COUPLED SECOND ORDER SINGULAR PERTURBATIONS FOR PHASE TRANSITIONS

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Abstract. The asymptotic behavior of a family of singular perturbations of a non-convex second order functional of the type

$$\int_\Omega f(x, u(x), \nabla u(x), \nabla^2 u(x))\, dx$$

is studied through Γ-convergence techniques as a variational model to address two-phase transition problems.

Keywords: Two-Phase Transition Problems, Singular Perturbations, Functions of Bounded Variation, Γ-Convergence.

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Contents

1. Introduction

The objective of this work is to study the asymptotic behavior, as a small parameter ε tends to zero, of a sequence of functionals of the form

$$\frac{1}{\varepsilon}\int_\Omega f(x, u(x), \varepsilon\nabla u(x), \varepsilon^2\nabla^2 u(x))\, dx$$

obtained as a singular perturbation of a non-convex second order functional of the type

$$\int_\Omega f(x, u(x), \nabla u(x), \nabla^2 u(x))\, dx$$

where $f(\cdot, u(\cdot), \nabla u(\cdot), \nabla^2 u(\cdot))$ represents the free energy of a mixture of N fluids ($N \in \mathbb{N},\ N \geq 2$), occupying a fixed container $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}, d \geq 2$), and is a function of the density $u = (u_1, ..., u_N)$ and its first and second order derivatives. The bulk energy density f is assumed to be continuous, positive and such that for all $x \in \Omega$ the function $f(x, \cdot, \mathrm{O}, \mathrm{O})$ achieves its minimum value zero at exactly two vectors $\alpha, \beta \in \mathbb{R}^N_+$, $\alpha \neq \beta$. In addition, each component u_i (the density of the ith-ingredient of the mixture) is considered to be nonnegative and the total amount of bulk material is assumed to be preserved, i.e., u belongs to the following set

$$\mathcal{V} := \left\{ u : \int_\Omega u(x)\, dx = V,\, V \in \mathbb{R}^N_+,\, |\Omega|\, \min(\alpha^i, \beta^i) < V^i < |\Omega|\, \max(\alpha^i, \beta^i),\, i = 1, ..., N \right\},$$

where V^i, α^i and β^i are the i-th components of V, α and β, respectively.

More specifically, our aim is to study, via Γ-convergence techniques, the family of minimum problems

$$\min \left\{ \frac{1}{\varepsilon} \int_\Omega f(x, u(x), \varepsilon \nabla u(x), \varepsilon^2 \nabla^2 u(x))\, dx, \quad u \in \mathcal{V} \right\}. \tag{1.1}$$

Starting with the works of Modica [30] and Sternberg [35], following ideas of Modica & Mortola [31], minimum problems involving singularly perturbed functionals of the type (1.1) have been extensively studied in the literature to solve the classical problem proposed by Gurtin [26] (see also Gurtin [27]) of minimizing

$$\int_\Omega W(u)\, dx,$$

subject to the constraint

$$\frac{1}{|\Omega|} \int_\Omega u(x)\, dx = \theta\alpha + (1 - \theta)\beta,$$

where $u : \Omega \to \mathbb{R}$, $0 < \theta < 1$ and W has two potential wells, that is, $W(u) = 0$ if and only if $u \in \{\alpha, \beta\}$.

Given that the above problem has infinitely many solutions, to resolve this non-uniqueness one considers the perturbed functionals

$$\frac{1}{\varepsilon} \int_\Omega f_\varepsilon(x, u, \nabla u)\, dx, \tag{1.2}$$

where the energy density has the form

$$f_\varepsilon(x, u, \nabla u) = W(u) + \varepsilon^2 |\nabla u|^2. \tag{1.3}$$

The Γ-limit of this sequence of functionals, identified in [30], shows that for a given volume fraction θ the physically prefered solution has minimal interfacial area.

Still considering $u : \Omega \to \mathbb{R}$, Bouchitté [8] and Owen & Sternberg [32] considered the more general coupled case where the energy density is given by

$$f_\varepsilon(x, u, \nabla u) = g(x, u, \varepsilon \nabla u)$$

and where the function g is taken to be convex in the last variable. In the vectorial case, $u : \Omega \to \mathbb{R}^N$, and for integrands given by (1.3), the study of the above minimization problem was undertaken by Fonseca & Tartar [25], and by Baldo [5] in the case of multiple wells, whereas Barroso & Fonseca [6] identified the Γ-limit of the rescaled energies (1.2) when the integrands are of the type

$$f_\varepsilon(x, u, \nabla u) = W(u) + \varepsilon^2 h^2(x, \nabla u),$$

and Fonseca & Popovici [24] did the same in the coupled case in which

$$f_\varepsilon(x,u,\nabla u)=g(x,u,\varepsilon\nabla u).$$

The characterization of the Γ-limit for functionals involving higher order terms is due to Conti, Fonseca & Leoni [13] for an energy of the form

$$\int_\Omega \frac{1}{\varepsilon}W(\nabla u)+\varepsilon|\nabla^2 u|^2\,dx.$$

We also mention the work of Chermisi, Dal Maso, Fonseca & Leoni [10], where, for scalar functions u, the authors addressed a model on pattern formation based on the Ginzburg-Landau energy

$$\int_\Omega W(u)-q|\nabla u|^2+|\nabla^2 u|^2\,dx$$

(see Coleman, Marcus & Mizel [12] and Seul & Andelman [33]) by considering a perturbed second order energy of the form

$$f_\varepsilon(x,u,\nabla u,\nabla^2 u)=W(u)-q\varepsilon^2|\nabla u|^2+\varepsilon^4|\nabla^2 u|^2,\quad q>0$$

(see also Fonseca & Mantegazza [21] for the case $q=0$, Cicalese, Spadaro & Zeppieri [11] for $q>0$ in the one-dimensional case, and Hilhorst, Peletier & Schätzle [29] for the case $q<0$ where $|\nabla^2 u|^2$ is replaced by $|\triangle u|^2$).

In this work we treat general non-convex second order singular perturbed problems of the form (1.1). For this purpose, and under some technical hypotheses (we refer to Section 3 for our list of hypotheses and to Section 4 for the complete statement of our main result), we prove in Theorem 4.1 that the family of functionals

$$F_\varepsilon(u):=\begin{cases}\dfrac{1}{\varepsilon}\displaystyle\int_\Omega f(x,u(x),\varepsilon\nabla u(x),\varepsilon^2\nabla^2 u(x))\,dx, & u\in W^{2,2}(\Omega;\mathbb{R}^N_+)\cap\mathcal{V},\\ +\infty, & \text{otherwise}\end{cases}\tag{1.4}$$

Γ-converges, with respect to the $L^1(\Omega;\mathbb{R}^N_+)$-convergence, to the functional $F:L^1(\Omega;\mathbb{R}^N_+)\to\mathbb{R}\cup\{+\infty\}$ defined by

$$F(u):=\begin{cases}\displaystyle\int_{J_u}\sigma\big(x,\nu_u(x)\big)\,d\mathcal{H}^{d-1}(x), & u\in BV\big(\Omega;\{\alpha,\beta\}\big)\cap\mathcal{V},\\ +\infty, & \text{otherwise,}\end{cases}\tag{1.5}$$

where $\sigma:\Omega\times\mathbb{S}^{d-1}\to[0,+\infty)$ is given by

$$\sigma(x,\nu):=\inf\left\{\int_{Q_\nu}\frac{1}{t}f\big(x,w(y),t\nabla w(y),t^2\nabla^2 w(y)\big)\,dy:\ t>0,\ w\in\mathcal{A}(\nu)\right\},\tag{1.6}$$

and where for every $\nu\in\mathbb{S}^{d-1}$, the class $\mathcal{A}(\nu)$ of admissible density functions, is defined by

$$\mathcal{A}(\nu):=\Big\{w\in W^{2,2}_{\text{loc}}(S_\nu;\mathbb{R}^N_+):\ w(y)=\alpha\ \text{if}\ y\cdot\nu=-\frac{1}{2},\ \ w(y)=\beta\ \text{if}\ y\cdot\nu=\frac{1}{2},$$
$$w(y)=w(y+l\nu_i),\ \text{for all}\ y\in S_\nu,\quad i=1,\dots,d-1,\ \text{and}\ l\in\mathbb{Z}\Big\}\tag{1.7}$$

where $\{\nu_1,\dots,\nu_{d-1},\nu\}$ is an orthonormal basis of $\mathbb{R}^d$ and S_ν is the strip

$$S_\nu:=\left\{x\in\mathbb{R}^d:|x\cdot\nu|<\frac{1}{2}\right\}.$$

The proof of our main result relies on the blow-up method, introduced by Fonseca & Müller (see e.g [22] and [23]), which allows us to consider the case where Ω is a small cube and the target function has planar interface. We will also use a slicing argument (cf. Lemma 3.3) enabling us to modify a sequence near the boundary of the cube without increasing the total energy. The construction of a recovering sequence to prove the upper bound inequality for the Γ-limit, for which we provide the full details, is done in several steps according to the geometry of Ω and to the interface of the target function. We follow the main ideas of the proofs presented in Barroso & Fonseca [6], in Fonseca & Popovici [24] and in Chermisi, Dal Maso, Fonseca & Leoni [10].

Due to a compactness result (see Theorem 3.1), and by the properties of Γ-convergence, which we mention in Subsection 2.5, a sequence of minimizers of the functionals F_ε defined in (1.4), assuming they exist, will converge (up to a subsequence) to a minimizer of the limiting functional F in (1.5) (see Corollary 3.2).

The overall plan of this work in the ensuing sections will be as follows: in Section 2 we set up the notation and state some preliminary results on measure theory, BV functions and Γ-convergence that will be used throughout the paper. In Section 3 we list our hypotheses and prove some auxiliary results which will be needed in the sequel. The statements and proofs of our main results can be found in Section 4.

2. Preliminaries

The purpose of this section is to give a brief overview of the concepts and results that are used in the sequel. Almost all these results are stated without proofs as they can be readily found in the references given below.

2.1. **Notation.**

Throughout the text, unless otherwise specified, $\Omega \subset \mathbb{R}^d$, $d \geq 2$, will denote an open bounded set with Lipschitz boundary and we will use the following notations:

- $|\Omega|$ denotes the Lebesgue measure of Ω.
- $\mathbb{R}^N_+ := [0, +\infty)^N$.
- $\mathcal{L}^d$ and $\mathcal{H}^{d-1}$ stand, respectively, for the d-dimensional Lebesgue measure and the $(d-1)$-dimensional Hausdorff measure in $\mathbb{R}^d$.
- Given $x \in \mathbb{R}^d$ we write $x = (x', x_d)$, where x' stands for its first $d-1$ coordinates and x_d for the d-th one. For any set $\Omega \subseteq \mathbb{R}^d$ we denote by Ω' the set of points $x' \in \mathbb{R}^{d-1}$ for which there exists $x_d \in \mathbb{R}$ such that $x = (x', x_d) \in \Omega$.
- Q is the open unit cube centered at the origin with faces normal to the coordinates axes.
- $B(x, r)$ denotes the open ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$.
- $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$.
- Given $\nu \in \mathbb{S}^{d-1}$ the set S_ν represents the strip

$$S_\nu := \left\{ x \in \mathbb{R}^d : |x \cdot \nu| < \frac{1}{2} \right\}$$

  and Q_ν denotes an open unit cube centered at the origin with two of its faces normal to ν, i.e., if $\{\nu_1, \ldots, \nu_{d-1}, \nu\}$ is an orthonormal basis of $\mathbb{R}^d$ then

$$Q_\nu := \left\{ x \in \mathbb{R}^d : |x \cdot \nu| < \frac{1}{2}, \, |x \cdot \nu_i| < \frac{1}{2}, \, i = 1, \ldots, d-1 \right\}. \tag{2.1}$$

- $Q_\nu(x_0, r) := x_0 + rQ_\nu$ for $x_0 \in \mathbb{R}^d$, $r > 0$ and $\nu \in \mathbb{S}^{d-1}$. If $\{e_1, \ldots, e_d\}$ is the canonical basis of $\mathbb{R}^d$ then $Q_{e_d}(x_0, r) = x_0 + rQ =: Q(x_0, r)$.
- $C_{\mathrm{per}}(Q)$ represents the set of all continuous and Q-periodic functions.

- $L^p(\Omega; C_{\rm per}(Q))$ is the space of all measurable functions $f:\Omega\to C_{\rm per}(Q)$ such that
$$||f||_{L^p(\Omega;C_{\rm per}(Q))} = \Big(\int_\Omega ||f(x)||^p_{C_{\rm per}(Q)}\,dx\Big)^{1/p} < +\infty$$
where, for $x\in\Omega$, $||f(x)||_{C_{\rm per}(Q)} := \sup_{y\in Q}|f(x,y)|$ and $f(x,y) := f(x)(y)$.
- Sym^N stands for the space of symmetric $N\times N$ matrices.
- $\mathcal{T}^{N\times N\times d}$ is the space of tensors $\Lambda = (\Lambda^1,\dots,\Lambda^d),\ \Lambda^i\in\mathrm{Sym}^N,\ i=1,\dots,d$.
- $\otimes$ and $\odot$ represent, respectively, the usual tensor product and symmetric tensor product of two tensors.
- C denotes a generic positive constant whose value might change from line to line.

2.2. **Some useful covering and convergence theorems.** The following covering theorem due to Whitney [36] (see also Stein [34, Chapter 1, Theorem 3] or Guzmán [28]) is used in the proof of our main result and gives the decomposition of the complement of a given closed set into a disjoint union of cubes whose sides are parallel to the axes.

Theorem 2.1 (Whitney's Covering Theorem). *Let $F\subset\mathbb{R}^d$ be a non-empty closed set. Then there exists a countable family of cubes of the form $Q_i = a_i + \delta_i Q$, $a_i\in\mathbb{R}^d$, $\delta_i>0$, such that the following properties hold:*

(i) $\mathbb{R}^d\setminus F = \bigcup_{i=1}^\infty \overline{Q}_i$;
(ii) *the cubes Q_i are mutually disjoint;*
(iii) $\operatorname{diam} Q_i \le \operatorname{dist}(Q_i,F)\le 4\operatorname{diam} Q_i$.

Given the periodicity of our admissible functions in the first $d-1$ variables, the following version of the Riemann-Lebesgue Lemma (see Lemma 5.2 in Allaire [2] and also Bensoussan, Lions & Papanicolaou [7] and Donato[17]) will be used in the construction of a recovering sequence.

Lemma 2.2 (Riemann-Lebesgue Lemma). *Let $f\in L^p(\Omega;C_{\rm per}(Q))$, $1\le p<+\infty$, and let $\{\varepsilon_n\}_n$ be a fixed sequence of positive real numbers converging to zero. Then, for every $n\in\mathbb{N}$, the function $f(\cdot,\frac{\cdot}{\varepsilon_n})$ is measurable in Ω,*
$$\left|\left|f\left(\cdot,\frac{\cdot}{\varepsilon_n}\right)\right|\right|_{L^p(\Omega)} \le ||f||_{L^p(\Omega;C_{\rm per}(Q))}$$
and
$$\lim_n \int_\Omega \left|f\left(x,\frac{x}{\varepsilon_n}\right)\right|^p dx = \int_\Omega\int_Q |f(x,y)|^p\,dydx.$$

2.3. **Remarks on measure theory.** In this section we recall some notations and well known results in measure theory (see e.g. Ambrosio, Fusco & Pallara [4] and Fonseca & Leoni [20], as well as the bibliography therein).

Let X be a locally compact separable metric space and let $C_c(X;\mathbb{R}^N)$, $N\ge1$, denote the set of continuous functions with compact support in X. We denote by $C_0(X;\mathbb{R}^N)$ the completion of $C_c(X;\mathbb{R}^N)$ with respect to the supremum norm. Let $\mathcal{B}(X)$ be the Borel σ-algebra of X. By the Riesz-Representation Theorem the dual of the Banach space $C_0(X;\mathbb{R}^N)$, denoted by $\mathcal{M}(X;\mathbb{R}^N)$, is the space of finite $\mathbb{R}^N$-valued Radon measures $\mu:\mathcal{B}(X)\to\mathbb{R}^N$ under the pairing
$$<\mu,\varphi>:=\int_X \varphi\,d\mu \equiv \sum_{i=1}^N\int_X\varphi_i\,d\mu_i$$

where $\varphi = (\varphi_1, ..., \varphi_N)$ and $\mu = (\mu_1, ..., \mu_N)$. The space $\mathcal{M}(X;\mathbb{R}^N)$ will be endowed with the weak*-topology deriving from this duality. In particular, a sequence $\{\mu_n\} \subset \mathcal{M}(X;\mathbb{R}^N)$ is said to weak*-converge to $\mu \in \mathcal{M}(X;\mathbb{R}^N)$ (indicated by $\mu_n \overset{\star}{\rightharpoonup} \mu$) if for all $\varphi \in C_0(X;\mathbb{R}^N)$

$$\lim_{n\to+\infty} \int_X \varphi \, d\mu_n = \int_X \varphi \, d\mu.$$

If $N = 1$ we write by simplicity $\mathcal{M}(X)$ and we denote by $\mathcal{M}^+(X)$ its subset of positive measures.

Given $\mu \in \mathcal{M}(X;\mathbb{R}^N)$ let $|\mu|$ denote its *total variation* and let supp μ denote its *support*. In addition, given $E \in \mathcal{B}(X)$ we denote by $\mu \llcorner E$ the measure given by $\mu \llcorner E(A) := \mu(E \cap A)$ for every $A \in \mathcal{B}(X)$.

The following result can be found in Fonseca & Leoni [20, Corollary 1.204].

Proposition 2.3. *Let $\mu_n \in \mathcal{M}(X)$ be such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(X)$ and $|\mu_n| \overset{*}{\rightharpoonup} \nu$ in $\mathcal{M}(X)$. If $A \subset X$ is open, $\bar{A}$ compact and $\nu(\partial A) = 0$ then*

$$\mu_n(A) \to \mu(A).$$

We recall that a measure μ is said to be *absolutely continuous* with respect to a positive measure ν, written $\mu << \nu$, if for every $E \in \mathcal{B}(X)$ the following implication holds:

$$\nu(E) = 0 \;\Rightarrow\; \mu(E) = 0.$$

Two positive measures μ and ν are said to be *mutually singular*, written $\mu \perp \nu$, if there exists $E \in \mathcal{B}(X)$ such that $\nu(E) = 0$ and $\mu(X \setminus E) = 0$. For general vector-valued measures μ and ν we say that $\mu \perp \nu$ if $|\mu| \perp |\nu|$.

Theorem 2.4 (Lebesgue-Radon-Nikodým Theorem)**.** *Let $\mu \in \mathcal{M}^+(X)$ and $\nu \in \mathcal{M}(X;\mathbb{R}^N)$. Then*

(i) *there exist two $\mathbb{R}^N$-valued measures ν_a and ν_s such that*

$$\nu = \nu_a + \nu_s \tag{2.2}$$

with $\nu_a << \mu$ and $\nu_s \perp \mu$. Moreover, the decomposition (2.2) is unique, that is, if $\nu = \bar{\nu}_a + \bar{\nu}_s$ for some measures $\bar{\nu}_a, \bar{\nu}_s$, with $\bar{\nu}_a << \mu$ and $\bar{\nu}_s \perp \mu$, then $\nu_a = \bar{\nu}_a$ and $\nu_s = \bar{\nu}_s$;

(ii) *there is a μ-measurable function $u \in L^1(\Omega;\mathbb{R}^N)$ such that*

$$\nu_a(E) = \int_E u \, d\mu$$

for every $E \in \mathcal{B}(\Omega)$. The function u is unique up to a set of μ measure zero.

The decomposition $\nu = \nu_a + \nu_s$ is called the *Lebesgue decomposition* of ν with respect to μ (see [20, Theorem 1.115]), ν_a and ν_s are called, respectively, the absolutely continuous part and the singular part of ν with respect to μ and the function u is called the *Radon-Nikodým derivative* of ν with respect to μ, denoted by $u = d\nu/d\mu$ (see [20, Theorem 1.101]).

The next result is a strong version of the Besicovitch Derivation Theorem due to Ambrosio and Dal Maso [3] (see also [4, Theorem 2.22 and Theorem 5.52] or [20, Theorem 1.155]).

Theorem 2.5. *Let $\mu \in \mathcal{M}^+(\Omega)$ and $\nu \in \mathcal{M}(\Omega;\mathbb{R}^N)$. Then there exists a Borel set $E \subset \Omega$ with $\mu(E) = 0$ such that for every $x \in$ (supp μ)$\setminus E$*

$$\frac{d\nu}{d\mu}(x) = \frac{d\nu_a}{d\mu}(x) = \lim_{\varepsilon\to 0^+} \frac{\nu\big((x+\varepsilon D)\cap\Omega\big)}{\mu\big((x+\varepsilon D)\cap\Omega\big)} \in \mathbb{R}$$

and

$$\frac{d\nu_s}{d\mu}(x) = \lim_{\varepsilon\to 0^+} \frac{\nu_s\big((x+\varepsilon D)\cap\Omega\big)}{\mu\big((x+\varepsilon D)\cap\Omega\big)} = 0,$$

where D is any bounded, convex, open set containing the origin and the exceptional set E is independent of the choice of D.

2.4. **BV-functions and some of their main properties.** Now we introduce some definitions and standard facts from the theory of BV-functions and we refer to Ambrosio, Fusco & Pallara [4] for an exhaustive exposition of the subject.

A function $u \in L^1(\Omega;\mathbb{R}^N)$ is said to be of *bounded variation*, and we write $u \in BV(\Omega;\mathbb{R}^N)$ (or $BV(\Omega)$ for $N = 1$), if all its first order distributional derivatives $D_j u_i$ belong to $\mathcal{M}(\Omega)$ for $i = 1, ..., N$ and $j = 1, ..., d$. The matrix-valued measure whose entries are $D_j u_i$ is denoted by Du.

Clearly, we have that any $u \in W^{1,1}(\Omega;\mathbb{R}^N)$ is a BV-function with $Du \in L^1(\Omega;\mathbb{R}^N)$ and the measures Du^i_j are absolutely continuous with respect to the Lebesgue measure.

The space $BV(\Omega;\mathbb{R}^N)$ is a Banach space when endowed with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + |Du|(\Omega).$$

Given $u \in BV(\Omega;\mathbb{R}^N)$, let Ω_u be the set of points $x \in \Omega$ where the approximate limit of u exists, i.e. such that there exists $z \in \mathbb{R}^N$ with

$$\lim_{\varepsilon\to 0^+} \fint_{B(x,\varepsilon)} |u(y) - z|\, dy = 0.$$

If $x \in \Omega_u$ and $z = u(x)$ we say that u is *approximately continuous* at x (or that x is a Lebesgue point of u). The function u is approximately continuous $\mathcal{L}^d$-a.e. $x \in \Omega_u$ and

$$\mathcal{L}^d(S_u) = 0$$

where we denote by S_u the set of points where u is not approximately continuous, ie., $S_u = \Omega \setminus \Omega_u$. We say that $x \in S_u$ is an approximate jump point of u if there exists $\nu_u(x) \in \mathbb{S}^{d-1}$ and $u^{\pm}(x) \in \mathbb{R}^N$ such that

$$\lim_{r\to 0^+} \frac{1}{r^d}\left(\int_{B^+(x,r)} \left|u(y) - u^+(x)\right| dy + \int_{B^-(x,r)} \left|u(y) - u^-(x)\right| dy\right) = 0,$$

with $B^{\pm}(x,r) := \{y \in B(x,r) : \pm(y-x)\cdot\nu_u(x) > 0\}$. The triple $(\nu_u(x), u^+(x), u^-(x))$ is unique up to a change of sign of $\nu_u(x)$ and a permutation of $u^+(x)$ and $u^-(x)$. The set of approximate jump points is denoted by J_u.

By the Lebesgue-Radon-Nikodým Theorem 2.4, if $u \in BV(\Omega;\mathbb{R}^N)$ then Du can be split into the sum of two mutually singular measures $D^a u$ and $D^s u$ (the absolutely continous part and singular part, respectively, of Du with respect to the Lebesgue measure $\mathcal{L}^d$). We denote by ∇u the Radon-Nikodým derivative of $D^a u$ with respect to $\mathcal{L}^d$, so that we can write

$$Du = \nabla u \mathcal{L}^d \lfloor \Omega + D^s u.$$

By the Calderón-Zygmund Theorem, the L^1-function ∇u coincides with the *approximate differential* of u (see, e.g., [4, Theorem 3.83 and Definition 3.70]).

We recall that an $\mathcal{H}^{d-1}$-measurable set $E \subset \mathbb{R}^d$ is said to be a *countably $\mathcal{H}^{d-1}$-rectifiable set* if it can be covered $\mathcal{H}^{d-1}$-almost everywhere by a countable family of $(d-1)$-dimensional surfaces of class C^1. The proof of the well known Structure Theorem for BV-functions can be found in [4, Theorem 3.78 (Federer-Vol'pert) and Proposition 3.92].

Theorem 2.6 (Structure Theorem for BV-functions)**.** *If $\Omega \subset \mathbb{R}^d$ is open and $u \in BV(\Omega;\mathbb{R}^N)$, then J_u is a countably $\mathcal{H}^{d-1}$-rectifiable set oriented by ν_u, $\mathcal{H}^{d-1}(S_u \setminus J_u) = |Du|(S_u \setminus J_u) = 0$ and $D^s u$ can be decomposed as $D^c u + D^j u$, where $|D^c u|(E) = 0$ for every Borel set E with $\mathcal{H}^{d-1}(E) < +\infty$, and*

$$D^j u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{d-1} \llcorner J_u.$$

$D^c u$ and $D^j u$ are called the *Cantor part* and the *jump part* of the measure Du, respectively.

We also recall that a $\mathcal{L}^d$-measurable subset $E \subset \mathbb{R}^d$ is a *set of finite perimeter* in Ω if the characteristic function χ_E of E is a function of bounded variation. In this case, the perimeter of E in Ω is given by the total variation of χ_E in Ω, i.e., $\mathrm{Per}_\Omega(E) := |D\chi_E|(\Omega)$.

Definition 2.7 (Reduced boundary)**.** Let E be a $\mathcal{L}^d$-measurable subset of $\mathbb{R}^d$ and Ω be the largest open set such that E is locally of finite perimeter in Ω, i.e., such that $\chi_E \in BV_{\mathrm{loc}}(\Omega)$. The *reduced boundary* of E, $\partial^* E$, is the collection of all points $x_0 \in \Omega$ such that

(i) $\big|D\chi_E\big|\big(B(x_0, r)\big) > 0$ for all $r > 0$, that is, $x_0 \in \mathrm{supp}|D\chi_E|$;

(ii) the limit $\nu_E(x_0) := \lim_{r\to 0^+} \dfrac{D\chi_E\big(B(x_0,r)\big)}{|D\chi_E|\big(B(x_0,r)\big)}$ exists in $\mathbb{R}^d$;

(iii) $|\nu_E(x_0)| = 1$.

The function $\nu_E : \partial^* E \to \mathbb{S}^{d-1}$ is called the *generalized unit inner normal* to E.

It can be easily checked that $\partial^* E$ is a Borel set and that ν_E is a Borel map. By the Besicovitch Derivation Theorem the measure $|D\chi_E|$ is concentrated on $\partial^* E$ and $D\chi_E = \nu_E |D\chi_E|$. In addition, by De Giorgi's Rectifiability Theorem, see [4, Theorem 3.59], $|D\chi_E|$ coincides with $\mathcal{H}^{d-1}\llcorner\partial^* E$, and for every $x \in \partial^* E$ the following properties hold

$$\lim_{r\to 0^+} \frac{1}{r^{d-1}} \mathcal{H}^{d-1}\Big(\partial^* E \cap Q_{\nu_E(x)}(x, r)\Big) = 1 \tag{2.3}$$

$$\lim_{r\to 0^+} \frac{1}{r^d} \mathcal{L}^d\big(\{y \in B(x,r) \setminus E : (y - x)\cdot \nu_E(x) \geq 0\}\big) = 0 \tag{2.4}$$

$$\lim_{r\to 0^+} \frac{1}{r^d} \mathcal{L}^d\big(\{y \in B(x,r) \cap E : (y - x)\cdot \nu_E(x) \leq 0\}\big) = 0 \tag{2.5}$$

(see also Evans & Gariepy [18, § 5.7.2, Corollary 1]).

Definition 2.8 (Essential boundary)**.** *For every $t \in [0,1]$ and every $\mathcal{L}^d$-measurable subset $E \subset \mathbb{R}^d$ let E^t be the set of all points where E has density t, i.e.,*

$$E^t := \left\{ x \in \mathbb{R}^d : \lim_{r\to 0^+} \frac{\mathcal{L}^d\big(E \cap B(x,r)\big)}{\mathcal{L}^d\big(B(x,r)\big)} = t \right\}.$$

The essential boundary of E is the set $\partial^e E := \mathbb{R}^d \setminus (E^0 \cap E^1)$ (set of points where the density is neither 0 nor 1).

We remark that every set E^t in Definition 2.8 is a Borel set, and that the sets E^0 and E^1 could be considered, respectively, as the measure theoretic exterior and interior of E motivating the definition of $\partial^e E$.

The following result is due to Federer and provides a way to compute the perimeter of a set of finite perimeter (see Federer [19] or [4, Theorem 3.61]).

Theorem 2.9. *Let E be a set of finite perimeter in Ω. Then*

$$\partial^* E \subset E^{1/2} \subset \partial^e E \qquad \text{and} \qquad \mathcal{H}^{d-1}\big(\Omega \setminus (E^0 \cup \partial^* E \cup E^1)\big) = 0.$$

In particular, E has density either 0 or $1/2$ or 1 at $\mathcal{H}^{d-1}$-a.e. $x \in \Omega$, and $\mathcal{H}^{d-1}$-a.e. $x \in \Omega \cap \partial^e E$ belongs to $\partial^* E$. Moreover,

$$\mathrm{Per}_\Omega(E) = \mathcal{H}^{d-1}(\Omega \cap E^{1/2}) = \mathcal{H}^{d-1}(\Omega \cap \partial^e E).$$

Example 2.10. (The set $BV(\Omega; \{\alpha, \beta\}))$ Given $\alpha, \beta \in \mathbb{R}^N$, $\alpha \neq \beta$, we denote by $BV\big(\Omega; \{\alpha, \beta\}\big)$ the set of all vector-valued functions u of bounded variation in Ω such that $u(x) \in \{\alpha, \beta\}$ for $\mathcal{L}^d$-a.e. $x \in \Omega$. If $u \in BV\big(\Omega; \{\alpha, \beta\}\big)$, that is, $u = \beta\chi_E + \alpha\chi_{\Omega\setminus E}$ for some $\mathcal{L}^d$-measurable set E of finite perimeter, then

S_u, the reduced boundary $\partial^* E$ and the jump set J_u of u have the same $\mathcal{H}^{d-1}$-measure in Ω. By (2.4) and (2.5), we also have $\nu_u(x) = \nu_E(x)$, $u^+(x) = \beta$ and $u^-(x) = \alpha$, for $\mathcal{H}^{d-1}$-a.e. $x \in \partial^* E$.

The following theorem is a variant of a well-known approximation result for sets of finite perimeter.

Theorem 2.11. *Let Ω be an open, bounded set with Lipschitz boundary and let E be a subset of Ω with $\mathrm{Per}_\Omega(E) < +\infty$. There exists a sequence $\{E_n\}$ of polyhedral sets (i.e., for each n, E_n is a bounded Lipschitz domain with $\partial E_n = H_{1,n} \cup H_{2,n} \cup \dots H_{L_n,n}$, where each $H_{j,n}$ is a closed subset of a hyperplane $\{x \in \mathbb{R}^d : x \cdot \nu_j = c_j\}$, for some $c_j \in \mathbb{R}$ and $\nu_j \in \mathbb{S}^{d-1}$, $j = 1, \dots, L_n$, $L_n \in \mathbb{N}$) satisfying the following properties:*

(i) $\chi_{E_n} \to \chi_E$ *in* $L^1(\Omega)$, *as* $n \to +\infty$,

(ii) $\lim_{n\to+\infty} \mathrm{Per}_\Omega(E_n) = \mathrm{Per}_\Omega(E)$,

(iii) $\mathcal{H}^{d-1}(\partial^* E_n \cap \partial\Omega) = 0$,

(iv) $\mathcal{L}^d(E_n) = \mathcal{L}^d(E)$.

Moreover, for every nonnegative continuous function ϕ on $\mathbb{R}^d$, we have

$$\lim_{n\to+\infty} \int_{\partial^* E_n \cap \Omega} \phi\big(\nu_n(x)\big)\, d\mathcal{H}^{d-1}(x) = \int_{\partial^* E \cap \Omega} \phi\big(\nu(x)\big)\, d\mathcal{H}^{d-1}(x), \tag{2.6}$$

where $\nu_n(x)$ and $\nu(x)$ denote the generalized unit normals to E_n and E, respectively, at x.

For the construction of the sets E_n in Theorem 2.11 we refer to Lemma 3.1 in Baldo [5]. Assertion (2.6) is due to Acerbi & Bouchitté [1, Lemma 3.6 ii].

2.5. **The notion of Γ-convergence of a family of functionals and related properties.** We briefly recall De Giorgi's notion of Γ-convergence and some of its basic properties. We refer to De Giorgi & Dal Maso [15] and De Giorgi & Franzoni [16] and to Braides [9] and Dal Maso [14] for a comprehensive treatment and bibliography on the subject.

Let X denote a metric space.

Definition 2.12. (Γ-convergence of a sequence of functionals) *Let $F_n, F : X \to \mathbb{R} \cup \{+\infty\}$. The functional F is said to be the* Γ-lim inf *(resp.* Γ-lim sup) *of $\{F_n\}_n$ with respect to the metric of X if for every $u \in X$*

$$F(u) = \inf_{\{u_n\}} \left\{ \liminf_{n\to+\infty} F_n(u_n) : \ u_n \in X,\, u_n \to u \text{ in } X \right\} \ (\text{resp. } \limsup_{n\to+\infty}).$$

In this case we write

$$F = \Gamma\text{-}\liminf_{n\to+\infty} F_n \qquad \left(\text{resp. } \ F = \Gamma\text{-}\limsup_{n\to+\infty} F_n\right).$$

Moreover, F is said to be the Γ-lim *of $\{F_n\}_n$ if*

$$F = \Gamma\text{-}\liminf_{n\to+\infty} F_n = \Gamma\text{-}\limsup_{n\to+\infty} F_n,$$

and in this case we write

$$F = \Gamma\text{-}\lim_{n\to+\infty} F_n.$$

For every $\varepsilon > 0$ let F_ε be a functional defined in X with values in $\mathbb{R} \cup \{+\infty\}$, $F_\varepsilon : X \to \mathbb{R} \cup \{+\infty\}$.

Definition 2.13. (Γ-convergence of a family of functionals)

A functional $F : X \to \mathbb{R} \cup \{+\infty\}$ is said to be the Γ-$\liminf$ *(resp.* Γ-$\limsup$ *or* Γ-$\lim$*) of $\{F_\varepsilon\}_\varepsilon$ with respect to the metric of X, as $\varepsilon \to 0^+$, if for every sequence $\varepsilon_n \to 0^+$,*

$$F = \Gamma\text{-}\liminf_{n\to+\infty} F_{\varepsilon_n} \quad \left(\text{resp.} \quad F = \Gamma\text{-}\limsup_{n\to+\infty} F_{\varepsilon_n} \ \textit{ or } \ F = \Gamma\text{-}\lim_{n\to+\infty} F_{\varepsilon_n}\right),$$

and we write

$$F = \Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon \quad \left(\text{resp.} \quad F = \Gamma\text{-}\limsup_{\varepsilon\to 0^+} F_\varepsilon \ \textit{ or } \ F = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon\right).$$

One of the most important properties of Γ-convergence is that under appropriate compactness assumptions it implies the convergence of minimizers of a family of functionals to the minimum of the limiting functional, as a consequence of the following result (see Corollary 7.20 in [14]).

Theorem 2.14. (Fundamental Theorem of Γ-convergence) *Let $\{F_\varepsilon\}_\varepsilon$ be a family of functionals defined in X and let*

$$F = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon.$$

If u_ε is a minimizer of F_ε in X and $u_\varepsilon \to u$ in X then u is a minimizer of F in X and

$$F(u) = \lim_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon).$$

3. Hypotheses and auxiliary results

In this section we list our hypotheses and we state and prove several results which will be used in the sequel.

We recall that our goal is to prove that the family of functionals F_ε defined in (1.4) Γ-converges, with respect to the $L^1(\Omega;\mathbb{R}^N_+)$-convergence, to the functional F given in (1.5). To simplify the notation throughout this work we define

$$E_\varepsilon(u;U) = \frac{1}{\varepsilon}\int_U f(x, u(x), \varepsilon\nabla u(x), \varepsilon^2\nabla^2 u(x))\,dx \tag{3.1}$$

for all $\varepsilon > 0$, $u \in W^{2,2}(U;\mathbb{R}^N_+)$ with $U \subset \Omega$ an open set.

We assume that the integrand f satisfies the following hypotheses

(H1) $f : \Omega \times \mathbb{R}^N_+ \times \mathbb{R}^{N\times d} \times \mathcal{T}^{N\times N\times d} \to [0,+\infty)$, $f = f(x,u,\xi,\Lambda)$ is continuous;

(H2) $f(x,u,\mathrm{O},\mathrm{O}) = 0$ if and only if $u \in \{\alpha,\beta\} \subset \mathbb{R}^N_+$;

(H3) there exists a continuous function $g : \overline{\Omega} \times \mathbb{R}^N_+ \to [0,+\infty)$ such that

$$\frac{1}{C}\Big(g(x,u) + |\xi|^2 + |\Lambda|^2\Big) \le f\big(x,u,\xi,\Lambda\big) \le C\Big(g(x,u) + |\xi|^2 + |\Lambda|^2\Big)$$

for all $(x,u,\xi,\Lambda) \in \Omega \times \mathbb{R}^N_+ \times \mathbb{R}^{N\times d} \times \mathcal{T}^{N\times N\times d}$, where g satisfies

$$\frac{1}{C}|u|^q - C \le g(x,u) \le C(1 + |u|^q) \tag{3.2}$$

for some $q \ge 2$, some $C > 0$ and for all $(x,u) \in \overline{\Omega} \times \mathbb{R}^N_+$;

(H4) for every $x_0 \in \Omega$ and every $\tau > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|f(x,u,\xi,\Lambda) - f(x_0,u,\xi,\Lambda)| \le \tau f(x,u,\xi,\Lambda)$$

for all $(u,\xi,\Lambda) \in \mathbb{R}^N_+ \times \mathbb{R}^{N\times d} \times \mathcal{T}^{N\times N\times d}$.

In addition, due to the constraint $u \in \mathcal{V}$, in order to obtain a recovering sequence to prove the upper bound inequality for our Γ-limit result, we need to assume further that

(H5) for every $M > 0$ there exists $C_M > 0$ such that for every $u_1, u_2 \in \mathbb{R}^N_+$ with $|u_1|, |u_2| \leq M$, and every $(x, \xi, \Lambda) \in \Omega \times \mathbb{R}^{N\times d} \times \mathcal{T}^{N\times N\times d}$,

$$\left|f(x, u_1, \xi, \Lambda) - f(x, u_2, \xi, \Lambda)\right| \leq C_M |u_1 - u_2|(1 + |\xi|^2 + |\Lambda|^2);$$

(H6) there exist $\delta_0 > 0$ and $C > 0$ such that

$$f(x, u, \mathrm{O}, \mathrm{O}) \leq C|u - \alpha|^2$$

whenever $|u - \alpha| < \delta_0$ and $x \in \Omega$, and

$$f(x, u, \mathrm{O}, \mathrm{O}) \leq C|u - \beta|^2$$

whenever $|u - \beta| \leq \delta_0$ and $x \in \Omega$.

As a consequence of our hypotheses we show that any sequence $\{u_\varepsilon\}$ which has uniformly bounded energy is relatively compact in L^1. Precisely, we have the following result.

Theorem 3.1. *Assume that hypotheses $(H1) - (H3)$ hold. Let $\{u_\varepsilon\} \subset W^{2,2}(\Omega; \mathbb{R}^N_+) \cap \mathcal{V}$ be such that*

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) < +\infty.$$

Then there exist a subsequence $\{u_{\varepsilon_k}\} \subset \{u_\varepsilon\}$ and a function $u \in BV(\Omega; \{\alpha, \beta\}) \cap \mathcal{V}$ such that $u_{\varepsilon_k} \to u$ in $L^1(\Omega; \mathbb{R}^N_+)$.

Proof. Assume, without loss of generality, that

$$\lim_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) = L < +\infty.$$

The lower bound on the function g in hypothesis $(H3)$ implies that there exist constants $R > C \geq 1$ and $\tilde{C} > 0$ such that

$$|u| > R \Rightarrow g(x, u) \geq \tilde{C}|u|.$$

Again by $(H3)$ we have

$$F_\varepsilon(u_\varepsilon) \geq \int_\Omega \frac{\tilde{C}}{\varepsilon}|u_\varepsilon(x)| + \varepsilon|\nabla u_\varepsilon(x)|^2\, dx.$$

Thus, by the compactness result in [25, Theorem 4.1] we conclude that $\{u_\varepsilon\}$ is relatively compact in $L^1(\Omega; \mathbb{R}^N_+)$. The argument given in Lemma 4.3 shows that any cluster point of $\{u_\varepsilon\}$ belongs to $BV(\Omega; \{\alpha, \beta\}) \cap \mathcal{V}$. □

It follows immediately from Theorem 3.1 and Theorem 2.14 that

Corollary 3.2. *Under hypotheses $(H1) - (H6)$, if $\{u_\varepsilon\}$ is a sequence of minimizers of F_ε satisfying the constraint*

$$\int_\Omega u_\varepsilon(x)\, dx = V,$$

then $\{u_\varepsilon\}$ is relatively compact in $L^1(\Omega; \mathbb{R}^N_+)$ and any cluster point of $\{u_\varepsilon\}$ belongs to $BV(\Omega; \{\alpha, \beta\}) \cap \mathcal{V}$ and is a solution of the minimization problem

$$\min\left\{F(u) : u \in L^1(\Omega; \mathbb{R}^N_+) \cap \mathcal{V}\right\}.$$

Our next auxiliary result is crucial to apply a blow-up argument in the proof of Theorem 4.1. It relies on a slicing argument applied in the cube Q_ν, for $\nu \in \mathbb{S}^{d-1}$ (see (2.1)), and to a target function of the form

$$u_0(x) := \begin{cases} \beta, & \text{if } x \in Q_\nu,\ x\cdot\nu > 0, \\ \alpha, & \text{if } x \in Q_\nu,\ x\cdot\nu < 0, \end{cases} \tag{3.3}$$

allowing us to replace a sequence $\{u_n\}$ converging to u_0 in $L^1(Q_\nu;\mathbb{R}^N_+)$ by a sequence $\{v_n\} \subset \mathcal{A}(\nu)$, still converging to u_0 in $L^1(Q_\nu;\mathbb{R}^N_+)$, without increasing the total energy.

To present this result we need to introduce the following notation. Given an even function $\Psi \in C_c^\infty\left(\mathbb{R}^d\right)$ with $\operatorname{supp}\Psi \subset B(0,1)$ and $\int_{\mathbb{R}^d} \Psi(x)\,dx = 1$, for every $\varepsilon > 0$ we define the standard mollifier

$$\Psi_\varepsilon(x) := \frac{1}{\varepsilon^d}\Psi\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^d, \tag{3.4}$$

and we observe that

$$(u_0 * \Psi_\varepsilon)(x) = \beta \quad \text{if} \quad x\cdot\nu > \varepsilon, \qquad (u_0 * \Psi_\varepsilon)(x) = \alpha \quad \text{if} \quad x\cdot\nu < -\varepsilon, \tag{3.5}$$

and

$$\nabla(u_0 * \Psi_\varepsilon)(x) = 0 \qquad \text{if} \quad |x\cdot\nu| > \varepsilon. \tag{3.6}$$

Note that $\operatorname{supp}\Psi_\varepsilon \subset B(0,\varepsilon)$ and that given $u \in L^1_{\text{loc}}\left(\mathbb{R}^d\right)$ the function

$$u_\varepsilon(x) := (u * \Psi_\varepsilon)(x) = \int_{\mathbb{R}^d} \Psi_\varepsilon(x-y)\,u(y)\,dy, \quad x \in \mathbb{R}^d, \tag{3.7}$$

belongs to $C^\infty\left(\mathbb{R}^d\right)$. In addition, if u is bounded, then for every $1 \le p < +\infty$,

$$u_\varepsilon \to u \text{ in } L^p_{\text{loc}}\left(\mathbb{R}^d\right),\ \|u_\varepsilon\|_\infty \le \|u\|_\infty,\ \|\nabla u_\varepsilon\|_\infty \le C\frac{\|u\|_\infty}{\varepsilon},\ \left\|\nabla^2 u_\varepsilon\right\|_\infty \le C\frac{\|u\|_\infty}{\varepsilon^2}. \tag{3.8}$$

Lemma 3.3. *Assuming that f is defined in Q_ν and satisfies hypotheses (H1)–(H3) in this domain, let $\{\varepsilon_n\}$ be a sequence such that $\varepsilon_n \to 0^+$ as $n \to +\infty$, let $\nu \in \mathbb{S}^{d-1}$ and u_0 be defined as in* (3.3). *If $\{u_n\}$ is a sequence in $W^{2,2}(Q_\nu;\mathbb{R}^N_+)$ converging in $L^1(Q_\nu;\mathbb{R}^N_+)$ to u_0, then there exists a sequence $\{v_n\} \subset W^{2,2}(Q_\nu;\mathbb{R}^N_+)$ such that*

i) *$v_n \to u_0$ in $L^1(Q_\nu;\mathbb{R}^N_+)$;*
ii) *$v_n = u_0 * \Psi_{\varepsilon_n}$ on ∂Q_ν;*
iii) *there is no increase in the total energy when u_n is replaced by v_n, that is,*

$$\begin{aligned}
\limsup_{n\to+\infty} E_{\varepsilon_n}(v_n;Q_\nu) &= \limsup_{n\to+\infty}\int_{Q_\nu}\frac{1}{\varepsilon_n}f\big(x,v_n(x),\varepsilon_n\nabla v_n(x),\varepsilon_n^2\nabla^2 v_n(x)\big)dx\\
&\le \liminf_{n\to+\infty}\int_{Q_\nu}\frac{1}{\varepsilon_n}f\big(x,u_n(x),\varepsilon_n\nabla u_n(x),\varepsilon_n^2\nabla^2 u_n(x)\big)dx\\
&= \liminf_{n\to+\infty} E_{\varepsilon_n}(u_n;Q_\nu)
\end{aligned}$$

where E_ε, $\varepsilon > 0$, is the functional given in (3.1) *with the integrals taken in Q_ν .*

Proof. For simplicity we assume that $\nu = e_d$ and we write $Q_\nu \equiv Q$. Conclusions i)-iii) clearly follow if

$$\liminf_{n\to\infty} E_{\varepsilon_n}(u_n;Q) = +\infty$$

so it suffices to consider the case where

$$\liminf_{n\to+\infty} E_{\varepsilon_n}(u_n;Q) \le C. \tag{3.9}$$

Passing to a subsequence if necessary, let us also assume that

$$u_n(x) \to u_0(x) \quad \mathcal{L}^d - \text{a.e. } x \in Q \tag{3.10}$$

and that

$$\liminf_{n\to+\infty} E_{\varepsilon_n}(u_n;Q) = \lim_{n\to+\infty} E_{\varepsilon_n}(u_n;Q), \tag{3.11}$$

which implies, from (3.9), that for n large enough

$$E_{\varepsilon_n}(u_n; Q) \le C. \tag{3.12}$$

We start by showing that

$$\limsup_{n\to+\infty} \int_Q |u_n(x) - u_0(x)|^q \, dx = 0, \tag{3.13}$$

where q is the exponent given in (3.2). In fact, by (H3) we have that

$$\big|u_n(x) - u_0(x)\big|^q \le C\left[f\big(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\big) + 1\right], \ x \in Q, \tag{3.14}$$

for some $C > 0$. In addition, by (3.9) and (3.11),

$$\lim_{n\to+\infty} \int_Q f\big(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\big) \, dx = 0 \tag{3.15}$$

and, consequently,

$$\lim_{n\to+\infty} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\big) \, dx = 0, \tag{3.16}$$

for a.e $x \in Q$.

Thus, using (3.15), (3.14), Fatou's Lemma, (3.16) and (3.10), we obtain

$$\begin{aligned} & C|Q| - \limsup_{n\to+\infty} \int_Q |u_n(x) - u_0(x)|^q \, dx \\ = \ & \liminf_{n\to+\infty} \int_Q \left[C f\big(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\big) + C - \big|u_n(x) - u_0(x)\big|^q\right] dx \\ \ge \ & \int_Q \liminf_{n\to+\infty} \left[C f\big(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\big) + C - \big|u_n(x) - u_0(x)\big|^q\right] dx \\ \ge \ & C|Q|, \end{aligned}$$

which proves equality (3.13).

Therefore, as $q \ge 2$, we also conclude that

$$u_n \underset{n\to+\infty}{\overset{L^2(Q;\mathbb{R}^N_+)}{\longrightarrow}} u_0. \tag{3.17}$$

For every $l, m \in \mathbb{N}$, with $l, m \ge 4$, define

$$L_{l,m} := \left\{x \in Q : \ \frac{1}{l} < \operatorname{dist}(x, \partial Q) \le \frac{1}{l} + \frac{1}{m}\right\}$$

and divide $L_{l,m}$ into $\lceil \varepsilon_n^{-1} \rceil$ pairwise disjoint layers of width $\dfrac{1}{m\lceil \varepsilon_n^{-1} \rceil}$, where $\lceil \varepsilon_n^{-1} \rceil$ denotes the smallest integer greater than ε_n^{-1}, i.e, we consider

$$L^{(i)}_{l,m,n} := \left\{x \in Q : \ \frac{1}{l} + \frac{i-1}{m\lceil \varepsilon_n^{-1} \rceil} < \operatorname{dist}\big(x, \partial Q\big) \le \frac{1}{l} + \frac{i}{m\lceil \varepsilon_n^{-1} \rceil}\right\}, \qquad i = 1, \dots, \lceil \varepsilon_n^{-1} \rceil.$$

We observe that for $\varepsilon_n < 1$,

$$\frac{\varepsilon_n}{2} \le \lceil \varepsilon_n^{-1} \rceil^{-1} \le \varepsilon_n. \tag{3.18}$$

Set

$$\widetilde{u}_n(x) := (u_0 * \Psi_{\varepsilon_n})(x), \ x \in Q$$

and note that if $u_n = \widetilde{u}_n$ for infinitely many n, then there is nothing to prove. Thus, without loss of generality, we may assume that for every $n \in \mathbb{N}$

$$\|u_n - \widetilde{u}_n\|_{L^2(Q;\mathbb{R}^N_+)} > 0.$$

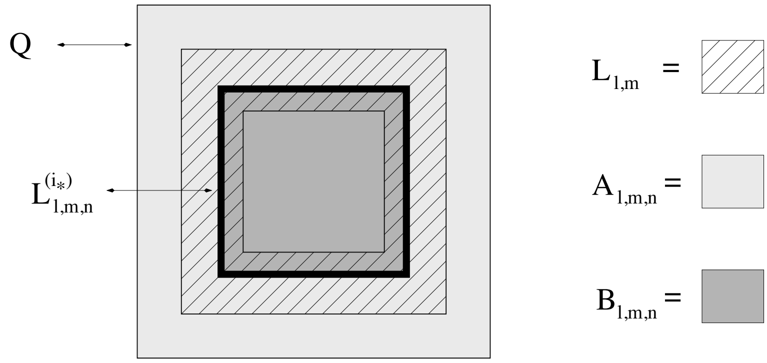

FIGURE 1. Geometry of the sets $A_{l,m,n}$, $B_{l,m,n}$ and $L^{(i_*)}_{l,m,n}$.

By (3.12), (3.17) and since $\widetilde{u}_n \to u_0$ in $L^2(Q;\mathbb{R}^N_+)$ (see (3.8)), it follows that

$$\sum_{i=1}^{\lceil \varepsilon_n^{-1}\rceil} \int_{L^{(i)}_{l,m,n}} \left[\frac{1}{\varepsilon_n} f(x,u_n(x),\varepsilon_n\nabla u_n(x),\varepsilon_n^2\nabla^2 u_n(x)) + \frac{|u_n(x)-\widetilde{u}_n(x)|^2}{\|u_n-\widetilde{u}_n\|_{L^2(Q;\mathbb{R}^N_+)}}\right] dx$$
$$= \int_{L_{l,m}} \left[\frac{1}{\varepsilon_n} f(x,u_n(x),\varepsilon_n\nabla u_n(x),\varepsilon_n^2\nabla^2 u_n(x)) + \frac{|u_n(x)-\widetilde{u}_n(x)|^2}{\|u_n-\widetilde{u}_n\|_{L^2(Q;\mathbb{R}^N_+)}}\right] dx \le C.$$

Using a contradiction argument and (3.18), this implies that there exists $i_* = i_*(m,n)$ such that

$$\int_{L^{(i_*)}_{l,m,n}} \frac{1}{\varepsilon_n} f(x,u_n(x),\varepsilon_n\nabla u_n(x),\varepsilon_n^2\nabla^2 u_n(x))\,dx + \int_{L^{(i_*)}_{l,m,n}} \frac{|u_n(x)-\widetilde{u}_n(x)|^2}{\|u_n-\widetilde{u}_n\|_{L^2(Q;\mathbb{R}^N_+)}}\,dx \le C\varepsilon_n,$$

which, together with (H3) and the fact that

$$\frac{1}{\varepsilon_n}\mathcal{L}^d\big(L^{(i_*)}_{l,m,n}\big) \le \frac{C}{m},$$

leads to the following inequalities

$$\frac{1}{\varepsilon_n}\int_{L^{(i_*)}_{l,m,n}} |u_n(x)|^q\,dx \le \frac{C}{\varepsilon_n}\mathcal{L}^d\big(L^{(i_*)}_{l,m,n}\big) + \frac{C}{\varepsilon_n}\int_{L^{(i_*)}_{l,m,n}} g(x,u_n(x))\,dx \le \frac{C}{m} + C\varepsilon_n, \tag{3.19}$$

$$\frac{1}{\varepsilon_n}\int_{L^{(i_*)}_{l,m,n}} \big[\varepsilon_n^2|\nabla u_n(x)|^2 + \varepsilon_n^4|\nabla^2 u_n(x)|^2\big]\,dx \le C\varepsilon_n, \tag{3.20}$$

and

$$\int_{L^{(i_*)}_{l,m,n}} \frac{|u_n(x)-\widetilde{u}_n(x)|^2}{\|u_n-\widetilde{u}_n\|_{L^2(Q;\mathbb{R}^N_+)}}\,dx \le C\varepsilon_n. \tag{3.21}$$

Let $\varphi_{l,m,n} \in C_c^\infty(Q;[0,1])$ be a family of cut-off functions such that

$$\varphi_{l,m,n}(x) = 0, \qquad x \in\ A_{l,m,n} := \left\{x\in Q:\ \mathrm{dist}(x,\partial Q) \le \frac{1}{l} + \frac{i_*-1}{m\lceil\varepsilon_n^{-1}\rceil}\right\},$$
$$\varphi_{l,m,n}(x) = 1, \qquad x \in\ B_{l,m,n} := \left\{x\in Q:\ \mathrm{dist}(x,\partial Q) > \frac{1}{l} + \frac{i_*}{m\lceil\varepsilon_n^{-1}\rceil}\right\},$$

(see Figure 1 above),

$$\|\nabla\varphi_{l,m,n}\|_\infty = O\left(\frac{m}{\varepsilon_n}\right) \quad \text{and} \quad \|\nabla^2\varphi_{l,m,n}\|_\infty = O\left(\frac{m^2}{\varepsilon_n^2}\right) \tag{3.22}$$

and define

$$v_{l,m,n}(x) := [\varphi_{l,m,n}u_n + (1-\varphi_{l,m,n})\widetilde{u}_n](x),\ x \in Q. \tag{3.23}$$

It can be easily verified that

$$\nabla v_{l,m,n} = \varphi_{l,m,n}\nabla u_n + (1-\varphi_{l,m,n})\nabla\widetilde{u}_n + (u_n - \widetilde{u}_n)\otimes\nabla\varphi_{l,m,n} \tag{3.24}$$

and

$$\nabla^2 v_{l,m,n} = \varphi_{l,m,n}\nabla^2 u_n + (1-\varphi_{l,m,n})\nabla^2\widetilde{u}_n + 2\nabla\varphi_{l,m,n}\odot(\nabla u_n - \nabla\widetilde{u}_n) + (u_n-\widetilde{u}_n)\otimes\nabla^2\varphi_{l,m,n}. \tag{3.25}$$

Note that Q is the disjoint union of $A_{l,m,n}$, $L^{(i_*)}_{l,m,n}$ and $B_{l,m,n}$, and that $v_{l,m,n} \in W^{2,2}(Q;\mathbb{R}^N_+)$. Since $u_n, \widetilde{u}_n \to u_0$ in $L^2(Q;\mathbb{R}^N_+)$ then

$$\lim_{n\to+\infty} \|v_{l,m,n} - u_0\|_{L^2(Q;\mathbb{R}^N_+)} = 0 \tag{3.26}$$

for all $l, m \geq 4$. Moreover,

$$\begin{aligned} E_{\varepsilon_n}\big(v_{l,m,n};Q\big) &= E_{\varepsilon_n}\big(\widetilde{u}_n;A_{l,m,n}\big) + E_{\varepsilon_n}\Big(v_{l,m,n};L^{(i_*)}_{l,m,n}\Big) + E_{\varepsilon_n}\big(u_n;B_{l,m,n}\big) \\ &\leq E_{\varepsilon_n}\big(\widetilde{u}_n;A_{l,m,n}\big) + E_{\varepsilon_n}\Big(v_{l,m,n};L^{(i_*)}_{l,m,n}\Big) + E_{\varepsilon_n}\big(u_n;Q\big). \end{aligned} \tag{3.27}$$

To estimate $E_{\varepsilon_n}\big(v_{l,m,n};Q\big)$ we start by noting that, by (H2), (3.5) and (3.6),

$$E_{\varepsilon_n}\big(\widetilde{u}_n;A_{l,m,n}\big) = E_{\varepsilon_n}\big(\widetilde{u}_n;A_{l,m,n}\cap\{|x_d|<\varepsilon_n\}\big) \tag{3.28}$$

and that, by (H3) and (3.8),

$$\begin{aligned} E_{\varepsilon_n}\big(\widetilde{u}_n;A_{l,m,n}\cap\{|x_d|<\varepsilon_n\}\big) &\leq \int_{A_{l,m,n}\cap\{|x_d|<\varepsilon_n\}} \frac{C}{\varepsilon_n}\big[1+|\widetilde{u}_n(x)|^q + \varepsilon_n^2|\nabla\widetilde{u}_n(x)|^2 + \varepsilon_n^4|\nabla^2\widetilde{u}_n(x)|^2\big]\,dx \\ &\leq \frac{C}{\varepsilon_n}\mathcal{L}^d\bigg(\bigg\{x\in Q:\ \operatorname{dist}(x,\partial Q)\leq \frac{1}{l}+\frac{1}{m},\ |x_d|<\varepsilon_n\bigg\}\bigg) \\ &\leq C\bigg(\frac{1}{m}+\frac{1}{l}\bigg), \end{aligned}$$

for all n sufficiently large. On the other hand, using the fact that $|\varphi_{l,m,n}| \leq 1$, and by (3.6), (H3), (3.22), (3.24), (3.25), (3.8) and (3.18)-(3.21), we have

$$\begin{aligned} E_{\varepsilon_n}\Big(v_{l,m,n}; L^{(i_*)}_{l,m,n}\Big) &\leq \int_{L^{(i_*)}_{l,m,n} \cap \{|x_d|<\varepsilon_n\}} \frac{C}{\varepsilon_n}\Big[|\widetilde{u}_n(x)|^q + (2m^2+1)\,\varepsilon_n^2 |\nabla \widetilde{u}_n(x)|^2 + \varepsilon_n^4 |\nabla^2 \widetilde{u}_n(x)|^2\Big]\,dx \\ &\quad + \int_{L^{(i_*)}_{l,m,n}} \frac{C}{\varepsilon_n}\Big[1 + |u_n(x)|^q + (2m^2+1)\,\varepsilon_n^2 |\nabla u_n(x)|^2 + \varepsilon_n^4 |\nabla^2 u_n(x)|^2\Big]\,dx \\ &\quad + \int_{L^{(i_*)}_{l,m,n}} \frac{C(m^4+m^2)}{\varepsilon_n} |u_n(x) - \widetilde{u}_n(x)|^2\,dx \\ &\leq \frac{C}{\varepsilon_n}(1+2m^2)\mathcal{L}^d\Big(L^{(i_*)}_{l,m,n} \cap \Big\{x \in Q:\, |x_d| < \varepsilon_n\Big\}\Big) + \frac{C}{m\lceil \varepsilon_n^{-1}\rceil \varepsilon_n} \\ &\quad + \int_{L^{(i_*)}_{l,m,n}} \frac{C}{\varepsilon_n}\Big[|u_n(x)|^q + m^2\,\varepsilon_n^2 |\nabla u_n(x)|^2 + \varepsilon_n^4 |\nabla^2 u_n(x)|^2\Big]\,dx \\ &\quad + \frac{Cm^4}{\varepsilon_n}\int_{L^{(i_*)}_{l,m,n}} |u_n(x) - \widetilde{u}_n(x)|^2\,dx \\ &\leq C(1+2m^2)\frac{\varepsilon_n}{m} + \frac{C}{m} + C\varepsilon_n + Cm^2\varepsilon_n + Cm^4 \|u_n - \widetilde{u}_n\|_{L^2(Q;\mathbb{R}^N_+)}, \end{aligned} \tag{3.29}$$

where we have also used the inequalities $2m^2 + 1 \leq Cm^2$ and $m^4 + m^2 \leq Cm^4$. Thus, by (3.27)-(3.29), we obtain

$$\lim_{l\to+\infty} \lim_{m\to+\infty} \lim_{n\to+\infty} E_{\varepsilon_n}\big(v_{l,m,n}; Q\big) \leq \lim_{n\to+\infty} E_{\varepsilon_n}\big(u_n; Q\big). \tag{3.30}$$

In view of (3.26) and (3.30), the result now follows by a standard diagonalization argument. □

Remark 3.4. By (3.23) we notice that if the sequence $\{u_n\}$ is such that $\|u_n - u_0\|_{L^1(Q_\nu;\mathbb{R}^N_+)} = O(\varepsilon_n)$, then $\|v_n - u_0\|_{L^1(Q_\nu;\mathbb{R}^N_+)} = O(\varepsilon_n)$. In addition, by (3.5) and Lemma 3.3 i)-ii), we have that the extension of v_n to S_ν is an element of $\mathcal{A}(\nu)$ for n sufficiently large.

Our next result analyses some continuity properties of the surface energy density σ which describes the limiting interface energy arising from our model (see (1.6)).

For this purpose, for $x \in \Omega$, we extend $\sigma(x,\cdot)$ to the whole $\mathbb{R}^d$ by setting

$$\tilde{\sigma}(x,z) := \begin{cases} |z|\,\sigma\left(x, \frac{z}{|z|}\right) & \text{for every } z \in \mathbb{R}^d,\ z \neq 0 \\ 0 & z = 0. \end{cases}$$

For simplicity of notation we will always denote $\tilde{\sigma}$ by σ.

Proposition 3.5. *Let Ω be an open and bounded subset of $\mathbb{R}^d$, and let f be an integrand such that hypotheses $(H1)-(H4)$ hold. Then the following properties are satisfied:*

(i) *For all $(x,\nu) \in \Omega \times \mathbb{S}^{d-1}$*

$$0 \leq \sigma(x,\nu) \leq C\big(1 + |\alpha|^q + |\beta|^q + |\alpha-\beta|^2\big).$$

(ii) *For all $(x_0,\nu) \in \Omega \times \mathbb{S}^{d-1}$ and $\tau > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$, $x \in \Omega$, implies that*

$$|\sigma(x,\nu) - \sigma(x_0,\nu)| \leq \tau C\big(1 + |\alpha|^q + |\beta|^q + |\alpha-\beta|^2\big).$$

(iii) *For all* $(x,\nu)\in\Omega\times\mathbb{S}^{d-1}$

$$\sigma(x,\nu)=\inf\bigg\{\int_Q \frac{1}{t}f\big(x,w(y),t\nabla w(y)R^T,t^2R\nabla^2w(y)R^T\big)\,dy: t>0,\ w\in\mathcal{A}(e_d),\ Re_d=\nu,\ R\in SO(d)\bigg\},\tag{3.31}$$

where $SO(d)$ *denotes the set of rotations in* $\mathbb{R}^d$.

(iv) *The function* $\sigma(x,\cdot)$ *is upper semicontinuous on the whole space* $\mathbb{R}^d$.

Proof. (i) Let $(x,\nu)\in\Omega\times\mathbb{S}^{d-1}$ and consider

$$w(y):=(\beta-\alpha)(y\cdot\nu)+\frac{\alpha+\beta}{2},\quad y\in Q_\nu.$$

Since $w\in\mathcal{A}(\nu)$, $f\geq 0$ and (H3) holds, we have

$$\begin{aligned}0\leq\sigma(x,\nu)&\leq\int_{Q_\nu}f\big(x,w(y),\nabla w(y),\nabla^2w(y)\big)\,dy\\&\leq\int_{Q_\nu}C\left(g(x,w(y))+|\nabla w(y)|^2+|\nabla^2w(y)|^2\right)\,dy\\&\leq\int_{Q_\nu}C\left(1+|w(y)|^q+\big|(\beta-\alpha)\otimes\nu\big|^2\right)\,dy\\&\leq C\big(1+|\alpha|^q+|\beta|^q+|\alpha-\beta|^2\big).\end{aligned}$$

(ii) Fix $(x_0,\nu)\in\Omega\times\mathbb{S}^{d-1}$ and $\tau>0$. Let δ be such that the conclusions of hypotheses $(H4)$ hold and consider $x\in\Omega$ with $|x-x_0|<\delta$. Let $t_n>0$ and $w_n\in\mathcal{A}(\nu)$ be such that

$$\int_{Q_\nu}\frac{1}{t_n}f(x,w_n(y),t_n\nabla w_n(y),t_n^2\nabla^2w_n(y))\,dy\leq\sigma(x,\nu)+\frac{1}{n}.$$

Then, by definition of $\sigma(x_0,\nu)$ and $(H4)$, we have

$$\begin{aligned}\sigma(x_0,\nu)-\sigma(x,\nu)&\leq\int_{Q_\nu}\frac{1}{t_n}f(x_0,w_n(y),t_n\nabla w_n(y),t_n^2\nabla^2w_n(y))\,dy\\&\quad-\int_{Q_\nu}\frac{1}{t_n}f(x,w_n(y),t_n\nabla w_n(y),t_n^2\nabla^2w_n(y))\,dy+\frac{1}{n}\\&\leq\tau\int_{Q_\nu}\frac{1}{t_n}f(x,w_n(y),t_n\nabla w_n(y),t_n^2\nabla^2w_n(y))\,dy+\frac{1}{n}\\&=\tau\left(\sigma(x,\nu)+\frac{1}{n}\right)+\frac{1}{n}\end{aligned}$$

and the desired inequality follows from (i) and by letting $n\to+\infty$. To obtain the reverse inequality it suffices to exchange the roles of $\sigma(x,\nu)$ and $\sigma(x_0,\nu)$.

(iii) Follows by a change of variable argument.

(iv) Since we have extended $\sigma(x,\cdot)$ as a positively homogeneous function of degree one (see Fonseca & Müller [23]), it suffices to prove that $\sigma(x,\cdot)$ is upper semicontinuous on $\mathbb{S}^{d-1}$. Fix $\nu\in\mathbb{S}^{d-1}$ and take $\nu_n\in\mathbb{S}^{d-1}$ such that $\nu_n\to\nu$. Using (iii), given $\varepsilon>0$ choose $t_\varepsilon>0$, $w_\varepsilon\in\mathcal{A}(e_d)$, and a rotation R_ε such that $R_\varepsilon e_d=\nu$ and

$$\left|\sigma(x,\nu)-\int_Q\frac{1}{t_\varepsilon}f\big(x,w_\varepsilon(y),t_\varepsilon\nabla w_\varepsilon(y)R_\varepsilon^T,t_\varepsilon^2R_\varepsilon\nabla^2w_\varepsilon(y)R_\varepsilon^T\big)\,dy\right|<\varepsilon.\tag{3.32}$$

Notice that, by (3.32),

$$\int_Q \frac{1}{t_\varepsilon} f\big(x, w_\varepsilon(y), t_\varepsilon \nabla w_\varepsilon(y) R_\varepsilon^T, t_\varepsilon^2 R_\varepsilon \nabla^2 w_\varepsilon(y) R_\varepsilon^T\big) dy < +\infty$$

since, from (i), $\sigma(x,\nu) < +\infty$. For every $n \in \mathbb{N}$, let $R_n \in SO(d)$ be such that $R_n e_d = \nu_n$ and $R_n \to R_\varepsilon$ as $n \to +\infty$. Since $w_\varepsilon \in \mathcal{A}(e_d)$, in view of (3.31), we have

$$\sigma(x,\nu_n) \le \int_Q \frac{1}{t_\varepsilon} f\big(x, w_\varepsilon(y), t_\varepsilon \nabla w_\varepsilon(y) R_n^T, t_\varepsilon^2 R_n \nabla^2 w_\varepsilon(y) R_n^T\big)\, dy. \tag{3.33}$$

As f is continuous,

$$\lim_{n\to+\infty} \frac{1}{t_\varepsilon} f\big(x, w_\varepsilon(y), t_\varepsilon \nabla w_\varepsilon(y) R_n^T, t_\varepsilon^2 R_n \nabla^2 w_\varepsilon(y) R_n^T\big) = \frac{1}{t_\varepsilon} f\big(x, w_\varepsilon(y), t_\varepsilon \nabla w_\varepsilon(y) R_\varepsilon^T, t_\varepsilon^2 R_\varepsilon \nabla^2 w_\varepsilon(y) R_\varepsilon^T\big),$$

for a.e. $y \in Q$. Moreover, by (H3), we have that for a.e. $y \in Q$,

$$\begin{aligned} f\big(x, w_\varepsilon(y), t_\varepsilon \nabla w_\varepsilon(y) R_n^T, t_\varepsilon^2 R_n \nabla^2 w_\varepsilon(y) R_n^T\big) &\le C\left(1 + |w_\varepsilon(y)|^q + t_\varepsilon^2 |\nabla w_\varepsilon(y)|^2 + t_\varepsilon^4 |\nabla^2 w_\varepsilon(y)|^2\right) \\ &\le C\left(1 + \|w_\varepsilon\|^q_{L^\infty(Q)} + t_\varepsilon^2 \|\nabla w_\varepsilon\|^2_{L^\infty(Q)} + t_\varepsilon^4 \|\nabla^2 w_\varepsilon\|^2_{L^\infty(Q)}\right). \end{aligned}$$

Thus, by Lebesgue's Dominated Convergence Theorem, we obtain that

$$\begin{aligned} &\lim_{n\to+\infty} \int_Q \frac{1}{t_\varepsilon} f\big(x, w_\varepsilon(y), t_\varepsilon \nabla w_\varepsilon(y) R_n^T, t_\varepsilon^2 R_n \nabla^2 w_\varepsilon(y) R_n^T\big)\, dy \\ &\quad = \int_Q \frac{1}{t_\varepsilon} f\big(x, w_\varepsilon(y), t_\varepsilon \nabla w_\varepsilon(y) R_\varepsilon^T, t_\varepsilon^2 R_\varepsilon \nabla^2 w_\varepsilon(y) R_\varepsilon^T\big)\, dy. \end{aligned}$$

Passing to the limsup as $n \to +\infty$ in (3.33), and taking into account (3.32), we have

$$\limsup_{n\to+\infty} \sigma(x,\nu_n) \le \sigma(x,\nu) + \varepsilon.$$

We conclude the proof by letting $\varepsilon \to 0^+$. □

4. Main result

Our goal is to prove that the family of functionals (1.4),

$$F_\varepsilon(u) := \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_\Omega f(x, u(x), \varepsilon \nabla u(x), \varepsilon^2 \nabla^2 u(x))\, dx, & u \in W^{2,2}(\Omega; \mathbb{R}^N_+) \cap \mathcal{V}, \\ +\infty, & \text{otherwise} \end{cases}$$

Γ-converges, with respect to the $L^1(\Omega;\mathbb{R}^N_+)$-convergence, to the functional $F : L^1(\Omega;\mathbb{R}^N_+) \to \mathbb{R} \cup \{+\infty\}$ given in (1.5),

$$F(u) := \begin{cases} \displaystyle\int_{J_u} \sigma\big(x, \nu_u(x)\big)\, d\mathcal{H}^{d-1}(x), & u \in BV\big(\Omega; \{\alpha,\beta\}\big) \cap \mathcal{V}, \\ +\infty, & \text{otherwise}, \end{cases}$$

where, for $\alpha, \beta \in \mathbb{R}^N_+$, $\alpha \neq \beta$, the space $BV\big(\Omega; \{\alpha,\beta\}\big)$ is defined in Example 2.10, the set $\mathcal{V}$ is given by

$$\mathcal{V} := \left\{ u : \int_\Omega u(x)\, dx = V,\, V \in \mathbb{R}^N_+,\, |\Omega| \min(\alpha^i,\beta^i) < V^i < |\Omega| \max(\alpha^i,\beta^i),\, i = 1,...,N \right\},$$

where V^i, α^i and β^i are the i-th components of V, α and β, respectively, the function $\sigma : \Omega \times \mathbb{S}^{d-1} \to [0, +\infty)$ is defined by

$$\sigma(x, \nu) := \inf \left\{ \int_{Q_\nu} \frac{1}{t} f\big(x, w(y), t\nabla w(y), t^2 \nabla^2 w(y)\big)\, dy : \, t > 0, \, w \in \mathcal{A}(\nu) \right\},$$

and

$$\mathcal{A}(\nu) := \Bigg\{ w \in W^{2,2}_{\mathrm{loc}}(S_\nu; \mathbb{R}^N_+) : \, w(y) = \alpha \; \text{ if } \; y \cdot \nu = -\tfrac{1}{2}, \quad w(y) = \beta \; \text{ if } \; y \cdot \nu = \tfrac{1}{2},$$
$$w(y) = w(y + l\nu_i), \; \text{for all } y \in S_\nu, \quad i = 1, \dots, d-1, \; \text{and } l \in \mathbb{Z} \Bigg\}$$

for every $\nu \in \mathbb{S}^{d-1}$ with $\{\nu_1, \dots, \nu_{d-1}, \nu\}$ an orthonormal basis of $\mathbb{R}^d$.

Precisely, our main result states the following.

Theorem 4.1. *Let Ω be an open and bounded subset of $\mathbb{R}^d$ with Lipschitz boundary, let F_ε and F be given by* (1.4) *and* (1.5), *respectively, and assume that hypotheses (H1)–(H6) hold. Then for all $u \in L^1(\Omega; \mathbb{R}^N_+)$ we have that*

$$F(u) = \Gamma - \lim_{\varepsilon \to 0^+} F_\varepsilon(u)$$

with respect to the $L^1(\Omega; \mathbb{R}^N_+)$-convergence.

Remark 4.2. As a byproduct of the proof of Theorem 4.1, under hypotheses $(H1)-(H4)$ it is immediate that the unconstrained functionals

$$F_\varepsilon(u) := \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_\Omega f(x, u(x), \varepsilon \nabla u(x), \varepsilon^2 \nabla^2 u(x))\, dx, & u \in W^{2,2}(\Omega; \mathbb{R}^N_+), \\ +\infty, & \text{otherwise} \end{cases}$$

Γ-converge, with respect to the $L^1(\Omega; \mathbb{R}^N_+)$-convergence, to the functional $F : L^1(\Omega; \mathbb{R}^N_+) \to \mathbb{R} \cup \{+\infty\}$ given by

$$F(u) := \begin{cases} \displaystyle\int_{J_u} \sigma\big(x, \nu_u(x)\big)\, d\mathcal{H}^{d-1}(x), & u \in BV\big(\Omega; \{\alpha, \beta\}\big), \\ +\infty, & \text{otherwise.} \end{cases}$$

The following lemma addresses the proof of Theorem 4.1 in the case where the function $u \in L^1(\Omega; \mathbb{R}^N_+) \setminus [BV\big(\Omega; \{\alpha, \beta\}\big) \cap \mathcal{V}]$.

Lemma 4.3. *Let $u \in L^1(\Omega; \mathbb{R}^N_+) \setminus [BV\big(\Omega; \{\alpha, \beta\}\big) \cap \mathcal{V}]$. Then*

$$\Gamma - \lim_{\varepsilon \to 0^+} F_\varepsilon(u) = +\infty \tag{4.1}$$

Proof. Given $u \in L^1(\Omega; \mathbb{R}^N_+) \setminus [BV\big(\Omega; \{\alpha, \beta\}\big) \cap \mathcal{V}]$, to show (4.1) it is enough to see that for every sequence $\{u_n\} \subset L^1(\Omega; \mathbb{R}^N_+)$ such that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^N_+)$ we have that

$$\liminf_{n \to +\infty} F_{\varepsilon_n}(u_n) = +\infty.$$

Without loss of generality we can consider the case $u \in [L^1(\Omega; \mathbb{R}^N_+) \cap \mathcal{V}] \setminus BV\big(\Omega; \{\alpha, \beta\}\big)$ and $u_n \to u$ in $L^1(\Omega; \mathbb{R}^N_+)$ with $\{u_n\} \subset W^{2,2}(\Omega; \mathbb{R}^N_+) \cap \mathcal{V}$ (otherwise there is nothing to show). Let us prove that

$$\liminf_{n \to +\infty} F_{\varepsilon_n}(u_n) = \liminf_{n \to +\infty} \int_\Omega \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\big)\, dx = +\infty. \tag{4.2}$$

Step 1. We consider first the case where

$$\mathcal{L}^d\big(\{x \in \Omega : \, u(x) \notin \{\alpha, \beta\}\,\}\big) > 0. \tag{4.3}$$

To show (4.2) we argue by contradiction. In fact, if for such a sequence $\{u_n\}$ we had that

$$\sup_{n\in\mathbb{N}} F_{\varepsilon_n}(u_n) = \sup_{n\in\mathbb{N}} \int_\Omega \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\big)\, dx \le C,$$

for some $C > 0$, then by (H3) and Fatou's Lemma we would get, in particular, that

$$\int_\Omega g(x, u(x))\, dx = 0,$$

and so $g(x, u(x)) = 0$ for $\mathcal{L}^d$-a.e. $x \in \Omega$. Thus, by (H2) and (H3), we would obtain $u(x) \in \{\alpha, \beta\}$ for $\mathcal{L}^d$-a.e. $x \in \Omega$, contradicting (4.3).

Step 2. Let us now assume that $u = \beta\chi_E + \alpha\big(1 - \chi_E\big)$, and $u \notin BV(\Omega; \mathbb{R}^N_+)$, i.e., $\mathrm{Per}_\Omega(E) = +\infty$. We argue again by contradiction and assume that there exists a sequence $\{u_n\}$, as in the previous step, such that

$$\sup_{n\in\mathbb{N}} \int_\Omega \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\big)\, dx \le C.$$

Then, by (H3), we would have that

$$\sup_{n\in\mathbb{N}} \int_\Omega \left[\frac{1}{\varepsilon_n} g\big(x, u_n(x)\big) + \varepsilon_n |\nabla u_n(x)|^2\right] dx \le C,$$

which would imply, by the Cauchy-Schwarz inequality, that

$$\sup_{n\in\mathbb{N}} \int_\Omega \sqrt{g\big(x, u_n(x)\big)}\, |\nabla u_n(x)|\, dx \le C. \tag{4.4}$$

Let

$$\overline{g}(v) := \min_{x\in\overline{\Omega}} g(x, v), \;\; v \in \mathbb{R}^N_+.$$

In view of (H2) and (H3), it follows that $\overline{g}(v) = 0$ if and only if $v \in \{\alpha, \beta\}$ and, in addition,

$$\overline{g}(v) \ge C|v|$$

for a suitable constant C and for $|v|$ large enough. As in Fonseca & Tartar [25], we consider the function

$$\Phi(v) = \inf\left\{\int_{-1}^1 \sqrt{\min\left\{\overline{g}\big(\varphi(s)\big), M\right\}}\, \big|\varphi'(s)\big|\, ds \colon \varphi \text{ continuous and piecewice } C^1,\, \varphi(-1) = \alpha,\, \varphi(1) = v\right\}$$

(see Lemma 3.7 in [25] for the definition of the constant M in this case) which is Lipschitz continuous and satisfies

$$\big|\nabla(\Phi \circ v)(x)\big| \le \sqrt{\overline{g}(v(x))}\, \big|\nabla v(x)\big|, \tag{4.5}$$

for every $v \in W^{2,2}(\Omega; \mathbb{R}^N_+)$, for $\mathcal{L}^d$-a.e. $x \in \Omega$. We observe that

$$(\Phi \circ u)(x) = \Phi(\beta)\chi_E(x), \;\; x \in \Omega,$$

and, consequently,

$$\big|D\big(\Phi \circ u\big)\big|(\Omega) = \Phi(\beta)\mathrm{Per}_\Omega(E). \tag{4.6}$$

In addition, we note that

$$\Phi(\beta) > 0. \tag{4.7}$$

In fact, to show (4.7), let φ be a continuous and piecewise C^1-function with $\varphi(-1) = \alpha$ and $\varphi(1) = \beta$. By continuity we can find $t_1 \in (-1, 1)$ such that

$$|\varphi(t_1) - \beta| = \frac{|\alpha - \beta|}{3} \qquad \text{and} \qquad |\varphi(s) - \beta| > \frac{|\alpha - \beta|}{3} \quad \text{for all } -1 \le s < t_1. \tag{4.8}$$

Let $t_0 := \max\{t \in (-1, t_1) : |\varphi(t) - \alpha| = |\alpha - \beta|/3\}$. By the Mean Value Theorem it follows that

$$|\varphi(s) - \alpha| > \frac{|\alpha - \beta|}{3} \qquad \text{for all } s \in (t_0, t_1). \tag{4.9}$$

Set now

$$T_M := \min\left\{ \sqrt{\min\{\overline{g}(z), M\}} : |z - \alpha| \geq \frac{|\alpha - \beta|}{3}, |z - \beta| \geq \frac{|\alpha - \beta|}{3} \right\}$$

which is positive since $\{\overline{g}\}^{-1}(0) = \{\alpha, \beta\}$. In view of (4.8) and (4.9) we have

$$\int_{-1}^{1} \sqrt{\min\{g(\varphi(s)), M\}}\, |\varphi'(s)|\, ds \geq T_M \int_{t_0}^{t_1} |\varphi'(s)|\, ds \geq T_M \left| \int_{t_0}^{t_1} \varphi'(s)\, ds \right|$$
$$= T_M\, |\varphi(t_1) - \varphi(t_0)| \geq T_M \frac{|\alpha - \beta|}{3}$$

and therefore $\Phi(\beta) \geq T_M \dfrac{|\alpha - \beta|}{3} > 0$.

Using (4.4) and (4.5) we conclude that

$$\sup_{n \in \mathbb{N}} \left\| \nabla(\Phi \circ u_n) \right\|_{L^1(\Omega; \mathbb{R}^N_+)} < +\infty.$$

Hence, by the lower semicontinuity of the total variation and the fact that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^N_+)$, it follows from (4.6) and (4.7) that $\mathrm{Per}_\Omega(E) < +\infty$, which is a contradiction.

□

We note that to complete the proof of Theorem 4.1 it suffices to show that

i) Lower bound: For every $u \in BV(\Omega; \{\alpha, \beta\}) \cap \mathcal{V}$, for every sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0^+$, and every sequence $\{u_n\} \subset W^{2,2}(\Omega; \mathbb{R}^N_+) \cap \mathcal{V}$ such that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^N_+)$,
$$F(u) \leq \liminf_{n \to +\infty} F_{\varepsilon_n}(u_n).$$

ii) Upper bound: For every $\eta > 0$, every $u \in BV(\Omega; \{\alpha, \beta\}) \cap \mathcal{V}$ and every sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0^+$, there exists $\{u_n^\eta\} \subset W^{2,2}(\Omega; \mathbb{R}^N_+) \cap \mathcal{V}$ such that $u_n^\eta \to u$ in $L^1(\Omega; \mathbb{R}^N_+)$ as $n \to +\infty$ and such that
$$\limsup_{n \to +\infty} F_{\varepsilon_n}(u_n^\eta) \leq F(u) + O(\eta).$$

Indeed, given $\eta > 0$, from i) and ii), for every sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0^+$, and every $u \in BV(\Omega; \{\alpha, \beta\}) \cap \mathcal{V}$, then

$$\Gamma - \limsup_{n \to +\infty} F_{\varepsilon_n}(u) - O(\eta) \leq F(u) \leq \Gamma - \liminf_{n \to +\infty} F_{\varepsilon_n}(u),$$

from where, letting $\eta \to 0^+$,

$$\Gamma - \liminf_{n \to +\infty} F_{\varepsilon_n}(u) = \Gamma - \limsup_{n \to +\infty} F_{\varepsilon_n}(u) = F(u)$$

so that the conclusion of Theorem 4.1 holds.

The proofs of properties i) and ii) can be found in Subsections 4.1 and 4.2 below.

4.1. **The lower bound inequality.**

Proposition 4.4 (Lower bound). *Let Ω be an open and bounded subset of $\mathbb{R}^d$ with Lipschitz boundary, and assume that hypotheses (H1)-(H4) hold. Let $u \in BV(\Omega; \{\alpha, \beta\}) \cap \mathcal{V}$ and let $\varepsilon_n \to 0^+$. Then for every sequence $\{u_n\} \subset W^{2,2}(\Omega; \mathbb{R}^N_+) \cap \mathcal{V}$ such that $u_n \underset{n \to +\infty}{\longrightarrow} u$ in $L^1(\Omega; \mathbb{R}^N_+)$ we have*

$$F(u) \leq \liminf_{n \to +\infty} F_{\varepsilon_n}(u_n). \tag{4.10}$$

Proof. Let $E \subset \Omega$ with $\operatorname{Per}_\Omega(E) < +\infty$ be such that $u = \beta\chi_E + \alpha\big(1-\chi_E\big)$ and let $\{u_n\} \subset W^{2,2}(\Omega;\mathbb{R}^N_+) \cap \mathcal{V}$ satisfy $u_n \to u$ in $L^1(\Omega;\mathbb{R}^N_+)$. Without loss of generality, let us assume that

$$\liminf_{n\to+\infty} \int_\Omega \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\big)\, dx < +\infty$$

and, up to a subsequence (still denoted by u_n), suppose also that

$$\liminf_{n\to+\infty} \int_\Omega \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\big)\, dx = \lim_{n\to+\infty} \int_\Omega \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\big)\, dx$$

and that, in addition, $u_n(x) \to u(x)$ $\mathcal{L}^d$-a.e. $x \in \Omega$. Recalling the definition of $F(u)$ (see (1.5)), to show (4.10) we must prove that

$$\int_{J_u} \sigma\big(x, \nu_u(x)\big)\, d\mathcal{H}^{d-1}(x) \le \lim_{n\to+\infty} \int_\Omega \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\big)\, dx. \tag{4.11}$$

Set

$$f_n := \frac{1}{\varepsilon_n} f\big(\cdot, u_n(\cdot), \varepsilon_n \nabla u_n(\cdot), \varepsilon_n^2 \nabla^2 u_n(\cdot)\big),\ \ n \in \mathbb{N}.$$

Since the sequence $\{f_n\}$ is bounded in $L^1\big(\Omega;[0,+\infty)\big)$, there exists a subsequence (still denoted by f_n) and a nonnegative finite Radon measure ζ such that

$$f_n\, \mathcal{L}^d \llcorner \Omega \ \underset{\mathcal{M}^+(\Omega)}{\overset{*}{\rightharpoonup}} \ \zeta. \tag{4.12}$$

Consider the nonnegative measure

$$\pi(A) := \mathcal{H}^{d-1}(A \cap J_u)$$

defined over all Borel subsets $A \subset \Omega$. Recall from Example 2.10 that

$$\mathcal{H}^{d-1}(J_u) = \mathcal{H}^{d-1}(\partial^* E) = \operatorname{Per}_\Omega(E) \tag{4.13}$$

and that

$$\nu_u(x) = \nu_E(x) \text{ for } \mathcal{H}^{d-1} - \text{a.e } x \in J_u. \tag{4.14}$$

Thus, in particular,

$$\pi(\Omega) = \operatorname{Per}_\Omega(E) < +\infty,$$

i.e, π is a nonnegative finite Radon measure. Hence, by the Lebesgue-Radon-Nikodým Theorem 2.4, we can decompose ζ as

$$\zeta = \zeta_a \pi + \zeta_s,$$

where ζ_a is a nonnegative integrable function and ζ_s is a nonnegative Radon measure such that π and ζ_s are mutually singular.

We claim that

$$\zeta_a(x) \ge \sigma\big(x, \nu_u(x)\big) \quad \text{for } \ \mathcal{H}^{d-1}\text{-a.e. } x \in J_u, \tag{4.15}$$

which immediately implies (4.11). Indeed, by (4.12) and (4.15) we obtain

$$\begin{aligned}
\lim_{n\to+\infty} \int_\Omega \frac{1}{\varepsilon_n} f\big(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\big)\, dx &\ge \zeta(\Omega) \\
&\ge \int_{J_u} \zeta_a(x)\, d\mathcal{H}^{d-1}(x) \\
&\ge \int_{J_u} \sigma\big(x, \nu_u(x)\big)\, d\mathcal{H}^{d-1}(x).
\end{aligned}$$

To conclude our argument let us show that (4.15) holds. By (2.3)-(2.5), (4.13) and (4.14) we know that for $\mathcal{H}^{d-1}$-a.e $x \in J_u$

$$\lim_{r\to 0^+} \frac{1}{r^{d-1}} \mathcal{H}^{d-1}\big(Q_{\nu_u(x)}(x,r) \cap J_u\big) = 1, \tag{4.16}$$

$$\lim_{r\to 0^+} \frac{1}{r^d} \mathcal{L}^d\big(\{y \in B(x,r) \setminus E :\, (y-x)\cdot \nu_u(x) \geq 0\}\big) = 0, \tag{4.17}$$

$$\lim_{r\to 0^+} \frac{1}{r^d} \mathcal{L}^d\big(\{y \in B(x,r) \cap E :\, (y-x)\cdot \nu_u(x) \leq 0\}\big) = 0. \tag{4.18}$$

Let $x \in J_u$ be such that (4.16)-(4.18) hold and set $\nu := \nu_u(x)$. In view of Theorem 2.5, we can also assume that

$$\zeta_a(x) = \lim_{r\to 0^+} \frac{\zeta(Q_\nu(x,r))}{\mathcal{H}^{d-1}\big(Q_\nu(x,r) \cap J_u\big)} < +\infty.$$

Then, choosing $r_k \to 0^+$ such that $\zeta\big(\partial(x + r_k Q_\nu)\big) = 0$, by (4.16) and (4.12) we have that (see Proposition 2.3),

$$\begin{aligned}
\zeta_a(x) &= \lim_{k\to+\infty} \frac{\zeta\big(Q_\nu(x,r_k)\big)}{r_k^{d-1}} \\
&= \lim_{k\to+\infty} \frac{1}{r_k^{d-1}} \lim_{n\to+\infty} \int_{Q_\nu(x,r_k)} \frac{1}{\varepsilon_n} f\big(z, u_n(z), \varepsilon_n \nabla u_n(z), \varepsilon_n^2 \nabla^2 u_n(z)\big)\, dz \\
&= \lim_{k\to+\infty} \lim_{n\to+\infty} \int_{Q_\nu} \frac{r_k}{\varepsilon_n} f\big(x + r_k y, u_n(x+r_k y), \varepsilon_n \nabla u_n(x+r_k y), \varepsilon_n^2 \nabla^2 u_n(x+r_k y)\big)\, dy \\
&= \lim_{k\to+\infty} \lim_{n\to+\infty} \int_{Q_\nu} \frac{r_k}{\varepsilon_n} f\left(x + r_k y, v_{n,k}(y), \frac{\varepsilon_n}{r_k} \nabla v_{n,k}(y), \frac{\varepsilon_n^2}{r_k^2} \nabla^2 v_{n,k}(y)\right) dy,
\end{aligned} \tag{4.19}$$

where $v_{n,k} \in W^{2,2}(Q_\nu; \mathbb{R}^N_+)$ is defined by $v_{n,k}(y) := u_n(x + r_k y)$, $y \in Q_\nu$. We claim that

$$\lim_{k\to+\infty} \lim_{n\to+\infty} \|v_{n,k} - u_0\|_{L^1(Q_\nu;\mathbb{R}^d_+)} = 0 \tag{4.20}$$

where, for $y \in Q_\nu$, u_0 is given by

$$u_0(y) := \begin{cases} \beta & \text{if } y\cdot\nu > 0, \\ \alpha & \text{if } y\cdot\nu < 0. \end{cases}$$

Indeed (4.20) holds since

$$\begin{aligned}
&\lim_{k\to+\infty} \lim_{n\to+\infty} \int_{Q_\nu} |v_{n,k}(y) - u_0(y)|\, dy \\
&= \lim_{k\to+\infty} \lim_{n\to+\infty} \left[\int_{Q_\nu \cap \{y:\, y\cdot\nu>0\}} |u_n(x+r_k y) - \beta|\, dy + \int_{Q_\nu \cap \{y:\, y\cdot\nu<0\}} |u_n(x+r_k y) - \alpha|\, dy\right] \\
&= \lim_{k\to+\infty} \left[\int_{Q_\nu \cap \{y:\, y\cdot\nu>0\}} |u(x+r_k y) - \beta|\, dy + \int_{Q_\nu \cap \{y:\, y\cdot\nu<0\}} |u(x+r_k y) - \alpha|\, dy\right] \\
&= \lim_{k\to+\infty} \frac{1}{r_k^d} \left[\int_{Q_\nu(x,r_k) \cap \{y:\, (y-x)\cdot\nu>0\} \setminus E} |\alpha - \beta|\, dy + \int_{Q_\nu(x,r_k) \cap \{y:\, (y-x)\cdot\nu<0\} \cap E} |\beta - \alpha|\, dy\right] \\
&= 0,
\end{aligned}$$

where we have first used the fact that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^N_+)$ and then (4.17)-(4.18) in the last equality.

By (4.19), (4.20) and a diagonalization argument we may find a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that

$$t_k := \frac{\varepsilon_{n_k}}{r_k} \underset{k\to+\infty}{\longrightarrow} 0^+, \qquad v_k := v_{n_k,k} \overset{L^1(Q_\nu;\mathbb{R}^N_+)}{\underset{k\to+\infty}{\longrightarrow}} u_0,$$

and

$$\zeta_a(x) = \lim_{k\to+\infty} \int_{Q_\nu} \frac{1}{t_k} f\Big(x + r_k y, v_k(y), t_k \nabla v_k(y), t_k^2 \nabla^2 v_k(y)\Big)\, dy. \tag{4.21}$$

Applying Lemma 3.3, see also Remark 3.4, to the sequences $\{v_k\}$ and $\{t_k\}$, we conclude that there exists a sequence $\{w_k\} \subset W^{2,2}(Q_\nu;\mathbb{R}^N_+) \cap \mathcal{A}(\nu)$ such that $w_k \to u_0$ in $L^1(Q_\nu;\mathbb{R}^N_+)$, and

$$\begin{aligned} &\lim_{k\to+\infty} \int_{Q_\nu} \frac{1}{t_k} f\Big(x, v_k(y), t_k \nabla v_k(y), t_k^2 \nabla^2 v_k(y)\Big)\, dy \\ &\geq \limsup_{k\to+\infty} \int_{Q_\nu} \frac{1}{t_k} f\big(x, w_k(y), t_k \nabla w_k(y), t_k^2 \nabla^2 w_k(y)\big) dy. \end{aligned} \tag{4.22}$$

Thus, by (4.21), (4.22), the fact that $w_k \in \mathcal{A}(\nu)$ and using (1.6) in the last inequality, we conclude that

$$\begin{aligned} \zeta_a(x) \geq\ &\lim_{k\to+\infty} \int_{Q_\nu} \frac{1}{t_k} f\Big(x, v_k(y), t_k \nabla v_k(y), t_k^2 \nabla^2 v_k(y)\Big)\, dy \\ &- \limsup_{k\to+\infty} \int_{Q_\nu} \frac{1}{t_k} \Big[f\big(x, v_k(y), t_k \nabla v_k(y), t_k^2 \nabla^2 v_k(y)\big) - f\big(x + r_k y, v_k(y), t_k \nabla v_k(y), t_k^2 \nabla^2 v_k(y)\big)\Big] dy \\ \geq\ &\limsup_{k\to+\infty} \int_{Q_\nu} \frac{1}{t_k} f\Big(x, w_k(y), t_k \nabla w_k(y), t_k^2 \nabla^2 w_k(y)\Big)\, dy \\ &- \limsup_{k\to+\infty} \int_{Q_\nu} \frac{1}{t_k} \Big[f\big(x, v_k(y), t_k \nabla v_k(y), t_k^2 \nabla^2 v_k(y)\big) - f\big(x + r_k y, v_k(y), t_k \nabla v_k(y), t_k^2 \nabla^2 v_k(y)\big)\Big] dy \\ \geq\ &\sigma\big(x, \nu_u(x)\big) \\ &- \limsup_{k\to+\infty} \int_{Q_\nu} \frac{1}{t_k} \Big[f\big(x, v_k(y), t_k \nabla v_k(y), t_k^2 \nabla^2 v_k(y)\big) - f\big(x + r_k y, v_k(y), t_k \nabla v_k(y), t_k^2 \nabla^2 v_k(y)\big)\Big] dy. \end{aligned} \tag{4.23}$$

Finally, hypothesis (H4) yields

$$\begin{aligned} &\limsup_{k\to+\infty} \int_{Q_\nu} \frac{1}{t_k} \Big[f\big(x, v_k(y), t_k \nabla v_k(y), t_k^2 \nabla^2 v_k(y)\big) - f\big(x + r_k y, v_k(y), t_k \nabla v_k(y), t_k^2 \nabla^2 v_k(y)\big)\Big] dy \\ &\leq \limsup_{k\to+\infty} \tau \int_{Q_\nu} \frac{1}{t_k} f\big(x + r_k y, v_k(y), t_k \nabla v_k(y), t_k^2 \nabla^2 v_k(y)\big) dy \\ &= \tau \zeta_a(x). \end{aligned} \tag{4.24}$$

Therefore (4.15) follows by (4.23) and (4.24), letting $\tau \to 0^+$.

□

4.2. **The upper bound inequality.** In this subsection we prove the upper bound inequality for the Γ-limit by constructing a recovering sequence. The proof is done in several steps according to the geometry of the set Ω and to the interface of the function u. We start with the simpler case where Ω is a cube with two of its faces normal to a given unit vector, the integrand function f does not depend explicitly on the variable x and the target function u has planar interface.

Lemma 4.5. *Let $\Omega = Q_\nu(x_0, r)$ with $\nu \in \mathbb{S}^{d-1}$, $x_0 \in \mathbb{R}^d$ and $r > 0$. Assume that the function f does not depend explicitly on x and that hypotheses (H1)–(H6) hold. Let*

$$u(x) := \begin{cases} \beta & \text{if } (x - x_0) \cdot \nu > 0 \\ \\ \alpha & \text{if } (x - x_0) \cdot \nu < 0. \end{cases}$$

Then, for every $\eta > 0$ and every sequence $\varepsilon_n \to 0^+$, there exists a sequence $\{u_n^\eta\} \equiv \{u_n\} \subseteq W^{2,2}(\Omega; \mathbb{R}^N_+)$ satisfying

$$\int_\Omega u_n(x)\,dx = \int_\Omega u(x)\,dx = V \tag{4.25}$$

such that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^N_+)$ with $\|u_n - u\|_{L^1(\Omega;\mathbb{R}^N_+)} = O(\varepsilon_n)$ and

$$\limsup_{n\to+\infty} F_{\varepsilon_n}(u_n) < r^{d-1}\sigma(\nu) + r^{d-1}\eta = F(u) + r^{d-1}\eta. \tag{4.26}$$

Proof. For simplicity, we assume that $x_0 = 0$ and that $\nu = e_d$, and let $\eta > 0$ and $\varepsilon_n \to 0^+$ be given. The proof is divided into two steps.
Step 1. Let $t_\eta > 0$ and $w_\eta \in \mathcal{A}(e_d)$ be such that

$$\int_Q \frac{1}{t_\eta} f\big(w_\eta(x), t_\eta \nabla w_\eta(x), t_\eta^2 \nabla^2 w_\eta(x)\big)\,dx < \sigma(e_d) + \eta. \tag{4.27}$$

We extend w_η outside of S_{e_d} by setting

$$w_\eta(x) = \alpha \ \text{ if } \ x_d < -1/2$$

and

$$w_\eta(x) = \beta \ \text{ if } \ x_d > 1/2,$$

and we define $v_{n,\eta} \in W^{2,2}(\Omega; \mathbb{R}^N_+)$ by

$$v_{n,\eta}(x) := \begin{cases} w_\eta\left(\dfrac{t_\eta x}{\varepsilon_n}\right) & \text{if } x = (x', x_d) \in \Omega' \times \left(-\dfrac{\varepsilon_n}{2t_\eta}, \dfrac{\varepsilon_n}{2t_\eta}\right), \\ u(x) & \text{if } x \in \Omega \text{ and } |x_d| \geq \dfrac{\varepsilon_n}{2t_\eta}. \end{cases} \tag{4.28}$$

We claim that

$$\|v_{n,\eta} - u\|_{L^1(\Omega;\mathbb{R}^N_+)} = O(\varepsilon_n). \tag{4.29}$$

Indeed, changing variables we have

$$\begin{aligned} \|v_{n,\eta}\|_{L^1(\{x\in\Omega:\,|x_d|\leq \frac{\varepsilon_n}{2t_\eta}\};\mathbb{R}^N_+)} &= \int_{\{x\in\Omega:|x_d|\leq\frac{\varepsilon_n}{2t_\eta}\}} \left| w_\eta\left(\frac{t_\eta x}{\varepsilon_n}\right)\right| dx \\ &= \frac{\varepsilon_n}{t_\eta}\int_{-1/2}^{1/2}\int_{\Omega'} \left| w_\eta\left(\frac{t_\eta x'}{\varepsilon_n}, y_d\right)\right| dx'\,dy_d, \end{aligned} \tag{4.30}$$

where, by the periodicity of w_η with respect to the first $d-1$ variables and the Riemann-Lebesgue Lemma,

$$\begin{aligned} \lim_{n\to+\infty}\int_{-1/2}^{1/2}\int_{\Omega'} \left| w_\eta\left(\frac{t_\eta x'}{\varepsilon_n}, y_d\right)\right| dx'\,dy_d &= \int_{-1/2}^{1/2}\int_{\Omega'}\int_{Q'} |w_\eta(z', y_d)|\,dz'dx'dy_d \\ &= \mathcal{H}^{d-1}(\Omega')||w_\eta||_{L^1(Q;\mathbb{R}^N_+)} = r^{d-1}||w_\eta||_{L^1(Q;\mathbb{R}^N_+)}. \end{aligned}$$

Thus, for n sufficiently large, we have that

$$\int_{-1/2}^{1/2}\int_{\Omega'} \left| w_\eta\left(\frac{t_\eta x'}{\varepsilon_n}, y_d\right)\right| dx'\,dy_d \leq r^{d-1}||w_\eta||_{L^1(Q;\mathbb{R}^N_+)} + 1$$

and so

$$\|v_{n,\eta}\|_{L^1(\{x\in\Omega:\,|x_d|\le\frac{\varepsilon_n}{2t_\eta}\};\mathbb{R}^N_+)} \le \frac{\varepsilon_n}{t_\eta}\left(r^{d-1}||w_\eta||_{L^1(Q;\mathbb{R}^N_+)}+1\right).$$

This inequality, together with (4.28), yields

$$\begin{aligned}||v_{n,\eta}-u||_{L^1(\Omega;\mathbb{R}^N_+)} &= ||v_{n,\eta}-u||_{L^1(\{x\in\Omega:|x_d|\le\frac{\varepsilon_n}{2t_\eta}\};\mathbb{R}^N_+)}\\ &\le ||v_{n,\eta}||_{L^1(\{x\in\Omega:|x_d|\le\frac{\varepsilon_n}{2t_\eta}\};\mathbb{R}^N_+)} + C_{\alpha,\beta}\mathcal{L}^d\Big(\Big\{x\in\Omega:|x_d|\le\frac{\varepsilon_n}{2t_\eta}\Big\}\Big)\\ &\le \frac{\varepsilon_n}{t_\eta}\left(r^{d-1}||w_\eta||_{L^1(Q;\mathbb{R}^N_+)}+1\right)+C_{\alpha,\beta}\mathcal{L}^d\Big(\Big\{x\in\Omega:|x_d|\le\frac{\varepsilon_n}{2t_\eta}\Big\}\Big)=O(\varepsilon_n),\end{aligned}$$

where $C_{\alpha,\beta}=\max\{|\alpha|,|\beta|\}$. Let us now estimate $E_{\varepsilon_n}(v_{n,\eta};\Omega)$. By (4.28), (H2), a change of variables and the Riemann-Lebesgue Lemma, we can write

$$\begin{aligned}\limsup_{n\to+\infty}E_{\varepsilon_n}(v_{n,\eta};\Omega) &= \limsup_{n\to+\infty}\frac{1}{\varepsilon_n}\int_\Omega f(v_{n,\eta}(x),\varepsilon_n\nabla v_{n,\eta}(x),\varepsilon_n^2\nabla^2 v_{n,\eta}(x))\,dx\\ &= \limsup_{n\to+\infty}\frac{1}{\varepsilon_n}\int_{-\frac{\varepsilon_n}{2t_\eta}}^{\frac{\varepsilon_n}{2t_\eta}}\int_{\Omega'} f\left(w_\eta\left(\frac{t_\eta x}{\varepsilon_n}\right),t_\eta\nabla w_\eta\left(\frac{t_\eta x}{\varepsilon_n}\right),t_\eta^2\nabla^2 w_\eta\left(\frac{t_\eta x}{\varepsilon_n}\right)\right)dx'\,dx_d\\ &= \limsup_{n\to+\infty}\int_{-1/2}^{1/2}\int_{\Omega'}\frac{1}{t_\eta}f\left(w_\eta\left(\frac{t_\eta x'}{\varepsilon_n},y_d\right),t_\eta\nabla w_\eta\left(\frac{t_\eta x'}{\varepsilon_n},y_d\right),t_\eta^2\nabla^2 w_\eta\left(\frac{t_\eta x'}{\varepsilon_n},y_d\right)\right)dx'\,dy_d\\ &= \int_{-1/2}^{1/2}\int_{\Omega'}\int_{Q'}\frac{1}{t_\eta}f\left(w_\eta(z',y_d),t_\eta\nabla w_\eta(z',y_d),t_\eta^2\nabla^2 w_\eta(z',y_d)\right)dz'\,dx'\,dy_d\\ &= r^{d-1}\int_Q\frac{1}{t_\eta}f\left(w_\eta(y),t_\eta\nabla w_\eta(y),t_\eta^2\nabla^2 w_\eta(y)\right)dy,\end{aligned}$$

and thus, recalling (4.27), we get

$$\limsup_{n\to+\infty}E_{\varepsilon_n}(v_{n,\eta};\Omega)<r^{d-1}\sigma(e_d)+r^{d-1}\eta. \tag{4.31}$$

If we did not require (4.25) to hold, a diagonalization argument and (4.27) would now yield (4.26). However, in order to meet the volume constraint we will need to modify the sequence $v_{n,\eta}$ to obtain a new sequence u_n converging to u in $L^1(\Omega;\mathbb{R}^N_+)$ and such that (4.25) and (4.26) are satisfied. This will be achieved in the following step.

Step 2. Define

$$u_{n,\eta}:=v_{n,\eta}+b_{n,\eta},\quad\text{where}\quad b_{n,\eta}:=⨍_\Omega u(x)-v_{n,\eta}(x)\,dx.$$

As seen in Step 1 (cf. (4.29)), $||v_{n,\eta}-u||_{L^1((\Omega;\mathbb{R}^N_+)}=O(\varepsilon_n)$ so

$$|b_{n,\eta}|\le ⨍_\Omega|u(x)-v_{n,\eta}(x)|\,dx=O(\varepsilon_n). \tag{4.32}$$

Thus, as $n\to+\infty$, $u_{n,\eta}\to u$ in $L^1(\Omega;\mathbb{R}^N_+)$, and

$$\int_\Omega u_{n,\eta}(x)\,dx=\int_\Omega v_{n,\eta}(x)+\left(⨍_\Omega u(y)-v_{n,\eta}(y)\,dy\right)dx=\int_\Omega u(x)\,dx.$$

We claim that, for fixed η, we have, for n sufficiently large,

$$E_{\varepsilon_n}(u_{n,\eta};\Omega)=E_{\varepsilon_n}(v_{n,\eta};\Omega)+O(\varepsilon_n). \tag{4.33}$$

Indeed, adding and subtracting the quantity $E_{\varepsilon_n}(v_{n,\eta};\Omega)$ and recalling (4.28), it follows that

$$E_{\varepsilon_n}(u_{n,\eta};\Omega) = E_{\varepsilon_n}(v_{n,\eta};\Omega) + \frac{1}{\varepsilon_n}\int_{\{x\in\Omega: x_d<-\frac{\varepsilon_n}{2t_\eta}\}} f(\alpha+b_{n,\eta},\mathrm{O},\mathrm{O})\,dx$$

$$+\frac{1}{\varepsilon_n}\int_{\{x\in\Omega: x_d>\frac{\varepsilon_n}{2t_\eta}\}} f(\beta+b_{n,\eta},\mathrm{O},\mathrm{O})\,dx - \frac{1}{\varepsilon_n}\int_{\{x\in\Omega: |x_d|>\frac{\varepsilon_n}{2t_\eta}\}} f(u(x),\mathrm{O},\mathrm{O})\,dx$$

$$+\frac{1}{\varepsilon_n}\int_{\{x\in\Omega: |x_d|<\frac{\varepsilon_n}{2t_\eta}\}} f(v_{n,\eta}(x)+b_{n,\eta},\varepsilon_n\nabla v_{n,\eta}(x),\varepsilon_n^2\nabla^2 v_{n,\eta}(x)) - f(v_{n,\eta}(x),\varepsilon_n\nabla v_{n,\eta}(x),\varepsilon_n^2\nabla^2 v_{n,\eta}(x))\,dx.$$

Notice that the integral with integrand $f(u(x),\mathrm{O},\mathrm{O})$ is zero by definition of u and by (H2). By (4.29) and (4.32), let $M>0$ be a suitable constant such that

$$|v_{n,\eta}+b_{n,\eta}|<M \ \text{ and } \ |v_{n,\eta}|<M$$

for n large enough. Then, by (H5), a change of variables, the periodicity of w_η in the first $d-1$ variables and the Riemann-Lebesgue Lemma and (4.32), we have, for n sufficiently large,

$$\begin{aligned}
&\frac{1}{\varepsilon_n}\int_{\{x\in\Omega: |x_d|<\frac{\varepsilon_n}{2t_\eta}\}} f(v_{n,\eta}(x)+b_{n,\eta},\varepsilon_n\nabla v_{n,\eta}(x),\varepsilon_n^2\nabla^2 v_{n,\eta}(x)) - f(v_{n,\eta}(x),\varepsilon_n\nabla v_{n,\eta}(x),\varepsilon_n^2\nabla^2 v_{n,\eta}(x))\,dx\\
&\le C_M\frac{|b_{n,\eta}|}{\varepsilon_n}\int_{\{x\in\Omega: |x_d|<\frac{\varepsilon_n}{2t_\eta}\}}\left(1+\varepsilon_n^2|\nabla v_{n,\eta}(x)|^2+\varepsilon_n^4|\nabla^2 v_{n,\eta}(x)|^2\right)dx\\
&= C_M|b_{n,\eta}|\left(\frac{r^{d-1}}{t_\eta}+\frac{1}{\varepsilon_n}\int_{-\frac{\varepsilon_n}{2t_\eta}}^{\frac{\varepsilon_n}{2t_\eta}}\int_{\Omega'} t_\eta^2\left|\nabla w_\eta\left(\frac{t_\eta x}{\varepsilon_n}\right)\right|^2+t_\eta^4\left|\nabla^2 w_\eta\left(\frac{t_\eta x}{\varepsilon_n}\right)\right|^2 dx'dx_d\right)\\
&= C_M|b_{n,\eta}|\left(\frac{r^{d-1}}{t_\eta}+\frac{1}{t_\eta}\int_{-\frac12}^{\frac12}\int_{\Omega'} t_\eta^2\left|\nabla w_\eta\left(\frac{t_\eta x'}{\varepsilon_n},y_d\right)\right|^2+t_\eta^4\left|\nabla^2 w_\eta\left(\frac{t_\eta x'}{\varepsilon_n},y_d\right)\right|^2 dx'dy_d\right)\\
&\le C_M|b_{n,\eta}|\left(\frac{r^{d-1}}{t_\eta}+\frac{1}{t_\eta}\int_{-\frac12}^{\frac12}\int_{\Omega'}\int_{Q'} t_\eta^2|\nabla w_\eta(z',y_d)|^2+t_\eta^4|\nabla^2 w_\eta(z',y_d)|^2\,dz'dx'dy_d+1\right)\\
&\le C|b_{n,\eta}|\left(\frac{1}{t_\eta}+1\right)=O(\varepsilon_n).
\end{aligned}$$

Furthermore, by (H6) and (4.32) we have that

$$\frac{1}{\varepsilon_n}\int_{\{x\in\Omega: x_d<-\frac{\varepsilon_n}{2t_\eta}\}} f(\alpha+b_{n,\eta},\mathrm{O},\mathrm{O})\,dx \le \frac{C}{\varepsilon_n}|b_{n,\eta}|^2\mathcal{L}^d(\Omega)=O(\varepsilon_n),$$

since $|\alpha+b_{n,\eta}-\alpha|=|b_{n,\eta}|=O(\varepsilon_n)<\delta_0$, for n large enough. Likewise,

$$\frac{1}{\varepsilon_n}\int_{\{x\in\Omega: x_d>\frac{\varepsilon_n}{2t_\eta}\}} f(\beta+b_{n,\eta},\mathrm{O},\mathrm{O})\,dx = O(\varepsilon_n),$$

and thus (4.33) is proved. Hence, $u_{n,\eta}\to u$ in $L^1(\Omega;\mathbb{R}^N_+)$, as $n\to+\infty$, with $||u_{n,\eta}-u||_{L^1(\Omega;\mathbb{R}^N_+)}=O(\varepsilon_n)$,

$$\int_\Omega u_{n,\eta}(x)\,dx=\int_\Omega u(x)\,dx$$

and, by (4.33) and (4.31),

$$\begin{aligned}\limsup_{n\to+\infty}\frac{1}{\varepsilon_n}\int_\Omega f(u_{n,\eta}(x),\varepsilon_n\nabla u_{n,\eta}(x),\varepsilon_n^2\nabla^2 u_{n,\eta}(x))\,dx &= \limsup_{n\to+\infty} E_{\varepsilon_n}(v_{n,\eta};\Omega)\\ &< r^{d-1}\sigma(e_d)+r^{d-1}\eta = F(u)+r^{d-1}\eta.\end{aligned}$$

□

Let us now address the general case.

Proposition 4.6 (Upper bound)**.** *Let Ω be an open and bounded subset of $\mathbb{R}^d$ with Lipschitz boundary. Assume that hypotheses (H1)–(H6) hold and let $u\in BV(\Omega;\{\alpha,\beta\})\cap\mathcal{V}$. Then, for every $\eta>0$ and every sequence $\varepsilon_n\to 0^+$, there exists a sequence $\{u_n^\eta\}\equiv\{u_n\}\subseteq W^{2,2}(\Omega;\mathbb{R}^N_+)\cap\mathcal{V}$ such that $u_n\to u$ in $L^1(\Omega;\mathbb{R}^N_+)$ with $\|u_n-u\|_{L^1(\Omega;\mathbb{R}^N_+)}=O(\varepsilon_n)$ and*

$$\limsup_{n\to+\infty} F_{\varepsilon_n}(u_n)\le F(u)+O(\eta). \tag{4.34}$$

Proof. Let $\eta>0$ and $\varepsilon_n\to 0^+$. The proof is divided into four steps. In the first three steps we will start, respectively, with target functions with planar and polyhedral interfaces, and we will need to consider an auxiliary compact subset of Ω satisfying certain geometrical properties. This restriction is removed in the final step where a general target function is also considered.

Step 1. We consider first the case where u has a planar interface, i.e., for $x\in\Omega$,

$$u(x)=\begin{cases}\beta & \text{if } (x-x_0)\cdot\nu>0,\\ \alpha & \text{if } (x-x_0)\cdot\nu<0,\end{cases}$$

for some $x_0\in\Omega$ and $\nu\in S^{d-1}$. For $r>0$ sufficiently small, we consider the set $U:=Q_\nu(x_0,r)$ such that $\overline{U}\subset\Omega$, and we assume, without loss of generality, that $\nu=e_d$ and $x_0=0$. We will prove that there exists a sequence $\{u_n^\eta\}\equiv\{u_n\}\subseteq W^{2,2}(U;\mathbb{R}^N_+)$ satisfying

$$\int_U u_n(x)\,dx=\int_U u(x)\,dx=V$$

such that $u_n\to u$ in $L^1(U;\mathbb{R}^N_+)$ with $\|u_n-u\|_{L^1(U;\mathbb{R}^N_+)}=O(\varepsilon_n)$ and

$$\limsup_{n\to+\infty}E_{\varepsilon_n}(u_n;U)\le\int_{J_u\cap U}\sigma(x,e_d)\,d\mathcal{H}^{d-1}(x)+O(\eta).$$

Substep 1.a) We begin by showing that there exists a sequence $\{v_n^\eta\}\equiv\{v_n\}\subset W^{2,2}(U;\mathbb{R}^N_+)$ such that $v_n\to u$ in $L^1(U;\mathbb{R}^N_+)$ with $\|v_n-u\|_{L^1(U;\mathbb{R}^N_+)}=O(\varepsilon_n)$ and

$$\limsup_{n\to+\infty}\int_U\frac{1}{\varepsilon_n}f(x,v_n(x),\varepsilon_n\nabla v_n(x),\varepsilon_n^2\nabla^2 v_n(x))\,dx\le\int_{J_u\cap U}\sigma(x,e_d)\,d\mathcal{H}^{d-1}(x)+O(\eta), \tag{4.35}$$

where

$$O(\eta)=\eta\Big[\int_{J_u\cap U}\sigma(x,e_d)\,d\mathcal{H}^{d-1}(x)+C(1+\eta)r^{d-1}\Big]$$

and C is a constant depending only on α, β and f.

Since $\overline{U}$ is a compact subset of Ω, it is possible to find a $\delta>0$ such that Proposition 3.5 (ii) and (H4) are satisfied uniformly in U, i.e.

$$x,y\in U,\ \ |x-y|<\delta\Rightarrow|\sigma(x,\mu)-\sigma(y,\mu)|\le\eta C\left(1+|\alpha|^q+|\beta|^q+|\alpha-\beta|^2\right),\ \ \forall\mu\in\mathbb{S}^{d-1} \tag{4.36}$$

and

$$x,y\in U,\ \ |x-y|<\delta\Rightarrow\big|f(x,u,\xi,\Lambda)-f(y,u,\xi,\Lambda)\big|\le\eta f(x,u,\xi,\Lambda),\ \ \forall(u,\xi,\Lambda)\in\mathbb{R}^N_+\times\mathbb{R}^{N\times d}\times\mathcal{T}^{N\times N\times d}. \tag{4.37}$$

Let $m\in\mathbb{N}$ be such that

$$\rho:=\frac{r}{m}<\min\{\delta(\eta),\eta\}, \tag{4.38}$$

and consider a partition of U' into m^{d-1} cubes of dimension $d-1$, aligned according to the coordinate axes and having mutually disjoint interiors

$$U'=\bigcup_{i=1}^{m^{d-1}}(a_i+\rho\overline{Q'}). \tag{4.39}$$

For the sake of simplicity in what follows we write $Q_i:=a_i+\rho Q$.

By Lemma 4.5 there exists a sequence $\{u_n^{(1)}\}\subset W^{2,2}(Q_1;\mathbb{R}^N_+)$ such that $u_n^{(1)}\to u$ in $L^1(Q_1;\mathbb{R}^N_+)$, with $\|u_n^{(1)}-u\|_{L^1(Q_1;\mathbb{R}^N_+)}=O(\varepsilon_n)$ and

$$\limsup_{n\to+\infty}\int_{Q_1}\frac{1}{\varepsilon_n}f\left(a_1,u_n^{(1)}(x),\varepsilon_n\nabla u_n^{(1)}(x),\varepsilon_n^2\nabla^2u_n^{(1)}(x)\right)\,dx\le\rho^{d-1}\sigma(a_1,e_d)+\rho^{d-1}\eta.$$

Now, by Lemma 3.3 and Remark 3.4, we can find a sequence $\{w_n^{(1)}\}\subset W^{2,2}(Q_1;\mathbb{R}^N_+)$ such that $w_n^{(1)}\to u$ in $L^1(Q_1;\mathbb{R}^N_+)$, with $\|w_n^{(1)}-u\|_{L^1(Q_1;\mathbb{R}^N_+)}=O(\varepsilon_n)$, $w_n^{(1)}(x)=u*\Psi_{\varepsilon_n}\left(\frac{x-a_1}{\rho}\right)$ on ∂Q_1 and

$$\begin{aligned}\limsup_{n\to+\infty}&\int_{Q_1}\frac{1}{\varepsilon_n}f\left(a_1,w_n^{(1)}(x),\varepsilon_n\nabla w_n^{(1)}(x),\varepsilon_n^2\nabla^2w_n^{(1)}(x)\right)\,dx\\&\le\liminf_{n\to+\infty}\int_{Q_1}\frac{1}{\varepsilon_n}f\left(a_1,u_n^{(1)}(x),\varepsilon_n\nabla u_n^{(1)}(x),\varepsilon_n^2\nabla^2u_n^{(1)}(x)\right)\,dx\\&\le\rho^{d-1}\sigma(a_1,e_d)+\rho^{d-1}\eta.\end{aligned}$$

Applying this reasoning in each cube Q_i, $i=1,...,m^{d-1}$, we obtain sequences $\{w_n^{(j)}\}\subset W^{2,2}(Q_j;\mathbb{R}^N_+)$ such that $w_n^{(j)}\to u$ in $L^1(Q_j;\mathbb{R}^N_+)$, with $\|w_n^{(j)}-u\|_{L^1(Q_j;\mathbb{R}^N_+)}=O(\varepsilon_n)$, $w_n^{(j)}(x)=u*\Psi_{\varepsilon_n}\left(\frac{x-a_j}{\rho}\right)$ on ∂Q_j and

$$\int_{Q_j}\frac{1}{\varepsilon_n}f\left(a_j,w_n^{(j)}(x),\varepsilon_n\nabla w_n^{(j)}(x),\varepsilon_n^2\nabla^2w_n^{(j)}(x)\right)\,dx\le\rho^{d-1}\sigma(a_j,e_d)+\rho^{d-1}\eta \tag{4.40}$$

for $j=1,\dots,m^{d-1}$ and for n sufficiently large.

For $x\in U$ define the sequence $v_{n,\eta}$ by

$$v_{n,\eta}(x):=\begin{cases}w_n^{(j)}(x), & \text{if } x\in Q_j,\\ \beta, & \text{if } x_d\ge\frac{\rho(\eta)}{2},\\ \alpha, & \text{if } x_d\le-\frac{\rho(\eta)}{2}.\end{cases}$$

Taking into account the definition of $w_n^{(j)}$ we clearly have that $v_{n,\eta}\in W^{2,2}(U;\mathbb{R}^N_+)$ and

$$||v_{n,\eta}-u||_{L^1(U;\mathbb{R}^N_+)}=O(\varepsilon_n). \tag{4.41}$$

Recall now that for each η we can find $\delta(\eta)$ such that (4.36) and (4.37) hold and so that $\rho(\eta) < \delta(\eta)$ (cf. (4.38)). By (4.37) and (4.40), we then get

$$\sum_{j=1}^{m^{d-1}} \frac{1}{\varepsilon_n} \int_{Q_j} \Big| f\left(x, w_n^{(j)}(x), \varepsilon_n \nabla w_n^{(j)}(x), \varepsilon_n^2 \nabla^2 w_n^{(j)}(x)\right) - f\left(a_j, w_n^{(j)}(x), \varepsilon_n \nabla w_n^{(j)}(x), \varepsilon_n^2 \nabla^2 w_n^{(j)}(x)\right) \Big| \, dx$$
$$\leq \eta \sum_{j=1}^{m^{d-1}} \frac{1}{\varepsilon_n} \int_{Q_j} f\left(a_j, w_n^{(j)}(x), \varepsilon_n \nabla w_n^{(j)}(x), \varepsilon_n^2 \nabla^2 w_n^{(j)}(x)\right) \, dx \leq \eta \sum_{j=1}^{m^{d-1}} \left(\rho^{d-1}\sigma(a_j, e_d) + \rho^{d-1}\eta\right) \tag{4.42}$$

for n sufficiently large, while by (4.36)

$$\Big| \int_{J_u \cap U} \sigma(x, e_d) \, d\mathcal{H}^{d-1}(x) - \rho^{d-1} \sum_{j=1}^{m^{d-1}} \sigma(a_j, e_d) \Big|$$
$$\leq \sum_{j=1}^{m^{d-1}} \int_{J_u \cap Q_j} \Big| \sigma(x, e_d) - \sigma(a_j, e_d) \Big| \, d\mathcal{H}^{d-1}(x) \leq C\eta r^{d-1}. \tag{4.43}$$

Combining the two previous inequalities, we conclude, in view of (4.40) and (H2), that

$$\int_U \frac{1}{\varepsilon_n} f\left(x, v_{n,\eta}(x), \varepsilon_n \nabla v_{n,\eta}(x), \varepsilon_n^2 \nabla^2 v_{n,\eta}(x)\right) \, dx$$
$$\leq \sum_{j=1}^{m^{d-1}} \int_{Q_j} \frac{1}{\varepsilon_n} f\left(a_j, w_n^{(j)}(x), \varepsilon_n \nabla w_n^{(j)}(x), \varepsilon_n^2 \nabla^2 w_n^{(j)}(x)\right) \, dx + \eta \sum_{j=1}^{m^{d-1}} \left(\rho^{d-1}\sigma(a_j, e_d) + \rho^{d-1}\eta\right)$$
$$\leq (1+\eta) \sum_{j=1}^{m^{d-1}} \left(\rho^{d-1}\sigma(a_j, e_d) + \rho^{d-1}\eta\right)$$
$$\leq (1+\eta) \Big[\int_{J_u \cap U} \sigma(x, e_d) \, d\mathcal{H}^{d-1}(x) + C \ll \eta r^{d-1} \Big]$$
$$\leq \int_{J_u \cap U} \sigma(x, e_d) \, d\mathcal{H}^{d-1}(x) + O(\eta)$$

for n sufficiently large, with

$$O(\eta) = \eta \Big[\int_{J_u \cap U} \sigma(x, e_d) \, d\mathcal{H}^{d-1}(x) + C(1+\eta) r^{d-1} \Big] \tag{4.44}$$

where C is a constant depending only on α, β and f, thus concluding the proof of (4.35).

Substep 1.b) In order to comply with the volume constraint we will now modify the sequence v_n obtained in Substep 1.a). Define

$$u_n := v_n + b_n, \ \text{ where } \ b_n := \fint_U \big(u(x) - v_n(x)\big) \, dx.$$

Clearly $\int_U u_n(x) \, dx = \int_U u(x) \, dx$. In addition, by (4.41)

$$|b_n| \leq \fint_U |(u - v_n)|(x) \, dx = \frac{1}{\mathcal{L}^d(U)} ||v_n - u||_{L^1(U; \mathbb{R}^N_+)} = O(\varepsilon_n). \tag{4.45}$$

It follows that $u_n \to u$ in $L^1(U;\mathbb{R}^N_+)$ with $||u_n - u||_{L^1(U;\mathbb{R}^N_+)} = O(\varepsilon_n)$. We claim that

$$\limsup_{n\to+\infty} \int_U \frac{1}{\varepsilon_n} f\left(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\right) dx \\ \leq \limsup_{n\to+\infty} \int_U \frac{1}{\varepsilon_n} f\left(x, v_n(x), \varepsilon_n \nabla v_n(x), \varepsilon_n^2 \nabla^2 v_n(x)\right) dx + O(\eta). \tag{4.46}$$

If we show that (4.46) holds, the conclusion of Step 1 will follow by (4.35). To prove (4.46) we add and subtract the quantity $E_{\varepsilon_n}(v_n;U)$ to obtain

$$\begin{aligned} E_{\varepsilon_n}(u_n;U) &= E_{\varepsilon_n}(v_n;U) \\ &\quad + \int_U \frac{1}{\varepsilon_n} f\left(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\right) - \frac{1}{\varepsilon_n} f\left(x, v_n(x), \varepsilon_n \nabla v_n(x), \varepsilon_n^2 \nabla^2 v_n(x)\right) dx \\ &= E_{\varepsilon_n}(v_n;U) + \frac{1}{\varepsilon_n} \int_{\{x\in U: x_d < -\frac{\rho}{2}\}} f\left(x, \alpha + b_n, \mathrm{O}, \mathrm{O}\right) dx \\ &\quad + \frac{1}{\varepsilon_n} \int_{\{x\in U: x_d > \frac{\rho}{2}\}} f\left(x, \beta + b_n, \mathrm{O}, \mathrm{O}\right) dx \\ &\quad + \frac{1}{\varepsilon_n} \int_{\{x\in U: |x_d| < \frac{\rho}{2}\}} f\left(x, v_n(x) + b_n, \varepsilon_n \nabla v_n(x), \varepsilon_n^2 \nabla^2 v_n(x)\right) - f\left(x, v_n(x), \varepsilon_n \nabla v_n(x), \varepsilon_n^2 \nabla^2 v_n(x)\right) dx \\ &\quad - \frac{1}{\varepsilon_n} \int_{\{x\in U: |x_d| > \frac{\rho}{2}\}} f\left(x, u(x), \mathrm{O}, \mathrm{O}\right) dx. \end{aligned}$$

By (H2) the last integral is zero. Let $M > 0$ be a suitable constant such that

$$|v_n + b_n| < M \;\text{ and }\; |v_n| < M,$$

for n large enough. Then, by (H5), (H3), (4.45) and (4.38), in this order, we have

$$\begin{aligned} &\frac{1}{\varepsilon_n} \int_{\{x\in U: |x_d| < \frac{\rho}{2}\}} f\left(x, v_n(x) + b_n, \varepsilon_n \nabla v_n(x), \varepsilon_n^2 \nabla^2 v_n(x)\right) - f\left(x, v_n(x), \varepsilon_n \nabla v_n(x), \varepsilon_n^2 \nabla^2 v_n(x)\right) dx \\ &\quad \leq \frac{C}{\varepsilon_n}|b_n| \int_{\{x\in U: |x_d| < \frac{\rho}{2}\}} \left(1 + \varepsilon_n^2 |\nabla v_n(x)|^2 + \varepsilon_n^4 |\nabla^2 v_n(x)|^2\right) dx \\ &\quad \leq \frac{C}{\varepsilon_n}|b_n| \int_{\{x\in U: |x_d| < \frac{\rho}{2}\}} \left(1 + f\left(x, v_n(x), \varepsilon_n \nabla v_n(x), \varepsilon_n^2 \nabla^2 v_n(x)\right)\right) dx \\ &\quad \leq O(\rho) + C\varepsilon_n \int_U \frac{1}{\varepsilon_n} f\left(x, v_n(x), \varepsilon_n \nabla v_n(x), \varepsilon_n^2 \nabla^2 v_n(x)\right) dx \leq O(\eta) + O(\varepsilon_n). \end{aligned}$$

Finally, by (H6) and (4.45) we have

$$\frac{1}{\varepsilon_n} \int_{\{x\in U: x_d < -\frac{\rho}{2}\}} f\left(x, \alpha + b_n, \mathrm{O}, \mathrm{O}\right) dx \leq \frac{C}{\varepsilon_n} |b_n|^2 \mathcal{L}^d(U) = O(\varepsilon_n),$$

since $|\alpha + b_n - \alpha| = |b_n| = O(\varepsilon_n) < \delta_0$, for n large enough. Likewise, we conclude that

$$\frac{1}{\varepsilon_n} \int_{\{x\in U: x_d > \frac{\rho}{2}\}} f\left(x, \beta + b_n, \mathrm{O}, \mathrm{O}\right) dx = O(\varepsilon_n).$$

Thus,

$$\limsup_{n\to+\infty} E_{\varepsilon_n}(u_n;U) \leq \limsup_{n\to+\infty} \left(E_{\varepsilon_n}(v_n;U) + O(\eta) + O(\varepsilon_n)\right) = \limsup_{n\to+\infty} E_{\varepsilon_n}(v_n;U) + O(\eta)$$

and (4.46) is proved.

Step 2. Let u be as in the previous step and, as before, without loss of generality, assume that $\nu = e_d$ and $x_0 = 0$. Now let U be an open subset of Ω satisfying the following condition

$$\lim_{j\to+\infty}\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\mathcal{L}^d\left(\left\{x\in U:\ \mathrm{dist}\,(x, J_u\cap\overline{U})<\varepsilon,\ \mathrm{dist}\,(x,\partial U)<\frac{1}{j}\right\}\right)=0. \tag{4.47}$$

Notice that, since $\partial U'\subseteq\partial U$ and by the triangle inequality, we also obtain

$$\lim_{j\to+\infty}\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\mathcal{L}^d\left(\left\{x\in U:\ \mathrm{dist}\,(x, J_u\cap\overline{U})<\varepsilon,\ \mathrm{dist}\,(x',\partial U')<\frac{1}{j}\right\}\right)=0, \tag{4.48}$$

where $x=(x',x_d)$ and we are using the notation established in Subsection 2.1 for U'.

We claim that the same conclusion derived in Step 1 holds.

Substep 2.a) As before, let us see that there exists a sequence $\{v_n^\eta\}\equiv\{v_n\}\subset W^{2,2}(U;\mathbb{R}^N_+)$ such that $v_n\to u$ in $L^1(U;\mathbb{R}^N_+)$ with $\|v_n-u\|_{L^1(U;\mathbb{R}^N_+)}=O(\varepsilon_n)$ and

$$\limsup_{n\to+\infty}\int_U\frac{1}{\varepsilon_n}f(x,v_n(x),\varepsilon_n\nabla v_n(x),\varepsilon_n^2\nabla^2 v_n(x))\,dx\le\int_{J_u\cap U}\sigma(x,e_d)\,d\mathcal{H}^{d-1}(x)+O(\eta), \tag{4.49}$$

where

$$O(\eta)=\eta\Big[\int_{J_u\cap U}\sigma(x,e_d)\,d\mathcal{H}^{d-1}(x)+C(1+\eta)|U'|\Big].$$

Indeed, by Theorem 2.1 we may write

$$U'=\bigcup_{i=1}^{+\infty}(a_i'+\delta_i\overline{Q}_\nu')=:\bigcup_{i=1}^{+\infty}\overline{Q_i}',$$

where

$$\mathrm{diam}\,Q_i'=C(d)\delta_i\le\mathrm{dist}\,(Q_i',\partial U')\le 4C(d)\delta_i. \tag{4.50}$$

Condition (4.50) implies that there exist constants $C_1(d), C_2(d)>0$ such that

$$C_1(d)\delta_i\le\mathrm{dist}\,(a_i',\partial U')\le C_2(d)\delta_i. \tag{4.51}$$

Let $C'(d)$ be such that the inequality

$$\mathrm{dist}\,(x',a_i')\le C'(d)\delta_i, \tag{4.52}$$

holds for every $x'\in Q_i'$ and every $i\in\mathbb{N}$. Now choose $L>0$ such that

$$\left(\frac{C'(d)}{C_1(d)}+1\right)\frac{1}{L}<1,$$

and let

$$\Omega_j':=\left\{x'\in U':\ \mathrm{dist}\,(x',\partial U')\ge\frac{1}{j}\right\},$$

$$\mathcal{G}_j:=\left\{Q_i':\ \mathrm{dist}\,(a_i',\partial U')<\frac{1}{Lj}\right\},$$

$$\mathcal{F}_j:=\left\{Q_i':\ \mathrm{dist}\,(a_i',\partial U')\ge\frac{1}{Lj}\right\}.$$

Notice that, for all $j\in\mathbb{N}$, $\mathcal{F}_j$ is a finite family of cubes. Indeed, if $Q_i'\in\mathcal{F}_j$, by (4.51), it follows that $\delta_i\ge\dfrac{1}{LC_2(d)j}$ and, if there were infinitely many cubes Q_i' in $\mathcal{F}_j$, we would have that

$$\mathcal{L}^{d-1}(U')\ge\sum_{Q_i'\in\mathcal{F}_j}\mathcal{L}^{d-1}(Q_i')=\sum_{Q_i'\in\mathcal{F}_j}\delta_i^{d-1}\ge C(\mathrm{card}\,\mathcal{F}_j)=+\infty,$$

contradicting the fact that U' is bounded. Also, by our choice of L, it follows that

$$\Omega'_j \subset \bigcup_{Q'_i \in \mathcal{F}_j} Q'_i.$$

Indeed, given $x' \in \Omega'_j$ we can find an index i_0 such that $x' \in \overline{Q'_{i_0}}$. Then necessarily $Q'_{i_0} \in \mathcal{F}_j$, because otherwise we would have $\mathrm{dist}(a'_{i_0}, \partial U') < 1/(Lj)$ which would imply, by (4.51) and (4.52), that

$$\begin{aligned}\mathrm{dist}(x', \partial U') &\leq \mathrm{dist}(a'_{i_0}, \partial U') + \mathrm{dist}(x', a'_{i_0}) \leq \mathrm{dist}(a'_{i_0}, \partial U') \left(1 + \frac{C'(d)}{C_1(d)}\right) \\ &< \frac{1}{Lj}\left(1 + \frac{C'(d)}{C_1(d)}\right) < \frac{1}{j},\end{aligned}$$

which is a contradiction. Thus Ω'_j is covered by finitely many cubes $Q'_i \in \mathcal{F}_j$.

We denote by Q_i the cube $Q_i := Q'_i \times \big(-\frac{\delta_i}{2}, \frac{\delta_i}{2}\big)$. By Substep 1.a) applied to the cube Q_1 there exists a sequence $\{v_n^{(1)}\} \subset W^{2,2}(Q_1; \mathbb{R}^N_+)$ such that $v_n^{(1)} \to u$ in $L^1(Q_1; \mathbb{R}^N_+)$, with $\|v_n^{(1)} - u\|_{L^1(Q_1;\mathbb{R}^N_+)} = O(\varepsilon_n)$ and

$$\limsup_{n\to+\infty} E_{\varepsilon_n}(v_n^{(1)}; Q_1) \leq \int_{J_u \cap Q_1} \sigma(x, e_d)\, d\mathcal{H}^{d-1}(x) + \eta\Big[\int_{J_u\cap Q_1} \sigma(x, e_d)\, d\mathcal{H}^{d-1}(x) + C(1+\eta)\delta_1^{d-1}\Big]. \tag{4.53}$$

Thus, by (4.53) and Lemma 3.3 there exists a sequence $\{w_n^{(1)}\} \subset W^{2,2}(Q_1; \mathbb{R}^N)$ such that $w_n^{(1)}(x) = u * \Psi_{\varepsilon_n}\left(\frac{x-a_1}{\delta_1}\right)$ on ∂Q_1 and

$$\begin{aligned}&\limsup_{n\to+\infty} \int_{Q_1} \frac{1}{\varepsilon_n} f\left(x, w_n^{(1)}(x), \varepsilon_n \nabla w_n^{(1)}(x), \varepsilon_n^2 \nabla^2 w_n^{(1)}(x)\right) dx \\ &\leq \liminf_{n\to+\infty} \int_{Q_1} \frac{1}{\varepsilon_n} f\left(x, v_n^{(1)}(x), \varepsilon_n \nabla v_n^{(1)}(x), \varepsilon_n^2 \nabla^2 v_n^{(1)}(x)\right) dx \\ &\leq \int_{J_u\cap Q_1} \sigma(x, e_d)\, d\mathcal{H}^{d-1}(x) + \eta\Big[\int_{J_u \cap Q_1} \sigma(x, e_d)\, d\mathcal{H}^{d-1}(x) + C(1+\eta)\delta_1^{d-1}\Big].\end{aligned} \tag{4.54}$$

We can also assume that $\|w_n^{(1)} - u\|_{L^1(Q_1;\mathbb{R}^N_+)} = O(\varepsilon_n)$ (cf. Remark 3.4).

By applying the above reasoning in each of the finitely many cubes Q'_i in $\mathcal{F}_j$, we obtain sequences $\{w_n^{(i)}\} \subset W^{2,2}(Q_i; \mathbb{R}^N)$ such that $w_n^{(i)}(x) = u * \Psi_{\varepsilon_n}\left(\frac{x-a_i}{\delta_i}\right)$ on ∂Q_i, $\|w_n^{(i)} - u\|_{L^1(Q_i;\mathbb{R}^N_+)} = O(\varepsilon_n)$, and

$$\limsup_{n\to+\infty} E_{\varepsilon_n}(w_n^{(i)}; Q_i) \leq \int_{J_u\cap Q_i} \sigma(x, e_d)\, d\mathcal{H}^{d-1}(x) + \eta\Big[\int_{J_u\cap Q_i} \sigma(x, e_d)\, d\mathcal{H}^{d-1}(x) + C(1+\eta)\delta_i^{d-1}\Big] \tag{4.55}$$

for every Q_i such that $Q'_i \in \mathcal{F}_j$.

We now define the sequence

$$v_n(x) := \begin{cases} w_n^{(i)}(x), & \text{if } x \in Q_i,\ Q'_i \in \mathcal{F}_j \\ \\ u * \Psi_{\varepsilon_n}(x), & \text{if } x \in U \setminus \{\cup_i Q_i : Q'_i \in \mathcal{F}_j\}. \end{cases} \tag{4.56}$$

Clearly $v_n \to u$ in $L^1(U; \mathbb{R}^N_+)$ with

$$\|v_n - u\|_{L^1(U;\mathbb{R}^N_+)} = O(\varepsilon_n).$$

Moreover, taking into account (3.8) and since $x \in U \setminus \{Q_i : Q_i' \in \mathcal{F}_j\}$ implies that $\operatorname{dist}(x', \partial U') < \dfrac{1}{j}$, by (H2), (H3) and (4.55) we have that

$$\begin{aligned}
&\limsup_{n\to+\infty} \int_U \frac{1}{\varepsilon_n} f\left(x, v_n(x), \varepsilon_n \nabla v_n(x), \varepsilon_n^2 \nabla^2 v_n(x)\right) dx \\
&\quad \leq \limsup_{n\to+\infty} \int_{U\setminus\{Q_i : Q_i'\in\mathcal{F}_j\}} \frac{1}{\varepsilon_n} f\left(x, (u*\Psi_{\varepsilon_n})(x), \varepsilon_n \nabla (u*\Psi_{\varepsilon_n})(x), \varepsilon_n^2 \nabla^2 (u*\Psi_{\varepsilon_n})(x)\right) dx \\
&\qquad\qquad + \sum_{Q_i'\in\mathcal{F}_j} \limsup_{n\to+\infty} \int_{Q_i} \frac{1}{\varepsilon_n} f\left(x, w_n^{(i)}(x), \varepsilon_n \nabla w_n^{(i)}(x), \varepsilon_n^2 \nabla^2 w_n^{(i)}(x)\right) dx \\
&\quad \leq C \limsup_{n\to+\infty} \frac{1}{\varepsilon_n} \mathcal{L}^d\left(\left\{x=(x',x_d)\in U : |x_d|<\varepsilon_n,\ \operatorname{dist}(x',\partial U') < \frac{1}{j}\right\}\right) \\
&\qquad\qquad + \sum_{Q_i'\in\mathcal{F}_j} \int_{J_u\cap Q_i} \sigma(x,e_d)\, d\mathcal{H}^{d-1}(x) + \eta\Big[\int_{J_u\cap U} \sigma(x,e_d)\, d\mathcal{H}^{d-1}(x) + C(1+\eta)|U'|\Big] \\
&\quad \leq C \limsup_{n\to+\infty} \frac{1}{\varepsilon_n} \mathcal{L}^d\left(\left\{x=(x',x_d)\in U : |x_d|<\varepsilon_n,\ \operatorname{dist}(x',\partial U') < \frac{1}{j}\right\}\right) \\
&\qquad\qquad + \int_{J_u\cap U} \sigma(x,e_d)\, d\mathcal{H}^{d-1}(x) + \eta\Big[\int_{J_u\cap U} \sigma(x,e_d)\, d\mathcal{H}^{d-1}(x) + C(1+\eta)|U'|\Big].
\end{aligned}$$

Taking the limit when $j \to +\infty$, by (4.48) we conclude that

$$\limsup_{n\to+\infty} \int_U \frac{1}{\varepsilon_n} f(x, v_n(x), \varepsilon_n \nabla v_n(x), \varepsilon_n^2 \nabla^2 v_n(x))\, dx \leq \int_{J_u\cap U} \sigma(x,e_d)\, d\mathcal{H}^{d-1}(x) + O(\eta),$$

where

$$O(\eta) = \eta\Big[\int_{J_u\cap U} \sigma(x,e_d)\, d\mathcal{H}^{d-1}(x) + C(1+\eta)|U'|\Big],$$

and C is a constant depending only on α, β and f.

Substep 2.b) The change of the sequence v_n in order to comply with the volume constraint follows from (4.56) and Substep 1.b) above.

Step 3. We consider the case where u has a polyhedral interface, i.e. $u = \chi_E\beta + (1-\chi_E)\alpha$, where $E = E' \cap \Omega, \partial^* E \cap \Omega = \partial^* E' \cap \Omega$ with E' a polyhedral set, that is, E' is a bounded strongly Lipschitz domain such that $\partial E' = \bigcup_{i=1}^M H_i$ where H_i are closed subsets of hyperplanes in $\mathbb{R}^d$. As in the previous step let $U \subset \Omega$ be such that (4.47) holds. Define $\bar u : \mathbb{R}^d \to \mathbb{R}^{\mathbb{N}}$ by $\bar u(\cdot) := \chi_{E'}\beta + (1-\chi_{E'})\alpha$ so that $\bar u$ coincides with u in U and let

$$\bar u_{\varepsilon_n}(\cdot) := \frac{1}{(\varepsilon_n)^d} \Psi\left(\frac{\cdot}{\varepsilon_n}\right) * \bar u, \quad x \in \mathbb{R}^d$$

where Ψ is the symmetric mollifier given in (3.4). Recall that $\{\bar u_{\varepsilon_n}\}$ is bounded in L^∞ and

$$||\nabla \bar u_{\varepsilon_n}||_\infty = O\left(\frac{1}{\varepsilon_n}\right),\ ||\nabla^2 \bar u_{\varepsilon_n}||_\infty = O\left(\frac{1}{\varepsilon_n^2}\right),\ \operatorname{supp} \nabla \bar u_{\varepsilon_n} \subset \{x\in\mathbb{R}^d :\ \operatorname{dist}(x, J_{\bar u}) < \varepsilon_n\}. \tag{4.57}$$

Notice that when the target function is extended to $\mathbb{R}^d$ it could happen that $\mathcal{H}^{d-1}(J_{\bar u} \cap \overline{\Omega}) > 0$. To avoid this situation, we work with a smaller set $U \subset \Omega$ satisfying the first condition in (4.47). The second condition that we impose on U ensures that ∂U does not contain cusps.

We have $J_u \cap U = \displaystyle\bigcup_{i=1}^M (H_i \cap U)$ where $\mathcal{H}^{d-1}(H_i \cap H_j) = 0$, if $i \neq j$. Let

$$I := \{i \in \{1, \dots M\} : \mathcal{H}^{d-1}(H_i \cap U) > 0\}.$$

As before we divide the argument in two substeps.

Substep 3.a) Using an induction argument on card I, we will first show that there exists a sequence $\{v_n^\eta\} \equiv \{v_n\} \subset W^{2,2}(U;\mathbb{R}^N_+)$ such that

$$\begin{aligned} &v_n \to u \text{ in } L^1(U;\mathbb{R}^N_+) \text{ with } \|v_n - u\|_{L^1(U;\mathbb{R}^N_+)} = O(\varepsilon_n), v_n = \bar{u}_{\varepsilon_n} \text{ on } \partial U \text{ and} \\ &\limsup_{n\to+\infty} \int_U \frac{1}{\varepsilon_n} f\left(x, v_n(x), \varepsilon_n \nabla v_n(x), \varepsilon_n^2 \nabla^2 v_n(x)\right) dx \le \int_{J_u\cap U} \sigma(x,\nu(x))\, d\mathcal{H}^{d-1}(x) + O(\eta), \end{aligned} \tag{4.58}$$

where

$$O(\eta) = \eta\Big[\int_{J_u\cap U} \sigma(x,\nu(x))\, d\mathcal{H}^{d-1}(x) + C(1+\eta)\mathcal{H}^{d-1}(J_u\cap U)\Big].$$

The case where card $I = 0$ is trivial and the case where card $I = 1$ was treated in the previous step. Recall that the sequence derived in Substep 2.a) satisfies all the conditions in (4.58). We assume that (4.58) is true for card $I = M-1$ and we prove that it still holds when card $I = M$. Let

$$S := \{x \in \mathbb{R}^d : \operatorname{dist}(x, H_1) = \operatorname{dist}(x, H_2 \cup \ldots \cup H_M)\}$$

and

$$\Omega_1 := \{x \in U : \operatorname{dist}(x, H_1) < \operatorname{dist}(x, H_2 \cup \ldots \cup H_M)\}.$$

Clearly Ω_1 is open and $\Omega_1 \cap (H_2 \cup \ldots \cup H_M) = \emptyset$ so that card $\{i \in \{1,\ldots M\} : \mathcal{H}^{d-1}(H_i \cap \Omega_1) > 0\} = 1$. Also, given $x \in \Omega_1$ we have $\operatorname{dist}(x, J_u \cap \overline{\Omega}_1) \ge \operatorname{dist}(x, J_u \cap \overline{U})$, and $\partial\Omega_1 \backslash (\partial\Omega_1 \cap \partial U)$ is contained in hyperplanes so, by (4.47), it follows that

$$\lim_{j\to+\infty} \lim_{\varepsilon\to0^+} \frac{1}{\varepsilon} \mathcal{L}^d \left(\left\{ x \in \Omega_1 : \operatorname{dist}(x, J_u \cap \overline{\Omega}_1) < \varepsilon,\, \operatorname{dist}(x, \partial\Omega_1) < \frac{1}{j} \right\} \right) = 0.$$

Likewise, letting

$$\Omega_2 := \{x \in U : \operatorname{dist}(x, H_1) > \operatorname{dist}(x, H_2 \cup \ldots \cup H_M)\},$$

we have that Ω_2 is open, $J_u \cap \Omega_2 = (H_2 \cap U) \cup \ldots \cup (H_M \cap U)$ and

$$\lim_{j\to+\infty} \lim_{\varepsilon\to0^+} \frac{1}{\varepsilon} \mathcal{L}^d \left(\left\{ x \in \Omega_2 : \operatorname{dist}(x, J_u \cap \overline{\Omega}_2) < \varepsilon,\, \operatorname{dist}(x, \partial\Omega_2) < \frac{1}{j} \right\} \right) = 0.$$

We can apply Substep 2.a) to Ω_1 since it contains only one interface. Hence, there exists a sequence $\{v_n^{(1)}\} \subset W^{2,2}(\Omega_1;\mathbb{R}^N_+)$ such that $v_n^{(1)} \to u$ in $L^1(\Omega_1;\mathbb{R}^N_+)$ with $\|v_n^{(1)} - u\|_{L^1(\Omega_1;\mathbb{R}^N_+)} = O(\varepsilon_n)$, $v_n^{(1)} = \bar{u}_{\varepsilon_n}$ on $\partial\Omega_1$ and

$$\limsup_{n\to+\infty} E_{\varepsilon_n}(v_n^{(1)};\Omega_1) \le \int_{J_u\cap\Omega_1} \sigma(x,\nu(x))\, d\mathcal{H}^{d-1}(x) + \eta\Big[\int_{J_u\cap\Omega_1} \sigma(x,\nu(x))\, d\mathcal{H}^{d-1}(x) + C(1+\eta)\mathcal{H}^{d-1}(J_u\cap\Omega_1)\Big]. \tag{4.59}$$

Since $J_u \cap \Omega_2$ contains at most $M-1$ flat interfaces we can apply the induction hypothesis in Ω_2 to obtain a sequence $\{v_n^{(2)}\} \subset W^{2,2}(\Omega_2;\mathbb{R}^N_+)$ such that $v_n^{(2)} \to u$ in $L^1(\Omega_2;\mathbb{R}^N_+)$ with $\|v_n^{(2)} - u\|_{L^1(\Omega_2;\mathbb{R}^N_+)} = O(\varepsilon_n)$, $v_n^{(2)} = \bar{u}_{\varepsilon_n}$ on $\partial\Omega_2$ and

$$\limsup_{n\to+\infty} E_{\varepsilon_n}(v_n^{(2)};\Omega_2) \le \int_{J_u\cap\Omega_2} \sigma(x,\nu(x))\, d\mathcal{H}^{d-1}(x) + \eta\Big[\int_{J_u\cap\Omega_2} \sigma(x,\nu(x))\, d\mathcal{H}^{d-1}(x) + C(1+\eta)\mathcal{H}^{d-1}(J_u\cap\Omega_2)\Big]. \tag{4.60}$$

We now define

$$v_n(x) := \begin{cases} v_n^{(1)}(x), & \text{if } x \in \overline{\Omega}_1 \cap U \\ v_n^{(2)}(x), & \text{if } x \in \overline{\Omega}_2 \cap U \\ \bar{u}_{\varepsilon_n}(x), & \text{if } x \in \operatorname{int} S \cap U. \end{cases} \tag{4.61}$$

We point out that in certain cases it may happen that $\mathcal{L}^d(\mathrm{int}S \cap U) > 0$.

Since $||v_n^{(1)} - u||_{L^1(\Omega_1;\mathbb{R}^N_+)} = O(\varepsilon_n)$, $||v_n^{(2)} - u||_{L^1(\Omega_2;\mathbb{R}^N_+)} = O(\varepsilon_n)$ and $||\bar{u}_{\varepsilon_n} - u||_{L^1(\mathrm{int}S\cap U;\mathbb{R}^N_+)} = O(\varepsilon_n)$, it follows that

$$||v_n - u||_{L^1(U;\mathbb{R}^N_+)} = O(\varepsilon_n). \tag{4.62}$$

On the other hand, by (4.59), (4.60), (4.57), (H3) and (H2) we have

$$\begin{aligned}
&\limsup_{n\to+\infty} \int_U \frac{1}{\varepsilon_n} f\left(x, v_n(x), \varepsilon_n \nabla v_n(x), \varepsilon_n^2 \nabla^2 v_n(x)\right) dx \\
&\quad\le \limsup_{n\to+\infty} \int_{\Omega_1} \frac{1}{\varepsilon_n} f\left(x, v_n^{(1)}(x), \varepsilon_n \nabla v_n^{(1)}(x), \varepsilon_n^2 \nabla^2 v_n^{(1)}(x)\right) dx \\
&\quad+ \limsup_{n\to+\infty} \int_{\Omega_2} \frac{1}{\varepsilon_n} f\left(x, v_n^{(2)}(x), \varepsilon_n \nabla v_n^{(2)}(x), \varepsilon_n^2 \nabla^2 v_n^{(2)}(x)\right) dx \\
&\quad+ \limsup_{n\to+\infty} \int_{\mathrm{int}S\cap U} \frac{1}{\varepsilon_n} f\left(x, \bar{u}_{\varepsilon_n}(x), \varepsilon_n \nabla \bar{u}_{\varepsilon_n}(x), \varepsilon_n^2 \nabla^2 \bar{u}_{\varepsilon_n}(x)\right) dx \\
&\quad\le \sum_{i=1}^2 \int_{\Omega_i\cap J_u} \sigma(x,\nu(x))\, d\mathcal{H}^{d-1}(x) + O(\eta) \\
&\quad+ \limsup_{n\to+\infty} \frac{1}{\varepsilon_n} \int_{\mathrm{int}S\cap U} f\left(x, \bar{u}_{\varepsilon_n}(x), \varepsilon_n \nabla \bar{u}_{\varepsilon_n}(x), \varepsilon_n^2 \nabla^2 \bar{u}_{\varepsilon_n}(x)\right) dx \\
&\quad\le \sum_{i=1}^2 \int_{\Omega_i\cap J_u} \sigma(x,\nu(x))\, d\mathcal{H}^{d-1}(x) + \limsup_{n\to+\infty} \frac{C}{\varepsilon_n} \mathcal{L}^d\{x \in \mathrm{int}S\cap U : \mathrm{dist}(x, J_u) < \varepsilon_n\} + O(\eta) \\
&\quad\le \int_{U\cap J_u} \sigma(x,\nu(x))\, d\mathcal{H}^{d-1}(x) + O(\eta),
\end{aligned}$$

where in the last step we used the fact that

$$\mathcal{L}^d\{x \in \mathrm{int}S\cap U : \mathrm{dist}(x, J_u) < \varepsilon_n\} = \mathcal{L}^d\{x \in \mathrm{int}S\cap U : \mathrm{dist}(x, \overline{\Omega}_1 \cap \overline{\Omega}_2) < \varepsilon_n\} \le C\varepsilon_n^{d-1}, \tag{4.63}$$

and where

$$O(\eta) = \eta\Big[\int_{J_u\cap U} \sigma(x,\nu(x))\, d\mathcal{H}^{d-1}(x) + C(1+\eta)\mathcal{H}^{d-1}(J_u\cap U)\Big]$$

and C is a constant depending only on α, β and f.

Substep 3.b) As before, in order to meet the volume constraint, we change the sequence v_n defined in (4.61) by setting

$$u_n := v_n + b_n, \quad b_n := \fint_U \big(u(x) - v_n(x)\big)\, dx.$$

By (4.62)

$$|b_n| \le O(\varepsilon_n) \tag{4.64}$$

and it follows that $u_n \to u$ in $L^1(U;\mathbb{R}^N_+)$ with $||u_n - u||_{L^1(U;\mathbb{R}^N_+)} = O(\varepsilon_n)$.

Using the same notation as in Substep 3.a) we will show, using an induction procedure on card I, that the sequence u_n defined above satisfies

$$\limsup_{n\to+\infty} \int_U \frac{1}{\varepsilon_n} f(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x))\, dx \le \int_{J_u\cap U} \sigma(x,\nu(x))\, d\mathcal{H}^{d-1}(x) + O(\eta). \tag{4.65}$$

When card $I = 1$, since $\mathcal{L}^d(\Omega_1) = \mathcal{L}^d(U)$, (4.65) follows immediately from Step 2.b). Assuming the result is true when card $I = M - 1$ we show that it also holds when card $I = M$. In this case, $J_u \cap \Omega_2$

contains at most $M-1$ flat interfaces and $J_u \cap \Omega_1$ only one. Therefore, using the induction hypothesis, we have

$$\begin{aligned}
&\limsup_{n\to+\infty} \int_U \frac{1}{\varepsilon_n} f\left(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\right) dx \\
&\quad \le \limsup_{n\to+\infty} \int_{\Omega_1} \frac{1}{\varepsilon_n} f\left(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\right) dx \\
&\quad + \limsup_{n\to+\infty} \int_{\Omega_2} \frac{1}{\varepsilon_n} f\left(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\right) dx \\
&\quad + \limsup_{n\to+\infty} \int_{\mathrm{int} S \cap U} \frac{1}{\varepsilon_n} f\left(x, u_n(x), \varepsilon_n \nabla u(x), \varepsilon_n^2 \nabla^2 u(x)\right) dx \\
&\quad \le \int_{J_u \cap \Omega_1} \sigma(x, \nu(x))\, d\mathcal{H}^{d-1}(x) + O(\eta) + \int_{J_u \cap \Omega_2} \sigma(x, \nu(x))\, d\mathcal{H}^{d-1}(x) + O(\eta) \\
&\quad + \limsup_{n\to+\infty} \int_{\mathrm{int} S \cap U} \frac{1}{\varepsilon_n} f\left(x, u_n(x), \varepsilon_n \nabla u(x), \varepsilon_n^2 \nabla^2 u(x)\right) dx
\end{aligned}$$

where by (H3), (4.57), (4.63), (4.64) and (H6),

$$\begin{aligned}
&\int_{\mathrm{int} S \cap U} \frac{1}{\varepsilon_n} f\left(x, u_n(x), \varepsilon_n \nabla u(x), \varepsilon_n^2 \nabla^2 u(x)\right) dx \\
&\quad \le \int_{\{x \in \mathrm{int} S \cap U : \mathrm{dist}(x, J_u) < \varepsilon_n\}} \frac{1}{\varepsilon_n} f\left(x, \bar{u}_{\varepsilon_n}(x) + b_n, \varepsilon_n \nabla \bar{u}_{\varepsilon_n}(x), \varepsilon_n^2 \nabla^2 \bar{u}_{\varepsilon_n}(x)\right) dx \\
&\quad + \int_{\{x \in \mathrm{int} S \cap U : \mathrm{dist}(x, J_u) \ge \varepsilon_n\}} \frac{1}{\varepsilon_n} f\left(x, u(x) + b_n, \mathrm{O}, \mathrm{O}\right) dx \\
&\quad \le \frac{C}{\varepsilon_n} \mathcal{L}^d \{x \in \mathrm{int} S \cap U : \mathrm{dist}(x, J_u) < \varepsilon_n\} + \frac{C}{\varepsilon_n} |b_n|^2 \mathcal{L}^d(U) = O(\varepsilon_n).
\end{aligned}$$

Thus,

$$\limsup_{n\to+\infty} \int_U \frac{1}{\varepsilon_n} f\left(x, u_n(x), \varepsilon_n \nabla u_n(x), \varepsilon_n^2 \nabla^2 u_n(x)\right) dx \le \int_{J_u \cap \Omega} \sigma(x, \nu(x))\, d\mathcal{H}^{d-1}(x) + O(\eta).$$

Step 4. We conclude with the general case where Ω is a bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary and $u = \beta\chi_E + \alpha(1-\chi_E)$ with $\mathrm{Per}_\Omega(E) < +\infty$. Applying Theorem 2.11 there exist polyhedral sets E_k such that $\chi_{E_k} \to \chi_E$ in $BV(\Omega)$, $\mathcal{L}^d(E_k) = \mathcal{L}^d(E)$ and $\mathcal{H}^{d-1}(\partial^* E_k \cap \partial\Omega) = 0$. Moreover, in order to apply the results of the previous step, we choose E_k such that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \mathcal{L}^d\left(\left\{x \in \Omega : \mathrm{dist}(x, \partial^* E_k \cap \overline{\Omega}) < \varepsilon\right\}\right) = 0. \tag{4.66}$$

Substep 4.a) Since $\partial\Omega$ is Lipschitz, and using (4.66), we may apply Substep 3.a) in Ω. Hence, for every k, we find a sequence

$$v_n^{(k)} \to u_k := \beta\chi_{E_k} + \alpha(1-\chi_{E_k})$$

as $n \to +\infty$ in $L^1(\Omega; \mathbb{R}^N_+)$ with $\|v_n^{(k)} - u_k\|_{L^1(\Omega;\mathbb{R}^N_+)} = O(\varepsilon_n)$ and such that

$$\limsup_{n\to+\infty} \int_\Omega \frac{1}{\varepsilon_n} f\left(x, v_n^{(k)}(x), \varepsilon_n \nabla v_n^{(k)}(x), \varepsilon_n^2 \nabla^2 v_n^{(k)}(x)\right) dx \le \int_{J_{u_k} \cap \Omega} \sigma(x, \nu_k(x))\, d\mathcal{H}^{d-1}(x) + C_k \eta,$$

where $\nu_k(x)$ is the measure theoretic unit inner normal to J_{u_k} at x,

$$C_k = \int_{J_{u_k} \cap \Omega} \sigma(x, \nu_k(x))\, d\mathcal{H}^{d-1}(x) + C(1+\eta)\mathcal{H}^{d-1}(J_{u_k} \cap \Omega)$$

and C is a constant depending only on α, β and f. Let $n(k)$ be such that

$$||v^{(k)}_{n(k)} - u_k||_{L^1(\Omega;\mathbb{R}^N_+)} \leq \frac{1}{k}$$

and

$$\left| \int_{J_{u_k}\cap\Omega} \sigma(x,\nu_k(x))\, d\mathcal{H}^{d-1}(x) - \int_\Omega \frac{1}{\varepsilon_k} f\left(x, v^{(k)}_{n(k)}(x), \varepsilon_k \nabla v^{(k)}_{n(k)}(x), \varepsilon_k^2 \nabla^2 v^{(k)}_{n(k)}(x)\right) dx \right| < \frac{1}{k} + C_k \eta.$$

Setting $v_k := v^{(k)}_{n(k)}$ it follows that $v_k \to u$ in $L^1(\Omega;\mathbb{R}^N_+)$. Moreover, by Theorem 2.11, every continuous function $h : \Omega \times \mathbb{R}^d \to [0,+\infty)$ satisfies

$$\lim_{k\to+\infty} \int_{J_{u_k}\cap\Omega} h(x,\nu_k(x))\, d\mathcal{H}^{d-1}(x) = \int_{J_u\cap\Omega} h(x,\nu(x))\, d\mathcal{H}^{d-1}(x).$$

Since σ is upper semicontinuous (cf. Proposition 3.5 - (iii)), there exist continuous functions $h_m : \Omega\times\mathbb{R}^d \to [0,+\infty)$ such that, for every $(x,\zeta) \in \Omega \times \mathbb{R}^d$,

$$\sigma(x,\zeta) \leq h_m(x,\zeta) \leq C|\zeta|,$$

and

$$\sigma(x,\zeta) = \inf_m h_m(x,\zeta),$$

where we have extended $\sigma(x,\cdot)$ as a homogeneous function of degree one. Thus, for all m it follows that

$$\begin{aligned}
\limsup_{k\to+\infty} \int_\Omega \frac{1}{\varepsilon_k} f\left(x, v_k(x), \varepsilon_k \nabla v_k(x), \varepsilon_k^2 \nabla^2 v_k(x)\right) dx &\leq (1+\eta) \limsup_{k\to+\infty} \int_{J_{u_k}\cap\Omega} \sigma(x,\nu_k(x))\, d\mathcal{H}^{d-1}(x) \\
&\quad + C\eta(1+\eta) \limsup_{k\to+\infty} \mathcal{H}^{d-1}(J_{u_k}\cap\Omega) \\
&\leq (1+\eta) \limsup_{k\to+\infty} \int_{J_{u_k}\cap\Omega} h_m(x,\nu_k(x))\, d\mathcal{H}^{d-1}(x) + C\eta(1+\eta)\mathcal{H}^{d-1}(J_u\cap\Omega) \\
&= (1+\eta) \int_{J_u\cap\Omega} h_m(x,\nu(x))\; d\mathcal{H}^{d-1}(x) + C\eta(1+\eta)\mathcal{H}^{d-1}(J_u\cap\Omega).
\end{aligned}$$

Taking the limit when $m \to +\infty$, by Lebesgue's monotone convergence theorem, we conclude that

$$\limsup_{k\to+\infty} \int_\Omega \frac{1}{\varepsilon_k} f\left(x, v_k(x), \varepsilon_k \nabla v_k(x), \varepsilon_k^2 \nabla^2 v_k(x)\right) dx \leq \int_{J_u\cap\Omega} \sigma(x,\nu(x))\, d\mathcal{H}^{d-1}(x) + O(\eta).$$

Substep 4.b) Following the argument given in Substep 3.b) above, we can change the sequence v_k above so that it satisfies the volume constraint without increasing the energy. This completes the proof of Theorem 4.1. □

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