Multiscale nonconvex relaxation and application to thin films

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Abstract

Γ-convergence techniques are used to give a characterization of the behavior of a family of heterogeneous multiple scale integral functionals. Periodicity, standard growth conditions and nonconvexity are assumed whereas a stronger uniform continuity hypothesis with respect to the macroscopic variable, normally required in the existing literature, is avoided. An application to dimension reduction problems in reiterated homogenization of thin films is presented.

Keywords: integral functionals, periodicity, homogenization, Γ-convergence, quasiconvexity, equi-integrability, dimension reduction, thin films.

MSC 2000 ($\mathcal{AMS}$): 35E99, 35M10, 49J45, 74B20, 74G65, 74Q05, 74K35.

1 Introduction

In this work we study the ε-limit behavior of an elastic body whose microstructure is periodic of period ε and ε^2, and whose volume may also depend on this small parameter ε, by a Γ-limit argument. We refer to the books of Braides [13], Braides and Defranceschi [14] and Dal Maso [19] for a comprehensive treatment on this kind of variational convergence. We seek to approximate in a Γ-convergence sense the microscopic behavior of such materials by a macroscopic, or average, description. The asymptotic analysis of media with multiple scale of homogenization is referred to as *Reiterated Homogenization.*

Let Ω_ε denote the reference configuration of this elastic body and assume Ω_ε to be a bounded and open subset of $\mathbb{R}^N$ ($N \geqslant 1$). In the sequel we identify $\mathbb{R}^{d\times N}$ (resp. $\mathbb{Q}^{d\times N}$) with the space of real (resp. rational)-valued $d \times N$ matrices and Q will stand for the unit cube $(0,1)^N$ of $\mathbb{R}^N$.

To take into account the heterogeneous periodic structure of this material, we suppose that its stored energy density, given by a function $f : \Omega_\varepsilon \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$, is Q-periodic with respect to its second and third variables, and we treat the nonconvex case under standard growth and coercivity conditions of order p, with $1 < p < \infty$. Under a deformation $u : \Omega_\varepsilon \to \mathbb{R}^d$ the elastic energy of this body is given by the functional

$$\int_{\Omega_\varepsilon} f\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}; \nabla u(x)\right) dx. \tag{1.1}$$

Its dependence upon the small parameter ε allow us to consider materials whose microscopic heterogeneities scales like ε and ε^2. The generalization of this study to any number of scales $k \geqslant 2$ follows by an iterated argument similar to the one used in Braides and Defranceschi (see Remark 22.8 in [14]).

We will address two independent problems: In Section 2 we will analyze the behavior of bodies with periodic microstructure whose volume does not depend on ε, yielding to a reiterated homogenization problem (Figure 1). The originality of this part is that we do not require any uniform continuity hypotheses on the first and second variables of f, as it is customary in the existing literature (see references below).

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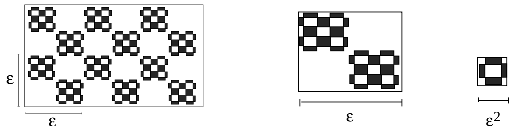

Figure 1: Example of a domain with periodic microstructure of order ε and ε^2

In Section 3 we consider three dimensional cylindrical bodies with similar periodic properties as before, whose thickness scales like ε, leading to a homogenization and dimension reduction problem. The main contribution here is that our arguments allow us to homogenize this material in the reducing direction.

More precisely, in Section 2 we describe the case where $\Omega_\varepsilon = \Omega$ and our family of energies is of the form

$$\int_\Omega f\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}; \nabla u(x)\right) dx. \tag{1.2}$$

This kind of asymptotic problems can be seen as a generalization of the *Iterated Homogenization Theorem* for linear integrands, proved by Bensoussan, Lions and Papanicolau [10], in which the homogenized operator is derived by a formal two-scale asymptotic expansion method. This result has been recovered in several ways via other types of convergence such as H-convergence, G-convergence, and multiscale convergence (see Lions, Lukkassen, Persson and Wall [28, 29] and references therein). In the framework of Γ-convergence, Braides and Lukkassen (see Theorem 1.1 in [15] and also [30]) investigated the nonlinear setting for integral functionals of the type

$$\int_\Omega f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}; \nabla u(x)\right) dx,$$

where the integrand $f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ is assumed to satisfy usual periodicity and growth conditions and

- $f(y, \cdot\,; \xi)$ is measurable for all $(y, \xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$;
- $f(y, z; \cdot)$ is continuous and convex for all $(y, z) \in \mathbb{R}^N \times \mathbb{R}^N$;
- there exist a locally integrable function b and a continuous positive real function ω, with $\omega(0) = 0$, such that
$$|f(y, z; \xi) - f(y', z; \xi)| \leqslant \omega(|y - y'|)\big[b(z) + f(y, z; \xi)\big] \tag{1.3}$$
for all y, y', $z \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{d\times N}$.

This result has been extended to the case of nonconvex integrands depending explicitly on the macroscopic variable x, as in (1.2), under the above strong uniform continuity condition (1.3) (see Theorem 22.1 and Remark 22.8 of Braides and Defranceschi [14]). Using techniques of multiscale convergence and restricting the argument to the convex and homogeneous case (no dependence on the variable x), Fonseca and Zappale were able to weaken the continuity condition (1.3). Namely, they only required f to be continuous (see Theorem 1.9 in [26]). Recently, in an independent work, Barchiesi [9] studied the Γ-limit of functionals of the type (1.2) in the convex case under weak regularity assumptions on f with respect to the oscillating variables.

All the above results share the same property: The homogenized functional, as it is referred in the literature for the Γ-limit of (1.2), is obtained by iterating twice the homogenization formula derived in the study of the Γ-limit of functionals of the type

$$\int_\Omega f\left(x, \frac{x}{\varepsilon}; \nabla u(x)\right) dx. \tag{1.4}$$

This can be seen from formula (14.12) in Braides and Defranceschi [14], or in Baía and Fonseca [8] where $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ is assumed to satisfy standard p-growth and p-coercivity conditions, Q-periodicity with respect to the oscillating variable, and

- $f(x,\cdot;\cdot)$ is continuous for a.e. $x \in \Omega$;
- $f(\cdot,y;\xi)$ is measurable for all $y \in \mathbb{R}^N$ and all $\xi \in \mathbb{R}^{d\times N}$

(see Theorem 1.1 in [8]; it will be of use for the proofs of Theorems 1.1 and 1.2 below).

As these previous results seem to show, it is not clear what is the natural regularity on f for the integral (1.2) to be well defined. Such problems have been discussed by Allaire (Section 5 in [1]). In particular, the measurability of the function $x \mapsto f(x, x/\varepsilon, x/\varepsilon^2; \xi)$ is ensured whenever f is continuous in its second and third variable. Following the lines of Baía and Fonseca [8] we will assume that

(H_1) $f(x,\cdot,\cdot;\cdot)$ is continuous for a.e. $x \in \Omega$;

(H_2) $f(\cdot,y,z;\xi)$ is measurable for all $(y,z,\xi) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$;

(H_3) $f(x,\cdot,z;\xi)$ is Q-periodic for all $(z,\xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and a.e. $x \in \Omega$, and $f(x,y,\cdot;\xi)$ is Q-periodic for all $(y,\xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and a.e. $x \in \Omega$;

(H_4) there exists $\beta > 0$ such that for all $(y,z,\xi) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and a.e. $x \in \Omega$

$$\frac{1}{\beta}|\xi|^p - \beta \leqslant f(x,y,z;\xi) \leqslant \beta(1+|\xi|^p).$$

From the applications point of view it would be interesting to consider functions that are continuous with respect to the first variable and only measurable with respect to some of the oscillating variables, as it is relevant, for instance, in the case of mixtures. Nevertheless, the arguments we use here do not allow us to treat this case.

In what follows we write $\Gamma(L^p(\Omega))$-limit whenever we refer to the Γ-convergence with respect to the usual metric in $L^p(\Omega;\mathbb{R}^d)$. The above considerations lead us to the main result of Section 2 that in particular recovers Theorem 1.9 in Fonseca and Zappale [26].

Theorem 1.1. *Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ be a function satisfying (H_1)-(H_4). For each $\varepsilon > 0$ define $\mathcal{F}_\varepsilon : L^p(\Omega;\mathbb{R}^d) \to \overline{\mathbb{R}}$ by*

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}; \nabla u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.5}$$

Then the $\Gamma(L^p(\Omega))$-limit of the family $\{\mathcal{F}_\varepsilon\}_{\varepsilon>0}$ is given by the functional

$$\mathcal{F}_{\rm hom}(u) := \begin{cases} \displaystyle\int_\Omega \overline{f}_{\rm hom}(x;\nabla u(x))\, dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\overline{f}_{\rm hom}$ is defined for all $\xi \in \mathbb{R}^{d\times N}$ and a.e. $x \in \Omega$ by

$$\overline{f}_{\rm hom}(x;\xi) := \lim_{T\to+\infty} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f_{\rm hom}(x,y;\xi + \nabla\phi(y))\, dy :\ \phi \in W_0^{1,p}((0,T)^N;\mathbb{R}^d) \right\}, \tag{1.6}$$

and

$$f_{\rm hom}(x,y;\xi) := \lim_{T\to+\infty} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(x,y,z;\xi + \nabla\phi(z))\, dz :\ \phi \in W_0^{1,p}((0,T)^N;\mathbb{R}^d) \right\} \tag{1.7}$$

for a.e. $x \in \Omega$ and all $(y,\xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$.

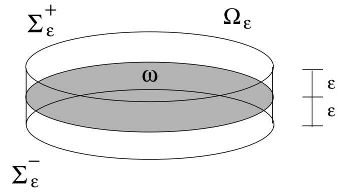

Figure 2: Cylindrical thin domain of thickness ε

As mentioned before one homogenizes first with respect to z, considering y as a parameter, and then one homogenizes with respect to y.

We remark that most of the proofs presented in this section follow the lines of the ones in Braides and Defranceschi [14] (Theorem 22.1 and Remark 22.8), and that our main contribution is to use arguments that allow us to weaken the strong uniform continuity hypothesis (1.3). Let us briefly describe how we proceed: The idea consists in proving the result for integrands which do not depend explicitly on x (see Theorem 2.2 in Subsection 2.2), and then to treat the general case by freezing this macroscopic variable (Subsection 2.3). To do this, we start by claiming that under hypotheses (H_1)-(H_4), f is uniformly continuous up to a small error. Indeed, since f is a Carathéodory integrand, Scorza-Dragoni's Theorem (see Ekeland and Temam [22]) implies that the restriction of f to $K\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^{d\times N}$ is continuous, for some compact set $K\subset\Omega$ whose complement has arbitrarily small Lebesgue measure. Then the periodicity of f with respect to its second and third variable leads f to be uniformly continuous on $K\times\mathbb{R}^N\times\mathbb{R}^N\times\overline{B}$, for some closed ball $\overline{B}$ of $\mathbb{R}^{d\times N}$ of sufficiently large radius. Finally, to ensure that the energy remains arbitrarily small on the complement of K and on the set of x's such that the gradient of the deformation does not belong to $\overline{B}$, we use the Decomposition Lemma (see Fonseca, Müller and Pedregal [25] or Fonseca and Leoni [24]) which allows us to select minimizing sequences with p-equi-integrable gradient. Thus, in view of the p-growth character of the integrand, the energy over sets of arbitrarily small Lebesgue measure tends to zero.

In Section 3 we consider the case where Ω_ε is a cylindrical thin domain of the form $\Omega_\varepsilon := \omega\times(-\varepsilon,\varepsilon)$ (Figure. 2), whose heterogeneities may depend periodically upon its thickness. We assume that its basis, ω, is a bounded open subset of $\mathbb{R}^2$ and we seek to characterize the behavior of the elastic energy (1.1) when ε tends to zero.

Two simultaneous features occur in this case: a reiterated homogenization and a dimension reduction process. As usual, in order to study this problem as $\varepsilon\to 0$ we rescale the ε-thin body into a reference domain of unit thickness (see e.g. Anzellotti, Baldo and Percivale [4], Le Dret and Raoult [27], and Braides, Fonseca and Francfort [16]), so that the resulting energy will be defined on a fixed body, while the dependence on ε turns out to be explicit in the transverse derivative. For this, we consider the change of variables

$$\Omega_\varepsilon\to\Omega:=\omega\times I,\quad (x_1,x_2,x_3)\mapsto\left(x_1,x_2,\frac{1}{\varepsilon}x_3\right),$$

and define $v(x_\alpha,x_3/\varepsilon)=u(x_\alpha,x_3)$ on the rescaled cylinder Ω, where $I:=(-1,1)$ and $x_\alpha:=(x_1,x_2)$ is the in-plane variable (Figure. 3).

In what follows we denote by $\nabla_i=\partial/\partial x_i$ for $i\in\{1,2,3\}$ and $\nabla_\alpha=(\nabla_1,\nabla_2)$. For all $\overline{\xi}=(z_1|z_2)\in\mathbb{R}^{3\times 2}$ and $z\in\mathbb{R}^3$, $(\overline{\xi}|z)$ is the matrix whose first two columns are z_1 and z_2 and whose last one is z. After replacing v by u in (1.1), changing variables and dividing by ε, our goal is to study the sequence of rescaled energies

$$\int_\Omega W\left(x,\frac{x}{\varepsilon},\frac{x_\alpha}{\varepsilon^2};\nabla_\alpha v(x)\middle|\frac{1}{\varepsilon}\nabla_3 v(x)\right)dx, \tag{1.8}$$

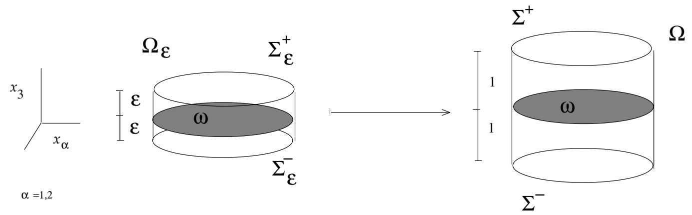

Figure 3: Rescaled domain of unit thickness

where to simplify notations we set

$$f(x, y, z; \xi) = W(x_\alpha, y_3, y_\alpha, z_3, z_\alpha; \xi) \tag{1.9}$$

for $W : \Omega \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times3} \to \mathbb{R}$ (or equivalently $W(x', y', z'_\alpha; \xi) = f(x'_\alpha, x_3, y'_\alpha, x'_3, z'_\alpha, y'_3; \xi)$).

Similar results have been already studied independently by Shu [34] (Theorem 3 (i)) and by Braides, Fonseca and Francfort [16] (Theorem 4.2) for energies of the type

$$\int_\Omega W\left(x_3, \frac{x_\alpha}{\varepsilon}; \nabla_\alpha v \Big| \frac{1}{\varepsilon}\nabla_3 v\right) dx$$

where $W : I \times \mathbb{R}^2 \times \mathbb{R}^{3\times3} \to \mathbb{R}$ is a Carathéodory integrand. Later Babadjian and Baía [6] (Theorem 1.2) treated the case where the integrand W depends also on the macroscopic in-plane variable x_α under measurability hypothesis with respect to $x = (x_\alpha, x_3)$, and a continuity requirement with respect to the oscillating variable. Integral functionals of the form

$$\int_\Omega W\left(\frac{x_\alpha}{\varepsilon^2}; \nabla_\alpha v \Big| \frac{1}{\varepsilon}\nabla_3 v\right) dx,$$

have been studied by Shu [34] (Theorem 5) under different length scales for the film thickness and the material microstructure. Homogenization in the transverse direction x_3 remained an open question and follows as a consequence of the main result of this section.

In the sequel we denote by $\mathcal{L}^N$ the N-dimensional Lebesgue measure in $\mathbb{R}^N$, by $Q' := (0,1)^2$ the unit cube in $\mathbb{R}^2$ and we will identify $W^{1,p}(\omega; \mathbb{R}^3)$ (resp. $L^p(\omega; \mathbb{R}^3)$) with those functions $u \in W^{1,p}(\Omega; \mathbb{R}^3)$ (resp. $L^p(\Omega; \mathbb{R}^3)$) such that $\nabla_3 u(x) = 0$ a.e. in Ω.

Following the lines of Babadjian and Baía [6] and Babadjian and Francfort [7] we assume that

(A_1) $W(x, \cdot\,, \cdot\,; \cdot)$ is continuous for a.e. $x \in \Omega$;

(A_2) $W(\cdot\,, y, z_\alpha; \xi)$ is measurable for all $(y, z_\alpha, \xi) \in \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times3}$;

$$(A_3) \quad \begin{cases} y_\alpha \mapsto W(x, y_\alpha, y_3, z_\alpha; \xi) \text{ is } Q'\text{-periodic for all } (z_\alpha, y_3, \xi) \in \mathbb{R}^3 \times \mathbb{R}^{3\times3} \text{ and a.e. } x \in \Omega, \\ (z_\alpha, y_3) \mapsto W(x, y_\alpha, y_3, z_\alpha; \xi) \text{ is } Q\text{-periodic for all } (y_\alpha, \xi) \in \mathbb{R}^2 \times \mathbb{R}^{3\times3} \text{ and a.e. } x \in \Omega; \end{cases}$$

(A_4) there exists $\beta > 0$ such that for all $(y, z_\alpha, \xi) \in \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times3}$ and a.e. $x \in \Omega$

$$\frac{1}{\beta}|\xi|^p - \beta \leqslant W(x, y, z_\alpha; \xi) \leqslant \beta(1 + |\xi|^p).$$

We prove the following theorem.

Theorem 1.2. *Let $W : \Omega \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ be a function satisfying (A_1)-(A_4). For each $\varepsilon > 0$, consider the functional $\mathcal{W}_\varepsilon : L^p(\Omega;\mathbb{R}^3) \to \overline{\mathbb{R}}$ defined by*

$$\mathcal{W}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega W\left(x, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}; \nabla_\alpha u(x)\Big|\frac{1}{\varepsilon}\nabla_3 u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^3),\\ +\infty & \text{otherwise.}\end{cases} \tag{1.10}$$

Then the $\Gamma(L^p(\Omega))$-limit of the family $\{\mathcal{W}_\varepsilon\}_{\varepsilon>0}$ is given by the functional

$$\mathcal{W}_{\rm hom}(u) := \begin{cases} \displaystyle 2\int_\omega \overline{W}_{\rm hom}(x_\alpha; \nabla_\alpha u(x_\alpha))\, dx_\alpha & \text{if } u \in W^{1,p}(\omega;\mathbb{R}^3),\\ +\infty & \text{otherwise,}\end{cases}$$

where $\overline{W}_{\rm hom}$ is defined, for all $\bar\xi \in \mathbb{R}^{3\times 2}$ and a.e. $x_\alpha \in \omega$, by

$$\begin{aligned}\overline{W}_{\rm hom}(x_\alpha;\bar\xi) \;:=\; \lim_{T\to+\infty} \inf_\phi \Bigg\{ &\frac{1}{2T^2}\int_{(0,T)^2\times I} W_{\rm hom}(x_\alpha, y_3, y_\alpha; \bar\xi + \nabla_\alpha\phi(y)|\nabla_3\phi(y))\, dy : \\ &\phi \in W^{1,p}((0,T)^2\times I;\mathbb{R}^3), \quad \phi = 0 \text{ on } \partial(0,T)^2 \times I \Bigg\}\end{aligned} \tag{1.11}$$

and

$$\begin{aligned}W_{\rm hom}(x, y_\alpha; \xi) \;:=\; \lim_{T\to+\infty} \inf_\phi \Bigg\{ &\frac{1}{T^3}\int_{(0,T)^3} W(x, y_\alpha, z_3, z_\alpha; \xi + \nabla\phi(z))\, dz : \\ &\phi \in W^{1,p}_0((0,T)^3;\mathbb{R}^3)\Bigg\},\end{aligned} \tag{1.12}$$

for a.e. $x \in \Omega$ and all $(y_\alpha, \xi) \in \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$.

Remark 1.3. It can be proved that the limits as $T \to +\infty$ in (1.11) and (1.12) can be replaced by an infimum taken for every $T > 0$ as in Braides and Defranceschi [14] or Baía and Fonseca [8].

Let us formally justify the periodicity assumptions (A_3): Since the volume of Ω_ε is of order ε and $\varepsilon^2 \ll \varepsilon$, in a first step, we can think of ε as being a fixed parameter and let ε^2 tend to zero. Then at this point dimension reduction is not occurring and (1.1) can be seen as a single one-scale homogenization problem as in (1.4), in which it is natural to assume $f(x_\alpha, y, \cdot;\xi)$, or equivalently, $(z_\alpha, y_3) \mapsto W(x, y_\alpha, y_3, z_\alpha;\xi)$ to be Q-periodic (see (1.9)). The homogenization formula for this case give us an homogenized stored energy density $W_{\rm hom}(x, y_\alpha;\xi)$ that, in a second step, is used as the integrand of a similar problem than the one treated in Babadjian and Baía [6]. In particular the required Q'- periodicity of $W_{\rm hom}(x,\cdot;\xi)$ can be obtained from the Q'-periodicity of $y_\alpha \mapsto W(x, y_\alpha, y_3, z_\alpha;\xi)$.

As before, we start by treating the case where the integrand does not depend explicitly on x (see Theorem 3.2 in Subsection 3.2). We observe that the proof of Theorem 3.2 is very close to the one of its N-dimensional analogue Theorem 2.2, the main difference being the use of the Scaled Gradients Decomposition Lemma derived by Bocea and Fonseca in place of the usual Decomposition Lemma (compare Theorem 1.1 of [11] with Fonseca and Leoni [24] or Fonseca, Müller and Pedregal [25]). As it has been noted in [6, 7], Theorem 1.2 and Theorem 1.1 cannot be treated similarly. In the former case we need to extend our Carathéodory integrands by a continuous function by means of Tietze's Extension Theorem (see e.g. DiBenedetto [20]). This argument was already used in Babadjian and Baía [6] and Babadjian and Francfort [7] where the authors used a weaker extension result (see Theorem 1, Section 1.2 in Evans-Gariepy [23] and Lemma 4.1 in [6]).

In the sequel, given $\lambda > 0$ we denote by $\overline{B}(0,\lambda)$ the closed ball of radius λ in $\mathbb{R}^{d\times N}$, that is the set $\{\xi \in \mathbb{R}^{d\times N} : |\xi| \leqslant \lambda\}$, and the letter C stands for a generic constant. Throughout the text $\lim\limits_{n,m} := \lim\limits_n \lim\limits_m$ while $\lim\limits_{m,n} := \lim\limits_m \lim\limits_n$ with obvious generalizations.

2 Reiterated homogenization

The main objective of this section is to prove Theorem 1.1. In Subsection 2.1 we state the main properties of $f_{\rm hom}$ and $\overline{f}_{\rm hom}$ that are basic for our analysis. In Subsection 2.2 we present some auxiliary results for the proof of the homogeneous counterpart of Theorem 1.1, Theorem 2.2, in which we assume that f does not depend explicitly on x. The proof of Theorem 1.1 in its fully generality is presented in Subsection 2.3. Finally, in Subsection 2.4 we remark an alternative proof for convex integrands.

Remark 2.1. In the sequel, and without loss of generality, we assume that f is non negative. Indeed, it suffices to replace f by $f+\beta$ which is non negative in view of (H_4).

As for notations $Q(a,\delta) := a+\delta(-1/2,1/2)^N$ (cube of center $a\in\mathbb{R}^N$ and edge length δ) and $Q:=(0,1)^N$ stands for the unit cube in $\mathbb{R}^N$.

2.1 Properties of $\overline{f}_{\rm hom}$

Repeating the argument used in Baía and Fonseca [8] (Theorem 1.1), we can see that the function $f_{\rm hom}$ given in (1.7) is well defined and it is (equivalent to) a Carathéodory function:

$$f_{\rm hom}(\cdot,\cdot;\xi) \text{ is } \mathcal{L}^N\otimes\mathcal{L}^N\text{-measurable for all } \xi\in\mathbb{R}^{d\times N}, \tag{2.1}$$

$$f_{\rm hom}(x,y;\cdot) \text{ is continuous for } \mathcal{L}^N\otimes\mathcal{L}^N\text{-a.e. } (x,y)\in\Omega\times\mathbb{R}^N. \tag{2.2}$$

By condition (H_3) it follows that

$$f_{\rm hom}(x,\cdot;\xi) \text{ is } Q\text{-periodic for a.e. } x\in\Omega \text{ and all } \xi\in\mathbb{R}^{d\times N}. \tag{2.3}$$

Moreover, $f_{\rm hom}$ is quasiconvex in the ξ variable and satisfies the same p-growth and p-coercivity condition (H_4) as f:

$$\frac{1}{\beta}|\xi|^p-\beta\leqslant f_{\rm hom}(x,y;\xi)\leqslant\beta(1+|\xi|^p) \quad \text{for a.e. } x\in\Omega \text{ and all } (y,\xi)\in\mathbb{R}^N\times\mathbb{R}^{d\times N}, \tag{2.4}$$

where β is the constant in (H_4).

As a consequence of (2.1), (2.2) and (2.4), the function $\overline{f}_{\rm hom}$ given in (1.6) is also well defined, and is (equivalent to) a Carathéodory function, which implies that the definition of $\mathcal{F}_{\rm hom}$ is well posed on $W^{1,p}(\Omega;\mathbb{R}^d)$.

Finally, $\overline{f}_{\rm hom}$ is also quasiconvex in the ξ variable and satisfies the same p-growth and p-coercivity condition (H_4) as f and $f_{\rm hom}$:

$$\frac{1}{\beta}|\xi|^p-\beta\leqslant \overline{f}_{\rm hom}(x;\xi)\leqslant\beta(1+|\xi|^p) \quad \text{for a.e. } x\in\Omega \text{ and all } \xi\in\mathbb{R}^{d\times N}, \tag{2.5}$$

where, as before, β is the constant in (H_4).

2.2 Independence of the macroscopic variable

We assume that f does not depend explicitly on x, namely $f:\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^{d\times N}\to\mathbb{R}^+$. In addition, according to (H_1) and (H_2), unless we specify the contrary, we assume f to be continuous and to satisfy hypotheses (H_3) and (H_4).

For each $\varepsilon>0$ consider the functional $\mathcal{F}_\varepsilon : L^p(\Omega;\mathbb{R}^d)\to[0,+\infty]$ defined by

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2};\nabla u(x)\right)dx & \text{if } u\in W^{1,p}(\Omega;\mathbb{R}^d),\\ +\infty & \text{otherwise.}\end{cases} \tag{2.6}$$

Our objective is to prove the following result.

Theorem 2.2. *Under the assumption that f is continuous and hypotheses (H_3) and (H_4) are satisfied the $\Gamma(L^p(\Omega))$-limit of the family $\{\mathcal{F}_\varepsilon\}_{\varepsilon>0}$ is given by*

$$\mathcal{F}_{\rm hom}(u) = \begin{cases} \displaystyle\int_\Omega \overline{f}_{\rm hom}(\nabla u(x))\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d),\\ +\infty & \text{otherwise,}\end{cases}$$

where $\overline{f}_{\rm hom}$ is defined by

$$\overline{f}_{\rm hom}(\xi) := \lim_{T\to+\infty}\inf_\phi \left\{\frac{1}{T^N}\int_{(0,T)^N} f_{\rm hom}(y;\xi+\nabla\phi(y))\,dy : \ \phi \in W^{1,p}_0((0,T)^N;\mathbb{R}^d)\right\} \tag{2.7}$$

for all $\xi \in \mathbb{R}^{d\times N}$, and where

$$f_{\rm hom}(y;\xi) := \lim_{T\to+\infty}\inf_\phi \left\{\frac{1}{T^N}\int_{(0,T)^N} f(y,z;\xi+\nabla\phi(z))\,dz : \ \phi \in W^{1,p}_0((0,T)^N;\mathbb{R}^d)\right\} \tag{2.8}$$

for all $(y,\xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$.

This result can be seen as a generalization of Theorem 1.9 in Fonseca and Zappale [26] (for $s = 1$), in which, as it is usual for the convex case, it is enough to consider variations that are periodic in the cell Q. Their multiscale argument (see Subsection 2.4 below) does not apply here since, as it is expected in the non convex case, the variations should be considered to be periodic over an infinite ensemble of cells, as it is seen from (2.7) and (2.8).

We start the proof of Theorem 2.2 by localizing the functionals given in (2.6) in order to highlight their dependence on the class of bounded, open subsets of $\mathbb{R}^N$, denoted by $\mathcal{A}_0$. As it will be clear from the proofs of Lemmas 2.8 and 2.9 below, it would not be sufficient to localize, as usual, on any open subset of Ω. Indeed, formulas (2.7) and (2.8) suggest to work in cubes of the type $(0,T)^N$, with T arbitrarily large, not necessarily contained in Ω.

For each $\varepsilon > 0$ consider $\mathcal{F}_\varepsilon : L^p(\mathbb{R}^N;\mathbb{R}^d)\times\mathcal{A}_0 \to [0,+\infty]$ defined by

$$\mathcal{F}_\varepsilon(u;A) := \begin{cases} \displaystyle\int_A f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2};\nabla u(x)\right)dx & \text{if } u \in W^{1,p}(A;\mathbb{R}^d),\\ +\infty & \text{otherwise.}\end{cases} \tag{2.9}$$

We will prove (Subsections 2.2.1 and 2.2.2 below) that the family of functionals $\{\mathcal{F}_\varepsilon(\cdot\,;A)\}_{\varepsilon>0}$, with $A \in \mathcal{A}_0$, Γ-converges with respect to the strong $L^p(A;\mathbb{R}^d)$-topology to the functional $\mathcal{F}_{\rm hom}(\cdot\,;A)$, where $\mathcal{F}_{\rm hom} : L^p(\mathbb{R}^N;\mathbb{R}^d)\times\mathcal{A}_0 \to [0,+\infty]$ is given by

$$\mathcal{F}_{\rm hom}(u;A) = \begin{cases} \displaystyle\int_A \overline{f}_{\rm hom}(\nabla u(x))\,dx & \text{if } u \in W^{1,p}(A;\mathbb{R}^d),\\ +\infty & \text{otherwise.}\end{cases}$$

As a consequence, taking $A = \Omega$ yields Theorem 2.2.

2.2.1 Existence and integral representation of the Γ-limit

Given $\{\varepsilon_j\} \searrow 0^+$ and $A \in \mathcal{A}_0$, consider the Γ-lower limit of $\{\mathcal{F}_{\varepsilon_j}(\cdot\,;A)\}_{j\in\mathbb{N}}$ for the $L^p(A;\mathbb{R}^d)$-topology defined for $u \in L^p(\mathbb{R}^N;\mathbb{R}^d)$ by

$$\mathcal{F}_{\{\varepsilon_j\}}(u;A) := \inf_{\{u_j\}}\left\{\liminf_{j\to+\infty}\mathcal{F}_{\varepsilon_j}(u_j;A) : \quad u_j \to u \text{ in } L^p(A;\mathbb{R}^d)\right\}.$$

In view of the p-coercivity condition (H_4) it follows that $\mathcal{F}_{\{\varepsilon_j\}}(u;A)$ is infinite whenever $u \in L^p(\mathbb{R}^N;\mathbb{R}^d) \setminus W^{1,p}(A;\mathbb{R}^d)$ for each $A \in \mathcal{A}_0$, so it suffices to study the case where $u \in W^{1,p}(A;\mathbb{R}^d)$.

The following result states the existence of Γ-convergent subsequences and can be proved by classical arguments on the lines of those of Braides, Fonseca and Francfort [16].

Lemma 2.3. *For any sequence $\{\varepsilon_j\} \searrow 0^+$, there exists a subsequence $\{\varepsilon_n\} \equiv \{\varepsilon_{j_n}\}$ such that $\mathcal{F}_{\{\varepsilon_n\}}(u;A)$ is the $\Gamma(L^p(A))$-limit of $\{\mathcal{F}_{\varepsilon_n}(u;A)\}_{n\in\mathbb{N}}$ for all $A \in \mathcal{A}_0$ and all $u \in W^{1,p}(A;\mathbb{R}^d)$.*

Our goal is to study the behavior of $\mathcal{F}_{\{\varepsilon_n\}}(u;\cdot)$ in $\mathcal{A}(A)$ (family of open subsets of A) for each $u \in W^{1,p}(A;\mathbb{R}^d)$. Following the proof of Lemma 2.10 in Baía and Fonseca [8], it is possible to show that $\mathcal{F}_{\{\varepsilon_n\}}(u;\cdot)$ is a measure on $\mathcal{A}(A)$ for all $A \in \mathcal{A}_0$. Namely, the following result holds.

Lemma 2.4. *For each $A \in \mathcal{A}_0$ and all $u \in W^{1,p}(A;\mathbb{R}^d)$, the restriction of $\mathcal{F}_{\{\varepsilon_n\}}(u;\cdot)$ to $\mathcal{A}(A)$ is a Radon measure, absolutely continuous with respect to the N-dimensional Lebesgue measure.*

For the moment, we are not in position to apply Buttazzo-Dal Maso Integral Representation Theorem (see Theorem 1.1 in [17]) because, a priori, the integrand would depend on the open set $A \in \mathcal{A}_0$. The following result prevents this dependence from holding since it leads to an homogeneous integrand as it will be seen in Lemma 2.6 below.

Lemma 2.5. *For all $\xi \in \mathbb{R}^{d\times N}$, y_0 and $z_0 \in \mathbb{R}^N$, and $\delta > 0$*

$$\mathcal{F}_{\{\varepsilon_n\}}(\xi\,\cdot\,;Q(y_0,\delta)) = \mathcal{F}_{\{\varepsilon_n\}}(\xi\,\cdot\,;Q(z_0,\delta)).$$

Proof. Clearly, it suffices to establish the inequality

$$\mathcal{F}_{\{\varepsilon_n\}}(\xi\,\cdot\,;Q(y_0,\delta)) \geqslant \mathcal{F}_{\{\varepsilon_n\}}(\xi\,\cdot\,;Q(z_0,\delta)).$$

Let $\{w_n\} \subset W_0^{1,p}(Q(y_0,\delta);\mathbb{R}^d)$, with $w_n \to 0$ in $L^p(Q(y_0,\delta);\mathbb{R}^d)$, be such that

$$\mathcal{F}_{\{\varepsilon_n\}}(\xi\cdot\,;Q(y_0,\delta)) = \lim_{n\to+\infty} \int_{Q(y_0,\delta)} f\Big(\frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi + \nabla w_n(x)\Big)\, dx$$

(see e.g. Proposition 11.7 in Braides and Defranceschi [14]). By hypothesis (H_4) and the Poincaré Inequality, we can suppose that the sequence $\{w_n\}$ is uniformly bounded in $W^{1,p}(Q(y_0,\delta);\mathbb{R}^d)$. Thus by the Decomposition Lemma (see Fonseca and Leoni [24] or Fonseca, Müller and Pedregal [25]), there exists a subsequence of $\{w_n\}$ (still denoted by $\{w_n\}$) and a sequence $\{u_n\} \subset W_0^{1,\infty}(Q(y_0,\delta);\mathbb{R}^d)$ such that $u_n \rightharpoonup 0$ in $W^{1,p}(Q(y_0,\delta);\mathbb{R}^d)$,

$$\{|\nabla u_n|^p\} \text{ is equi-integrable} \tag{2.10}$$

and

$$\mathcal{L}^N(\{y \in Q(y_0,\delta) : \quad u_n(y) \neq w_n(y)\}) \to 0. \tag{2.11}$$

Then, in view of (2.10), (2.11) and the p-growth condition (H_4),

$$\begin{aligned} \mathcal{F}_{\{\varepsilon_n\}}(\xi\cdot\,;Q(y_0,\delta)) &\geqslant \limsup_{n\to+\infty} \int_{Q(y_0,\delta)\cap\{u_n=w_n\}} f\Big(\frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi + \nabla u_n(x)\Big)\, dx \\ &= \limsup_{n\to+\infty} \int_{Q(y_0,\delta)} f\Big(\frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi + \nabla u_n(x)\Big)\, dx \end{aligned} \tag{2.12}$$

For all $n \in \mathbb{N}$ we write $\frac{y_0 - z_0}{\varepsilon_n} = m_{\varepsilon_n} + s_{\varepsilon_n}$ with $m_{\varepsilon_n} \in \mathbb{Z}^N$ and $s_{\varepsilon_n} \in [0,1)^N$,

$$\frac{m_{\varepsilon_n}}{\varepsilon_n} = \theta_{\varepsilon_n} + l_{\varepsilon_n} \tag{2.13}$$

with $\theta_{\varepsilon_n} \in \mathbb{Z}^N$ and $l_{\varepsilon_n} \in [0,1)^N$, and we define

$$x_{\varepsilon_n} := m_{\varepsilon_n}\varepsilon_n - \varepsilon_n^2 l_{\varepsilon_n}. \tag{2.14}$$

Note that $x_{\varepsilon_n} = y_0 - z_0 - \varepsilon_n s_{\varepsilon_n} - \varepsilon_n^2 l_{\varepsilon_n} \to y_0 - z_0$ as $n \to +\infty$. For all $n \in \mathbb{N}$, extend u_n by zero to the whole $\mathbb{R}^N$ and set $v_n(x) = u_n(x + x_{\varepsilon_n})$ for $x \in Q(z_0,\delta)$. Then $\{v_n\} \subset W^{1,p}(Q(z_0,\delta);\mathbb{R}^d)$, $v_n \to 0$ in $L^p(Q(z_0,\delta);\mathbb{R}^d)$, and the sequence $\{|\nabla v_n|^p\}$ is equi-integrable by the translation invariance of the Lebesgue measure. In view of (2.12), (2.13), (2.14) and (H_3),

$$\begin{aligned}
\mathcal{F}_{\{\varepsilon_n\}}(\xi\cdot\,;Q(y_0,\delta)) &\geqslant \limsup_{n\to+\infty}\int_{Q(y_0-x_{\varepsilon_n},\delta)} f\Big(\frac{x+x_{\varepsilon_n}}{\varepsilon_n},\frac{x+x_{\varepsilon_n}}{\varepsilon_n^2};\xi+\nabla u_n(x+x_{\varepsilon_n})\Big)\,dx\\
&= \limsup_{n\to+\infty}\int_{Q(y_0-x_{\varepsilon_n},\delta)} f\Big(\frac{x}{\varepsilon_n}-\varepsilon_n l_{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla v_n(x)\Big)\,dx\\
&\geqslant \limsup_{n\to+\infty}\int_{Q(z_0,\delta)} f\Big(\frac{x}{\varepsilon_n}-\varepsilon_n l_{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla v_n(x)\Big)\,dx\\
&\qquad -\limsup_{n\to+\infty}\int_{Q(z_0,\delta)\setminus Q(y_0-x_{\varepsilon_n},\delta)} f\Big(\frac{x}{\varepsilon_n}-\varepsilon_n l_{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla v_n(x)\Big)\,dx.
\end{aligned}$$

Since $v_n \equiv 0$ outside $Q(y_0 - x_{\varepsilon_n},\delta)$, the p-growth condition (H_4) and the fact that $\mathcal{L}^N(Q(z_0,\delta)\setminus Q(y_0 - x_{\varepsilon_n},\delta)) \to 0$ yield

$$\begin{aligned}
&\limsup_{n\to+\infty}\int_{Q(z_0,\delta)\setminus Q(y_0-x_{\varepsilon_n},\delta)} f\Big(\frac{x}{\varepsilon_n}-\varepsilon_n l_{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla v_n(x)\Big)\,dx\\
&\qquad\qquad \leqslant \limsup_{n\to+\infty}\beta(1+|\xi|^p)\mathcal{L}^N(Q(z_0,\delta)\setminus Q(y_0-x_{\varepsilon_n},\delta)) = 0,
\end{aligned}$$

and therefore

$$\mathcal{F}_{\{\varepsilon_n\}}(\xi\cdot\,;Q(y_0,\delta)) \geqslant \limsup_{n\to+\infty}\int_{Q(z_0,\delta)} f\Big(\frac{x}{\varepsilon_n}-\varepsilon_n l_{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla v_n(x)\Big)\,dx. \tag{2.15}$$

To eliminate the term $\varepsilon_n\, l_{\varepsilon_n}$ in (2.15), and thus to recover $\mathcal{F}_{\{\varepsilon_n\}}(\xi\cdot\,;Q(z_0,\delta))$, we would like to apply a uniform continuity argument. Since f is continuous on $\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^{d\times N}$ and separately Q-periodic with respect to its two first variables, by hypothesis (H_3), then f is uniformly continuous on $\mathbb{R}^N\times\mathbb{R}^N\times\overline{B}(0,\lambda)$ for any $\lambda > 0$. We define

$$R_n^\lambda := \{x \in Q(z_0,\delta) : |\xi+\nabla v_n(x)| \leqslant \lambda\},$$

and we note that by Chebyshev's inequality

$$\mathcal{L}^N(Q(z_0,\delta)\setminus R_n^\lambda) \leqslant C/\lambda^p, \tag{2.16}$$

for some constant $C > 0$ independent of λ or n. Thus, in view of (2.15) and the fact that f is nonnegative,

$$\mathcal{F}_{\{\varepsilon_n\}}(\xi\cdot\,;Q(y_0,\delta)) \geqslant \limsup_{\lambda,n}\int_{R_n^\lambda} f\Big(\frac{x}{\varepsilon_n}-\varepsilon_n l_{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla v_n(x)\Big)\,dx.$$

Denoting by $\omega_\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ the modulus of continuity of f on $\mathbb{R}^N\times\mathbb{R}^N\times\overline{B}(0,\lambda)$, we get that for any $x \in R_n^\lambda$

$$\left|f\Big(\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla v_n(x)\Big) - f\Big(\frac{x}{\varepsilon_n}-\varepsilon_n l_{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla v_n(x)\Big)\right| \leqslant \omega_\lambda(\varepsilon_n l_{\varepsilon_n}).$$

Then, the continuity of ω_λ and the fact that $\omega_\lambda(0) = 0$ yield

$$\begin{aligned}
\mathcal{F}_{\{\varepsilon_n\}}(\xi\cdot\,;Q(y_0,\delta)) &\geqslant \limsup_{\lambda,n}\left\{\int_{R_n^\lambda} f\Big(\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla v_n(x)\Big)\,dx - \delta^N\omega_\lambda(\varepsilon_n l_{\varepsilon_n})\right\}\\
&= \limsup_{\lambda,n}\int_{R_n^\lambda} f\Big(\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla v_n(x)\Big)\,dx.
\end{aligned}$$

The equi-integrability of $\{|\nabla v_n|^p\}$, the p-growth condition (H_4) and (2.16), imply that

$$\limsup_{\lambda,n}\int_{Q(z_0,\delta)\setminus R_n^\lambda} f\Big(\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla v_n(x)\Big)\,dx \leqslant \beta\limsup_{\lambda\to+\infty}\sup_{n\in\mathbb{N}}\int_{Q(z_0,\delta)\setminus R_n^\lambda}(1+|\nabla v_n(x)|^p)dx=0,$$

and since $v_n\to 0$ in $L^p(Q(z_0,\delta);\mathbb{R}^d)$,

$$\begin{aligned}\mathcal{F}_{\{\varepsilon_n\}}(\xi\cdot\,;Q(y_0,\delta)) &\geqslant \limsup_{n\to+\infty}\int_{Q(z_0,\delta)} f\Big(\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla v_n(x)\Big)\,dx\\ &\geqslant \mathcal{F}_{\{\varepsilon_n\}}(\xi\cdot\,;Q(z_0,\delta)).\end{aligned}$$

■

As a consequence of this lemma, we derive the following result.

Lemma 2.6. *There exists a continuous function $f_{\{\varepsilon_n\}}:\mathbb{R}^{d\times N}\to\mathbb{R}^+$ such that for all $A\in\mathcal{A}_0$ and all $u\in W^{1,p}(A;\mathbb{R}^d)$,*

$$\mathcal{F}_{\{\varepsilon_n\}}(u;A)=\int_A f_{\{\varepsilon_n\}}(\nabla u(x))\,dx.$$

Proof. Let $A\in\mathcal{A}_0$. By Buttazzo-Dal Maso Integral Representation Theorem (Theorem 1.1 in [17]), there exists a Carathéodory function $f^A_{\{\varepsilon_n\}}:A\times\mathbb{R}^{d\times N}\to\mathbb{R}^+$ satisfying

$$\mathcal{F}_{\{\varepsilon_n\}}(u;U)=\int_U f^A_{\{\varepsilon_n\}}(x;\nabla u(x))\,dx$$

for all $U\in\mathcal{A}(A)$ and all $u\in W^{1,p}(U;\mathbb{R}^d)$. Furthermore, for a.e. $x\in A$ and all $\xi\in\mathbb{R}^{d\times N}$

$$f^A_{\{\varepsilon_n\}}(x;\xi)=\lim_{\delta\to0}\frac{\mathcal{F}_{\{\varepsilon_n\}}(\xi\cdot\,;Q(x,\delta))}{\delta^N}.$$

Define $f_{\{\varepsilon_n\}}:\mathbb{R}^{d\times N}\to\mathbb{R}^+$ by

$$f_{\{\varepsilon_n\}}(\xi)=\lim_{\delta\to0}\frac{\mathcal{F}_{\{\varepsilon_n\}}(\xi\cdot\,;Q(0,\delta))}{\delta^N}.$$

As a consequence of Lemma 2.5, $f^A_{\{\varepsilon_n\}}(x;\xi)=f_{\{\varepsilon_n\}}(\xi)$ for a.e. $x\in A$ and for all $\xi\in\mathbb{R}^{d\times N}$. Then

$$\mathcal{F}_{\{\varepsilon_n\}}(u;A)=\int_A f_{\{\varepsilon_n\}}(\nabla u(x))\,dx$$

for all $u\in W^{1,p}(A;\mathbb{R}^d)$. ■

2.2.2 Characterization of the Γ-limit

Our next objective is to show that $\mathcal{F}_{\{\varepsilon_n\}}(u;A)=\mathcal{F}_{\rm hom}(u;A)$ for any $A\in\mathcal{A}_0$ and all $u\in W^{1,p}(A;\mathbb{R}^d)$. In view of Lemma 2.6, we only need to prove that $\overline{f}_{\rm hom}(\xi)=f_{\{\varepsilon_n\}}(\xi)$ for all $\xi\in\mathbb{R}^{d\times N}$, and thus it suffices to work with affine functions instead of general Sobolev functions. In order to estimate $f_{\{\varepsilon_n\}}$ from below in terms of $\overline{f}_{\rm hom}$, we will need the following result, close in spirit to Proposition 22.4 in Braides and Defranceschi [14].

Proposition 2.7. *Let $f:\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^{d\times N}\to\mathbb{R}^+$ be a (not necessarily continuous) function such that $f(x,\cdot\,;\cdot)$ is continuous, $f(\cdot\,,y;\xi)$ is measurable, and (H_3) and (H_4) hold. Let A be an open, bounded, connected and Lipschitz subset of $\mathbb{R}^N$. Given M and η two positive numbers, and $\varphi:[0,+\infty)\to[0,+\infty]$ a* *continuous and increasing function satisfying* $\varphi(t)/t \to +\infty$ *as* $t \to +\infty$*, there exists* $\varepsilon_0 \equiv \varepsilon_0(\varphi, M, \eta) > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$*, every* $a \in \mathbb{R}^N$ *and every* $u \in W^{1,p}(a + A; \mathbb{R}^d)$ *with*

$$\int_{a+A} \varphi(|\nabla u|^p)\, dx \leqslant M,$$

there exists $v \in W_0^{1,p}(a + A; \mathbb{R}^d)$ *with* $\|v\|_{L^p(a+A;\mathbb{R}^d)} \leqslant \eta$ *satisfying*

$$\int_{a+A} f\Big(x, \frac{x}{\varepsilon}; \nabla u\Big)\, dx \geqslant \int_{a+A} f_{\hom}(x; \nabla u + \nabla v)\, dx - \eta.$$

Proof. The proof is divided into two steps. First, we prove this proposition under the additional hypothesis that a belong to a compact set of $\mathbb{R}^N$. Then, we conclude the result in its full generality replacing a by its decimal part $a - [\![a]\!]$ and using the periodicity of the integrands f and $f_{\hom}$.

Step 1. For $a \in [-1, 1]^N$, the claim of Proposition 2.7 holds. Indeed, if not we may find φ, M and η as above, and sequences $\{\varepsilon_n\} \to 0^+$, $\{a_n\} \subset [-1, 1]^N$ and $\{u_n\} \subset W^{1,p}(a_n + A; \mathbb{R}^d)$ with

$$\int_{a_n+A} \varphi(|\nabla u_n|^p)\, dx \leqslant M \tag{2.17}$$

such that, for every $n \in \mathbb{N}$

$$\int_{a_n+A} f\left(x, \frac{x}{\varepsilon_n}; \nabla u_n\right) dx < \inf_{v \in W_0^{1,p}(a_n+A;\mathbb{R}^d)} \left\{ \int_{a_n+A} f_{\hom}(x; \nabla u_n + \nabla v)\, dx : \|v\|_{L^p(a_n+A;\mathbb{R}^d)} \leqslant \eta \right\} - \eta. \tag{2.18}$$

From (2.17) and the Poincaré-Wirtinger Inequality, up to a translation argument, we can suppose that the sequence $\{\|u_n\|_{W^{1,p}(a_n+A;\mathbb{R}^d)}\}$ is uniformly bounded. From this fact and since the set $a_n + A$ is an extension domain, there is no loss of generality in assuming that $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N; \mathbb{R}^d)$ and that, due to (2.17),

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} \varphi(|\nabla u_n|^p)\, dx \leqslant M_1 \tag{2.19}$$

for some constant $M_1 > 0$ depending only on M (see the proof of the Extension Theorem for Sobolev functions, Theorem 1, Section 4.4 in Evans and Gariepy [23]). Passing to a subsequence, we can also assume that $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N; \mathbb{R}^d)$. Let B be a ball of sufficiently large radius such that $a_n + A \subset B$ for all $n \in \mathbb{N}$. De La Vallée Poussin criterion (see e.g. Proposition 1.27 in Ambrosio, Fusco and Pallara [3]) and (2.19) guarantee that the sequence $\{|\nabla u_n|^p\}$ is equi-integrable on B. This implies that there exists $\delta = \delta(\eta)$ such that

$$\sup_{n \in \mathbb{N}} \beta \int_E (1 + |\nabla u|^p + |\nabla u_n|^p)\, dx \leqslant \frac{\eta}{2} \tag{2.20}$$

whenever E is a measurable subset of B satisfying $\mathcal{L}^N(E) \leqslant \delta$ and where β is the constant given in (H_4). As $\{a_n\} \subset [-1, 1]^N$ we may suppose, without loss of generality, that $a_n \to a \in [-1, 1]^N$, and that for fixed $0 < \rho < 1$, with $\rho^N \ll \delta$, the following hold for n large enough:

$$\begin{cases} a + (1-\rho)A \subset a_n + A \subset a + (1+\rho)A, \\ \mathcal{L}^N(S_n) \leqslant \delta, \text{ where } S_n := [a_n + A] \setminus [a + (1-\rho)A] \subset B, \\ \text{and } \ \|u_n - u\|_{L^p(a+(1+\rho)A;\mathbb{R}^d)} \leqslant \eta. \end{cases} \tag{2.21}$$

Take now a sequence of cut-off functions $\varphi_n \in \mathcal{C}_c^\infty(\mathbb{R}^N; [0, 1])$ such that

$$\varphi_n = \begin{cases} 1 & \text{on } a + (1-\rho)A, \\ 0 & \text{outside } a_n + A, \end{cases}$$

and $\|\nabla\varphi_n\|_{L^\infty(\mathbb{R}^N)} \leqslant C/\rho$ for some constant $C > 0$. Let $w_n = \varphi_n u + (1-\varphi_n)u_n$. Then $w_n - u_n \in W_0^{1,p}(a_n + A;\mathbb{R}^d)$ and

$$\int_{a_n+A} |w_n - u_n|^p\, dx \leqslant \int_{a_n+A} \varphi_n |u-u_n|^p\, dx \leqslant \int_{a+(1+\rho)A} |u-u_n|^p\, dx \leqslant \eta^p.$$

Then, taking $v := w_n - u_n$ as test function in (2.18), it follows from (2.4), (2.20), and (2.21) that

$$\begin{aligned}\int_{a_n+A} f\left(x, \frac{x}{\varepsilon_n}; \nabla u_n\right) dx &< \int_{a_n+A} f_{\hom}(x;\nabla w_n)\, dx - \eta \\ &\leqslant \int_{a+(1-\rho)A} f_{\hom}(x;\nabla u)\, dx \\ &\qquad + \beta \int_{S_n} \left(1 + |\nabla u|^p + |\nabla u_n|^p + \frac{C}{\rho}|u-u_n|^p\right) dx - \eta \\ &\leqslant \int_{a+(1-\rho)A} f_{\hom}(x;\nabla u)\, dx - \frac{\eta}{2} + \frac{\beta\, C}{\rho}\int_{S_n} |u-u_n|^p\, dx. \qquad (2.22)\end{aligned}$$

Since $u_n \to u$ in $L^p(\mathbb{R}^N;\mathbb{R}^d)$, by (2.21) and (2.22) we have

$$\limsup_{n\to+\infty} \int_{a_n+A} f\left(x, \frac{x}{\varepsilon_n}; \nabla u_n\right) dx \leqslant \int_{a+(1-\rho)A} f_{\hom}(x;\nabla u)\, dx - \frac{\eta}{2}, \qquad (2.23)$$

and as $u_n \rightharpoonup u$ in $W^{1,p}(a + (1-\rho)A;\mathbb{R}^d)$, by Theorem 1.1 in Baía and Fonseca [8] and (2.23), we get

$$\begin{aligned}\int_{a+(1-\rho)A} f_{\hom}(x;\nabla u)\, dx &\leqslant \liminf_{n\to+\infty} \int_{a+(1-\rho)A} f\left(x, \frac{x}{\varepsilon_n}; \nabla u_n\right) dx \\ &\leqslant \liminf_{n\to+\infty} \int_{a_n+A} f\left(x, \frac{x}{\varepsilon_n}; \nabla u_n\right) dx \\ &\leqslant \int_{a+(1-\rho)A} f_{\hom}(x;\nabla u)\, dx - \frac{\eta}{2}\end{aligned}$$

which is a contradiction.

Step 2. (General case) Let $a \in \mathbb{R}^N$. Then $a - [\![a]\!] \in [-1,1]^N$. Given $u \in W^{1,p}(a + A;\mathbb{R}^d)$, set $\tilde{u}(x) := u(x + [\![a]\!])$ and thus $\tilde{u} \in W^{1,p}(a - [\![a]\!] + A;\mathbb{R}^d)$. Applying Step 1 with $\eta/3$, we get the existence of $0 < \varepsilon_0' \equiv \varepsilon_0'(M,\varphi,\eta)$ such that, for all $0 < \varepsilon < \varepsilon_0'$, there exist $\tilde{v} \in W_0^{1,p}(a - [\![a]\!] + A;\mathbb{R}^d)$ satisfying $\|\tilde{v}\|_{L^p(a-[\![a]\!]+A;\mathbb{R}^d)} \leqslant \eta/3$ and

$$\int_{a-[\![a]\!]+A} f\left(x, \frac{x}{\varepsilon}; \nabla\tilde{u}(x)\right) dx \geqslant \int_{a-[\![a]\!]+A} f_{\hom}(x;\nabla\tilde{u}(x) + \nabla\tilde{v}(x))\, dx - \frac{\eta}{3}.$$

Setting $v(x) := \tilde{v}(x - [\![a]\!])$, then $v \in W_0^{1,p}(a + A;\mathbb{R}^d)$ and $\|v\|_{L^p(a+A;\mathbb{R}^d)} \leqslant \frac{\eta}{3} \leqslant \eta$. Therefore, by a change of variables

$$\int_{a+A} f\left(x, \frac{x-[\![a]\!]}{\varepsilon}; \nabla u(x)\right) dx \geqslant \int_{a+A} f_{\hom}(x;\nabla u(x) + \nabla v(x))\, dx - \frac{\eta}{3}, \qquad (2.24)$$

where we have used condition (H_3) and (2.3). Writing

$$\frac{[\![a]\!]}{\varepsilon} =: m_\varepsilon + r_\varepsilon, \qquad \text{with } m_\varepsilon \in \mathbb{Z}^N \text{ and } |r_\varepsilon| < \sqrt{N}\varepsilon$$

by (H_3) the inequality (2.24) reduces to

$$\int_{a+A} f\left(x, \frac{x}{\varepsilon} - r_\varepsilon; \nabla u(x)\right) dx \geqslant \int_{a+A} f_{\hom}(x;\nabla u(x) + \nabla v(x))\, dx - \frac{\eta}{3}. \qquad (2.25)$$

Choose $\lambda > 0$ large enough (depending on η) so that

$$\beta \int_{\{|\nabla u|>\lambda\}\cap[a+A]} (1+|\nabla u|^p)\, dx \leqslant \frac{\eta}{6}. \tag{2.26}$$

Fixed $\rho > 0$, by Scorza-Dragoni's Theorem there exists a compact set $K_\rho \subset a+A$ with $\mathcal{L}^N([a+A]\setminus K_\rho) \leqslant \rho$ such that $f : K_\rho \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ is continuous. Take $\rho \equiv \rho(\eta)$ small enough as for

$$\beta \int_{[a+A]\setminus K_\rho} (1+|\nabla u|^p)\, dx \leqslant \frac{\eta}{6}. \tag{2.27}$$

Then, from (2.26), (2.27) and the p-growth condition (H_4)

$$\int_{a+A} f\left(x, \frac{x}{\varepsilon} - r_\varepsilon; \nabla u(x)\right) dx \leqslant \int_{\{|\nabla u|\leqslant\lambda\}\cap K_\rho} f\left(x, \frac{x}{\varepsilon} - r_\varepsilon; \nabla u(x)\right) dx + \frac{\eta}{3}. \tag{2.28}$$

But since f is Q-periodic in its second variable, then f is uniformly continuous on $K_\rho \times \mathbb{R}^N \times \overline{B}(0,\lambda)$. Thus, as $r_\varepsilon \to 0$, for any $\eta > 0$ there exists $\varepsilon_0'' \equiv \varepsilon_0''(\eta) > 0$ such that for all $\varepsilon < \varepsilon_0''$ and all $x \in \{|\nabla u| \leqslant \lambda\} \cap K_\rho$,

$$\left| f\left(x, \frac{x}{\varepsilon} - r_\varepsilon; \nabla u(x)\right) - f\left(x, \frac{x}{\varepsilon}; \nabla u(x)\right)\right| < \frac{\eta}{3\mathcal{L}^N(A)}.$$

Hence

$$\int_{\{|\nabla u|\leqslant\lambda\}\cap K_\rho} f\left(x, \frac{x}{\varepsilon} - r_\varepsilon; \nabla u(x)\right) dx \leqslant \int_{\{|\nabla u|\leqslant\lambda\}\cap K_\rho} f(x, \frac{x}{\varepsilon}, \nabla u)\, dx + \frac{\eta}{3}, \tag{2.29}$$

and consequently, by (2.25), (2.28) and (2.29), for all $\varepsilon < \varepsilon_0 := \min\{\varepsilon_0', \varepsilon_0''\}$ we have

$$\int_{a+A} f\left(x, \frac{x}{\varepsilon}; \nabla u(x)\right) dx \geqslant \int_{a+A} f_{\rm hom}(x; \nabla u(x) + \nabla v(x))\, dx - \eta.$$

■

We are now in position to prove that $f_{\{\varepsilon_n\}} = \overline{f}_{\rm hom}$.

Lemma 2.8. *For all $\xi \in \mathbb{R}^{d\times N}$, $\overline{f}_{\rm hom}(\xi) \leqslant f_{\{\varepsilon_n\}}(\xi)$.*

Proof. By Lemma 2.6, given $\xi \in \mathbb{R}^{d\times N}$ there exists a sequence $\{w_n\} \subset W_0^{1,p}(Q;\mathbb{R}^d)$ such that $w_n \to 0$ in $L^p(Q;\mathbb{R}^d)$ and

$$f_{\{\varepsilon_n\}}(\xi) = \lim_{n\to+\infty} \int_Q f\left(\frac{x}{\varepsilon_n}, \frac{x}{{\varepsilon_n}^2}; \xi + \nabla w_n(x)\right) dx \tag{2.30}$$

(see e.g. Proposition 11.7 in Braides and Defranceschi [14]). Following the same argument as in Lemma 2.5, by the Decomposition Lemma, there is no loss of generality in assuming that $\{|\nabla w_n|^p\}$ is equi-integrable. Thus, from De La Vallée Poussin criterion (see e.g. Proposition 1.27 in Ambrosio, Fusco and Pallara [3]) there exists an increasing continuous function $\varphi : [0,+\infty) \to [0,+\infty]$ satisfying $\varphi(t)/t \to +\infty$ as $t \to +\infty$ and such that

$$\sup_{n\in\mathbb{N}} \int_Q \varphi(|\nabla w_n|^p)\, dx \leqslant 1.$$

Changing variables (2.30) yields

$$f_{\{\varepsilon_n\}}(\xi) = \lim_{n\to+\infty} \frac{1}{T_n^N} \int_{(0,T_n)^N} f\left(x, \frac{x}{\varepsilon_n}; \xi + \nabla z_n(x)\right) dx$$

and

$$\sup_{n\in\mathbb{N}} \frac{1}{T_n^N} \int_{(0,T_n)^N} \varphi(|\nabla z_n|^p)\, dx \leqslant 1, \tag{2.31}$$

where we set $T_n := 1/\varepsilon_n$ and $z_n(x) := T_n w_n(x/T_n)$ with $z_n \in W_0^{1,p}((0,T_n)^N;\mathbb{R}^d)$. For any $n \in \mathbb{N}$ define $I_n := \left\{1, ..., [\![T_n]\!]^N\right\}$, and for $i \in I_n$ take $a_i^n \in \mathbb{Z}^N$ such that

$$\bigcup_{i\in I_n} (a_i^n + Q) \subseteq (0,T_n)^N. \tag{2.32}$$

Thus

$$f_{\{\varepsilon_n\}}(\xi) \geqslant \limsup_{n\to+\infty} \frac{1}{T_n^N} \sum_{i\in I_n} \int_{a_i^n+Q} f\left(x, \frac{x}{\varepsilon_n}; \xi + \nabla z_n(x)\right) dx. \tag{2.33}$$

Let $M > 2$ and $\eta > 0$. For $n \in \mathbb{N}$ define

$$I_n^M := \left\{ i \in I_n : \int_{a_i^n+Q} \varphi(|\nabla z_n|^p)\, dx \leqslant M \right\}.$$

We note that for any $M > 2$, there exists $n(M) \in \mathbb{N}$ such that for all $n \geqslant n(M)$ sufficiently large $T_n > M$ and $I_n^M \neq \emptyset$. In fact, if not we may find $M > 2$ and a subsequence $n_k \in \mathbb{N}$ satisfying

$$\int_{a_i^{n_k}+Q} \varphi(|\nabla z_{n_k}|^p) dx > M,$$

for all $i \in I_{n_k}$. Summation in i and (2.32) would yield to

$$\int_{(0,T_{n_k})^N} \varphi(|\nabla z_{n_k}|^p) dx > M[\![T_{n_k}]\!]^N$$

which is in contradiction with (2.31). We also note that in view of (2.31)

$$\operatorname{Card}(I_n \setminus I_n^M) M \leqslant \sum_{i\in I_n\setminus I_n^M} \int_{a_i^n+Q} \varphi(|\nabla z_n|^p)\, dx \leqslant \int_{(0,T_n)^N} \varphi(|\nabla z_n|^p)\, dx \leqslant T_n^N,$$

and so

$$\operatorname{Card}(I_n \setminus I_n^M) \leqslant \frac{T_n^N}{M}. \tag{2.34}$$

By Lemma 2.7 there exists $\varepsilon_0 \equiv \varepsilon_0(M,\eta)$ such that, for any n large enough satisfying $0 \leqslant \varepsilon_n < \varepsilon_0$ and for any $i \in I_n^M$, we can find $v_i^{n,M,\eta} \in W_0^{1,p}(a_i^n + Q;\mathbb{R}^d)$ with $\|v_i^{n,M,\eta}\|_{L^p(a_i^n+Q;\mathbb{R}^d)} \leqslant \eta$ and

$$\int_{a_i^n+Q} f\left(x, \frac{x}{\varepsilon_n}; \xi + \nabla z_n(x)\right) dx \geqslant \int_{a_i^n+Q} f_{\text{hom}}\left(x; \xi + \nabla z_n + \nabla v_i^{n,M,\eta}\right) dx - \eta.$$

Consequently, for n large enough

$$\sum_{i\in I_n} \int_{a_i^n+Q} f\left(x, \frac{x}{\varepsilon_n}; \xi + \nabla z_n(x)\right) dx \geqslant \sum_{i\in I_n^M} \int_{a_i^n+Q} f_{\text{hom}}(x; \xi + \nabla z_n + \nabla v_i^{n,M,\eta})\, dx - \eta \operatorname{card}(I_n^M).$$

As $\operatorname{Card}(I_n^M) \leqslant [\![T_n]\!]^N$, dividing by T_n^N and passing to the limit when $n \to +\infty$ we obtain from (2.33)

$$f_{\{\varepsilon_n\}}(\xi) \geqslant \limsup_{M,\eta,n} \frac{1}{T_n^N} \sum_{i\in I_n^M} \int_{a_i^n+Q} f_{\text{hom}}(x; \xi + \nabla\phi^{n,M,\eta})\, dx \tag{2.35}$$

where $\phi^{n,M,\eta} \in W_0^{1,p}((0;T_n)^N;\mathbb{R}^d)$ is defined by

$$\phi^{n,M,\eta}(x) := \begin{cases} z_n(x) + v_i^{n,M,\eta}(x) & \text{if } x \in a_i^n + Q \text{ and } i \in I_n^M, \\ z_n(x) & \text{otherwise.} \end{cases}$$

Now, in view of the definition of $\phi^{n,M,\eta}$, the p-growth condition (2.4) and (2.34),

$$\begin{aligned}\frac{1}{T_n^N}\sum_{i\in I_n\setminus I_n^M}\int_{a_i^n+Q} f_{\rm hom}(x;\xi+\nabla\phi^{n,M,\eta})\,dx &= \frac{1}{T_n^N}\sum_{i\in I_n\setminus I_n^M}\int_{a_i^n+Q} f_{\rm hom}(x;\xi+\nabla z_n)\,dx\\ &\leqslant \frac{\beta}{T_n^N}\sum_{i\in I_n\setminus I_n^M}\int_{a_i^n+Q}(1+|\nabla z_n|^p)\,dx\\ &\leqslant \frac{\beta}{M}+\frac{\beta}{T_n^N}\sum_{i\in I_n\setminus I_n^M}\int_{a_i^n+Q}|\nabla z_n|^p\,dx\\ &\leqslant \frac{\beta}{M}+\beta\int_{\bigcup\limits_{i\in I_n\setminus I_n^M}\frac{1}{T_n}(a_i^n+Q)}|\nabla w_n|^p\,dx. \qquad (2.36)\end{aligned}$$

By (2.34) we have that

$$\mathcal{L}^N\left(\bigcup_{i\in I_n\setminus I_n^M}\frac{1}{T_n}(a_i^n+Q)\right)\leqslant\frac{1}{M}.$$

Consequently, in view of the equi-integrability of $\{|\nabla w_n|^p\}$ and (2.36), we get

$$\limsup_{M,\eta,n}\frac{1}{T_n^N}\sum_{i\in I_n\setminus I_n^M}\int_{a_i^n+Q} f_{\rm hom}(x;\xi+\nabla\phi^{n,M,\eta})\,dx=0. \qquad (2.37)$$

Therefore, (2.35) and (2.37) imply

$$f_{\{\varepsilon_n\}}(\xi)\geqslant\limsup_{M,\eta,n}\frac{1}{T_n^N}\sum_{i\in I_n}\int_{a_i^n+Q} f_{\rm hom}(x;\xi+\nabla\phi^{n,M,\eta})\,dx. \qquad (2.38)$$

Similarly, since

$$\mathcal{L}^N\left(Q\setminus\left[\bigcup_{i\in I_n}\frac{1}{T_n}(a_i^n+Q)\right]\right)\to 0$$

as $n\to+\infty$, we get that

$$\limsup_{M,\eta,n}\frac{1}{T_n^N}\int_{\left(0,T_n^N\right)\setminus\left[\bigcup\limits_{i\in I_n}(a_i^n+Q)\right]} f_{\rm hom}(x;\xi+\nabla\phi^{n,M,\eta})\,dx=0. \qquad (2.39)$$

Hence by (2.38), (2.39) and (2.7) we get that

$$f_{\{\varepsilon_n\}}(\xi)\geqslant\limsup_{M,\eta,n}\frac{1}{T_n^N}\int_{(0,T_n)^N} f_{\rm hom}(x;\xi+\nabla\phi^{n,M,\eta})\,dx\geqslant\overline{f}_{\rm hom}(\xi).$$

■

Let us now prove the converse inequality.

Lemma 2.9. *For all $\xi\in\mathbb{R}^{d\times N}$, $\overline{f}_{\rm hom}(\xi)\geqslant f_{\{\varepsilon_n\}}(\xi)$.*

Proof. In view of (2.7), for $\delta>0$ fixed take $T\equiv T_\delta\in\mathbb{N}$, with $T_\delta\to+\infty$ as $\delta\to 0$, and $\phi\equiv\phi_\delta\in W_0^{1,p}((0,T)^N;\mathbb{R}^d)$ such that

$$\overline{f}_{\rm hom}(\xi)+\delta\geqslant\frac{1}{T^N}\int_{(0,T)^N} f_{\rm hom}(x;\xi+\nabla\phi(x))\,dx. \qquad (2.40)$$

By Theorem 1.1 in Baía and Fonseca [8] and, for instance, Proposition 11.7 in Braides and Defranceschi [14], there exists a sequence $\{\phi_n\} \subset W_0^{1,p}((0,T)^N;\mathbb{R}^d)$ with $\phi_n \to \phi$ in $L^p((0,T)^N;\mathbb{R}^d)$ such that

$$\int_{(0,T)^N} f_{\hom}(x;\xi+\nabla\phi(x))\,dx = \lim_{n\to+\infty}\int_{(0,T)^N} f\left(x,\frac{x}{\varepsilon_n};\xi+\nabla\phi_n(x)\right)dx. \tag{2.41}$$

Further, in view of the Decomposition Lemma (see Fonseca and Leoni [24] or Fonseca, Müller and Pedregal [25]), we can assume – upon extracting a subsequence still denoted by $\{\phi_n\}$ – $\{|\nabla\phi_n|^p\}$ to be equi-integrable. Fix $n \in \mathbb{N}$ such that $\varepsilon_n \ll 1$. For all $i \in \mathbb{Z}^N$ let $a_i^n \in \varepsilon_n\mathbb{Z}^N \cap (i(T+1)+[0,\varepsilon_n)^N)$ (uniquely defined) (see Example in Figure 4).

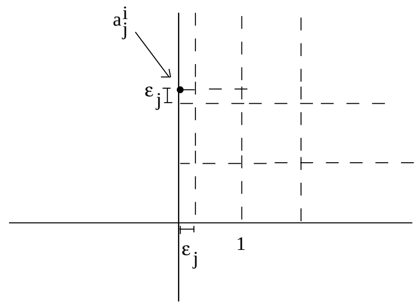

Figure 4: Example for $T=1$, $i=(0,1)$ and $N=2$

In particular, the cubes $a_i^n+(0,T)^N$ are not overlapping because if $i,j\in\mathbb{Z}^N$ with $i\neq j$, then $|i-j|\geqslant 1$ and thus $|a_i^n-a_j^n|>T$. Set

$$\tilde\phi_n(x) := \begin{cases} \phi_n(x-a_i^n) & \text{if } x\in a_i^n+(0,T)^N \text{ and } i\in\mathbb{Z}^N,\\ 0 & \text{otherwise,}\end{cases}$$

then $\tilde\phi_n \in W^{1,p}(\mathbb{R}^N;\mathbb{R}^d)$. Let $I_n := \{i\in\mathbb{Z}^N : (0,T/\varepsilon_n)^N \cap (a_i^n+(0,T)^N)\neq\emptyset\}$. Note that

$$\mathrm{Card}(I_n) \leqslant \left(\left[\!\left[\frac{1}{\varepsilon_n}\right]\!\right]+1\right)^N. \tag{2.42}$$

If $\psi_n(x) := \varepsilon_n\tilde\phi_n(x/\varepsilon_n)$ then $\psi_n\to 0$ in $L^p((0,T)^N;\mathbb{R}^d)$, as $n\to+\infty$, because

$$\begin{aligned}\int_{(0,T)^N}|\psi_n(x)|^p dx &= \varepsilon_n^p\int_{(0,T)^N}\left|\tilde\phi_n\left(\frac{x}{\varepsilon_n}\right)\right|^p dx\\ &= \varepsilon_n^{p+N}\int_{(0,T/\varepsilon_n)^N}|\tilde\phi_n(x)|^p dx\\ &\leqslant \varepsilon_n^{p+N}\sum_{i\in I_n}\int_{a_i^n+(0,T)^N}|\phi_n(x-a_i^n)|^p dx\\ &= \varepsilon_n^{p+N}\mathrm{Card}(I_n)\int_{(0,T)^N}|\phi_n(x)|^p dx\\ &\leqslant \varepsilon_n^{p+N}\left(\left[\!\left[\frac{1}{\varepsilon_n}\right]\!\right]+1\right)^N\int_{(0,T)^N}|\phi_n(x)|^p dx \to 0,\end{aligned}$$

where we have used the fact that $\tilde\phi_n\equiv 0$ on $(0,T/\varepsilon_n)^N\setminus\bigcup_{i\in I_n}(a_i^n+(0,T)^N)$ and

$$\sup_{n\in\mathbb{N}}\|\phi_n\|_{L^p((0,T)^N;\mathbb{R}^d)}<+\infty$$

by the Poincaré Inequality and (2.41). Consequently,

$$\begin{aligned}\mathcal{F}_{\{\varepsilon_n\}}(\xi\cdot\,;(0,T)^N) &\leqslant \liminf_{n\to+\infty}\int_{(0,T)^N} f\left(\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla\psi_n(x)\right)dx\\ &= \liminf_{n\to+\infty}\int_{(0,T)^N} f\left(\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla\tilde\phi_n\left(\frac{x}{\varepsilon_n}\right)\right)dx\\ &= \liminf_{n\to+\infty}\varepsilon_n^N\int_{(0,T/\varepsilon_n)^N} f\left(x,\frac{x}{\varepsilon_n};\xi+\nabla\tilde\phi_n(x)\right)dx.\end{aligned}$$

Note that since

$$\varepsilon_n^N\mathcal{L}^N\left(\left(0,\frac{T}{\varepsilon_n}\right)^N\setminus\bigcup_{i\in I_n}(a_i^n+(0,T)^N)\right) \leqslant T^N\left(1-\varepsilon_n^N\left[\!\!\left[\frac{T}{\varepsilon_n(T+1)}\right]\!\!\right]^N\right),$$

from the p-growth condition (H_4) it follows that

$$\begin{aligned}\mathcal{F}_{\{\varepsilon_n\}}(\xi\cdot\,;(0,T)^N) \leqslant{}& \liminf_{n\to+\infty}\varepsilon_n^N\sum_{i\in I_n}\int_{a_i^n+(0,T)^N} f\left(x,\frac{x}{\varepsilon_n};\xi+\nabla\tilde\phi_n(x)\right)dx\\ &+\beta(1+|\xi|^p)T^N\left(1-\left(\frac{T}{T+1}\right)^N\right).\end{aligned}$$

By a change of variables, for all $i\in I_n$

$$\begin{aligned}&\int_{a_i^n+(0,T)^N} f\left(x,\frac{x}{\varepsilon_n};\xi+\nabla\tilde\phi_n(x)\right)dx\\ &\qquad=\int_{(0,T)^N} f\left(x+a_i^n,\frac{x+a_i^n}{\varepsilon_n};\xi+\nabla\phi_n(x)\right)dx\\ &\qquad=\int_{(0,T)^N} f\left(x+a_i^n-i(T+1),\frac{x}{\varepsilon_n};\xi+\nabla\phi_n(x)\right)dx,\end{aligned}$$

where we have used (H_3), the fact that $T\in\mathbb{N}$ and $a_i^n/\varepsilon_n\in\mathbb{Z}^N$. By a similar uniform continuity argument than the one used in Lemma 2.5 and since (2.42) holds, $|a_i^n-i(T+1)|\leqslant\varepsilon_n$, and $\{|\nabla\phi_n|^p\}$ is equi-integrable. It follows that

$$\mathcal{F}_{\{\varepsilon_n\}}(\xi\cdot\,;(0,T)^N)\leqslant\liminf_{n\to+\infty}\int_{(0,T)^N} f\left(x,\frac{x}{\varepsilon_n};\xi+\nabla\tilde\phi_n(x)\right)dx+cT^N\left(1-\left(\frac{T}{T+1}\right)^N\right). \tag{2.43}$$

Consequently by (2.40), (2.41, (2.43) and Lemma 2.6

$$f_{\{\varepsilon_n\}}(\xi)\leqslant\overline{f}_{\text{hom}}(\xi)+\delta+c\left(1-\left(\frac{T}{T+1}\right)^N\right).$$

The result follows by letting δ tend to zero. ■

Proof of Theorem 2.2. From Lemma 2.8 and Lemma 2.9, we conclude that $\overline{f}_{\text{hom}}(\xi)=f_{\{\varepsilon_n\}}(\xi)$ for all $\xi\in\mathbb{R}^{d\times N}$. As a consequence, $\mathcal{F}_{\{\varepsilon_n\}}(u;A)=\mathcal{F}_{\text{hom}}(u;A)$ for all $A\in\mathcal{A}_0$ and all $u\in W^{1,p}(A;\mathbb{R}^d)$. Since the Γ-limit does not depend upon the extracted subsequence, Proposition 8.3 in Dal Maso [19] implies that the whole sequence $\mathcal{F}_\varepsilon(\cdot\,;A)$ $\Gamma(L^p(A))$-converges to $\mathcal{F}_{\text{hom}}(\cdot\,;A)$. ■

2.3 The general case

Our aim here is to prove Theorem 1.1.

2.3.1 Existence and integral representation of the Γ-limit

The idea in this case is to freeze the macroscopic variable and to use Theorem 2.2 through a blow up argument. This leads us to work on small cubes centered at convenient Lebesgue points of Ω which, contrary to Subsection 2.2, allow us to localize our functionals on $\mathcal{A}(\Omega)$, the family of open subsets of Ω. We define $\mathcal{F}_\varepsilon : L^p(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega)\to[0,+\infty]$ by

$$\mathcal{F}_\varepsilon(u;A) := \begin{cases} \displaystyle\int_A f\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2};\nabla u(x)\right)dx & \text{if } u\in W^{1,p}(A;\mathbb{R}^d),\\ +\infty & \text{otherwise,}\end{cases}$$

and we introduce the functional $\mathcal{F}_{\rm hom} : L^p(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega)\to[0,+\infty]$

$$\mathcal{F}_{\rm hom}(u;A) := \begin{cases} \displaystyle\int_A \overline{f}_{\rm hom}(x;\nabla u(x))\,dx & \text{if } u\in W^{1,p}(A;\mathbb{R}^d),\\ +\infty & \text{otherwise.}\end{cases}$$

Given $\{\varepsilon_j\}\searrow 0^+$ and $A\in\mathcal{A}(\Omega)$, consider the Γ-lower limit of $\{\mathcal{F}_{\varepsilon_j}(\cdot\,;A)\}_{j\in\mathbb{N}}$ for the $L^p(A;\mathbb{R}^d)$-topology defined, for $u\in W^{1,p}(\Omega;\mathbb{R}^d)$, by

$$\mathcal{F}_{\{\varepsilon_j\}}(u;A) := \inf_{\{u_j\}}\left\{\liminf_{j\to+\infty}\mathcal{F}_{\varepsilon_j}(u_j;A): \quad u_j\to u \text{ in } L^p(A;\mathbb{R}^d)\right\}.$$

Due to the p-coercivity condition in (H_4), to prove Theorem 1.1 it suffices to show that for all $u\in W^{1,p}(\Omega;\mathbb{R}^d)$

$$\Gamma(L^p(\Omega))\text{-}\lim_{\varepsilon\to0}\mathcal{F}_\varepsilon(u) = \int_\Omega \overline{f}_{\rm hom}(x,\nabla u(x))\,dx.$$

As a consequence of Theorem 8.5 in Dal Maso [19], there exists a subsequence $\{\varepsilon_n\}\equiv\{\varepsilon_{j_n}\}$ such that for any $A\in\mathcal{A}(\Omega)$, $\mathcal{F}_{\{\varepsilon_n\}}(\cdot\,;A)$ is the $\Gamma(L^p(A))$-limit of $\mathcal{F}_{\varepsilon_n}(\cdot\,;A)$ and, for all $u\in W^{1,p}(\Omega;\mathbb{R}^d)$, the set function $\mathcal{F}_{\{\varepsilon_n\}}(u;\cdot)$ is the restriction of a Radon measure to $\mathcal{A}(\Omega)$. Furthermore, from Buttazzo-Dal Maso Integral Representation Theorem (see Theorem 1.1 in [17]) it follows that

Lemma 2.10. *There exists a Carathéodory function $f_{\{\varepsilon_n\}}:\Omega\times\mathbb{R}^{d\times N}\to\mathbb{R}$, quasiconvex in its second variable, satisfying the same coercivity and growth conditions than f, such that*

$$\mathcal{F}_{\{\varepsilon_n\}}(u;A) = \int_A f_{\{\varepsilon_n\}}(x;\nabla u(x))\,dx$$

for every $A\in\mathcal{A}(\Omega)$ and $u\in W^{1,p}(\Omega;\mathbb{R}^d)$. Moreover, for all $\xi\in\mathbb{R}^{d\times N}$ and a.e. $x\in\Omega$,

$$f_{\{\varepsilon_n\}}(x;\xi) = \lim_{\delta\to0}\frac{\mathcal{F}_{\{\varepsilon_n\}}(\xi\cdot\,;Q(x,\delta))}{\delta^N}.$$

2.3.2 Characterization of the Γ-limit

Like in Section 4, we only need to prove that $f_{\{\varepsilon_n\}}(x;\xi)=\overline{f}_{\rm hom}(x;\xi)$ for a.e. x and all ξ. For this purpose let L be the set of Lebesgue points x_0 for all functions $f_{\{\varepsilon_n\}}(\cdot\,;\xi)$ and $\overline{f}_{\rm hom}(\cdot\,;\xi)$, for all $\xi\in\mathbb{Q}^{d\times N}$. We have $\mathcal{L}^N(\Omega\setminus L)=0$ and we will first show in Lemma 2.11 and 2.12 below that the equality $f_{\{\varepsilon_n\}}(x;\xi)=\overline{f}_{\rm hom}(x;\xi)$ holds for all $x\in L$ and all $\xi\in\mathbb{Q}^{d\times N}$. By definition of the set L it is enough to show that

$$\int_{Q(x_0,\delta)} f_{\{\varepsilon_n\}}(x;\xi)\,dx = \int_{Q(x_0,\delta)}\overline{f}_{\rm hom}(x;\xi)\,dx, \tag{2.44}$$

for every $x_0\in L$ and each $\delta>0$ small enough so that $Q(x_0,\delta)\in\mathcal{A}(\Omega)$.

Lemma 2.11. *For all $\xi \in \mathbb{Q}^{d\times N}$ and all $x_0 \in L$,*

$$\int_{Q(x_0,\delta)} f_{\{\varepsilon_n\}}(x;\xi)\,dx \geqslant \int_{Q(x_0,\delta)} \overline{f}_{\rm hom}(x;\xi)\,dx.$$

Proof. Let $\xi \in \mathbb{Q}^{d\times N}$ and $x_0 \in L$. Let $\{u_n\} \subset W^{1,p}(Q(x_0,\delta);\mathbb{R}^d)$ be a recovering sequence for $\mathcal{F}_{\{\varepsilon_n\}}(\xi\cdot\,;Q(x_0;\delta))$, that is, a sequence $\{u_n\}$ such that $u_n \to 0$ in $L^p(Q(x_0,\delta);\mathbb{R}^d)$ and

$$\int_{Q(x_0,\delta)} f_{\{\varepsilon_n\}}(x;\xi)\,dx = \mathcal{F}_{\{\varepsilon_n\}}(\xi\cdot\,;Q(x_0;\delta)) = \lim_{n\to+\infty}\int_{Q(x_0,\delta)} f\left(x,\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla u_n(x)\right)dx.$$

As before, the Decomposition Lemma (see Fonseca and Leoni [24] or Fonseca, Müller and Pedregal [25]) let us to assume that $\{|\nabla u_n|^p\}$ is equi-integrable. We split $Q(x_0,\delta)$ into h^N small disjoint cubes $Q_{i,h}$ such that

$$Q(x_0,\delta) = \bigcup_{i=1}^{h^N} Q_{i,h} \quad \text{and} \quad \mathcal{L}^N(Q_{i,h}) = (\delta/h)^N. \tag{2.45}$$

Then

$$\int_{Q(x_0,\delta)} f_{\{\varepsilon_n\}}(x;\xi)\,dx = \lim_{h,n}\sum_{i=1}^{h^N}\int_{Q_{i,h}} f\left(x,\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla u_n(x)\right)dx.$$

Let $\eta > 0$. By Scorza-Dragoni's Theorem (see Ekeland and Temam [22]), there exists a compact set $K_\eta \subset \Omega$ such that

$$\mathcal{L}^N(\Omega\setminus K_\eta) < \eta, \tag{2.46}$$

and the restriction of f to $K_\eta \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$ is a continuous function. Given $\lambda > 0$, we introduce

$$R_n^\lambda := \{x \in \Omega : |\xi+\nabla u_n(x)| \leqslant \lambda\},$$

for all $n \in \mathbb{N}$ and we note that due to Chebyshev's inequality, we have

$$\mathcal{L}^N(\Omega\setminus R_n^\lambda) \leqslant \frac{C}{\lambda^p}, \tag{2.47}$$

for some constant $C > 0$ independent of n and λ. Then

$$\int_{Q(x_0,\delta)} f_{\{\varepsilon_n\}}(x;\xi)\,dx \geqslant \limsup_{\lambda,\eta,h,n}\sum_{i=1}^{h^N}\int_{Q_{i,h}\cap K_\eta\cap R_n^\lambda} f\left(x,\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla u_n(x)\right)dx.$$

In view of condition (H_3), f is uniformly continuous on $K_\eta \times \mathbb{R}^N \times \mathbb{R}^N \times \overline{B}(0,\lambda)$. Denoting by $\omega_{\eta,\lambda} : \mathbb{R}^+ \to \mathbb{R}^+$ the modulus of continuity of f on $K_\eta \times \mathbb{R}^N \times \mathbb{R}^N \times \overline{B}(0,\lambda)$, for every $(x,x') \in [Q_{i,h}\cap K_\eta \cap R_n^\lambda] \times [Q_{i,h}\cap K_\eta]$,

$$\begin{aligned}\left|f\left(x,\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla u_n(x)\right) - f\left(x',\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla u_n(x)\right)\right| &\leqslant \omega_{\eta,\lambda}(|x-x'|)\\ &\leqslant \omega_{\eta,\lambda}\left(\frac{\sqrt{N}\delta}{h}\right).\end{aligned} \tag{2.48}$$

From (2.45) and (2.48), after integrating in (x,x') over $[Q_{i,h}\cap K_\eta\cap R_n^\lambda]\times[Q_{i,h}\cap K_\eta]$, we get since $\omega_{\eta,\lambda}$ is continuous and satisfies $\omega_{\eta,\lambda}(0) = 0$

$$\begin{aligned}\sum_{i=1}^{h^N}\frac{h^N}{\delta^N}\int_{Q_{i,h}\cap K_\eta}\left\{\int_{Q_{i,h}\cap K_\eta\cap R_n^\lambda}\right|&f\left(x,\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla u_n(x)\right)\\ &\left.\left.-f\left(x',\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla u_n(x)\right)\right|dx\right\}dx'\\ &\leqslant \delta^N\omega_{\eta,\lambda}\left(\frac{\sqrt{N}\delta}{h}\right)\xrightarrow[h\to+\infty]{} 0,\end{aligned}$$

uniformly in $n \in \mathbb{N}$, for all $\eta > 0$ and $\lambda > 0$. Hence, by Fubini's Theorem

$$\int_{Q(x_0,\delta)} f_{\{\varepsilon_n\}}(x;\xi)\,dx$$
$$\geqslant \limsup_{\lambda,\eta,h,n} \frac{h^N}{\delta^N}\sum_{i=1}^{h^N}\int_{Q_{i,h}\cap K_\eta\cap R_n^\lambda}\left\{\int_{Q_{i,h}\cap K_\eta} f\left(x', \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi+\nabla u_n(x)\right)dx'\right\}dx. \tag{2.49}$$

However, as a consequence of (H_4) and (2.46) we have that for all $\lambda > 0$,

$$\frac{h^N}{\delta^N}\sum_{i=1}^{h^N}\int_{Q_{i,h}\cap K_\eta\cap R_n^\lambda}\left\{\int_{Q_{i,h}\setminus K_\eta} f\left(x', \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi+\nabla u_n(x)\right)dx'\right\}dx$$
$$\leqslant \beta\frac{h^N}{\delta^N}\sum_{i=1}^{h^N}\frac{\delta^N}{h^N}(1+\lambda^p)\mathcal{L}^N(Q_{i,h}\setminus K_\eta)$$
$$\leqslant \beta(1+\lambda^p)\mathcal{L}^N(\Omega\setminus K_\eta) \xrightarrow[\eta\to 0]{} 0 \tag{2.50}$$

uniformly in $n \in \mathbb{N}$ and $h \in \mathbb{N}$, and similarly

$$\frac{h^N}{\delta^N}\sum_{i=1}^{h^N}\int_{Q_{i,h}\cap R_n^\lambda\setminus K_\eta}\left\{\int_{Q_{i,h}} f\left(x', \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi+\nabla u_n(x)\right)dx'\right\}dx \leqslant \beta(1+\lambda^p)\eta \xrightarrow[\eta\to 0]{} 0, \tag{2.51}$$

uniformly in $n \in \mathbb{N}$ and $h \in \mathbb{N}$. Moreover, (2.45) and (2.47), together with the equi-integrability of $\{|\nabla u_n|^p\}$, yield

$$\sup_{n,h,\eta}\frac{h^N}{\delta^N}\sum_{i=1}^{h^N}\int_{Q_{i,h}\setminus R_n^\lambda}\left\{\int_{Q_{i,h}} f\left(x', \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi+\nabla u_n(x)\right)dx'\right\}dx$$
$$\leqslant \sup_{n\in\mathbb{N}}\beta\int_{Q(x_0,\delta)\setminus R_n^\lambda}(1+|\nabla u_n(x)|^p)\,dx \xrightarrow[\lambda\to+\infty]{} 0. \tag{2.52}$$

Finally, (2.49)-(2.52) and Fubini's Theorem lead to

$$\int_{Q(x_0,\delta)} f_{\{\varepsilon_n\}}(x;\xi)\,dx$$
$$\geqslant \limsup_{h,n}\frac{h^N}{\delta^N}\sum_{i=1}^{h^N}\int_{Q_{i,h}}\left\{\int_{Q_{i,h}} f\left(x', \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi+\nabla u_n(x)\right)dx\right\}dx'$$
$$\geqslant \limsup_{h\to+\infty}\frac{h^N}{\delta^N}\sum_{i=1}^{h^N}\int_{Q_{i,h}}\left\{\liminf_{n\to+\infty}\int_{Q_{i,h}} f\left(x', \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi+\nabla u_n(x)\right)dx\right\}dx', \tag{2.53}$$

where we have used Fatou's Lemma. Fix $x' \in Q_{i,h}$ such that $\overline{f}_{\rm hom}(x';\xi)$ is well defined and apply Theorem 2.2 to the continuous function $(y,z,\xi)\mapsto f(x',y,z;\xi)$. Since $u_n \to 0$ in $L^p(Q(x_0,\delta);\mathbb{R}^d)$, we can use the Γ-$\liminf$ inequality to get

$$\liminf_{n\to+\infty}\int_{Q_{i,h}} f\left(x', \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi+\nabla u_n(x)\right)dx \geqslant \frac{\delta^N}{h^N}\overline{f}_{\rm hom}(x';\xi).$$

Then, in view of (2.53) we conclude (2.44). ■

Lemma 2.12. *For all $\xi \in \mathbb{Q}^{d\times N}$ and all $x_0 \in L$,*

$$\int_{Q(x_0,\delta)} f_{\{\varepsilon_n\}}(x;\xi)\,dx \leqslant \int_{Q(x_0,\delta)} \overline{f}_{\rm hom}(x;\xi)\,dx.$$

Proof. As in Lemma 2.11, we decompose $Q(x_0,\delta)$ into h^N small disjoints cubes $Q_{i,h}$ satisfying (2.45). Since f and $\overline{f}_{\rm hom}$ are Carathéodory functions, by Scorza-Dragoni's Theorem (see Ekeland and Temam [22]) for each $\eta>0$, we can find a compact set $K_\eta \subset Q(x_0,\delta)$ such that

$$\mathcal{L}^N(Q(x_0,\delta)\setminus K_\eta)<\eta, \tag{2.54}$$

f is continuous on $K_\eta \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and $\overline{f}_{\rm hom}$ is continuous on $K_\eta \times \mathbb{R}^{d\times N}$. Let

$$I_{h,\eta} := \left\{ i \in \{1,\cdots,h^N\} : \quad K_\eta \cap Q_{i,h} \neq \emptyset \right\}.$$

For $i \in I_{h,\eta}$, choose $x_i^{h,\eta} \in K_\eta \cap Q_{i,h}$. Theorem 2.2, together with e.g. Proposition 11.7 in Braides and Defranceschi [14], implies the existence of a sequence $\{u_i^{n,h,\eta}\} \subset W_0^{1,p}(Q_{i,h};\mathbb{R}^d)$ such that $u_i^{n,h,\eta} \to 0$ in $L^p(Q_{i,h};\mathbb{R}^d)$ as $n\to+\infty$ and

$$\int_{Q_{i,h}} \overline{f}_{\rm hom}(x_i^{h,\eta};\xi)\,dx = \left(\frac{\delta}{h}\right)^N \overline{f}_{\rm hom}(x_i^{h,\eta};\xi) = \lim_{n\to+\infty}\int_{Q_{i,h}} f\left(x_i^{h,\eta},\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla u_i^{n,h,\eta}\right)dx.$$

Set

$$u_n^\eta(x) := \begin{cases} u_i^{n,h,\eta}(x) & \text{if } x\in Q_{i,h} \text{ and } i \in I_{h,\eta},\\ 0 & \text{otherwise.}\end{cases} \tag{2.55}$$

Then $\{u_n^\eta\}\subset W_0^{1,p}(Q(x_0,\delta);\mathbb{R}^d)$, $u_n^\eta \to 0$ in $L^p(Q(x_0,\delta);\mathbb{R}^d)$ as $n\to+\infty$ and

$$\liminf_{h\to+\infty}\sum_{i\in I_{h,\eta}}\int_{Q_{i,h}} \overline{f}_{\rm hom}(x_i^{h,\eta};\xi)\,dx = \liminf_{h\to+\infty}\lim_{n\to+\infty}\sum_{i\in I_{h,\eta}}\int_{Q_{i,h}} f\left(x_i^{h,\eta},\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla u_n^\eta\right)dx. \tag{2.56}$$

In view of (2.5) and (2.54) we have

$$\sup_{h\in\mathbb{N}}\sum_{i\in I_{h,\eta}}\int_{Q_{i,h}\setminus K_\eta} \overline{f}_{\rm hom}(x_i^{h,\eta};\xi)\,dx \leqslant \beta(1+|\xi|^p)\mathcal{L}^N(Q(x_0,\delta)\setminus K_\eta) \xrightarrow[\eta\to 0]{} 0, \tag{2.57}$$

thus from (2.56) and (2.57) it comes that

$$\begin{aligned}\liminf_{\eta,h}\sum_{i\in I_{h,\eta}}\int_{Q_{i,h}\cap K_\eta} &\overline{f}_{\rm hom}(x_i^{h,\eta};\xi)\,dx\\ &\geqslant \liminf_{\eta,h,n}\sum_{i\in I_{h,\eta}}\int_{Q_{i,h}\cap K_\eta} f\left(x_i^{h,\eta},\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla u_n^\eta\right)dx.\end{aligned} \tag{2.58}$$

Since $\overline{f}_{\rm hom}(\cdot\,;\xi)$ is continuous on K_η, it is uniformly continuous. Thus, denoting by ω_η its modulus of continuity on K_η, we have for all $x\in Q_{i,h}\cap K_\eta$,

$$|\overline{f}_{\rm hom}(x;\xi)-\overline{f}_{\rm hom}(x_i^{h,\eta};\xi)| \leqslant \omega_\eta(|x-x_i^{h,\eta}|) \leqslant \omega_\eta\left(\frac{\sqrt{N}\delta}{h}\right) \xrightarrow[h\to+\infty]{} 0. \tag{2.59}$$

In view of (2.54), (2.59) and (2.58), we get since $Q_{i,h}\cap K_\eta=\emptyset$ for $i\notin I_{h,\eta}$,

$$\begin{aligned}\int_{Q(x_0,\delta)} \overline{f}_{\rm hom}(x;\xi)\,dx &= \lim_{\eta\to 0}\int_{K_\eta}\overline{f}_{\rm hom}(x;\xi)\,dx\\ &= \liminf_{\eta,h}\sum_{i\in I_{h,\eta}}\int_{Q_{i,h}\cap K_\eta}\overline{f}_{\rm hom}(x;\xi)\,dx\\ &= \liminf_{\eta,h}\sum_{i\in I_{h,\eta}}\int_{Q_{i,h}\cap K_\eta}\overline{f}_{\rm hom}(x_i^{h,\eta};\xi)\,dx\\ &\geqslant \liminf_{\lambda,\eta,h,n}\sum_{i\in I_{h,\eta}}\int_{Q_{i,h}\cap K_\eta\cap R_{n,\eta}^\lambda} f\left(x_i^{h,\eta},\frac{x}{\varepsilon_n},\frac{x}{\varepsilon_n^2};\xi+\nabla u_n^\eta\right)dx,\end{aligned} \tag{2.60}$$

where $R^\lambda_{n,\eta} := \{x \in Q(x_0,\delta) : |\xi + \nabla u^\eta_n(x)| \leqslant \lambda\}$. From (2.56) and the fact that $u^\eta_n \equiv 0$ on $Q(x_0,\delta) \setminus \bigcup_{i\in I_{h,\eta}} Q_{i,h}$, we get

$$\sup_{n\in\mathbb{N},\eta>0} \int_{Q(x_0,\delta)} |\nabla u^\eta_n|^p dx < +\infty. \tag{2.61}$$

In particular, according to Chebyshev's inequality, we have

$$\mathcal{L}^N(Q(x_0,\delta) \setminus R^\lambda_{n,\eta}) \leqslant \frac{C}{\lambda^p}, \tag{2.62}$$

for some constant $C > 0$ independent of n, η and λ. Since f is continuous on $K_\eta \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and separately Q-periodic in its second and third variable (see assumption (H_3)), it is uniformly continuous on $K_\eta \times \mathbb{R}^N \times \mathbb{R}^N \times \overline{B}(0,\lambda)$. Thus, denoting by $\omega_{\eta,\lambda}$ its modulus of continuity on $K_\eta \times \mathbb{R}^N \times \mathbb{R}^N \times \overline{B}(0,\lambda)$, we have for all $x \in Q_{i,h} \cap K_\eta \cap R^\lambda_{n,\eta}$,

$$\begin{aligned}\left| f\left(x, \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi + \nabla u^\eta_n(x)\right) - f\left(x^{h,\eta}_i, \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi + \nabla u^\eta_n(x)\right)\right| \\ \leqslant \omega_{\eta,\lambda}(|x - x^{h,\eta}_i|) \\ \leqslant \omega_{\eta,\lambda}\left(\frac{\sqrt{N}\delta}{h}\right) \xrightarrow[h\to+\infty]{} 0.\end{aligned}$$

Then, according to (2.60) and the fact that $Q_{i,h} \cap K_\eta = \emptyset$ for $i \notin I_{h,\eta}$,

$$\begin{aligned}\int_{Q(x_0,\delta)} \overline{f}_{\rm hom}(x;\xi)\, dx &\geqslant \liminf_{\lambda,\eta,h,n} \sum_{i\in I_{h,\eta}} \int_{Q_{i,h}\cap K_\eta\cap R^\lambda_{n,\eta}} f\left(x, \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi + \nabla u^\eta_n\right) dx, \\ &= \liminf_{\lambda,\eta,n} \int_{K_\eta\cap R^\lambda_{n,\eta}} f\left(x, \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi + \nabla u^\eta_n\right) dx.\end{aligned}$$

In view of the p-growth condition (H_4), (2.54) and the definition of $R^\lambda_{n,\eta}$,

$$\sup_{n\in\mathbb{N}} \int_{R^\lambda_{n,\eta}\setminus K_\eta} f\left(x, \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi + \nabla u^\eta_n\right) dx \leqslant \beta(1+\lambda^p)\eta \xrightarrow[\eta\to 0]{} 0,$$

then

$$\int_{Q(x_0,\delta)} \overline{f}_{\rm hom}(x;\xi)\, dx \geqslant \liminf_{\lambda,\eta,n} \int_{R^\lambda_{n,\eta}} f\left(x, \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi + \nabla u^\eta_n\right) dx.$$

Let $\lambda_k \nearrow +\infty$ and $\eta_k \searrow 0^+$, by a diagonalization procedure, it is possible to find a subsequence $\{n_k\}$ of $\{n\}$ such that, upon setting $v_k := u^{\eta_k}_{n_k}$ and $R_k := R^{\lambda_k}_{n_k,\eta_k}$, then $v_k \in W^{1,p}_0(Q(x_0,\delta);\mathbb{R}^d)$, $v_k \to 0$ in $L^p(Q(x_0,\delta);\mathbb{R}^d)$ and

$$\int_{Q(x_0,\delta)} \overline{f}_{\rm hom}(x;\xi)\, dx \geqslant \liminf_{k\to+\infty} \int_{R_k} f\left(x, \frac{x}{\varepsilon_{n_k}}, \frac{x}{\varepsilon_{n_k}^2}; \xi + \nabla v_k\right) dx.$$

By (2.61) and the Poincaré Inequality, the sequence $\{v_k\}$ is bounded in $W^{1,p}(Q(x_0,\delta);\mathbb{R}^d)$ uniformly with respect to $k \in \mathbb{N}$ so that, according to the Decomposition Lemma (see Fonseca and Leoni [24] or Fonseca, Müller and Pedregal [25]), there is no loss of generality to assume that $\{|\nabla v_k|^p\}$ is equi-integrable. It turns out, in view of the p-growth condition (H_4) and (2.62) that

$$\int_{Q(x_0,\delta)\setminus R_k} f\left(x, \frac{x}{\varepsilon_{n_k}}, \frac{x}{\varepsilon_{n_k}^2}; \xi + \nabla v_k\right) dx \leqslant \beta \sup_{l\in\mathbb{N}} \int_{Q(x_0,\delta)\setminus R_k} (1 + |\nabla v_l|^p)\, dx \xrightarrow[k\to+\infty]{} 0.$$

Thus, using the Γ-lim inf inequality,

$$\begin{aligned}\int_{Q(x_0,\delta)} \overline{f}_{\rm hom}(x;\xi)\,dx &\geqslant \liminf_{k\to+\infty}\int_{Q(x_0,\delta)} f\left(x,\frac{x}{\varepsilon_{n_k}},\frac{x}{\varepsilon_{n_k}^2};\xi+\nabla v_k\right)dx\\ &\geqslant \int_{Q(x_0,\delta)} f_{\{\varepsilon_n\}}(x;\xi)\,dx.\end{aligned}$$

■

Proof of Theorem 1.1. As a consequence of Lemma 2.11 and 2.12, we have $\overline{f}_{\rm hom}(x;\xi) = f_{\{\varepsilon_n\}}(x;\xi)$ for all $x \in L$ and all $\xi \in \mathbb{Q}^{d\times N}$. By Lemma 2.10 and the fact that $\overline{f}_{\rm hom}$ is (equivalent to) a Carathéodory function, it follows that the equality holds for all $\xi \in \mathbb{R}^{d\times N}$ and a.e. $x \in \Omega$. Therefore, we have $\mathcal{F}_{\{\varepsilon_n\}}(u;A) = \mathcal{F}_{\rm hom}(u;A)$ for all $A \in \mathcal{A}(\Omega)$ and all $u \in W^{1,p}(A;\mathbb{R}^d)$. Since the result does not depend upon the specific choice of the subsequence, we conclude thanks to Proposition 8.3 in Dal Maso [19] that the whole sequence $\mathcal{F}_\varepsilon(\cdot\,;A)$ $\Gamma(L^p(A))$-converges to $\mathcal{F}_{\rm hom}(\cdot\,;A)$. Taking $A=\Omega$ we conclude the proof of Theorem 1.1. ■

2.4 Some remarks in the convex case

We start this section by noticing that under the additional hypothesis that $f(x,y,z,\cdot)$ is convex for a.e. x and all (y,z), in which case (H_1) is equivalent to requiring that $f(x,\cdot,\cdot,\xi)$ is continuous for a.e. x and all ξ, equality (1.6) and (1.7) simplify to read

$$\overline{f}_{\rm hom}(x;\xi) = \inf_{\phi}\left\{\int_Q f_{\rm hom}(x,y;\xi+\nabla\phi(y))\,dy,\quad \phi\in W^{1,p}_0(Q;\mathbb{R}^d)\ \ \big(\text{or equivalently } \phi \in W^{1,p}_{\rm per}(Q;\mathbb{R}^d)\big)\right\}$$

for all $\xi \in \mathbb{R}^{d\times N}$ and a.e. $x\in\Omega$, and

$$f_{\rm hom}(x,y;\xi) = \inf_{\phi}\left\{\int_Q f(x,y,z;\xi+\nabla\phi(z))\,dz,\quad \phi\in W^{1,p}_0(Q;\mathbb{R}^d)\ \ \big(\text{or equivalently } \phi \in W^{1,p}_{\rm per}(Q;\mathbb{R}^d)\big)\right\}$$

for a.e $x\in\Omega$ and all $(y,\xi)\in\mathbb{R}^N\times\mathbb{R}^{d\times N}$ (see Braides and Defranceschi [14] and Müller [33]).

Our objective here is to present an alternative proof of Lemma 2.12 in the convex case. Namely we would like to show that $f_{\{\varepsilon_n\}}(x_0;\xi) \leqslant \overline{f}_{\rm hom}(x_0;\xi)$ a.e. $x_0\in\Omega$ and all $\xi\in\mathbb{R}^{d\times N}$, without appealing to Theorem 2.2. For this purpose let us denote by $\mathcal{S}$ (resp. $\mathcal{C}$) a countable set of functions in $\mathcal{C}^\infty_c(Q;\mathbb{R}^d)$ (resp. $\mathcal{C}^\infty_c(Q\times Q;\mathbb{R}^d)$) dense in $W^{1,p}_0(Q;\mathbb{R}^d)$ (resp. $L^p(Q;W^{1,p}_0(Q;\mathbb{R}^d))$). Define L to be the set of Lebesgue points x_0 for all functions

$$f_{\{\varepsilon_n\}}(\cdot\,;\xi),\quad \overline{f}_{\rm hom}(\cdot\,;\xi) \tag{2.63}$$

and

$$x \to \int_Q\int_Q f(x,y,z;\xi+\nabla_y\phi(y)+\nabla_z\psi(y,z))\,dy\,dz, \tag{2.64}$$

with $\phi\in\mathcal{S}$, $\psi\in\mathcal{C}$ and $\xi\in\mathbb{Q}^{d\times N}$, and for which $\overline{f}_{\rm hom}(x_0;\,\cdot\,)$ is well defined. Note that $\mathcal{L}^N(\Omega\setminus L)=0$. Let $x_0\in L$ and $\xi\in\mathbb{Q}^{d\times N}$, then

$$f_{\{\varepsilon_n\}}(x_0;\xi) = \lim_{\delta\to0}\frac{1}{\delta^N}\int_{Q(x_0,\delta)} f_{\{\varepsilon_n\}}(x;\xi)\,dx = \lim_{\delta\to0}\frac{\mathcal{F}_{\{\varepsilon_n\}}(\xi\,\cdot\,;Q(x_0,\delta))}{\delta^N}. \tag{2.65}$$

Given $m\in\mathbb{N}$ consider $\phi_m\in\mathcal{S}$ such that

$$\overline{f}_{\rm hom}(x_0;\xi)+\frac{1}{m} \geqslant \int_Q f_{\rm hom}(x_0,y;\xi+\nabla\phi_m(y))\,dy. \tag{2.66}$$

Then by Theorem III.30 in Castaing and Valadier [18] and following a similar argument as in Lemma 4.6 in Fonseca and Zappale [26], there exist $\Phi_m \in L^p\big(Q; W_0^{1,p}(Q;\mathbb{R}^d)\big)$ such that

$$f_{\text{hom}}(x_0, y; \xi + \nabla\phi_m(y)) + \frac{1}{m} \geqslant \int_Q f(x_0, y, z; \xi + \nabla_y\phi_m(y) + \nabla_z\Phi_m(y,z))\,dz.$$

We now choose $\Phi_{m,k} \in \mathcal{C}$ such that

$$\|\Phi_{m,k} - \Phi_m\|_{L^p(Q;W_0^{1,p}(Q;\mathbb{R}^d))} \xrightarrow[k\to+\infty]{} 0, \tag{2.67}$$

and we extend ϕ_m and $\Phi_{m,k}$ periodically to $\mathbb{R}^N$ and $\mathbb{R}^N \times \mathbb{R}^N$, respectively. For each $x \in \mathbb{R}^N$ define

$$u^n_{m,k}(x) := \xi\cdot x + \varepsilon_n\phi_m\Big(\frac{x}{\varepsilon_n}\Big) + \varepsilon_n^2\Phi_{m,k}\Big(\frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}\Big)$$

and consider $\delta > 0$ small enough so that $Q(x_0,\delta) \in \mathcal{A}(\Omega)$. For fixed m and k we have $u^n_{m,k} \to v$ in $L^p(Q(x_0;\delta);\mathbb{R}^d)$ as $n \to +\infty$, where $v(x) = \xi\cdot x$. Hence by (2.65) and the p-Lipschitz property of $f(x,y,\cdot)$ (see Marcellini [32])

$$\begin{aligned}
f_{\{\varepsilon_n\}}(x_0;\xi) \;&\leqslant\; \liminf_{k,\delta,n}\frac{1}{\delta^N}\int_{Q(x_0;\delta)} f\left(x, \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi + \nabla_y\phi_m\left(\frac{x}{\varepsilon_n}\right)\right.\\
&\qquad\qquad \left. + \varepsilon_n\nabla_y\Phi_{m,k}\Big(\frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}\Big) + \nabla_z\Phi_{m,k}\Big(\frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}\Big)\right)\,dx\\
&\leqslant\; \liminf_{k,\delta,n}\frac{1}{\delta^N}\int_{Q(x_0;\delta)} f\left(x, \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi + \nabla_y\phi_m\left(\frac{x}{\varepsilon_n}\right)\right.\\
&\qquad\qquad \left. + \nabla_z\Phi_{m,k}\Big(\frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}\Big)\right)\,dx.
\end{aligned} \tag{2.68}$$

Arguing as in Proposition 4.10 in Baía and Fonseca [8], we define

$$h_{m,k}(x,y,z) := f\big(x,y,z;\xi + \nabla_y\phi_m(y) + \nabla_z\Phi_{m,k}(y,z)\big).$$

Then (see e.g. Allaire and Briane [2] or Donato [21]) since $h_{m,k} \in L^p(Q(x_0,\delta);\mathcal{C}_{\text{per}}(Q\times Q))$, we get

$$\begin{aligned}
&\liminf_{n\to+\infty}\int_{Q(x_0;\delta)} f\left(x, \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi + \nabla_y\phi_m\left(\frac{x}{\varepsilon_n}\right) + \nabla_z\Phi_{m,k}\Big(\frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}\Big)\right)\,dx\\
&= \lim_{n\to+\infty}\int_{Q(x_0;\delta)} h_{m,k}\left(x, \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}\right)\,dx\\
&= \int_{Q(x_0;\delta)}\int_Q\int_Q h_{m,k}(x,y,z)\,dz\,dy\,dx\\
&= \int_{Q(x_0;\delta)}\int_Q\int_Q f\big(x,y,z;\xi + \nabla_y\phi_m(y) + \nabla_z\Phi_{m,k}(y,z)\big)dz\,dy\,dx.
\end{aligned} \tag{2.69}$$

Therefore by (2.64)

$$\liminf_{\delta\to 0}\;\liminf_{n\to+\infty}\frac{1}{\delta^N}\int_{Q(x_0;\delta)} f\left(x, \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}; \xi + \nabla_y\phi_m\left(\frac{x}{\varepsilon_n}\right) + \nabla_z\Phi_{m,k}\Big(\frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2}\Big)\right)dz\,dy\,dx$$

$$= \int_Q\int_Q f(x_0,y,z;\xi + \nabla_y\phi_m(y) + \nabla_z\Phi_{m,k}(y,z))\,dz\,dy,$$

and thus, by (2.67), (H_4), (2.68), (2.66) and Fubini's Theorem, we obtain

$$\begin{aligned} f_{\{\varepsilon_n\}}(x_0;\xi) &\leqslant \int_Q \int_Q f(x_0,y,z;\xi+\nabla_y\phi_m(y)+\nabla_z\Phi_m(y,z))\,dz\,dy \\ &= \int_Q \left[\int_Q f(x_0,y,z;\xi+\nabla_y\phi_m(y)+\nabla_z\Phi_m(y,z))\,dz\right] dy \\ &\leqslant \int_Q f_{\rm hom}(x_0,y;\xi+\nabla\phi_m(y))\,dy+\frac{1}{m} \\ &\leqslant \overline{f}_{\rm hom}(x_0;\xi)+\frac{2}{m}. \end{aligned}$$

Letting $m\to+\infty$ we deduce that $f_{\{\varepsilon_n\}}(x_0;\xi)\leqslant \overline{f}_{\rm hom}(x_0;\xi)$. ■

3 Application to thin films

This part is devoted to the study of a reiterated homogenization problem in the framework of 3D-2D dimensional reduction. Our main result is stated in Theorem 1.2. We organize this section as follows. In Subsection 3.1 we discuss the main properties of $W_{\rm hom}$ and $\overline{W}_{\rm hom}$. Then, in Subsection 3.2 we address the case where W is independent of the macroscopic in-plane variable x_α (Theorem 3.2). Finally, Theorem 1.2 is proved in Subsection 3.3.

Remark 3.1. Without loss of generality, we assume that W is non negative upon replacing W by $W+\beta$ which is non negative in view of (A_4).

3.1 Properties of $\overline{W}_{\rm hom}$

As in Subsection 2.1 we can see that the function $W_{\rm hom}$ given in (1.12) is well defined and it is (equivalent to) a Carathéodory function:

$$W_{\rm hom}(\cdot,\cdot;\xi) \text{ is } \mathcal{L}^3\otimes\mathcal{L}^2\text{-measurable for all } \xi\in\mathbb{R}^{3\times 3}, \tag{3.1}$$

$$W_{\rm hom}(x,y_\alpha;\cdot) \text{ is continuous for } \mathcal{L}^3\otimes\mathcal{L}^2\text{-a.e. } (x,y_\alpha)\in\Omega\times\mathbb{R}^2. \tag{3.2}$$

By condition (A_3) it follows that

$$W_{\rm hom}(x,\cdot;\xi) \text{ is } Q'\text{-periodic for a.e. } x\in\Omega \text{ and all } \xi\in\mathbb{R}^{3\times 3}. \tag{3.3}$$

Moreover, $W_{\rm hom}$ is quasiconvex in the ξ variable and satisfies the same p-growth and p-coercivity condition (A_4) as W:

$$\frac{1}{\beta}|\xi|^p-\beta\leqslant W_{\rm hom}(x,y_\alpha;\xi)\leqslant\beta(1+|\xi|^p) \quad \text{for a.e. } x\in\Omega \text{ and all } (y_\alpha,\xi)\in\mathbb{R}^2\times\mathbb{R}^{3\times 3}, \tag{3.4}$$

where β is the constant in (A_4). Arguing as in Remark 2.2 in Babadjian and Baía [6], (3.1), (3.2) and (3.4) imply that the function $\overline{W}_{\rm hom}$ given in (1.11) is also well defined, and is (equivalent to) a Carathéodory function, which implies that the definition of $\mathcal{W}_{\rm hom}$ makes sense on $W^{1,p}(\Omega;\mathbb{R}^3)$. Finally, $\overline{W}_{\rm hom}$ is also quasiconvex in the $\overline{\xi}$ variable and satisfies the same p-growth and p-coercivity condition (A_4) as W and $W_{\rm hom}$:

$$\frac{1}{\beta}|\overline{\xi}|^p-\beta\leqslant \overline{W}_{\rm hom}(x_\alpha;\overline{\xi})\leqslant\beta(1+|\overline{\xi}|^p) \quad \text{for a.e. } x_\alpha\in\omega \text{ and all } \overline{\xi}\in\mathbb{R}^{3\times 2}, \tag{3.5}$$

where, as before, β is the constant in (A_4).

3.2 Independence of the in-plane macroscopic variable

In this section, we assume that W does not depend explicitly on x_α, namely $W : I\times\mathbb{R}^3\times\mathbb{R}^2\times\mathbb{R}^{3\times 3} \to \mathbb{R}^+$. For each $\varepsilon > 0$, consider the functional $\mathcal{W}_\varepsilon : L^p(\Omega;\mathbb{R}^3) \to [0,+\infty]$ defined by

$$\mathcal{W}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega W\left(x_3, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}; \nabla_\alpha u(x)\Big|\frac{1}{\varepsilon}\nabla_3 u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.6}$$

Our objective is to prove the following result.

Theorem 3.2. *Under assumptions (A_1)-(A_4) the $\Gamma(L^p(\Omega))$-limit of the family $\{\mathcal{W}_\varepsilon\}_{\varepsilon>0}$ is given by*

$$\mathcal{W}_{\rm hom}(u) = \begin{cases} 2\displaystyle\int_\omega \overline{W}_{\rm hom}(\nabla_\alpha u(x_\alpha))\, dx_\alpha & \text{if } u \in W^{1,p}(\omega;\mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\overline{W}_{\rm hom}$ is defined, for all $\bar\xi \in \mathbb{R}^{3\times 2}$, by

$$\overline{W}_{\rm hom}(\bar\xi) := \lim_{T\to+\infty} \inf_\phi \Big\{ \frac{1}{2T^2}\int_{(0,T)^2\times I} W_{\rm hom}(y_3, y_\alpha; \bar\xi + \nabla_\alpha\phi(y)|\nabla_3\phi(y))\, dy : \\ \phi \in W^{1,p}((0,T)^2\times I;\mathbb{R}^3) \text{ and } \phi = 0 \text{ on } \partial(0,T)^2\times I\Big\}, \tag{3.7}$$

and

$$W_{\rm hom}(y_3, y_\alpha; \xi) := \lim_{T\to+\infty} \inf_\phi \Big\{ \frac{1}{T^3}\int_{(0,T)^3} W(y_3, y_\alpha, z_3, z_\alpha; \xi + \nabla\phi(z))\, dz : \\ \phi \in W^{1,p}_0((0,T)^3;\mathbb{R}^3)\Big\}, \tag{3.8}$$

for all $(y,\xi) \in \mathbb{R}^3\times\mathbb{R}^{3\times 3}$.

Since the proofs are very similar to that of Section 2.2, we just sketch them highlighting the main differences. For the detailed proofs we refer to Chapter 2 in Babadjian [5] .

3.2.1 Existence and integral representation of the Γ-limit

For the same reason than in the proof of Theorem 2.2 in Section 2.2, we localize the functionals given in (3.6) on the class of bounded open subsets of $\mathbb{R}^2$, denoted by $\mathcal{A}_0$. For each $\varepsilon > 0$, consider $\mathcal{W}_\varepsilon$: $L^p(\mathbb{R}^2\times I;\mathbb{R}^3)\times\mathcal{A}_0 \to [0,+\infty]$ defined by

$$\mathcal{W}_\varepsilon(u;A) := \begin{cases} \displaystyle\int_{A\times I} W\left(x_3, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}; \nabla_\alpha u(x)\Big|\frac{1}{\varepsilon}\nabla_3 u(x)\right) dx & \text{if } u \in W^{1,p}(A\times I;\mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.9}$$

Given $\{\varepsilon_j\} \searrow 0^+$ and $A \in \mathcal{A}_0$, consider the Γ-lower limit of $\{\mathcal{W}_{\varepsilon_j}(\cdot\,;A)\}_{j\in\mathbb{N}}$ for the $L^p(A\times I;\mathbb{R}^3)$-topology, defined for $u \in L^p(\mathbb{R}^2\times I;\mathbb{R}^3)$, by

$$\mathcal{W}_{\{\varepsilon_j\}}(u;A) := \inf_{\{u_j\}}\left\{\liminf_{j\to+\infty} \mathcal{W}_{\varepsilon_j}(u_j;A) : \quad u_j \to u \text{ in } L^p(A\times I;\mathbb{R}^3)\right\}.$$

In view of the p-coercivity condition (A_4), for each $A \in \mathcal{A}_0$ it follows that $\mathcal{W}_{\{\varepsilon_j\}}(u;A)$ is infinite whenever $u \in L^p(\mathbb{R}^2\times I;\mathbb{R}^3)\setminus W^{1,p}(A;\mathbb{R}^3)$, so it suffices to study the case where $u \in W^{1,p}(A;\mathbb{R}^3)$. Arguing as in the proof of Lemma 2.6 in Braides, Fonseca and Francfort [16], we can prove the existence of a subsequence $\{\varepsilon_{j_n}\} \equiv \{\varepsilon_n\}$ such that $\mathcal{W}_{\{\varepsilon_n\}}(\cdot\,;A)$ is the $\Gamma(L^p(A\times I)$-limit of $\{\mathcal{W}_{\varepsilon_n}(\cdot\,;A)\}_{n\in\mathbb{N}}$ for each $A \in \mathcal{A}_0$. In addition, following the lines of Lemma 2.5 in Braides, Fonseca and Francfort [16], it is possible to show the following result.

Lemma 3.3. *For each $A \in \mathcal{A}_0$ and all $u \in W^{1,p}(A;\mathbb{R}^3)$, the restriction of $\mathcal{W}_{\{\varepsilon_n\}}(u;\cdot)$ to $\mathcal{A}(A)$ is a Radon measure, absolutely continuous with respect to the two-dimensional Lebesgue measure.*

But as in Section 2.2.1, one has to ensure that the integral representation given by Theorem 1.1 in Buttazzo-Dal Maso [17] is independent of the open set $A \in \mathcal{A}_0$. The following result, prevents this dependence from holding since it leads to an homogeneous integrand as it will be seen in Lemma 3.5 below.

Lemma 3.4. *For all $\overline{\xi} \in \mathbb{R}^{3\times 2}$, y^0_α and $z^0_\alpha \in \mathbb{R}^2$, and $\delta > 0$*

$$\mathcal{W}_{\{\varepsilon_n\}}(\overline{\xi}\,\cdot\,; Q'(y^0_\alpha,\delta)) = \mathcal{W}_{\{\varepsilon_n\}}(\overline{\xi}\,\cdot\,; Q'(z^0_\alpha,\delta)).$$

Proof. It is obviously enough to show that

$$\mathcal{W}_{\{\varepsilon_n\}}(\overline{\xi}\,\cdot\,; Q'(y^0_\alpha,\delta)) \geqslant \mathcal{W}_{\{\varepsilon_n\}}(\overline{\xi}\,\cdot\,; Q'(z^0_\alpha,\delta)).$$

According to Theorem 1.1 in Bocea and Fonseca [11] together with Lemma 2.6 in Braides, Fonseca and Francfort [16], there exists a sequence $\{u_n\} \subset W^{1,p}(Q'(y^0_\alpha,\delta)\times I;\mathbb{R}^3)$ such that $\left\{\left|\left(\nabla_\alpha u_n\middle|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right|^p\right\}$ is equi-integrable, $u_n = 0$ on $\partial Q'(y^0_\alpha,\delta)\times I$, $u_n \to 0$ in $L^p(Q'(y^0_\alpha,\delta)\times I;\mathbb{R}^3)$ and

$$\mathcal{W}_{\{\varepsilon_n\}}(\overline{\xi}\cdot; Q'(y^0_\alpha,\delta)) = \lim_{n\to+\infty}\int_{Q'(y^0_\alpha,\delta)\times I} W\left(x_3,\frac{x}{\varepsilon_n},\frac{x_\alpha}{\varepsilon_n^2};\overline{\xi}+\nabla_\alpha u_n(x)\middle|\frac{1}{\varepsilon_n}\nabla_3 u_n(x)\right)dx.$$

We argue exactly as in the proof of Lemma 2.5 with y^0_α and z^0_α in place of y_0 and z_0. For all $n \in \mathbb{N}$, extend u_n by zero to the whole $\mathbb{R}^2\times I$ and set $v_n(x_\alpha,x_3) = u_n(x_\alpha + x^{\varepsilon_n}_\alpha, x_3)$ for $(x_\alpha,x_3) \in Q'(z^0_\alpha,\delta)\times I$, where $x^{\varepsilon_n}_\alpha := m_{\varepsilon_n}\varepsilon_n - \varepsilon_n^2 l_{\varepsilon_n}$. Then $\{v_n\} \subset W^{1,p}(Q'(z^0_\alpha,\delta)\times I;\mathbb{R}^3)$, $v_n \to 0$ in $L^p(Q'(z^0_\alpha,\delta)\times I;\mathbb{R}^3)$, the sequence $\left\{\left|\left(\nabla_\alpha v_n\middle|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right|^p\right\}$ is equi-integrable and

$$\begin{aligned}
&\mathcal{W}_{\{\varepsilon_n\}}(\overline{\xi}\cdot; Q'(y^0_\alpha,\delta))\\
&\qquad = \limsup_{n\to+\infty}\int_{Q'(z^0_\alpha,\delta)\times I} W\left(x_3,\frac{x_\alpha}{\varepsilon_n}-\varepsilon_n l_{\varepsilon_n},\frac{x_3}{\varepsilon_n},\frac{x_\alpha}{\varepsilon_n^2};\overline{\xi}+\nabla_\alpha v_n(x)\middle|\frac{1}{\varepsilon_n}\nabla_3 v_n(x)\right)dx, \quad (3.10)
\end{aligned}$$

where we have used the p-growth condition (A_4) and the fact that $\mathcal{L}^2(Q'(z^0_\alpha,\delta)\setminus Q'(y^0_\alpha - x^{\varepsilon_n}_\alpha,\delta)) \to 0$. To eliminate the term $\varepsilon_n\, l_{\varepsilon_n}$ in (3.10), we would like to apply a uniform continuity argument. Since for a.e. $x_3 \in I$ the function $W(x_3,\cdot,\cdot;\cdot)$ is continuous on $\mathbb{R}^3\times\mathbb{R}^2\times\mathbb{R}^{3\times 3}$, then (A_3) implies that it is uniformly continuous on $\mathbb{R}^3\times\mathbb{R}^2\times\overline{B}(0,\lambda)$ for any $\lambda > 0$. We define

$$R^\lambda_n := \left\{x \in Q'(z^0_\alpha,\delta)\times I : \left|\left(\overline{\xi}+\nabla_\alpha v_n(x)\middle|\frac{1}{\varepsilon_n}\nabla_3 v_n(x)\right)\right| \leqslant \lambda\right\},$$

and we note that by Chebyshev's inequality

$$\mathcal{L}^3([Q'(z^0_\alpha,\delta)\times I]\setminus R^\lambda_n) \leqslant C/\lambda^p, \tag{3.11}$$

for some constant $C > 0$ independent of λ or n. Thus, in view of (3.10) and the fact that W is nonnegative,

$$\mathcal{W}_{\{\varepsilon_n\}}(\overline{\xi}\cdot; Q'(y^0_\alpha,\delta)) \geqslant \limsup_{\lambda,n}\int_{R^\lambda_n} W\left(x_3,\frac{x_\alpha}{\varepsilon_n}-\varepsilon_n l_{\varepsilon_n},\frac{x_3}{\varepsilon_n},\frac{x_\alpha}{\varepsilon_n^2};\overline{\xi}+\nabla_\alpha v_n(x)\middle|\frac{1}{\varepsilon_n}\nabla_3 v_n(x)\right)dx.$$

Denoting by $\omega_\lambda(x_3,\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ the modulus of continuity of $W(x_3,\cdot,\cdot;\cdot)$ on $\mathbb{R}^3\times\mathbb{R}^2\times\overline{B}(0,\lambda)$, we can check that for a.e. $x_3 \in I$, the function $t \mapsto \omega_\lambda(x_3,t)$ is continuous, increasing and satisfies $\omega_\lambda(x_3,0) = 0$ while, for all $t \in \mathbb{R}^+$, the function $x_3 \mapsto \omega_\lambda(x_3,t)$ is measurable (as the supremum of measurable functions). We get, for any $x \in R^\lambda_n$

$$\begin{aligned}
&\left|W\left(x_3,\frac{x}{\varepsilon_n},\frac{x_\alpha}{\varepsilon_n^2};\overline{\xi}+\nabla_\alpha v_n(x)\middle|\frac{1}{\varepsilon_n}\nabla_3 v_n(x)\right)\right.\\
&\qquad\left.-W\left(x_3,\frac{x_\alpha}{\varepsilon_n}-\varepsilon_n l_{\varepsilon_n},\frac{x_3}{\varepsilon_n},\frac{x_\alpha}{\varepsilon_n^2};\overline{\xi}+\nabla_\alpha v_n(x)\middle|\frac{1}{\varepsilon_n}\nabla_3 v_n(x)\right)\right|\\
&\leqslant \omega_\lambda(x_3,\varepsilon_n l_{\varepsilon_n}).
\end{aligned}$$

The properties of ω_λ, Beppo-Levi's Monotone Convergence Theorem and (3.10) yield

$$\begin{aligned}
\mathcal{W}_{\{\varepsilon_n\}}(\bar{\xi}\cdot\,; Q'(y^0_\alpha,\delta)) &\geqslant \limsup_{\lambda,n}\left\{\int_{R_n^\lambda} W\left(x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{\xi} + \nabla_\alpha v_n(x)\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n(x)\right) dx \right. \\
&\qquad\qquad \left. -\delta^2 \int_{-1}^{1} \omega_\lambda(x_3, \varepsilon_n l_{\varepsilon_n})\, dx_3\right\} \\
&= \liminf_{n\to+\infty} \int_{Q'(z^0_\alpha,\delta)\times I} W\left(x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{\xi} + \nabla_\alpha v_n(x)\Big|\frac{1}{\varepsilon_n}\nabla_3 v_n(x)\right) dx \\
&\geqslant \mathcal{W}_{\{\varepsilon_n\}}(\bar{\xi}\cdot\,; Q'(z^0_\alpha,\delta)),
\end{aligned}$$

where we have used the equi-integrability of $\left\{\left|\left(\nabla_\alpha v_n\big|\frac{1}{\varepsilon_n}\nabla_3 v_n\right)\right|^p\right\}$, the p-growth condition (A_4), (3.11) and the fact that $v_n \to 0$ in $L^p(Q'(z^0_\alpha,\delta)\times I;\mathbb{R}^3)$. ■

As a consequence of this lemma and adapting the argument used in the proof of Lemma 2.6, we deduce that

Lemma 3.5. *There exists a continuous function $W_{\{\varepsilon_n\}} : \mathbb{R}^{3\times 2} \to \mathbb{R}^+$ such that for all $A \in \mathcal{A}_0$ and all $u \in W^{1,p}(A;\mathbb{R}^3)$,*

$$\mathcal{W}_{\{\varepsilon_n\}}(u;A) = 2\int_A W_{\{\varepsilon_n\}}(\nabla_\alpha u(x_\alpha))\, dx_\alpha.$$

3.2.2 Characterization of the Γ-limit

In view of Lemma 3.5, we only need to prove that $\overline{W}_{\rm hom}(\bar{\xi}) = W_{\{\varepsilon_n\}}(\bar{\xi})$ for all $\bar{\xi} \in \mathbb{R}^{3\times 2}$, and thus it suffices to work with affine functions instead of with general Sobolev functions.

We state, without proof, an equivalent result to Proposition 2.7 for the dimension reduction case.

Proposition 3.6. *Given $M > 0$, $\eta > 0$, and $\varphi : [0,+\infty) \to [0,+\infty]$ a continuous and increasing function satisfying $\varphi(t)/t \to +\infty$ as $t \to +\infty$, there exists $\varepsilon_0 \equiv \varepsilon_0(M,\eta) > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, every $a \in \mathbb{R}^2$ and every $u \in W^{1,p}((a+Q')\times I;\mathbb{R}^3)$ with*

$$\int_{(a+Q')\times I} \varphi(|\nabla u|^p)\, dx \leqslant M,$$

there exists $v \in W^{1,p}_0((a+Q')\times I;\mathbb{R}^3)$ with $\|v\|_{L^p((a+Q')\times I;\mathbb{R}^3)} \leqslant \eta$ satisfying

$$\int_{(a+Q')\times I} W\Big(x_3, x_\alpha, \frac{x_3}{\varepsilon}, \frac{x_\alpha}{\varepsilon}; \nabla u\Big)\, dx \geqslant \int_{(a+Q')\times I} W_{\rm hom}(x_3, x_\alpha; \nabla u + \nabla v)\, dx - \eta.$$

Lemma 3.7. *For all $\bar{\xi} \in \mathbb{R}^{3\times 2}$, $\overline{W}_{\rm hom}(\bar{\xi}) \leqslant W_{\{\varepsilon_n\}}(\bar{\xi})$.*

Proof. From Lemma 3.5, Theorem 1.1 in Bocea and Fonseca [11] and Lemma 2.6 in Braides, Fonseca and Francfort [16], we may find a sequence $\{w_n\} \subset W^{1,p}(Q'\times I;\mathbb{R}^3)$ such that $\left\{\left|\left(\nabla_\alpha w_n\big|\frac{1}{\varepsilon_n}\nabla_3 w_n\right)\right|^p\right\}$ is equi-integrable, $w_n = 0$ on $\partial Q'\times I$, $w_n \to 0$ in $L^p(Q'\times I;\mathbb{R}^3)$ and

$$2W_{\{\varepsilon_n\}}(\bar{\xi}) = \lim_{n\to+\infty} \int_{Q'\times I} W\left(x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{{\varepsilon_n}^2}; \bar{\xi} + \nabla_\alpha w_n(x)\Big|\frac{1}{\varepsilon_n}\nabla_3 w_n(x)\right) dx.$$

Thus, from De La Vallée Poussin criterion (see e.g. Proposition 1.27 in Ambrosio, Fusco and Pallara [3]) there exists an increasing continuous function $\varphi : [0,+\infty) \to [0,+\infty]$ satisfying $\varphi(t)/t \to +\infty$ as $t \to +\infty$ and such that

$$\sup_{n\in\mathbb{N}} \int_{Q'\times I} \varphi\left(\left|\left(\nabla_\alpha w_n\Big|\frac{1}{\varepsilon_n}\nabla_3 w_n\right)\right|^p\right) dx \leqslant 1.$$

Changing variables yields

$$W_{\{\varepsilon_n\}}(\bar{\xi}) = \lim_{n\to+\infty} \frac{1}{2T_n^2} \int_{(0,T_n)^2\times I} W\left(x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{\xi} + \nabla_\alpha z_n(x) | \nabla_3 z_n(x)\right) dx$$

and

$$\sup_{n\in\mathbb{N}} \frac{1}{T_n^2} \int_{(0,T_n)^2\times I} \varphi(|\nabla z_n|^p)\, dx \leqslant 1,$$

where we set $T_n := 1/\varepsilon_n$ and $z_n(x) := T_n w_n(x_\alpha/T_n, x_3)$. Note that $z_n \in W^{1,p}((0,T_n)^2 \times I;\mathbb{R}^3)$ and $z_n = 0$ on $\partial(0,T_n)^2 \times I$. For all $n \in \mathbb{N}$, define $I_n := \left\{1, \cdots, [\![T_n]\!]^2\right\}$ and for any $i \in I_n$, take $a_i^n \in \mathbb{Z}^2$ such that

$$\bigcup_{i\in I_n} (a_i^n + Q') \subset (0,T_n)^2.$$

Moreover, for all $M > 0$, let

$$I_n^M := \left\{ i \in I_n : \int_{(a_i^n+Q')\times I} \varphi(|\nabla z_n|^p)\, dx \leqslant M \right\}.$$

Applying Proposition 3.6, we get for any $\eta > 0$ and any $i \in I_n^M$ the existence of $v_i^{n,M,\eta} \in W_0^{1,p}((a_i^n + Q') \times I;\mathbb{R}^3)$ with $\|v_i^{n,M,\eta}\|_{L^p((a_i^n+Q')\times I;\mathbb{R}^3)} \leqslant \eta$ and

$$\begin{aligned}
&\int_{(a_i^n+Q'\times I} W\left(x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + \nabla_\alpha z_n | \nabla_3 z_n\right) dx\\
&\qquad \geqslant \frac{1}{T_n^2} \int_{(a_i^n+Q')\times I} W_{\rm hom}\left(x_3, x_\alpha; \bar{\xi} + \nabla_\alpha z_n + \nabla_\alpha v_i^{n,M,\eta} | \nabla_3 z_n + \nabla_3 v_i^{n,M,\eta}\right) dx - \eta.
\end{aligned}$$

Hence,

$$W_{\{\varepsilon_n\}}(\bar{\xi}) \geqslant \limsup_{M,\eta,n} \frac{1}{2T_n^2} \sum_{i\in I_n^M} \int_{(a_i^n+Q')\times I} W_{\rm hom}\left(x_3, x_\alpha; (\bar{\xi}|0) + \nabla \phi^{n,M,\eta}\right) dx \tag{3.12}$$

where $\phi^{n,M,\eta} \in W^{1,p}((0;T_n)^2 \times I;\mathbb{R}^3)$ is defined by

$$\phi^{n,M,\eta}(x) := \begin{cases} z_n(x) + v_i^{n,M,\eta}(x) & \text{if } x \in (a_i^n + Q') \times I \text{ and } i \in I_n^M,\\[2mm] z_n(x) & \text{otherwise} \end{cases}$$

and satisfies $\phi^{n,M,\eta} = 0$ on $\partial(0,T_n)^2 \times I$. In view of the definition of $\phi^{n,M,\eta}$, the p-growth condition (3.4) and the equi-integrability of $\left\{\left|\left(\nabla_\alpha w_n\middle|\frac{1}{\varepsilon_n}\nabla_3 w_n\right)\right|^p\right\}$, we get arguing exactly as in Lemma 2.8,

$$W_{\{\varepsilon_n\}}(\bar{\xi}) \geqslant \limsup_{M,\eta,n} \frac{1}{2T_n^2} \int_{(0,T_n)^2\times I} W_{\rm hom}(x_3, x_\alpha; \bar{\xi} + \nabla_\alpha \phi^{n,M,\eta} | \nabla_3 \phi^{n,M,\eta})\, dx \geqslant \overline{W}_{\rm hom}(\bar{\xi}).$$

■

Let us now prove the converse inequality.

Lemma 3.8. *For all $\xi \in \mathbb{R}^{3\times 2}$, $\overline{W}_{\rm hom}(\bar{\xi}) \geqslant W_{\{\varepsilon_n\}}(\bar{\xi})$.*

Proof. In view of (3.7), for $\delta > 0$ fixed take $T \equiv T_\delta \in \mathbb{N}$, with $T_\delta \to +\infty$ as $\delta \to 0$, and $\phi \equiv \phi_\delta \in W^{1,p}((0,T)^2 \times I;\mathbb{R}^3)$ be such that $\phi = 0$ on $\partial(0,T)^2 \times I$ and

$$\overline{W}_{\rm hom}(\bar{\xi}) + \delta \geqslant \frac{1}{2T^2} \int_{(0,T)^2\times I} W_{\rm hom}(x_3, x_\alpha; \bar{\xi} + \nabla_\alpha \phi(x) | \nabla_\alpha \phi(x))\, dx. \tag{3.13}$$

From Theorem 1.1 in Baía and Fonseca [8] (with $f(y,z;\xi) = W(y_3, y_\alpha, z_3, z_\alpha; \xi)$), Proposition 11.7 in Braides and Defranceschi [14] and Decomposition Lemma (see Fonseca and Leoni [24] or Fonseca, Müller and Pedregal [25]) there exists $\{\phi_n\} \subset W_0^{1,p}((0,T)^2 \times I; \mathbb{R}^3)$ such that $\{|\nabla\phi_n|^p\}$ is equi-integrable, $\phi_n \to \phi$ in $L^p((0,T)^2 \times I; \mathbb{R}^3)$ and

$$\int_{(0,T)^2\times I} W_{\rm hom}(x_3, x_\alpha; \bar\xi + \nabla_\alpha\phi(x)|\nabla_3\phi(x))\,dx$$
$$= \lim_{n\to+\infty}\int_{(0,T)^2\times I} W\left(x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \bar\xi + \nabla_\alpha\phi_n(x)|\nabla_3\phi_n(x)\right) dx. \tag{3.14}$$

Fix $n \in \mathbb{N}$ such that $\varepsilon_n \ll 1$. For all $i \in \mathbb{Z}^2$ let $a_i^n \in \varepsilon_n\mathbb{Z}^2 \cap (i(T+1) + [0,\varepsilon_n)^2)$ (uniquely defined). Set

$$\tilde\phi_n(x) := \begin{cases} \phi_n(x_\alpha - a_i^n, x_3) & \text{if } x \in (a_i^n + (0,T)^2)\times I \text{ and } i \in \mathbb{Z}^2,\\ 0 & \text{otherwise,}\end{cases}$$

then $\tilde\phi_n \in W^{1,p}(\mathbb{R}^2 \times I; \mathbb{R}^3)$. Let $I_n := \{i \in \mathbb{Z}^2 : (0, T/\varepsilon_n)^2 \cap (a_i^n + (0,T)^2) \neq \emptyset\}$. If $\psi_n(x) := \varepsilon_n\tilde\phi_n(x_\alpha/\varepsilon_n, x_3)$ then $\psi_n \to 0$ in $L^p((0,T)^2 \times I; \mathbb{R}^3)$, as $n \to +\infty$. Consequently, the p-growth condition (A_4) implies that

$$\begin{aligned}
&\mathcal{W}_{\{\varepsilon_n\}}(\bar\xi\cdot\,; (0,T)^2)\\
&\leqslant \liminf_{n\to+\infty}\int_{(0,T)^2\times I} W\left(x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar\xi + \nabla_\alpha\psi_n(x)\Big|\frac{1}{\varepsilon_n}\nabla_3\psi_n(x)\right) dx\\
&\leqslant \liminf_{n\to+\infty}\varepsilon_n^2\left\{\sum_{i\in I_n}\int_{(a_i^n+(0,T)^2)\times I} W\left(x_3, x_\alpha, \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \bar\xi + \nabla_\alpha\tilde\phi_n(x)|\nabla_3\tilde\phi_n(x)\right) dx\right.\\
&\qquad\left. + c\mathcal{L}^2\left(\left(0, \frac{T}{\varepsilon_n}\right)^2 \setminus \bigcup_{i\in I_n}(a_i^n + (0,T)^2)\right)\right\}\\
&= \liminf_{n\to+\infty}\varepsilon_n^2\sum_{i\in I_n}\int_{(0,T)^2\times I} W\left(x_3, x_\alpha + a_i^n - i(T+1), \frac{x_3}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n}; \bar\xi + \nabla_\alpha\phi_n(x)|\nabla_3\phi_n(x)\right) dx\\
&\qquad + cT^2\left(1 - \left(\frac{T}{T+1}\right)^2\right)
\end{aligned} \tag{3.15}$$

where we have used (A_3), the fact that $T \in \mathbb{N}$ and $a_i^n/\varepsilon_n \in \mathbb{Z}^2$. We now use the same uniform continuity argument than in the proof of Lemmas 3.4 and 2.9, we get

$$W_{\{\varepsilon_n\}}(\bar\xi) \leqslant \overline{W}_{\rm hom}(\bar\xi) + \delta + c\left(1 - \left(\frac{T}{T+1}\right)^2\right).$$

The result follows by letting δ tend to zero. ■

Proof of Theorem 3.2. From Lemma 3.7 and Lemma 3.8, we conclude that $\overline{W}_{\rm hom}(\bar\xi) = W_{\{\varepsilon_n\}}(\bar\xi)$ for all $\bar\xi \in \mathbb{R}^{3\times 2}$. As a consequence, $\mathcal{W}_{\{\varepsilon_n\}}(u; A) = \mathcal{W}_{\rm hom}(u; A)$ for all $A \in \mathcal{A}_0$ and all $u \in W^{1,p}(A; \mathbb{R}^3)$. Since the Γ-limit does not depend upon the extracted subsequence, Proposition 8.3 in Dal Maso [19] implies that the whole sequence $\mathcal{W}_\varepsilon(\cdot\,; A)$ $\Gamma(L^p(A \times I))$-converges to $\mathcal{W}_{\rm hom}(\cdot\,; A)$. ■

3.3 The general case

Our aim here is to study the case where the function W depends also on the in-plane variable.

3.3.1 Existence and integral representation of the Γ-limit

As in Subsection 2.3, to prove Theorem 1.2 it is convenient to localize the functionals $\mathcal{W}_\varepsilon$ in (1.10) on the class of all bounded open subsets of ω, denoted by $\mathcal{A}(\omega)$. For each $\varepsilon > 0$ we consider the family of functionals $\mathcal{W}_\varepsilon : L^p(\Omega;\mathbb{R}^3) \times \mathcal{A}(\omega) \to [0,+\infty]$ defined by

$$\mathcal{W}_\varepsilon(u;A) := \begin{cases} \displaystyle\int_{A\times I} W\left(x, \frac{x}{\varepsilon}, \frac{x_\alpha}{\varepsilon^2}; \nabla_\alpha u(x) \Big| \frac{1}{\varepsilon}\nabla_3 u(x)\right) dx & \text{if } u \in W^{1,p}(A\times I;\mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.16}$$

Given $\{\varepsilon_j\} \searrow 0^+$ and $A \in \mathcal{A}(\omega)$ we define the Γ-lower limit of $\{\mathcal{W}_{\varepsilon_j}(\cdot\,;A)\}_{j\in\mathbb{N}}$ with respect to the $L^p(A\times I;\mathbb{R}^3)$-topology by

$$\mathcal{W}_{\{\varepsilon_j\}}(u;A) := \inf_{\{u_j\}} \left\{ \liminf_{j\to+\infty} \mathcal{W}_{\varepsilon_j}(u_j;A) : \quad u_j \to u \text{ in } L^p(A\times I;\mathbb{R}^3) \right\}$$

for all $u \in L^p(\Omega;\mathbb{R}^3)$. Our main objective is to show that

$$\mathcal{W}_{\{\varepsilon_j\}} = \mathcal{W}_{\rm hom} \tag{3.17}$$

where $\mathcal{W}_{\rm hom} : L^p(\Omega;\mathbb{R}^3) \times \mathcal{A}(\omega) \to [0,+\infty]$ is given by

$$\mathcal{W}_{\rm hom}(u;A) = \begin{cases} 2\displaystyle\int_A \overline{W}_{\rm hom}(x_\alpha; \nabla_\alpha u(x_\alpha))\, dx_\alpha & \text{if } u \in W^{1,p}(A;\mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

The conclusion of Theorem 1.2 would follow taking $A = \omega$. By hypotheses (A_4) it follows that $\mathcal{W}_{\{\varepsilon_j\}}(u;A) = +\infty$ for each $A \in \mathcal{A}(\omega)$ whenever $u \in L^p(\Omega;\mathbb{R}^3) \setminus W^{1,p}(A;\mathbb{R}^3)$. As a consequence of Theorem 2.5 in Braides, Fonseca and Francfort [16], given $\{\varepsilon_j\} \searrow 0^+$ there exists a subsequence $\{\varepsilon_{j_n}\} \equiv \{\varepsilon_n\}$ of $\{\varepsilon_j\}$ for which the functional $\mathcal{W}_{\{\varepsilon_n\}}(\cdot\,;A)$ is the $\Gamma(L^p(A\times I))$-limit of $\{\mathcal{W}_{\varepsilon_n}(\cdot\,;A)\}_{n\in\mathbb{N}}$ for each $A \in \mathcal{A}(\omega)$. Moreover given $u \in W^{1,p}(A;\mathbb{R}^3)$

$$\mathcal{W}_{\{\varepsilon_n\}}(u;A) = 2\int_A W_{\{\varepsilon_n\}}(x_\alpha; \nabla_\alpha(x_\alpha))\, dx_\alpha,$$

for some Carathéodory function $W_{\{\varepsilon_n\}} : \omega \times \mathbb{R}^{3\times 2} \to \mathbb{R}$. Accordingly, to prove equality (3.17) it suffices to show that $W_{\{\varepsilon_n\}}(x_\alpha;\bar\xi) = \overline{W}_{\rm hom}(x_\alpha;\bar\xi)$ for a.e. $x_\alpha \in \omega$ and all $\bar\xi \in \mathbb{R}^{3\times 2}$, which allow us to work with affine functions instead of with general Sobolev functions.

The following proposition, that is of use in the sequel, allow us to extend continuously Carathéodory integrands. It relies on Scorza-Dragoni's Theorem (see Ekeland and Temam [22]) and on Tietze's Extension Theorem (see Theorem 3.1 in DiBenedetto [20]).

Proposition 3.9. *Let $W : \Omega \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfying (A_1)-(A_4). Then for any $m \in \mathbb{N}$, there exists a compact set $C_m \subset \Omega$ and a continuous function $W^m : \Omega \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ such that $W^m(x,\cdot,\cdot\,;\cdot) = W(x,\cdot,\cdot\,;\cdot)$ for all $x \in C_m$ and*

$$\mathcal{L}^3(\Omega \setminus C_m) < \frac{1}{m}. \tag{3.18}$$

Moreover,

- $y_\alpha \mapsto W^m(x, y_\alpha, y_3, z_\alpha; \xi)$ *is Q'-periodic for all $(z_\alpha, y_3, \xi) \in \mathbb{R}^3 \times \mathbb{R}^{3\times 3}$ and a.e. $x \in \Omega$,*
- $(z_\alpha, y_3) \mapsto W^m(x, y_\alpha, y_3, z_\alpha; \xi)$ *is Q-periodic for all $(y_\alpha, \xi) \in \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$ and a.e. $x \in \Omega$;*

and for some $\beta > 0$, we have

$$-\beta \leqslant W^m(x,y,z_\alpha;\xi) \leqslant \beta(1+|\xi|^p) \quad \textit{for all } (y,z_\alpha,\xi) \in \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3} \textit{ and a.e } x \in \Omega. \tag{3.19}$$

Proof. By Scorza-Dragoni's Theorem (see Ekeland and Temam [22]) for any $m \in \mathbb{N}$ there exists a compact set $C_m \subset \Omega$ with $\mathcal{L}^3(\Omega \setminus C_m) < 1/m$ such that W is continuous on $C_m \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$. Since $C_m \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$ is a closed set, according to Tietze's Extension Theorem (see DiBenedetto [20]) one can extend W into a continuous function W^m outside $C_m \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$. By the construction of W^m it can be seen that it satisfies the same periodicity and growth condition than W and that it is bounded from below by $-\beta$. ■

We remark that the above result improve Lemma 4.1 in Babadjian and Baía [6] in which we only obtained a separately continuous function.

3.3.2 Characterization of the Γ-limit

For each $T > 0$ consider $\mathcal{S}_T$ a countable set of functions in $\mathcal{C}^\infty([0,T]^2 \times [-1,1];\mathbb{R}^3)$ that is dense in

$$\{\varphi \in W^{1,p}((0,T)^2 \times I;\mathbb{R}^3) : \ \varphi = 0 \ \text{ on } \ \partial(0,T)^2 \times I\}.$$

Let L be the set of Lebesgue points x_α^0 for all functions $W_{\{\varepsilon_n\}}(\cdot;\overline{\xi})$, $\overline{W}_{\text{hom}}(\cdot;\overline{\xi})$ and

$$x_\alpha \mapsto \int_{(0,T)^2\times I} W_{\text{hom}}(x_\alpha, y_3, y_\alpha; \overline{\xi} + \nabla_\alpha \varphi(y)|\nabla_3\varphi(y))\, dy,$$

with $T \in \mathbb{N}$, $\varphi \in \mathcal{S}_T$ and $\overline{\xi} \in \mathbb{Q}^{3\times 2}$, and for which $\overline{W}_{\text{hom}}(x_\alpha^0; \cdot\,)$ is well defined. Note that $\mathcal{L}^2(\omega \setminus L) = 0$.

We start by proving the following inequality.

Lemma 3.10. *For all $x_\alpha^0 \in L$ and all $\overline{\xi} \in \mathbb{Q}^{3\times 2}$, we have $W_{\{\varepsilon_n\}}(x_\alpha^0;\overline{\xi}) \geqslant \overline{W}_{\text{hom}}(x_\alpha^0;\overline{\xi})$.*

Proof. Let $\delta > 0$ small enough so that $Q'(x_\alpha^0,\delta) \in \mathcal{A}(\omega)$. By Theorem 1.1 in Bocea and Fonseca [11] we can find a sequence $\{u_n\} \subset W^{1,p}(Q'(x_\alpha^0,\delta) \times I;\mathbb{R}^3)$ with $u_n \to 0$ in $L^p(Q'(x_\alpha^0,\delta) \times I;\mathbb{R}^3)$, such that the sequence of scaled gradients $\left\{\left(\nabla_\alpha u_n|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right\}$ is p-equi-integrable and

$$\begin{aligned}
\mathcal{W}_{\{\varepsilon_n\}}(\overline{\xi}\cdot\,; Q'(x_\alpha^0,\delta)) &= 2\int_{Q'(x_\alpha^0,\delta)} W_{\{\varepsilon_n\}}(x_\alpha;\overline{\xi})\, dx_\alpha \\
&= \lim_{n\to+\infty} \int_{Q'(x_\alpha^0,\delta)\times I} W\left(x, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \overline{\xi} + \nabla_\alpha u_n(x)\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n(x)\right) dx.
\end{aligned}$$

Given $m \in \mathbb{N}$ let C_m and W^m be given by Proposition 3.9. Then since $W \geqslant 0$ and $W = W^m$ on $C_m \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$ we get

$$\mathcal{W}_{\{\varepsilon_n\}}(\overline{\xi}\cdot\,; Q'(x_\alpha^0,\delta)) \geqslant \limsup_{m,n} \int_{[Q'(x_\alpha^0,\delta)\times I]\cap C_m} W^m\left(x, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \overline{\xi} + \nabla_\alpha u_n(x)\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n(x)\right) dx.$$

By the p-growth condition (3.19), the equi-integrability of $\left\{\left|\left(\nabla_\alpha u_n|\frac{1}{\varepsilon_n}\nabla_3 u_n\right)\right|^p\right\}$ and relation (3.18), we obtain

$$\begin{aligned}
&\int_{[Q'(x_\alpha^0,\delta)\times I]\setminus C_m} W^m\left(x, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \overline{\xi} + \nabla_\alpha u_n(x)\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n(x)\right) dx \\
&\qquad \leqslant \beta \int_{[Q'(x_\alpha^0,\delta)\times I]\setminus C_m} \left(1 + \left|\left(\overline{\xi} + \nabla_\alpha u_n(x)\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n(x)\right)\right|^p\right) dx \xrightarrow[m\to+\infty]{} 0,
\end{aligned}$$

uniformly with respect to $n \in \mathbb{N}$. Then, we get that

$$\mathcal{W}_{\{\varepsilon_n\}}(\overline{\xi}\cdot\,; Q'(x_\alpha^0,\delta)) \geqslant \limsup_{m,n} \int_{Q'(x_\alpha^0,\delta)\times I} W^m\left(x, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \overline{\xi} + \nabla_\alpha u_n(x)\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n(x)\right) dx.$$

For any $h \in \mathbb{N}$, we split $Q'(x_\alpha^0, \delta)$ into h^2 disjoint cubes $Q'_{i,h}$ of side length δ/h so that

$$Q'(x_\alpha^0, \delta) = \bigcup_{i=1}^{h^2} Q'_{i,h}$$

and

$$\begin{aligned}\mathcal{W}_{\{\varepsilon_n\}}(\bar{\xi}\cdot\,; Q'(x_\alpha^0, \delta)) &\geqslant \limsup_{m,h,n} \sum_{i=1}^{h^2} \int_{Q'_{i,h}\times I} W^m\left(x, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{\xi} + \nabla_\alpha u_n(x)\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n(x)\right) dx \\ &\geqslant \limsup_{m,\lambda,h,n} \sum_{i=1}^{h^2} \int_{[Q'_{i,h}\times I]\cap R_n^\lambda} W^m\left(x, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{\xi} + \nabla_\alpha u_n(x)\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n(x)\right) dx\end{aligned}$$

where given $\lambda > 0$ we define

$$R_n^\lambda := \left\{ x \in Q'(x_\alpha^0, \delta) \times I : \quad \left|\left(\bar{\xi} + \nabla_\alpha u_n(x)\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n(x)\right)\right| \leqslant \lambda \right\}.$$

Since W^m is continuous and separately periodic it is in particular uniformly continuous on $\overline{\Omega} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \overline{B}(0, \lambda)$. With similar arguments to that used in the proof of Lemma 3.5 in Babadjian and Baía [6] (with W^m in place of $W^{m,\lambda}$), we obtain

$$\begin{aligned}&\mathcal{W}_{\{\varepsilon_n\}}(\bar{\xi}\cdot\,; Q'(x_\alpha^0, \delta)) \\ &\geqslant \limsup_{h,n} \frac{h^2}{\delta^2} \sum_{i=1}^{h^2} \int_{Q'_{i,h}} \int_{Q'_{i,h}\times I} W\left(x'_\alpha, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{\xi} + \nabla_\alpha u_n(x)\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n(x)\right) dx\, dx'_\alpha \\ &\geqslant \limsup_{h\to+\infty} \frac{h^2}{\delta^2} \sum_{i=1}^{h^2} \int_{Q'_{i,h}} \liminf_{n\to+\infty} \int_{Q'_{i,h}\times I} W\left(x'_\alpha, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{\xi} + \nabla_\alpha u_n(x)\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n(x)\right) dx\, dx'_\alpha, \quad (3.20)\end{aligned}$$

where we have used Fatou's Lemma. We now fix $x'_\alpha \in Q'_{i,h}$ such that $\overline{W}_{\text{hom}}(x'_\alpha; \bar{\xi})$ is well defined, then by Theorem 3.2 we get that

$$\liminf_{n\to+\infty} \int_{Q'_{i,h}\times I} W\left(x'_\alpha, x_3, \frac{x}{\varepsilon_n}, \frac{x_\alpha}{\varepsilon_n^2}; \bar{\xi} + \nabla_\alpha u_n(x)\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n(x)\right) dx \geqslant 2\frac{\delta^2}{h^2}\overline{W}_{\text{hom}}(x'_\alpha; \bar{\xi}). \quad (3.21)$$

Gathering (3.20) and (3.21), it turns out that

$$\int_{Q'(x_\alpha^0,\delta)} W_{\{\varepsilon_n\}}(x_\alpha; \bar{\xi})\, dx_\alpha \geqslant \int_{Q'(x_\alpha^0,\delta)} \overline{W}_{\text{hom}}(x'_\alpha; \bar{\xi})\, dx'_\alpha.$$

As a consequence the claim follows by the choice of x_α^0, after dividing the previous inequality by δ^2 and letting $\delta \to 0$. ■

We now prove the converse inequality.

Lemma 3.11. *For all $\xi \in \mathbb{Q}^{3\times 2}$ and all $x_\alpha^0 \in L$, $W_{\{\varepsilon_n\}}(x_\alpha^0; \bar{\xi}) \leqslant \overline{W}_{\text{hom}}(x_\alpha; \bar{\xi})$.*

Proof. For every $m \in \mathbb{N}$, consider the set C_m and the function W^m given by Proposition 3.9, and define $\overline{(W^m)}_{\text{hom}}$ and $(W^m)_{\text{hom}}$ as (1.11) and (1.12), with W^m in place of W. For fixed $\eta > 0$ and any $m \in \mathbb{N}$ let K_η^m be a compact subset of ω given by Scorza-Dragoni's Theorem (see Ekeland and Temam [22]) with $\mathcal{L}^2(\omega \setminus K_\eta^m) \leqslant \eta$ and such that $\overline{(W^m)}_{\text{hom}} : K_\eta^m \times \mathbb{R}^{3\times 2} \to \mathbb{R}$ is continuous .

Step 1. We claim that

$$2\liminf_{m\to+\infty}\int_{Q'(x_\alpha^0,\delta)}\overline{(W^m)}_{\hom}(x_\alpha;\overline{\xi})\,dx_\alpha \quad\geqslant\quad 2\int_{Q'(x_\alpha^0,\delta)}W_{\{\varepsilon_n\}}\left(x_\alpha,\overline{\xi}\right)\,dx_\alpha. \tag{3.22}$$

To show this inequality we follow a similar argument to that of Lemma 2.12. As before, we first decompose $Q'(x_\alpha^0,\delta)$ into h^2 small disjoint cubes $Q'_{i,h}$ and we set

$$I_{h,\eta}^m := \left\{i\in\{1,\cdots,h^2\}:\quad K_\eta^m\cap Q'_{i,h}\neq\emptyset\right\}.$$

For $i\in I_{h,\eta}^m$ choose $x_i^{h,\eta,m}\in K_\eta^m\cap Q'_{i,h}$. By Theorem 3.2 together with Lemma 2.6 in Braides, Fonseca and Francfort [16] there exists a sequence $\{u_i^{n,h,\eta,m}\}\subset W^{1,p}(Q'_{i,h}\times I;\mathbb{R}^3)$ with $u_i^{n,h,\eta,m}=0$ on $\partial Q'_{i,h}\times I$, $u_i^{n,h,\eta,m}\xrightarrow[n\to+\infty]{}0$ in $L^p(Q'_{i,h}\times I;\mathbb{R}^3)$, and such that

$$\begin{aligned}2\int_{Q'_{i,h}}&\overline{(W^m)}_{\hom}(x_i^{h,\eta,m};\overline{\xi})\,dx_\alpha\\ &=\lim_{n\to+\infty}\int_{Q'_{i,h}\times I}W^m\left(x_i^{h,\eta,m},x_3,\frac{x}{\varepsilon_n},\frac{x_\alpha}{\varepsilon_n^2};\overline{\xi}+\nabla_\alpha u_i^{n,h,\eta,m}\Big|\frac{1}{\varepsilon_n}\nabla_3 u_i^{n,h,\eta,m}\right)dx.\end{aligned}$$

Setting

$$u_n^{\eta,m}(x):=\begin{cases}u_i^{n,h,\eta,m}(x) & \text{if } x_\alpha\in Q'_{i,h} \text{ and } i\in I_{h,\eta}^m,\\ 0 & \text{otherwise,}\end{cases}$$

it follows that $\{u_n^{\eta,m}\}\subset W^{1,p}(Q'(x_\alpha^0,\delta)\times I;\mathbb{R}^3)$ and $u_n^{\eta,m}\xrightarrow[n\to+\infty]{}0$ in $L^p(Q'(x_\alpha^0,\delta)\times I;\mathbb{R}^3)$. Thus

$$\begin{aligned}2\liminf_{\eta,h}&\sum_{i\in I_{h,\eta}^m}\int_{Q'_{i,h}}\overline{(W^m)}_{\hom}(x_i^{h,\eta,m};\overline{\xi})\,dx_\alpha\\ &\geqslant\liminf_{\eta,h,n}\sum_{i\in I_{h,\eta}^m}\int_{Q'_{i,h}\times I}W^m\left(x_i^{h,\eta,m},x_3,\frac{x}{\varepsilon_n},\frac{x_\alpha}{\varepsilon_n^2};\overline{\xi}+\nabla_\alpha u_n^{\eta,m}\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n^{\eta,m}\right)dx.\end{aligned}$$

As in Lemma 2.12 we obtain

$$\begin{aligned}2\int_{Q'(x_\alpha^0,\delta)}&\overline{(W^m)}_{\hom}(x_\alpha;\overline{\xi})\,dx_\alpha\\ &\geqslant\liminf_{\lambda,\eta,h,n}\sum_{i\in I_{h,\eta}^m}\int_{(Q'_{i,h}\times I)\cap R^\lambda_{n,\eta,m}}W^m\left(x_i^{h,\eta,m},x_3,\frac{x}{\varepsilon_n},\frac{x_\alpha}{\varepsilon_n^2};\overline{\xi}+\nabla_\alpha u_n^{\eta,m}\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n^{\eta,m}\right)dx,\end{aligned}$$

where

$$R^\lambda_{n,\eta,m}:=\left\{x\in Q'(x_\alpha^0,\delta)\times I:\left|\left(\overline{\xi}+\nabla_\alpha u_n^{\eta,m}(x)\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n^{\eta,m}(x)\right)\right|\leqslant\lambda\right\}$$

with

$$\mathcal{L}^3\left(\left[Q'(x_\alpha^0,\delta)\times I\right]\setminus R^\lambda_{n,\eta,m}\right)\leqslant\frac{C}{\lambda^p}, \tag{3.23}$$

for some constant $C>0$ independent of n, η, m and λ. Taking into account that W^m is continuous we get that

$$2\int_{Q'(x_\alpha^0,\delta)}\overline{(W^m)}_{\hom}(x_\alpha;\overline{\xi})\,dx_\alpha\geqslant\liminf_{\lambda,\eta,n}\int_{R^\lambda_{n,\eta,m}}W^m\left(x,\frac{x}{\varepsilon_n},\frac{x_\alpha}{\varepsilon_n^2};\overline{\xi}+\nabla_\alpha u_n^{\eta,m}\Big|\frac{1}{\varepsilon_n}\nabla_3 u_n^{\eta,m}\right)dx.$$

By a diagonalization argument, given $\lambda_m \nearrow +\infty$, and $\eta_m \searrow 0^+$ there exists $n_m \nearrow +\infty$ such that

$$2 \liminf_{m\to+\infty} \int_{Q'(x_\alpha^0,\delta)} \overline{(W^m)}_{\rm hom}(x_\alpha;\overline{\xi})\,dx_\alpha \geqslant \liminf_{m\to+\infty} \int_{R_m} W^m\left(x, \frac{x}{\varepsilon_{n_m}}, \frac{x_\alpha}{\varepsilon_{n_m}^2}; \overline{\xi} + \nabla_\alpha v_m \Big| \frac{1}{\varepsilon_{n_m}} \nabla_3 v_m\right) dx,$$

where $v_m := u_{n_m}^{\eta_m,m} \in W^{1,p}(Q'(x_\alpha^0,\delta)\times I;\mathbb{R}^3)$ with $v_m \to 0$ in $L^p(Q'(x_\alpha^0,\delta)\times I;\mathbb{R}^3)$, and where $R_m := R^{\lambda_m}_{n_m,\eta_m,m}$. Using the Bocea and Fonseca decomposition lemma for scaled gradients (Theorem 1.1 in [11]) we can assume, without loss of generality, that the sequence $\left\{\left|\left(\nabla_\alpha v_m \big| \frac{1}{\varepsilon_{n_m}} \nabla_3 v_m\right)\right|^p\right\}$ is equi-integrable. Then, since $W^m = W$ on $C_m \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$ it comes that

$$\begin{aligned}
2 \liminf_{m\to+\infty} &\int_{Q'(x_\alpha^0,\delta)} \overline{(W^m)}_{\rm hom}(x_\alpha;\overline{\xi})\,dx_\alpha \\
&\geqslant \liminf_{m\to+\infty} \int_{Q'(x_\alpha^0,\delta)\times I} W^m\left(x, \frac{x}{\varepsilon_{n_m}}, \frac{x_\alpha}{\varepsilon_{n_m}^2}; \overline{\xi} + \nabla_\alpha v_m \Big| \frac{1}{\varepsilon_{n_m}} \nabla_3 v_m\right) dx \\
&\geqslant \liminf_{m\to+\infty} \int_{[Q'(x_\alpha^0,\delta)\times I]\cap C_m} W\left(x, \frac{x}{\varepsilon_{n_m}}, \frac{x_\alpha}{\varepsilon_{n_m}^2}; \overline{\xi} + \nabla_\alpha v_m \Big| \frac{1}{\varepsilon_{n_m}} \nabla_3 v_m\right) dx \\
&= \liminf_{m\to+\infty} \int_{Q'(x_\alpha^0,\delta)\times I} W\left(x, \frac{x}{\varepsilon_{n_m}}, \frac{x_\alpha}{\varepsilon_{n_m}^2}; \overline{\xi} + \nabla_\alpha v_m \Big| \frac{1}{\varepsilon_{n_m}} \nabla_3 v_m\right) dx
\end{aligned}$$

by the growth conditions on W, the p-equi-integrability of the above sequence of scaled gradients, (3.18) and (3.23). As a result we get inequality (3.22).

Step 2. Fixed $\rho > 0$, let $T \in \mathbb{N}$ and $\varphi \in \mathcal{S}_T$ be such that

$$\overline{W}_{\rm hom}(x_\alpha^0;\overline{\xi}) + \rho \geqslant \frac{1}{2T^2} \int_{(0,T)^2\times I} W_{\rm hom}(x_\alpha^0, y_3, y_\alpha; \overline{\xi} + \nabla_\alpha\varphi(y)|\nabla_3\varphi(y))\,dy. \tag{3.24}$$

Taking (T,φ) in the definition of $\overline{(W^m)}_{\rm hom}$ and recalling Remark 1.3 (with W^m in place of W) it follows that

$$\begin{aligned}
&\int_{Q'(x_\alpha^0,\delta)} \overline{(W^m)}_{\rm hom}(x_\alpha;\overline{\xi})\,dx_\alpha \\
&\qquad \leqslant \frac{1}{2T^2} \int_{Q'(x_\alpha^0,\delta)} \int_{(0,T)^2\times I} (W^m)_{\rm hom}(x_\alpha, y_3, y_\alpha; \overline{\xi} + \nabla_\alpha\varphi(y)|\nabla_3\varphi(y))\,dy\,dx_\alpha.
\end{aligned} \tag{3.25}$$

Define $E_m := \{(x_\alpha, y_\alpha, y_3) \in Q'(x_\alpha^0,\delta)\times(0,T)^2\times I : \quad (x_\alpha, y_3) \in C_m\}$. From (3.18) it follows that

$$\mathcal{L}^2 \otimes \mathcal{L}^3\big([Q'(x_\alpha^0,\delta)\times(0,T)^2\times I]\setminus E_m\big) \leqslant T^2/m. \tag{3.26}$$

Since $(W^m)_{\rm hom} = W_{\rm hom}$ on $C_m \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$ it comes that

$$\begin{aligned}
\int_{Q'(x_\alpha^0,\delta)} &\int_{(0,T)^2\times I} (W^m)_{\rm hom}(x_\alpha, y_3, y_\alpha; \overline{\xi} + \nabla_\alpha\varphi(y)|\nabla_3\varphi(y))\,dy\,dx_\alpha \\
&= \int_{E_m} W_{\rm hom}(x_\alpha, y_3, y_\alpha; \overline{\xi} + \nabla_\alpha\varphi(y)|\nabla_3\varphi(y))\,dy\,dx_\alpha \\
&\qquad + \int_{[Q'(x_\alpha^0,\delta)\times(0,T)^2\times I]\setminus E_m} (W^m)_{\rm hom}(x_\alpha, y_3, y_\alpha; \overline{\xi} + \nabla_\alpha\varphi(y)|\nabla_3\varphi(y))\,dy\,dx_\alpha \\
&\leqslant \int_{Q'(x_\alpha^0,\delta)\times(0,T)^2\times I} W_{\rm hom}(x_\alpha, y_3, y_\alpha; \overline{\xi} + \nabla_\alpha\varphi(y)|\nabla_3\varphi(y))\,dy\,dx_\alpha \\
&\qquad + C \int_{[Q'(x_\alpha^0,\delta)\times(0,T)^2\times I]\setminus E_m} (1 + |\nabla\varphi(y)|^p)\,dy\,dx_\alpha
\end{aligned} \tag{3.27}$$

by property (3.4) with W^m in place of W. Passing to the limit as $m \to +\infty$, relations (3.25), (3.26) and (3.27) yield to

$$\limsup_{m\to+\infty} \int_{Q'(x_\alpha^0,\delta)} \overline{(W^m)}_{\rm hom}(x_\alpha;\overline{\xi})\,dx_\alpha \leqslant \frac{1}{2T^2}\int_{Q'(x_\alpha^0,\delta)}\int_{(0,T)^2\times I} W_{\rm hom}(x_\alpha,y_3,y_\alpha;\overline{\xi}+\nabla_\alpha\varphi(y)|\nabla_3\varphi(y))\,dy\,dx_\alpha.$$

Hence by (3.22) we obtain

$$\int_{Q'(x_\alpha^0,\delta)} W_{\{\varepsilon_n\}}\left(x_\alpha,\overline{\xi}\right)\,dx_\alpha \leqslant \frac{1}{2T^2}\int_{Q'(x_\alpha^0,\delta)}\int_{(0,T)^2\times I} W_{\rm hom}(x_\alpha,y_3,y_\alpha;\overline{\xi}+\nabla_\alpha\varphi(y)|\nabla_3\varphi(y))\,dy\,dx_\alpha.$$

As a consequence, by the choice of x_α^0 together with (3.24) we finally get, after dividing the previous inequality by δ^2 and letting $\delta \to 0$, that

$$\begin{aligned} W_{\{\varepsilon_n\}}\left(x_\alpha^0,\overline{\xi}\right) &\leqslant \frac{1}{2T^2}\int_{(0,T)^2\times I} W_{\rm hom}(x_\alpha^0,y_3,y_\alpha;\overline{\xi}+\nabla_\alpha\varphi(y)|\nabla_3\varphi(y))\,dy \\ &\leqslant \overline{W}_{\rm hom}(x_\alpha^0;\overline{\xi})+\rho \end{aligned}$$

and the result follows by letting $\rho \to 0$. ■

Proof of Theorem 1.2. As a consequence of Lemmas 3.11 and 3.10, we have $\overline{W}_{\rm hom}(x_\alpha;\overline{\xi}) = W_{\{\varepsilon_n\}}(x_\alpha;\overline{\xi})$ for all $x_\alpha \in L$ and all $\overline{\xi} \in \mathbb{Q}^{3\times 2}$. Since $\overline{W}_{\rm hom}$ and $W_{\{\varepsilon_n\}}$ are Carathéodory functions, it follows that the equality holds for all $\overline{\xi} \in \mathbb{R}^{3\times 2}$ and a.e. $x_\alpha \in \omega$. Therefore, we have $\mathcal{W}_{\{\varepsilon_n\}}(u;A) = \mathcal{W}_{\rm hom}(u;A)$ for all $A \in \mathcal{A}(\omega)$ and all $u \in W^{1,p}(A;\mathbb{R}^3)$. Since the result does not depend upon the specific choice of the subsequence, we conclude thanks to Proposition 8.3 in Dal Maso [19] that the whole sequence $\mathcal{W}_\varepsilon(\cdot\,;A)$ $\Gamma(L^p(A\times I))$-converges to $\mathcal{W}_{\rm hom}(\cdot\,;A)$. Taking $A=\omega$ we conclude the proof of Theorem 1.2. ■

To conclude, let us state an interesting consequence of Theorem 1.2.

Corollary 3.12. *Let $W:\Omega\times\mathbb{R}^3\times\mathbb{R}^{3\times 3}\to\mathbb{R}$ be a function satisfying (A_1), (A_2) and (A_4), and such that $W(x,\cdot\,;\xi)$ is Q-periodic for all $\xi\in\mathbb{R}^{3\times3}$ and a.e. $x\in\Omega$. Define the functional $\mathcal{W}_\varepsilon : L^p(\Omega;\mathbb{R}^3)\to[0,+\infty]$ by*

$$\mathcal{W}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega W\left(x,\frac{x}{\varepsilon};\nabla_\alpha u(x)\Big|\frac{1}{\varepsilon}\nabla_3 u(x)\right)dx & \text{if } u\in W^{1,p}(\Omega;\mathbb{R}^3),\\ +\infty & \text{otherwise.}\end{cases}$$

Then the $\Gamma(L^p(\Omega))$-limit of the family $\{\mathcal{W}_\varepsilon\}_{\varepsilon>0}$ is given by $\mathcal{W}_{\rm hom}: L^p(\Omega;\mathbb{R}^3)\to[0,+\infty]$ with

$$\mathcal{W}_{\rm hom}(u) := \begin{cases} \displaystyle 2\int_\omega \overline{W}_{\rm hom}(x_\alpha,\nabla_\alpha u(x_\alpha))\,dx_\alpha & \text{if } u\in W^{1,p}(\omega;\mathbb{R}^3),\\ +\infty & \text{otherwise,}\end{cases}$$

where, for all $\overline{\xi}\in\mathbb{R}^{3\times2}$ and a.e. $x_\alpha\in\omega$

$$\begin{aligned}\overline{W}_{\rm hom}(x_\alpha,\overline{\xi}) := \lim_{T\to+\infty}\inf_{\phi}\Big\{&\frac{1}{2T^2}\int_{(0,T)^2\times I} W_{\rm hom}(x_\alpha,y_3,y_\alpha;\overline{\xi}+\nabla_\alpha\phi(y)|\nabla_3\phi(y))\,dy : \\ &\phi\in W^{1,p}((0,T)^2\times I;\mathbb{R}^3),\quad \phi=0 \text{ on } \partial(0,T)^2\times I\Big\}\end{aligned}$$

and, for all $y_\alpha\in\mathbb{R}^2$ and a.e. $x\in\Omega$

$$W_{\hom}(x,y_\alpha;\xi) := \lim_{T\to+\infty} \inf_\phi \left\{ \frac{1}{T^3}\int_{(0,T)^3} W(x,y_\alpha,z_3;\xi+\nabla\phi(z))\,dz : \ \phi\in W^{1,p}_0((0,T)^3;\mathbb{R}^3)\right\}.$$

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