times I} W_{\rm hom}(x_\alpha,y_3,y_\alpha;\overline{\xi}+\nabla_\alpha\phi(y)|\nabla_3\phi(y))\,dy :\\ &\phi\in W^{1,p}((0,T)^2\times I;\mathbb{R}^3),\quad \phi=0 \text{ on } \partial(0,T)^2\times I\Bigg\}\end{aligned}$$

*and, for all $y_\alpha\in\mathbb{R}^2$ and a.e. $x\in\Omega$*

$$W_{\rm hom}(x,y_\alpha;\xi) := \lim_{T\to\infty}\inf_\phi\left\{\frac{1}{T^3}\int_{(0,T)^3} W(x,y_\alpha,z_3;\xi+\nabla\phi(z))\,dz :\ \phi\in W^{1,p}_0((0,T)^3;\mathbb{R}^3)\right\}.$$

# 6. GENERALIZATIONS AND FURTHER WORK

There are several ways to generalize or improve the results we have presented. First we could think of energies whose integrands depend also on the function $u$ and not only on its gradient $\nabla u$. In this direction, we have extended Theorem 4.1.1 by proving the following result.

**Theorem 6.0.13.** *Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}$ be a function such that*

$(H_1)$ *$f(x, \cdot, \cdot, \cdot)$ is continuous a.e. $x \in \Omega$;*

$(H_2)$ *$f(\cdot, y, s, \xi)$ is measurable for all $y \in \mathbb{R}^N$, $s \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^{d\times N}$;*

$(H_3)$ *$f(x, \cdot, s, \xi)$ is $Q$-periodic for a.e. $x \in \Omega$, all $s \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^{d\times N}$;*

$(H_4)$ *there exist number $p > 1$ and a constant $\beta > 0$ such that*

$$\frac{|\xi|^p}{\beta} - \beta \leqslant f(x, y, s, \xi) \leqslant \beta(1 + |s|^p + |\xi|^p),$$

*for a.e. $x \in \Omega$, all $y \in \mathbb{R}^N$, $s \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^{d\times N}$.*

*For each $\varepsilon > 0$ define the functional $\mathcal{I}_\varepsilon : L^p(\Omega; \mathbb{R}^d) \to [0, \infty]$ by*

$$\mathcal{I}_\varepsilon(u) := \begin{cases} \int_\Omega f\left(x, \frac{x}{\varepsilon}, u(x), \nabla u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^d), \\ \\ +\infty & \text{otherwise.} \end{cases}$$

*If $u \in L^p(\Omega; \mathbb{R}^d)$ then*

$$\mathcal{I}_{\text{hom}}(u) := \Gamma(L^p(\Omega))\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u) = \begin{cases} \int_\Omega f_{\text{hom}}(x, u(x), \nabla u(x))\, dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^d), \\ \\ +\infty & \text{otherwise,} \end{cases}$$

*where the integrand $f_{\text{hom}}$ is given by*

$$f_{\text{hom}}(x,s,\xi) := \lim_{T\to+\infty} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(x,y,s,\xi+\nabla\phi(y))\,dy,\ \phi \in W_0^{1,p}((0,T)^N;\mathbb{R}^d) \right\}$$

*for a.e. $x \in \Omega$, all $s \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{d\times N}$. It turns out that $f_{\text{hom}}$ is (equivalent to) a Carathéodory function and satisfies $p$-coercivity and $p$-growth conditions similar to those of $f$. Moreover $f_{\text{hom}}(x,s,\cdot)$ is quasiconvex for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^d$.*

Another possible generalization is to prove a similar result for integrands that are continuous with respect to the first variable $x$ and measurable with respect to the second variable $y$. In this direction, we were able to prove the following version of Theorem 4.1.1.

**Theorem 6.0.14.** *Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ be a function such that*

$(H_1)$ *$f(x,\cdot,\xi)$ is measurable for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^{d\times N}$;*

$(H_2)$ *$f(\cdot,y,\xi) \in C(\overline{\Omega})$ for all $y \in \mathbb{R}^N$ and all $\xi \in \mathbb{R}^{d\times N}$;*

$(H_3)$ *$f(x,\cdot,\xi)$ is $Q$-periodic for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^{d\times N}$;*

$(H_4)$ *there exist a real number $p > 1$ and a constant $\beta > 0$ such that*

$$\frac{|\xi|^p}{\beta} - \beta \leqslant f(x,y,s,\xi) \leqslant \beta(1+|\xi|^p),$$

*for a.e. $x \in \Omega$, all $y \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{d\times N}$.*
*For each $\varepsilon > 0$ define the functional $\mathcal{I}_\varepsilon : L^p(\Omega;\mathbb{R}^d) \to [0,\infty]$ by*

$$\mathcal{I}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\left(x,\frac{x}{\varepsilon},\nabla u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\ \infty & \text{otherwise.} \end{cases}$$

*If $u \in L^p(\Omega;\mathbb{R}^d)$ then*

$$\mathcal{I}_{\text{hom}}(u) := \Gamma(L^p(\Omega))\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega f_{\text{hom}}(x,\nabla u(x))\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\ \infty & \text{otherwise,} \end{cases}$$

*where the integrand $f_{\text{hom}}$ is given by*

$$f_{\text{hom}}(x,\xi) := \lim_{T\to\infty} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(x,y,\xi+\nabla\phi(y))\,dy,\ \phi \in W_0^{1,p}((0,T)^N;\mathbb{R}^d) \right\}$$

*for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^{d\times N}$. It turns out that $f_{\text{hom}}$ is (equivalent to) a Carathéodory function and satisfies $p$-coercivity and $p$-growth conditions similar to those of $f$. Moreover $f_{\text{hom}}(x,\cdot)$ is quasiconvex for a.e. $x \in \Omega$.*

Concerning future work, it would be interesting to extend this study to multiscale gradient Young measures, and to apply this characterization in different settings, including relaxation problems. This is the objective of a current collaboration with J. F. Babadjian.

# APPENDIX

# A Auxiliary lemmas for periodic homogenization

We start by recalling an auxiliary lemma by Licht and Michaille [59] that allowed us to justify that the function $f_{\text{hom}}$ given in (4.3) is well defined (see Lemma 4.1.4 above).

**Lemma A.1.** *Let $N \in \mathbb{N}$ with $N \geqslant 1$ and let $S : \mathcal{A}(\mathbb{R}^N) \to \mathbb{R}^+$ be such that*

i) $S(A) \leqslant \beta\mathcal{L}^N(A)$, *for all $A \in \mathcal{A}(\mathbb{R}^N)$, where $\beta$ is a positive constant,*

ii) $S(C) \leqslant S(A) + S(B)$ *for all $A, B, C \in \mathcal{A}(\mathbb{R}^N)$, with $A \cap B \neq \emptyset$, $\overline{C} = \overline{A} \cup \overline{B}$,*

iii) *there exists $T \subset \mathbb{R}^N$ and $M > 0$ such that $T + [0, M)^N = \mathbb{R}^N$ and $S(A + \tau) = S(A)$ for all $A \in \mathcal{A}(\mathbb{R}^N)$ and $\tau \in T$.*

*Then, for any cube $A$ of the form $[a, b)^N$ there exists the limit of the sequence $\left\{\frac{S(sA)}{\mathcal{L}^N(sA)}\right\}$ as $s \to +\infty$ and*

$$\lim_{s\to+\infty} \frac{S(sA)}{\mathcal{L}^N(sA)} = \lim_{s\to+\infty} \frac{S([0, s)^N)}{s^N}.$$

*Furthermore, if $\{S_L\}_L$ is a family of set functions satisfying $i), ii), iii)$ for $C, T$ and $M$ independent of $L$, then the above limits are attained uniformly with respect to $L$.*

Next, we recall an auxiliary lemma that was stated in [20] and is useful when diagonalization arguments are required (see Lemma 4.1.11).

**Lemma A.2.** *Let $a_{k,j}$ be a doubly indexed sequence of real numbers $(k, j \nearrow +\infty)$. If*

$$\lim_{k\to\infty} \lim_{j\to\infty} a_{k,j} = L,$$

*then there exists a subsequence $\{k(j)\}_j \nearrow +\infty$ such that*

$$\lim_{j\to\infty} a_{k(j),j} = L.$$

# B Continuous extension results for the applications to thin films

We prove here a technical result of extension of Carathéodory functions useful in the proof of Lemma 5.1.13. It was obtained in collaboration with J. F. Babadjian [9] and uses arguments analogous to those of Theorem 1, Section 1.2 in Evans and Gariepy [46].

**Lemma B.1.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{d\times N} \to \mathbb{R}$ a function such that*

$$\begin{cases} f(x, \cdot\,; \cdot\,) \text{ is continuous for a.e. } x \in \Omega; \\ \\ f(\,\cdot\,, y\,; \xi) \text{ is } \mathcal{L}^N\text{-measurable for all } y \in \mathbb{R}^N \text{ and } \xi \in \mathbb{R}^N; \\ \\ f(x, \cdot\,; \xi) \text{ is } (0,1)^m\text{-periodic for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^{d\times N}. \end{cases}$$

*Assume also that there exists $\beta > 0$ and $1 \leqslant p < \infty$ such that*

$$\frac{1}{\beta}|\xi|^p - \beta \leqslant f(x,y;\xi) \leqslant \beta(1+|\xi|^p), \quad \text{for a.e. } x \in \Omega \text{ and all } (y,\xi) \in \mathbb{R}^m \times \mathbb{R}^{d\times N}.$$

*Then for any $\eta > 0$ and $\lambda > 0$ there exist a compact set $K_\eta \subset \Omega$ and a function $f^{\eta,\lambda} : \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^{d\times N} \to \mathbb{R}$ such that*

$$\begin{cases} \mathcal{L}^N(\Omega \setminus K_\eta) < \eta, \\ \\ f^{\eta,\lambda}(x,y;\xi) = f(x,y;\xi) \text{ for all } (x,y;\xi) \in K_\eta \times \mathbb{R}^m \times \overline{B}(0,\lambda), \\ \\ f^{\eta,\lambda}(\,\cdot\,, y\,; \cdot\,) \text{ is continuous for all } y \in \mathbb{R}^m, \\ \\ f^{\eta,\lambda}(x, \cdot\,; \xi) \text{ is continuous and } (0,1)^m\text{-periodic for all } (x,\xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}, \end{cases}$$

*and*

$$-\beta \leqslant f^{\eta,\lambda}(x,y;\xi) \leqslant \beta(1+\lambda^p) \quad \text{for all } (x,y,\xi) \in \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^{d\times N}. \tag{0.1}$$

*Proof.* Since $f$ is a Carathéodory function, by Scorza-Dragoni's Theorem for all $\eta > 0$ there exists a compact set $K_\eta \subset \Omega$ satisfying $\mathcal{L}^N(\Omega \setminus K_\eta) < \eta$ and such that $f$ is continuous in $K_\eta \times \mathbb{R}^m \times \mathbb{R}^{d\times N}$. Let $C^{\eta,\lambda} := K_\eta \times \overline{B}(0,\lambda) \equiv C$ (to simplify notation) and $U^{\eta,\lambda} := (\mathbb{R}^N \times \mathbb{R}^{d\times N}) \setminus C^{\eta,\lambda} \equiv U$. Fix $(s,\gamma) \in C$, and for all $(x,\xi) \in U$ set

$$u^{\eta,\lambda}_{(s,\gamma)}(x,\xi) := \max\left\{2 - \frac{|(s,\gamma)-(x,\xi)|}{\operatorname{dist}((x,\xi),C)}, 0\right\} \equiv u_{(s,\gamma)}(x,\xi).$$

Clearly,

$$\begin{cases} u_{(s,\gamma)} \text{ is continuous in } U, \\ \\ 0 \leqslant u_{(s,\gamma)} \leqslant 1, \\ \\ u_{(s,\gamma)}(x,\xi) = 0 \text{ if and only if } |(s,\gamma) - (x,\xi)| \geqslant 2\text{dist}((x,\xi), C). \end{cases}$$

Let $\{s_j^\eta\}_{j\geqslant 1} \equiv \{s_j\}_{j\geqslant 1}$ and $\{\gamma_j^\lambda\}_{j\geqslant 1} \equiv \{\gamma_j\}_{j\geqslant 1}$ be a countable dense family in $K_\eta$ and $\overline{B}(0,\lambda)$, respectively. Define

$$\sigma^{\eta,\lambda}(x,\xi) := \sum_{j\geqslant 1} 2^{-j} u_{(s_j,\gamma_j)}(x,\xi) \equiv \sigma(x,\xi) \qquad \text{for all } (x,\xi) \in U.$$

Since $\sigma$ is the uniform limit of a sequence of continuous functions in $U$, then $\sigma$ is continuous in $U$. Moreover, for all $(x,\xi) \in U$,

$$0 < \sigma(x,\xi) \leqslant 1.$$

Indeed, assume that $\sigma(x,\xi) = 0$ for some $(x,\xi) \in U$. Then, for all $j \geqslant 1$, $u_{(s_j,\gamma_j)}(x,\xi) = 0$ and thus $|(s_j,\gamma_j) - (x,\xi)| \geqslant 2\,\text{dist}((x,\xi), C)$. The density of $\{s_j, \gamma_j\}$ in $C$ yields that

$$|(s,\gamma) - (x,\xi)| \geqslant 2\,\text{dist}((x,\xi), C)$$

for all $(s,\gamma) \in C$. We obtain a contradiction if we choose $(s,\gamma)$ to be a point of $C$ such that

$$\text{dist}((x,\xi), C) = \text{dist}((x,\xi), (s,\gamma))$$

so $\sigma(x,\xi) > 0$ for all $(x,\xi) \in U$. Consequently, the function

$$(x,\xi) \mapsto v_k(x,\xi) \equiv v_k^{\eta,\lambda}(x,\xi) := \frac{2^{-k} u_{(s_k,\gamma_k)}(x,\xi)}{\sigma(x,\xi)}$$

is well defined and continuous in $U$. Moreover it satisfies

$$0 \leqslant v_k(x,\xi) \leqslant 1, \quad \sum_{k\geqslant 1} v_k(x,\xi) = 1 \qquad \text{for all } (x,\xi) \in U. \tag{0.2}$$

Fix $y \in \mathbb{R}^m$ and define the continuous extension of $f(\cdot, y; \cdot)$ outside $C$ as

$$f^{\eta,\lambda}(x,y;\xi) = \begin{cases} f(x,y,\xi) & \text{if } (x,\xi) \in C, \\ \displaystyle\sum_{k\geqslant 1} v_k(x;\xi)\, f(s_k, y; \gamma_k) & \text{if } (x,\xi) \in U. \end{cases}$$

Obviously, we have $f^{\eta,\lambda}(x,y;\xi) = f(x,y;\xi)$ for all $(x,y;\xi) \in K_\eta \times \mathbb{R}^m \times \overline{B}(0,\lambda)$. On the other hand, if $(x,y,\xi)$ is such that $(x,\xi) \in U$, in view of the $p$-growth and the $p$-coercivity condition on $f$ we get

$$-\beta \leqslant f^{\eta,\lambda}(x,y;\xi) \leqslant \sum_{k\geqslant 1} v_k(x;\xi)\beta(1+|\gamma_k|^p) \leqslant \beta(1+\lambda^p).$$

Since we have

$$\sup_{y\in\mathbb{R}^m,\,(x,\xi)\in U} \Big[\sum_{k\geqslant n} \Big|2^{-k} u_{(s_k,\gamma_k)}(x,\xi) f(s_k,y;\gamma_k)\Big|\Big] \leqslant \beta(1+\lambda^p)\sum_{k\geqslant n} 2^{-k} \xrightarrow[n\to+\infty]{} 0, \tag{0.3}$$

then the function

$$(x,y;\xi) \mapsto \sum_{k\geqslant 1} 2^{-k} u_{(s_k,\gamma_k)}(x,\xi) f(s_k,y;\gamma_k)$$

is continuous on $\{(x,y,\xi) : (x,\xi) \in U,\ y \in \mathbb{R}^m\}$. In particular, for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$ the function $f^{\eta,\lambda}(x,\,\cdot\,;\xi)$ is continuous. Further, $f^{\eta,\lambda}(x,\,\cdot\,;\xi)$ it is $(0,1)^m$-periodic because if $\mathbf{i} \in \mathbb{Z}^m$ then for $(x,\xi) \in U$

$$f^{\eta,\lambda}(x,y+\mathbf{i};\xi) = \sum_{k\geqslant 1} v_k(x;\xi)\, f(s_k,y+\mathbf{i};\gamma_k) = \sum_{k\geqslant 1} v_k(x;\xi)\, f(s_k,y;\gamma_k) = f^{\eta,\lambda}(x,y;\xi).$$

Finally, we prove the continuity of $f^{\eta,\lambda}(\cdot,y;\cdot)$. By (0.3) it suffices to show that for all $(a,A) \in C$

$$\lim_{U\ni(x,\xi)\to(a,A)} f^{\eta,\lambda}(x,y;\xi) = f(a,y;A).$$

As $\{(s_j,\gamma_j)\}_{j\geqslant 1}$ is dense in $C$ and $f(\cdot,y;\cdot)$ is continuous on $C$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(a,y;A) - f(s_j,y;\gamma_j)| < \varepsilon$$

for all $j \geqslant 1$ with $|(a,A) - (s_j,\gamma_j)| < \delta$. Assume that $|(x,\xi) - (a,A)| < \delta/4$ and suppose that $j \geqslant 1$ is such that $|(a,A) - (s_j,\gamma_j)| \geqslant \delta$. Then

$$\delta \leqslant |(a,A) - (s_j,\gamma_j)| \leqslant |(a,A) - (x,\xi)| + |(x,\xi) - (s_j,\gamma_j)| \leqslant \frac{\delta}{4} + |(x,\xi) - (s_j,\gamma_j)|,$$

and thus

$$|(x,\xi) - (s_j,\gamma_j)| \geqslant \frac{3\delta}{4} > 2|(a,A) - (x,\xi)| \geqslant 2\,\mathrm{dist}((x,\xi),C).$$

Consequently, $v_j(x,\xi)=0$ if $j$ is such that $|(a,A)-(s_j,\gamma_j)|\geqslant\delta$, and so by (0.2)

$$|f^{\eta,\lambda}(x,y;\xi)-f(a,y;A)|\leqslant\sum_{j\geqslant 1,\,|(a,A)-(s_j,\gamma_j)|<\delta} v_j(x,\xi)|f(s_j,y;\gamma_j)-f(a,y;A)|<\varepsilon,$$

since the non zero terms of the sum are those which satisfy $|f(a,y;A)-f(s_j,y;\gamma_j)|<\varepsilon$. The continuity of $f^{\eta,\lambda}(\cdot,y;\cdot)$ now follows.

■

We recall here the Tietze Extension theorem.

**Theorem B.2.** *(Tietze Extension Theorem)(see DiBenedetto [42]) Let $X$ be a normal topological space. A continuous function $f$ from a closed subset $\mathcal{C}$ of $X$ into $\mathbb{R}$ has a continuous extension on $X$, i.e., there exists a continuous real-valued function $\tilde{f}$ defined on the whole $X$, such that $f=\tilde{f}$ on $\mathcal{C}$. Moreover, if $f$ is bounded, i.e., if*

$$|f(x)|\leqslant C$$

*for all $x\in\mathcal{C}$ for some $C>0$, then $\tilde{f}$ satisfies the same bound.*

The following proposition is used in Subsection 5.2.3. It allows us to extend Carathéodory integrands continuously. It relies on Scorza-Dragoni's Theorem and on Tietze's Extension Theorem.

**Lemma B.3.** *Let $W:\Omega\times\mathbb{R}^3\times\mathbb{R}^2\times\mathbb{R}^{3\times 3}\to\mathbb{R}$ satisfy $(A_1)$-$(A_4)$ in Section 5.2. Then for any $m\in\mathbb{N}$, there exists a compact set $\mathcal{C}_m\subset\Omega$ and a continuous function $W^m:\Omega\times\mathbb{R}^3\times\mathbb{R}^2\times\mathbb{R}^{3\times 3}\to\mathbb{R}$ such that $W^m(x,\cdot,\cdot;\cdot)=W(x,\cdot,\cdot;\cdot)$ for all $x\in\mathcal{C}_m$ and*

$$\mathcal{L}^3(\Omega\setminus\mathcal{C}_m)<\frac{1}{m}. \tag{0.4}$$

*Moreover,*

- *$y_\alpha\mapsto W^m(x,y_\alpha,y_3,z_\alpha;\xi)$ is $Q'$-periodic for all $(z_\alpha,y_3,\xi)\in\mathbb{R}^3\times\mathbb{R}^{3\times 3}$ and a.e. $x\in\Omega$,*
- *$(z_\alpha,y_3)\mapsto W^m(x,y_\alpha,y_3,z_\alpha;\xi)$ is $Q$-periodic for all $(y_\alpha,\xi)\in\mathbb{R}^2\times\mathbb{R}^{3\times 3}$ and a.e. $x\in\Omega$;*

*and for some $\beta>0$, we have*

$$-\beta\leqslant W^m(x,y,z_\alpha;\xi)\leqslant\beta(1+|\xi|^p)\quad\text{for all }(y,z_\alpha,\xi)\in\mathbb{R}^3\times\mathbb{R}^2\times\mathbb{R}^{3\times 3}\text{ and a.e }x\in\Omega. \tag{0.5}$$

*Proof.* By Scorza-Dragoni's Theorem for any $m \in \mathbb{N}$ there exists a compact set $\mathcal{C}_m \subset \Omega$ with $\mathcal{L}^3(\Omega \setminus \mathcal{C}_m) < 1/m$ such that $W$ is continuous on $\mathcal{C}_m \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$. Since $\mathcal{C}_m \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$ is a closed set, according to Tietze's Extension Theorem one can extend $W$ to a continuous function $W^m$ outside $\mathcal{C}_m \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$. By the construction of $W^m$ it can be seen that it satisfies the same periodicity and growth condition as $W$ and that it is bounded from below by $-\beta$. ■

We remark that the above result improves Lemma B.1.

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