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A note on concentrations for integral two-scale problems

Margarida Baía · Pedro Miguel Santos

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Abstract This work is devoted to the characterization of the asymptotic behavior, as $\{\varepsilon_n\}$ goes to zero, of a family of integral functionals of the form $\int_\Omega f(x, \langle x/\varepsilon_n\rangle, \nabla u_n(x))\, dx$ in terms of measures of oscillation and concentration associated to the sequence $\{(\langle x/\varepsilon_n\rangle, \nabla u_n(x))\}$.

Keywords Concentrations · Oscillations · Two-scale gradient Young measure · Homogenization

1 Introduction

Young measures, introduced by L. C. Young [33] in optimal control theory, have been successfully used throughout the literature to study the limit of nonlinear quantities of oscillating sequences. For the general theory of Young measures, as well as some applications, we refer to Balder [4], Ball [5], Braides [8], Kinderlehrer & Pedregal [16], Müller [25], Pedregal [26], Roubíček [27], Tartar [30], Valadier [31] and references therein.

Under suitable hypotheses, given an open and bounded subset $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$, an integrand $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ $(d \geq 1)$ and a sequence $\{a_n\}$, $a_n : \Omega \to \mathbb{R}^d$, it is well know that

$$\lim_{n\to\infty} \int_\Omega f\big(x, a_n(x)\big)\, dx = \int_\Omega \int_{\mathbb{R}^{d\times N}} f(x, z)\, d\nu_x(z)\, dx, \tag{1}$$

where ν is the Young measure generated by the sequence $\{a_n\}$, if and only if the sequence $\{f(x, a_n(x))\}$ is equi-integrable (see Ball [5]). The measure ν has been characterized by many authors when $\{a_n\}$ satisfies a certain constraint (see Bocea and Fonseca [10], Kinderlehrer and Pedregal [16], Fonseca and Müller [20], Santos [28] and Babadjian, Baía and Santos [6]).

Margarida Baía
Instituto Superior Técnico
Av. Rovisco Pais, 1049-001 Lisboa
Portugal
Tel.: +351-21-8417130
Fax: +351-21-8417598
E-mail: mbaia@math.ist.utl.pt

Pedro Miguel Santos
Instituto Superior Técnico
Av. Rovisco Pais, 1049-001 Lisboa
Portugal
Tel.: +351-21-8417130
Fax: +351-21-8417598
E-mail: pmsantos@math.ist.utl.pt

In many physical situations energy densities of the form $\{f(x, a_n(x))\}$ develop concentration effects so that equality (1) doesn't hold anymore. Many extensions of the notion of Young measures have been developed to address problems of concentrations through the notions of weak-star defect measures (see Di Perna & Majda [12], Federer & Ziemer [17], Lions [22,23]), reduced defect measures (see Di Perna & Majda [13]), generalized Young measures (see Di Perna & Majda [14], Dal Maso & De simone & Mora & Morini [11]), varifold measures (Allard [1], Almgren [3], Fonseca, Pedregal and Müller [21]) and indicator measures (Fonseca [19] and Reshetnyak [29]).

In this work, Ω stands for the reference configuration of a nonlinear elastic body with periodic microstructure and whose heterogeneities scale like a small parameter $\varepsilon > 0$. We assume that the energy of a deformation $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ of Ω is of the form

$$\int_\Omega f\left(x, \left\langle \frac{x}{\varepsilon} \right\rangle, \nabla u(x)\right) dx,$$

where $f : \Omega \times Q \times \mathbb{R}^{d\times N} \to [0, \infty)$, with $Q := (0,1)^N$, is the stored energy density of this body and is assumed to satisfy standard p-growth conditions with $p > 1$. The presence of the term $\langle x/\varepsilon \rangle$ (fractional part of the vector x/ε componentwise) takes into account the periodic microstructure of the body.

Given $\varepsilon_n \to 0$ and a bounded sequence $\{u_n\}$ of deformations our goal is to study the limit behavior, as $n \to \infty$, of

$$\int_\Omega f\left(x, \left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x)\right) dx. \tag{2}$$

Namely, we want to record the concentration effects (in each direction) developed by the sequence

$$\left\{ f\left(x, \left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x)\right) \right\},$$

and for this reason we derive the limit of (2) for a class of integrands in a separable space of continuous functions obtained through an appropriate compactification of $\mathbb{R}^{d\times N}$. Precisely, we define $\overline{\mathbb{R}^{d\times N}}$ to be the (sphere) compactification of $\mathbb{R}^{d\times N}$ through the homeomorphism $i : \mathbb{R}^{d\times N} \hookrightarrow B$ given by

$$i(\xi) = \frac{\xi}{1 + |\xi|},$$

where B denotes the unit open ball in $\mathbb{R}^{d\times N}$, and we consider the space

$$\mathcal{E}_p(\Omega) := \left\{ f : \overline{\Omega} \times \overline{Q} \times \mathbb{R}^{d\times N} \to \mathbb{R}, \quad \frac{f(x,y,\xi)}{1 + |\xi|^p} \in \mathcal{C}\left(\overline{\Omega} \times \overline{Q} \times \overline{\mathbb{R}^{d\times N}}\right) \right\} \tag{3}$$

(see Alibert and Bouchitté [2] and Fonseca, Pedregal and Müller [21]; see Dugundji [15] or Folland [18] for the notion and basic properties of compactifications).

We prove the following result on existence of Young measures (see Section 2 for notations).

Theorem 1 *Let Ω be a bounded open subset of $\mathbb{R}^N$. Let $\varepsilon_n \to 0$ and let $\{u_n\}$ be a bounded sequence in $W^{1,p}(\Omega; \mathbb{R}^d)$ with $p > 1$. Then there exists a subsequence (still denoted by $\{u_n\}$) and a pair $(\nu, \Lambda) \in L^\infty_w(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d\times N})) \times \mathcal{M}^+(\overline{\Omega} \times \overline{Q} \times S)$, $S := S^{d\times N-1}$, with $\nu_{(x,y)} \in \mathcal{P}(\mathbb{R}^{d\times N})$ for a.e. $(x,y) \in \Omega \times Q$, such that*

$$\lim_{n\to\infty} \int_\Omega f\left(x, \left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n\right) dx = \int_\Omega \int_Q \langle \nu_{(x,y)}, f(x,y,.)\rangle \, dy \, dx + \langle \Lambda, f^\infty \rangle$$

for all $f \in \mathcal{E}_p(\Omega)$, where

$$f^\infty(x,y,\xi) = \lim_{t\to\infty} \frac{f(x,y,t\xi)}{t^p}.$$

The proof of Theorem 1 is similar to the one of Theorem 2.5 of Alibert and Bouchitté [2] and is based on the identification of $\mathcal{E}_p(\Omega)$ with the space $C(\overline{\Omega} \times \overline{Q} \times \overline{B})$, through the compactification above, and the Banach-Alaoglu-Bourbaki Theorem (see e.g. Brezis [9]).

The pair (ν, Λ) is what we call a two-scale gradient Young measure for oscillations and concentrations (see Definition 2). In Babadjian, Baía and Santos [6] the authors characterized the measure $\nu \equiv \{\nu_{(x,y)}\}$, what is called a two-scale gradient Young measure (for oscillations), in terms of a Jensen's inequality with test functions in the space

$$E_p := \left\{ g : \overline{Q} \times \mathbb{R}^{d\times N} \to \mathbb{R}, \quad \frac{g(y,\xi)}{1+|\xi|^p} \in C\Big(\overline{Q} \times \overline{\mathbb{R}^{d\times N}}^{\mathcal{A}}\Big) \right\}$$

where here $\overline{\mathbb{R}^{d\times N}}^{\mathcal{A}}$ stands for the standard Alexandroff one-point compactification of $\mathbb{R}^{d\times N}$. The objective of our second theorem is to derive a similar characterization for the two-scale gradient Young measure for concentrations Λ, which turns out to be a Jensen's type of inequality in the space $\mathcal{H}_p \subset \mathcal{E}_p(\Omega)$ defined by

$$\mathcal{H}_p := \left\{ \psi \in C(\overline{Q} \times \mathbb{R}^{d\times N}) : \quad \psi(y, t\xi) = t^p \psi(y,\xi), \ t > 0 \right\}. \tag{4}$$

Remark 1 By the p-homogeneity we observe that $\mathcal{H}_p$ is isomorphic to $C(\overline{Q} \times S)$. Note also that if $\psi \in \mathcal{H}_p$ then $\psi_{\text{hom}}(0) = 0$ or $-\infty$, where

$$\psi_{\text{hom}}(0) := \lim_{T\to\infty} \inf \left\{ \fint_{(0,T)^N} \psi(\langle y\rangle, \nabla\phi(y))\, dy : \ \phi \in W_0^{1,p}\big((0,T)^N; \mathbb{R}^d\big) \right\}.$$

The main theorem of this work asserts the following.

Theorem 2 *Let Ω be a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary and let $(\nu, \Lambda) \in L^\infty_w(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d\times N})) \times \mathcal{M}^+(\overline{\Omega} \times \overline{Q} \times S)$ with $\nu_{(x,y)} \in \mathcal{P}(\mathbb{R}^{d\times N})$ for a.e. $(x,y) \in \Omega \times Q$. Let $\lambda_x \in \mathcal{P}(\overline{Q} \times S)$ and $\pi \in \mathcal{M}(\overline{\Omega})$ be such that $\Lambda = \lambda_x \otimes \pi$. Then (ν, Λ) is a two-scale gradient Young measure for oscillations and concentrations if and only if for some $p > 1$ the following conditions hold:*

i) there exist $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ and $u_1 \in L^p(\Omega; W^{1,p}_{per}(Q; \mathbb{R}^d))$ such that

$$\int_{\mathbb{R}^{d\times N}} \xi \, d\nu_{(x,y)}(\xi) = \nabla u(x) + \nabla_y u_1(x,y) \quad \textit{for a.e. } (x,y) \in \Omega \times Q;$$

ii) there exists $\Omega_o \subset \Omega$ with $\mathcal{L}^N(\Omega \setminus \Omega_o) = 0$ such that for every $g \in E_p$

$$\int_Q \int_{\mathbb{R}^{d\times N}} g(y,\xi)\, d\nu_{(x,y)}(\xi)\, dy \geq g_{\text{hom}}(\nabla u(x)) \quad \textit{for } x \in \Omega_o,$$

where

$$g_{\text{hom}}(\xi) := \lim_{T\to\infty} \inf \left\{ \fint_{(0,T)^N} g(\langle y\rangle, \xi + \nabla\phi(y))\, dy : \ \phi \in W_0^{1,p}\big((0,T)^N; \mathbb{R}^d\big) \right\};$$

iii)

$$(x,y) \mapsto \int_{\mathbb{R}^{d\times N}} |\xi|^p d\nu_{(x,y)}(\xi) \in L^1(\Omega \times Q);$$

iv) there exists $\Omega_c \subset \Omega$ with $\pi(\Omega \setminus \Omega_c) = 0$ such that for every $\psi \in \mathcal{H}_p$ with $\psi_{\text{hom}}(0) = 0$ we have that

$$\langle \lambda_x, \psi \rangle \geq 0 \quad \textit{for } x \in \Omega_c.$$

Remark 2 The decomposition of the measure Λ above comes from the Disintegration Theorem (see Valadier [32]).

The proof of i)-iii) was obtained in Babadjian, Baía and Santos [6] (Theorem 1.1), while the proof of iv) relies on the ideas of Theorem 1.1 in Fonseca, Pedregal and Müller [21], where a characterization of varifolds (concentrations measures) associated to uniformly bounded sequences of gradients of $W^{1,p}$- functions, with $p > 1$, is obtained.

The overall plan of this work in the ensuing sections will be as follows: Section 2 collects the main notations and results used throughout. Section 3 is devoted to the proofs of Theorem 1 and Theorem 2.

2 Preliminaries

The purpose of this section is to give a brief overview of the concepts and results that are used throughout this work. Almost all these results are stated without proofs as they can be readily found in the literature.

In what follows Ω is an open bounded subset of $\mathbb{R}^N$, and given U a subset of $\mathbb{R}^m$, we use the following notations:

- $\mathcal{C}_c(U)$ is the space of continuous functions $f : U \to \mathbb{R}$ with compact support.
- $\mathcal{C}_0(U)$ is the closure of $\mathcal{C}_c(U)$ for the uniform convergence; it coincides with the space of all continuous functions $f : U \to \mathbb{R}$ such that, for every $\eta > 0$, there exists a compact set $K_\eta \subset U$ with $|f| < \eta$ on $U \setminus K_\eta$.
- $\mathcal{M}(U)$ is the space of real-valued Radon measures with finite total variation. We recall that by the Riesz Representation Theorem $\mathcal{M}(U)$ can be identified with the dual space of $\mathcal{C}_0(U)$ through the duality

$$\langle \mu, \phi \rangle = \int_U \phi \, d\mu, \qquad \mu \in \mathcal{M}(U), \quad \phi \in \mathcal{C}_0(U).$$

- $\mathcal{M}^+(U)$ is the subset of $\mathcal{M}^(U)$ of positive measures.
- $\mathcal{P}(U)$ denotes the space of probability measures on U, *i.e.* the space of all $\mu \in \mathcal{M}^+(U)$ such that $\mu(U) = 1$.
- $L^1(\Omega; \mathcal{C}_0(U))$ is the space of maps $\phi : \Omega \to \mathcal{C}_0(U)$ such that
  i) ϕ is strongly measurable, *i.e.* there exists a sequence of simple functions $s_n : \Omega \to \mathcal{C}_0(U)$ such that $\|s_n(x) - \phi(x)\|_{\mathcal{C}_0(U)} \to 0$ for a.e. $x \in \Omega$;
  ii) $x \mapsto \|\phi(x)\|_{\mathcal{C}_0(U)} \in L^1(\Omega)$.
  We recall that the linear space spanned by $\{\varphi \otimes \psi : \ \varphi \in L^1(\Omega) \text{ and } \psi \in \mathcal{C}_0(U)\}$ is dense in $L^1(\Omega; \mathcal{C}_0(U))$.
- $L^\infty_w(\Omega; \mathcal{M}(U))$ is the space of maps $\nu : \Omega \to \mathcal{M}(U)$ such that
  i) ν is weak* measurable, *i.e.* $x \mapsto \langle \nu_x, \phi \rangle$ is measurable for every $\phi \in \mathcal{C}_0(U)$;
  ii) $x \mapsto \|\nu_x\|_{\mathcal{M}(U)} \in L^\infty(\Omega)$.
  The space $L^\infty_w(\Omega; \mathcal{M}(U))$ can be identified with the dual space of $L^1(\Omega; \mathcal{C}_0(U))$ through the duality

$$\langle \mu, \phi \rangle = \int_\Omega \int_U \phi(x, \xi) \, d\mu_x(\xi) \, dx, \qquad \mu \in L^\infty_w(\Omega; \mathcal{M}(U)), \quad \phi \in L^1(\Omega; \mathcal{C}_0(U)),$$

  where $\phi(x, \xi) := \phi(x)(\xi)$ for all $(x, \xi) \in \Omega \times U$. Hence it can endowed with the weak* topology (see *e.g.* Theorem 2.11 in Málek, Nečas, Rokyta & Růžička [24]).
- The space $W^{1,p}_{\text{per}}(Q; \mathbb{R}^d)$ stands for the $W^{1,p}$-closure of all functions $f \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R}^d)$ which are Q-periodic.

We start by recalling the notion of two-scale Young measure as well as a characterization of these kind of measures derived in Babadjian, Baía and Santos [6] that we use below.

Definition 1 Let Ω be a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary and let $\nu \in L^\infty_w(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d \times N}))$. The family $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ is said to be a two-scale gradient Young measure (for oscillations) if $\nu_{(x,y)} \in \mathcal{P}(\mathbb{R}^{d \times N})$ for a.e. $(x, y) \in \Omega \times Q$ and if there exists $\varepsilon_n \to 0$ and a bounded sequence $\{u_n\} \subset W^{1,p}(\Omega; \mathbb{R}^d)$ such that $\{(\langle \cdot / \varepsilon_n \rangle, \nabla u_n)\}$ generates the Young measure $\{\nu_{(x,y)} \otimes dy\}_{x \in \Omega}$ *i.e.* for every $z \in L^1(\Omega)$ and $\varphi \in \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^{d \times N})$,

$$\lim_{n \to +\infty} \int_\Omega z(x) \, \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x) \right) dx = \int_\Omega \int_Q \int_{\mathbb{R}^{d \times N}} z(x) \, \varphi(y, \xi) \, d\nu_{(x,y)}(\xi) \, dy \, dx.$$

In this case we say that $\{\nabla u_n\}$ generates $\nu \equiv \{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$, or that, $\nu \equiv \{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ is the two-scale Young measure associated to $\{\nabla u_n\}$.

Theorem 3 *(see [6]) Let $\nu \in L^\infty_w(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d \times N}))$ be such that $\nu_{(x,y)} \in \mathcal{P}(\mathbb{R}^{d \times N})$ for a.e. $(x, y) \in \Omega \times Q$. The family $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ is a* two-scale gradient Young measure *if and only if for some $p > 1$ the three conditions below hold:*

i) there exist $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ *and* $u_1 \in L^p(\Omega; W^{1,p}_{per}(Q;\mathbb{R}^d))$ *such that*

$$\int_{\mathbb{R}^{d\times N}} \xi \, d\nu_{(x,y)}(\xi) = \nabla u(x) + \nabla_y u_1(x,y) \quad \text{for a.e. } (x,y) \in \Omega \times Q;$$

ii) for every $g \in E_p$

$$\int_Q \int_{\mathbb{R}^{d\times N}} g(y,\xi)\, d\nu_{(x,y)}(\xi)\, dy \geq g_{\hom}(\nabla u(x)) \quad \text{for a.e. } x \in \Omega,$$

where

$$g_{\hom}(\xi) := \lim_{T\to+\infty} \inf \left\{ ⨍_{(0,T)^N} g(\langle y\rangle, \xi + \nabla\phi(y))\, dy : \ \phi \in W^{1,p}_0\big((0,T)^N;\mathbb{R}^d\big) \right\};$$

iii)

$$(x,y) \mapsto \int_{\mathbb{R}^{d\times N}} |\xi|^p d\nu_{(x,y)}(\xi) \in L^1(\Omega\times Q).$$

Here in order to take into account concentrations effects we extend Definition 1 as follows.

Definition 2 Let $(\nu, \Lambda) \in L^\infty_w(\Omega\times Q; \mathcal{M}(\mathbb{R}^{d\times N})) \times \mathcal{M}^+(\overline{\Omega}\times\overline{Q}\times S)$. The pair (ν,Λ) is said to be a two-scale gradient Young measure for oscillations and concentrations if $\nu_{(x,y)} \in \mathcal{P}(\mathbb{R}^{d\times N})$ for a.e. $(x,y)\in\Omega\times Q$ and there exist $\varepsilon_n \to 0$ and $\{u_n\} \subset W^{1,p}(\Omega;\mathbb{R}^d)$ bounded such that

$$\lim_{n\to\infty} \int_\Omega z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, \nabla u_n(x)\right) dx = \int_\Omega\int_Q\int_{\mathbb{R}^{d\times N}} z(x)\varphi(y,\xi)\, d\nu_{(x,y)}(\xi)\, dy\, dx + \langle \Lambda, z\otimes\varphi^\infty\rangle \tag{5}$$

for every $z \in C(\overline{\Omega})$ and $\varphi \in \mathcal{E}_p$, where

$$\mathcal{E}_p := \left\{ \varphi : \overline{Q}\times\mathbb{R}^{d\times N} \to \mathbb{R}, \quad \frac{\varphi(y,\xi)}{1+|\xi|^p} \in \mathcal{C}(\overline{Q}\times\overline{\mathbb{R}^{d\times N}}) \right\}.$$

In this case we say that $\{\nabla u_n\}$ generates (ν,Λ), or that, (ν,Λ) is the two-scale gradient Young measure for oscillations and concentrations associated to $\{\nabla u_n\}$.

Remark 3 The uniqueness of the pair (ν,Λ) is clear from Definition 2.

Lemma 1 *The pair* $(\nu,\Lambda) \in L^\infty_w(\Omega\times Q;\mathcal{M}(\mathbb{R}^{d\times N})) \times \mathcal{M}^+(\overline{\Omega}\times\overline{Q}\times S)$, $\nu_{(x,y)} \in \mathcal{P}(\mathbb{R}^{d\times N})$, *is a two-scale gradient Young measure for oscillations and concentrations if and only if there exist* $\varepsilon_n \to 0$ *and* $\{u_n\}\subset W^{1,p}(\Omega;\mathbb{R}^d)$ *bounded such that*

i) for all $z \in C(\overline{\Omega})$ *and* $\varphi \in C_0(\overline{Q}\times\mathbb{R}^{d\times N})$ *then*

$$\lim_{n\to\infty} \int_\Omega z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, \nabla u_n(x)\right) dx = \int_\Omega\int_Q\int_{\mathbb{R}^{d\times N}} z(x)\varphi(y,\xi)\, d\nu_{(x,y)}(\xi)\, dy\, dx;$$

ii) for all $z \in C(\overline{\Omega})$ *and* $\varphi \in \mathcal{H}_p$ *then*

$$\lim_{n\to\infty} \int_\Omega z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, \nabla u_n(x)\right) dx = \int_\Omega\int_Q\int_{\mathbb{R}^{d\times N}} z(x)\varphi(y,\xi)\, d\nu_{(x,y)}(\xi)\, dy\, dx + \langle\Lambda, z\otimes\varphi\rangle.$$

Proof Necessity is obvious. For the sufficiency, we start by remarking that i) implies $\nu_{(x,y)}\otimes dy$ to be the Young measure generated by $\left\{\left(\left\langle \frac{\cdot}{\varepsilon_n}\right\rangle, \nabla u_n\right)\right\}$. Let $z\in C(\overline{\Omega})$ and $\varphi\in\mathcal{E}_p$. Since φ can be decomposed as $\varphi = (\varphi-\varphi^\infty)+\varphi^\infty$ then (5) follows immediately from i)-ii) because $\varphi^\infty \in \mathcal{H}_p$ and $\left\{(\varphi-\varphi^\infty)\left(\left\langle\frac{\cdot}{\varepsilon_n}\right\rangle, \nabla u_n\right)\right\}$ is equi-integrable.

Let us see two examples.

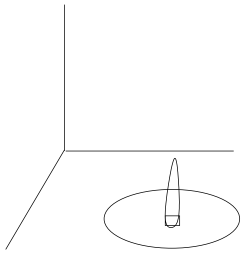

Fig. 1 Concentration at one point

Example 1 (Concentration at one point) Let $u \in W_0^{1,p}((0,T)^N;\mathbb{R}^d)$, $T \in \mathbb{N}$, $x_0 \in \Omega$ and $\varepsilon_n \to 0$. We extend u by zero outside of $(0,T)^N$ and define $u_n : \Omega \to \mathbb{R}$ (see Figure 1) by

$$u_n(x) := \frac{\varepsilon_n}{(\varepsilon_n T)^{N/p}}\, u\left(\frac{x - x_{\varepsilon_n}}{\varepsilon_n}\right),$$

where $x_{\varepsilon_n} := \varepsilon_n \left[\frac{x_0}{\varepsilon_n}\right]$ and $[\cdot]$ stands for the integer part of a vector componentwise. Let us see that the two-scale gradient Young measure for oscillations and concentrations (ν, Λ) generated by $\{\nabla u_n\}$ is given by $\nu_{(x,y)} = \delta_0$ and $\Lambda = \lambda_x \otimes \pi$ where, denoting $||\nabla u||_{L^p} \equiv ||\nabla u||_{L^p(0,T)^N}$,

$$\pi = \frac{||\nabla u||_{L^p}^p}{T^N}\delta_{x_0}$$

and

$$\langle \lambda_x, \varphi\rangle = \frac{1}{||\nabla u||_{L^p}^p}\int_{[0,T]^N} \varphi\left(\langle y\rangle, \frac{\nabla u(y)}{|\nabla u(y)|}\right)|\nabla u(y)|^p\, dy$$

for all $\varphi \in C_0(\overline{Q} \times S)$.

To use Lemma 1, let $z \in C(\overline{\Omega})$ and $\varphi \in C_0(\overline{Q} \times \mathbb{R}^{d\times N})$. We first remark that

$$\begin{aligned}\int_\Omega z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, \nabla u_n(x)\right) dx &= \int_{\Omega\setminus x_{\varepsilon_n}+[0,\varepsilon_n T]^N} z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, 0\right) dx \\ &\quad + \int_{x_{\varepsilon_n}+[0,\varepsilon_n T]^N} z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, \nabla u_n(x)\right) dx.\end{aligned}$$

Since

$$\left|x_{\varepsilon_n} + [0,\varepsilon_n T]^N\right| \underset{n\to\infty}{\longrightarrow} 0 \tag{6}$$

then

$$\lim_{n\to\infty}\int_{x_{\varepsilon_n}+[0,\varepsilon_n T]^N} z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, \nabla u_n(x)\right) dx = 0.$$

Thus

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, \nabla u_n(x)\right) dx &= \lim_{n\to\infty}\int_{\Omega\setminus x_{\varepsilon_n}+[0,\varepsilon_n T]^N} z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, 0\right) dx \\ &= \int_\Omega\int_Q z(x)\varphi(y,0)\, dy\, dx\end{aligned}$$

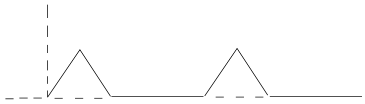

Fig. 2 The graph of the function u_n

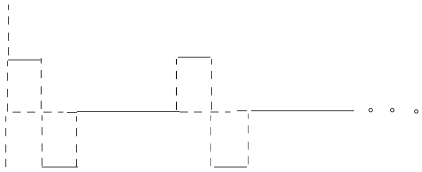

Fig. 3 The graph of the function ∇u_n

by (6) and Riemann-Lebesgue's Lemma, so that $\nu_{(x,y)} = \delta_0$. Finally, let $z \in C(\overline{\Omega})$ and $\varphi \in \mathcal{H}_p$. Then, by the p-homogeneity of φ

$$\begin{aligned}\int_\Omega z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, \nabla u_n(x)\right)\,dx &= \int_\Omega z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, \nabla u\Big(\frac{x-x_{\varepsilon_n}}{\varepsilon_n}\Big)(\varepsilon_n T)^{-N/p}\right)\,dx \\ &= (\varepsilon_n T)^{-N}\int_\Omega z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, \nabla u\Big(\frac{x-x_{\varepsilon_n}}{\varepsilon_n}\Big)\right)\,dx \\ &= (\varepsilon_n T)^{-N}\int_{x_{\varepsilon_n}+\varepsilon_n(0,T)^N} z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, \nabla u\Big(\frac{x-x_{\varepsilon_n}}{\varepsilon_n}\Big)\right)\,dx\end{aligned}$$

since $\varphi(y,0) = 0$ for all $y \in \overline{Q}$. Thus, changing variables

$$\int_\Omega z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, \nabla u_n(x)\right)\,dx = \fint_{(0,T)^N} z(x_{\varepsilon_n}+\varepsilon_n y)\varphi\left(\langle y\rangle, \nabla u(y)\right)\,dy$$

and so

$$\lim_{n\to\infty}\int_\Omega z(x)\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, \nabla u_n(x)\right)\,dx = z(x_0)\fint_{(0,T)^N}\varphi\left(\langle y\rangle, \nabla u(y)\right)\,dy.$$

Example 2 (Spatially distributed concentrations) Let $\Omega := (0,1)$ and let $v_n : [0,1] \to \mathbb{R}$ be the function given by

$$v_n(x) := \begin{cases} x n^{\frac{1}{p}-1} & 0 \le x \le \frac{1}{2n} \\ -n^{\frac{1}{p}-1}(x-1/n) & \frac{1}{2n} \le x \le \frac{1}{n} \\ 0 & \frac{1}{n} \le x \le 1. \end{cases}$$

We extend v_n by periodicity to $\mathbb{R}$ and define (see Figures 2 and 3)

$$u_n(x) := v_n(nx), \quad x \in \mathbb{R}.$$

It is easy to check that u_n is bounded in $W^{1,p}(\Omega)$ and that $\{\nabla u_n\}$ converge to zero in measure. Thus the two-scale gradient Young measure for oscillations associated to $\{\nabla u_n\}$ is given by $\nu_{(x,y)} = \delta_0$.

In order to compute the two-scale gradient Young measure for concentrations, we use again Lemma 1. Let $\varphi : [0,1]\times\mathbb{R} \to \mathbb{R}$, $\varphi \in \mathcal{H}_p$, and $z \in C(\overline{\Omega})$. By the p-homogeneity of φ and the fact that $\varphi(y,0) = 0$, for all $y \in [0,1]$, we have that

$$\int_0^1 z(x)\varphi(\langle nx\rangle, \nabla u_n(x))\,dx = \sum_{j=1}^{n}\left[\int_{\frac{j-1}{n}}^{\frac{j-1}{n}+\frac{1}{2n^2}} z(x)\varphi(\langle nx\rangle, n^{1/p})\,dx + \int_{\frac{j-1}{n}+\frac{1}{2n^2}}^{\frac{j-1}{n}+\frac{1}{n^2}} z(x)\varphi(\langle nx\rangle, -n^{1/p})\,dx\right]$$

$$= n\sum_{j=1}^{n}\left[\int_{\frac{j-1}{n}}^{\frac{j-1}{n}+\frac{1}{2n^2}} z(x)\varphi(\langle nx\rangle, 1)\,dx + \int_{\frac{j-1}{n}+\frac{1}{2n^2}}^{\frac{j-1}{n}+\frac{1}{n^2}} z(x)\varphi(\langle nx\rangle, -1)\,dx\right].$$

Hence, by considering the change of variables $y = nx + 1 - j$ in each interval $\left[\frac{j-1}{n}, \frac{j-1}{n} + \frac{1}{n^2}\right]$, $j = 1, ..., n$, we get that

$$\int_0^1 z(x)\varphi(\langle nx\rangle, \nabla u_n(x))\,dx = \sum_{j=1}^{n}\left[\int_0^{\frac{1}{2n}} z\left(\frac{y+j-1}{n}\right)\varphi(y,1)\,dy + \int_{\frac{1}{2n}}^{\frac{1}{n}} z\left(\frac{y+j-1}{n}\right)\varphi(y,-1)\,dy\right].$$

Finally, by the continuity of z and φ, it follows that

$$\lim_{n\to\infty}\int_0^1 z(x)\varphi(nx, \nabla u_n(x))\,dx = \lim_{n\to\infty}\left[\int_0^{\frac{1}{2n}} \varphi(y,1)\,dy + \int_{\frac{1}{2n}}^{\frac{1}{n}} \varphi(y,-1)\,dy\right]\left(\sum_{j=1}^{n} z\left(\frac{j-1}{n}\right)\right)$$

$$= \frac{\varphi(0,1)+\varphi(0,-1)}{2}\int_0^1 z(x)\,dx,$$

and consequently $\Lambda = \delta_0 \otimes \frac{\delta_1+\delta_{-1}}{2} \otimes \mathcal{L}^1_{|[0,1]}$.

The following lemmas are useful to prove Theorem 2, and their proofs are similar to the ones in Fonseca, Pedregal and Müller [21] (see Step 2 and 3 of Section 5). For the readers convenience we include these proofs here.

Lemma 2 *Let Ω be a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary and let (ν, Λ) be a two-scale gradient Young measure for oscillations and concentrations generated by $\{\nabla u_n\} \subset W^{1,p}(\Omega;\mathbb{R}^d)$. Then there exist bounded sequences $v_n,\ z_n \in W^{1,p}(\Omega;\mathbb{R}^d)$ such that (up to a subsequence) $u_n = v_n + z_n$, $\{|\nabla v_n|^p\}$ is equi-integrable, $|\{x \in \Omega :\ u_n \neq v_n \text{ or } \nabla u_n \neq \nabla v_n\}| \underset{n\to\infty}{\longrightarrow} 0$, $\{\nabla z_n\}$ generates (δ_0, Λ) and $\{\nabla v_n\}$ generates $(\nu, 0)$.*

Proof *By the Decomposition Lemma (see Lemma 1.2 in Fonseca, Müller and Pedregal [21]) there exists $v_n,\ z_n \in W^{1,p}(\Omega;\mathbb{R}^d)$ such that (up to a subsequence) $u_n = v_n + z_n$ where $\{|\nabla v_n|^p\}$ is equi-integrable and*

$$|\Omega_n| \underset{n\to\infty}{\longrightarrow} 0 \tag{7}$$

where

$$\Omega_n = \{x \in \Omega :\ u_n \neq v_n \text{ or } \nabla u_n \neq \nabla v_n\}.$$

By the properties of Young measures it follows that $\{\nabla v_n\}$ generates $(\nu, 0)$. Indeed i) of Lemma 1 holds by (7) and ii) holds by this equality too together with the fact that $\left\{\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, \nabla v_n(x)\right)\right\}$ is equi-integrable for $\varphi \in \mathcal{H}_p$.

To show that $\{\nabla z_n\}$ generates (δ_0, Λ) we remark that i) of Lemma 1 follows by Riemann-Lebesgue's Lemma and (7) with $\nu = \delta_0$. Let us prove ii) of this lemma. It is enough to see that for all $\varphi \in \mathcal{H}_p$ then

$$\lim_{n\to\infty}\int_\Omega z(x)\left[\varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, \nabla z_n(x)\right) + \varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, \nabla v_n(x)\right) - \varphi\left(\left\langle \frac{x}{\varepsilon_n}\right\rangle, \nabla u_n(x)\right)\right]dx = 0. \tag{8}$$

We start by proving (8) for $\varphi \in \mathcal{H}_p$ such that $\varphi|_{\overline{Q}\times S}$ is a polynomial function. Let φ be one of such functions. Then

$$\begin{aligned}
&\left|\int_\Omega z(x)\left[\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla z_n(x)\right)+\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla v_n(x)\right)-\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla u_n(x)\right)\right]dx\right|\\
&=\left|\int_{\Omega_n} z(x)\left[\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla z_n(x)\right)+\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla v_n(x)\right)-\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla u_n(x)\right)\right]dx\right|\\
&\le C\int_{\Omega_n}|\nabla v_n|^p\,dx+C\int_{\Omega_n}(|\nabla v_n|^{p-1}+|\nabla z_n|^{p-1})|\nabla v_n|\,dx
\end{aligned}$$

(see Remark 4 below). Equality (8) follows then from Hölder and the equi-integrability of $\{|\nabla v_n|^p\}$. The general case follows by density (Weierstrass Approximation Theorem).

Remark 4 Let $\varphi \in \mathcal{H}_p$ be a C^1-function. By the Mean-Value Theorem and the p-homogeneity of φ in the second variable it follows that

$$|\varphi(y,\xi_1)-\varphi(y,\xi_2)|\le C\left(|\xi_1|^{p-1}+|\xi_2|^{p-1}\right)|\xi_1-\xi_2|$$

for all $y \in \overline{Q}$ and $\xi_1,\xi_2 \in \mathbb{R}^{d\times N}$.

Lemma 3 *Let $\nu \in L^\infty_w(\Omega\times Q;\mathcal{M}(\mathbb{R}^{d\times N}))$ and $\Lambda \in \mathcal{M}^+(\overline{\Omega}\times\overline{Q}\times S)$. Let $\{v_n\}$, $\{z_n\} \subset W^{1,p}(\Omega;\mathbb{R}^d)$ be two sequences such that $\{|\nabla v_n|^p\}$ is equi-integrable, $\{\nabla z_n\}$ generates (δ_0,Λ) and $\{\nabla v_n\}$ generates $(\nu,0)$. Then $\{\nabla u_n\}$, $u_n = v_n + z_n$, generates (ν,Λ).*

Proof By density to prove i) of Lemma 1 it is enough to show that for all $z \in C(\overline{\Omega})$ and $\varphi \in C_0^\infty(\overline{Q}\times\mathbb{R}^{d\times N})$ then

$$\lim_{n\to\infty}\int_\Omega z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla u_n(x)\right)dx=\int_\Omega\int_Q\int_{\mathbb{R}^{d\times N}}z(x)\varphi(y,\xi)\,d\nu_{(x,y)}(\xi)\,dy\,dx. \tag{9}$$

As

$$\lim_{n\to\infty}\left|\int_\Omega z(x)\left[\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla u_n(x)\right)-\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla v_n(x)\right)\right]\right|dx\le C\int_\Omega|\nabla z_n|\,dx\to 0$$

since the two-scale gradient Young measure for oscillations associated to $\{\nabla z_n\}$ is δ_0 and this sequence is equi-integrable, then (9) follows by the hypothesis on $\{\nabla v_n\}$.

Now given $z \in C(\overline{\Omega})$ and $\varphi \in \mathcal{H}_p$ let us see that ii) of Lemma 1 hold, i.e,

$$\lim_{n\to\infty}\int_\Omega z(x)\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla u_n(x)\right)dx=\int_\Omega\int_Q\int_{\mathbb{R}^{d\times N}}z(x)\varphi(y,\xi)\,d\nu_{(x,y)}(\xi)\,dy\,dx+\langle\Lambda,z\otimes\varphi^\infty\rangle$$

or, equivalently, that

$$\lim_{n\to\infty}\int_\Omega z(x)\left[\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla u_n(x)\right)-\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla z_n(x)\right)-\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla v_n(x)\right)\right]dx=0. \tag{10}$$

Let $\lambda > 0$ and $\varphi \in \mathcal{H}_p$ such that $\varphi|_{\overline{Q}\times S}$ is a polynomial function. By Remark 4 and the p-homogeneity we have that

$$\begin{aligned}
&\left|\int_\Omega z(x)\left[\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla u_n(x)\right)-\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla z_n(x)\right)-\varphi\left(\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla v_n(x)\right)\right]dx\right|\\
&\le C\int_{\Omega\cap\{|\nabla z_n|\le\lambda\}}\left(|\nabla v_n|^{p-1}+|\nabla z_n|^{p-1}\right)|\nabla z_n|\,dx+C\int_{\Omega\cap\{|\nabla z_n|\le\lambda\}}|\nabla z_n|^p\,dx
\end{aligned}$$

$$+C\int_{\Omega\cap\{|\nabla z_n|\geq\lambda\}}\left(|\nabla v_n|^{p-1}+|\nabla z_n|^{p-1}\right)|\nabla v_n|\,dx+C\int_{\Omega\cap\{|\nabla z_n|\geq\lambda\}}|\nabla v_n|^p\,dx.$$

Thus, using Hölder inequality, the fact that $|\{|\nabla z_n|\geq\lambda\}|\underset{n\to\infty}{\longrightarrow}0$ and the p-equi-integrability of $\{\nabla v_n\}$, equality (10) follows.

3 Proofs

The aim of this section is to prove Theorem 1 and Theorem 2.

3.1 Proof of Theorem 1

To prove Theorem 1 we have to show that given $\varepsilon_n\to 0$ and $\{u_n\}$ a bounded sequence in $W^{1,p}(\Omega;\mathbb{R}^d)$, there exist $(\nu,\Lambda)\in L^\infty_w(\Omega\times Q;\mathcal{M}(\mathbb{R}^{d\times N}))\times\mathcal{M}^+(\overline{\Omega}\times\overline{Q}\times S)$ and a subsequence (still denoted by $\{u_n\}$) such that for all $f\in\mathcal{E}^p(\Omega)$

$$\lim_{n\to\infty}\int_\Omega f\left(x,\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla u_n\right)\,dx=\int_\Omega\int_Q\langle\nu_{(x,y)},f(x,y,.)\rangle\,dy\,dx+\langle\Lambda,f^\infty\rangle.$$

We start by observing that the space $\mathcal{E}_p(\Omega)$ in (3) can be identified with $C(\overline{\Omega}\times\overline{Q}\times\overline{B})$. Precisely, to each function $f\in\mathcal{E}_p(\Omega)$ we associate the function $\varphi_f\in C(\overline{\Omega}\times\overline{Q}\times\overline{B})$ given by

$$\varphi_f(x,y,\xi):=\begin{cases} f\left(x,y,\frac{\xi}{1-|\xi|}\right)\left[1+\left(\frac{|\xi|}{1-|\xi|}\right)^p\right]^{-1} & \xi\in B,\\ f^\infty(x,y,\xi) & \xi\in S.\end{cases}\tag{11}$$

Let $f\in\mathcal{E}_p(\Omega)$. By (11) we can write

$$\lim_{n\to\infty}\int_\Omega f\left(x,\left\langle\frac{x}{\varepsilon_n}\right\rangle,\nabla u_n\right)\,dx=\lim_{n\to\infty}\int_\Omega\varphi_f\left(x,\left\langle\frac{x}{\varepsilon_n}\right\rangle,\frac{\nabla u_n}{1+|\nabla u_n|}\right)(1+|\nabla u_n|^p)\,dx.\tag{12}$$

To derive (12) we use the Banach-Alaoglu-Bourbaki Theorem in $\mathcal{M}(\overline{\Omega}\times\overline{Q}\times\overline{B})$. For this purpose, we define a sequence of measures μ_n by

$$\langle\mu_n,\varphi\rangle:=\int_\Omega\varphi\left(x,\left\langle\frac{x}{\varepsilon_n}\right\rangle,\frac{\nabla u_n}{1+|\nabla u_n|}\right)(1+|\nabla u_n|^p)\,dx$$

for $\varphi\in C(\overline{\Omega}\times\overline{Q}\times\overline{B})$. It is easy to check that $\{\mu_n\}$ is uniformly bounded. Thus, up to a subsequence, still denoted by $\{\mu_n\}$, we can find $\mu\in\mathcal{M}(\overline{\Omega}\times\overline{Q}\times\overline{B})$ such that for all $\varphi\in C(\overline{\Omega}\times\overline{Q}\times\overline{B})$

$$\langle\mu_n,\varphi\rangle\to\langle\mu,\varphi\rangle.\tag{13}$$

By the Disintegration Theorem (see e.g. Valadier [32]) we can write

$$\mu=\mu_x\otimes\pi$$

where $\mu_x\in\mathcal{M}(\overline{Q}\times\overline{B})$ is a probability measure and $\pi\in\mathcal{M}(\overline{\Omega})$. We note that $\pi\in\mathcal{M}^+(\overline{\Omega})$ since

$$(1+|\nabla u_n|^p)\overset{*}{\rightharpoonup}\pi$$

(it is enough to consider in (13) test functions of the type $\varphi\equiv\varphi(x)\in C(\overline{\Omega})$). In addition, by the Radon-Nikodym Theorem it follows that

$$\pi=\pi_a dx+\pi_s$$

with $\pi_a \in L^1(\Omega)$ and $\pi_s \in \mathcal{M}(\overline{\Omega})$. We also remark that

$$\mu_x|_{\overline{Q}\times B} = 0 \text{ for } \pi_s\text{-a.e } x \in \overline{\Omega}. \tag{14}$$

Indeed, to see (14) let $\mathcal{D}$ be a countable dense subset of $C_0(\overline{Q}\times B)$ and let $\alpha \in C(\overline{\Omega})$. For all $\theta \in \mathcal{D}$ the sequence $\left\{\theta\left(\left\langle\frac{\cdot}{\varepsilon_n}\right\rangle, \frac{\nabla u_n}{1+|\nabla u_n|}\right)(1+|\nabla u_n|^p)\right\}$ is equi-integrable and then

$$\int_\Omega \alpha(x)\theta\left(x, \left\langle\frac{x}{\varepsilon_n}\right\rangle, \frac{\nabla u_n}{1+|\nabla u_n|}\right)(1+|\nabla u_n|^p)\,dx \underset{n\to\infty}{\longrightarrow} \int_\Omega \alpha(x) g_\theta(x)\,dx$$

for some $g_\theta \in L^1(\Omega)$. Thus by (13)

$$\int_{\overline{\Omega}} \alpha(x)\langle\mu_x, \theta\rangle\, d\pi_s(x)\, dx = 0$$

for all $\theta \in \mathcal{D}$, and so, since α is arbitrary,

$$\langle\mu_x, \theta\rangle = 0 \quad \pi_s\text{-a.e } x \in \overline{\Omega}$$

for all $\theta \in \mathcal{D}$. By density (14) follows.

Accordingly, by (14), for all $\varphi \in C(\overline{\Omega}\times\overline{Q}\times\overline{B})$ we get that

$$\begin{aligned}\langle\mu, \varphi\rangle &= \int_{\overline{\Omega}}\int_{\overline{Q}\times\overline{B}} \varphi(x,y,\xi)d\mu_x(y,\xi)d\pi(x)\\ &= \int_\Omega \pi_a(x)\int_{\overline{Q}\times B} \varphi(x,y,\xi)d\mu_x(y,\xi)\,dx + \int_{\overline{\Omega}}\int_{\overline{Q}\times S} \varphi(x,y,\xi)d\mu_x(y,\xi)d\pi(x).\end{aligned}$$

To characterize (12), that is, $\langle\mu, \varphi_f\rangle$, we define for a.e $x \in \Omega$ a measure $\nu_x \in \mathcal{M}(\overline{Q}\times\mathbb{R}^{d\times N})$ by

$$\langle\nu_x, \phi\rangle := \pi_a(x)\int_{\overline{Q}\times B} \varphi_\phi(y,\xi)\,d\mu_x(y,\xi)$$

for $\phi \in C_0(\overline{Q}\times\mathbb{R}^{d\times N})$. We take $\{\nu_{(x,y)}\}$ to be the disintegration of $\{\nu_x\} \subset \mathcal{M}(\overline{Q}\times\mathbb{R}^{d\times N})$ with respect to the Lebesgue measure on Q. Finally, we define

$$\langle\lambda_x, \psi\rangle := \int_{\overline{Q}\times\partial B} \psi(y,\gamma)d\mu_x(y,\gamma)$$

for $\psi \in C(\overline{Q}\times S)$ and take $\Lambda = \lambda_x \otimes \pi$.

3.2 Proof of Theorem 2

The objective of this section is to prove Theorem 2.

3.2.1 **Necessity**

Let (ν, Λ) be a two-scale gradient Young measure for oscillations and concentrations associated to a sequence $\{\nabla u_n\}$, $\{u_n\} \subset W^{1,p}(\Omega;\mathbb{R}^d)$ bounded. We have to show that for every $\psi \in \mathcal{H}_p$ with $\psi_{\text{hom}}(0) = 0$ the inequality

$$\langle \lambda_x, \psi \rangle \geq 0$$

holds for $\pi -$ a.e x.

Let $\mathcal{D}$ be a dense subset of $\{\psi \in \mathcal{H}_p : \ \psi_{\text{hom}}(0) = 0\}$ and let $a \in \Omega$ be a Lebesgue point (with respect to π) of

$$x \to \langle \lambda_x, \psi \rangle \tag{15}$$

for all $\psi \in \mathcal{D}$. It is enough to see that

$$\langle \lambda_a, \psi \rangle \geq 0 \tag{16}$$

for all $\psi \in \mathcal{D}$.

Using Lemma 2 (up to a subsequence) we can decompose $u_n = v_n + z_n$ with $v_n,\ z_n \in W^{1,p}(\Omega;\mathbb{R}^d)$ such that $\{|\nabla v_n|^p\}$ is equi-integrable, $\{\nabla z_n\}$ generates (δ_0, Λ), $\{\nabla v_n\}$ generates $(\nu, 0)$ and (7) is satisfied.

Let $\delta > 0$ and $\theta \in C_c\big(Q(a,\delta)\big)$, where $Q(a,\delta) := a + \delta Q$. Using standard gamma convergence results on homogenization (see Theorem 1.1 in Baía & Fonseca [7]) and since by (7) $z_n \underset{n\to\infty}{\rightharpoonup} 0$ in $W^{1,p}$ then

$$\liminf_{n\to\infty} \int_{Q(a,\delta)} \theta(x)\psi(\langle x/\varepsilon_n \rangle, \nabla z_n)\, dx \geq \int_{Q(a,\delta)} \theta(x)\psi_{\text{hom}}(0)\, dx = 0$$

for every $\psi \in \mathcal{H}_p$ with $\psi_{\text{hom}}(0) = 0$. On the other hand, as ∇z_n generates (δ_0, Λ) then for every $\psi \in \mathcal{H}_p$

$$\lim_{n\to\infty} \int_{Q(a,\delta)} \theta(x)\psi(\langle x/\varepsilon_n \rangle, \nabla z_n)\, dx = \int_{Q(a,\delta)} \theta(x)\langle \lambda_x, \psi \rangle d\pi(x).$$

Thus, in particular, for all $\psi \in \mathcal{D}$ we have that

$$\int_{Q(a,\delta)} \theta(x)\langle \lambda_x, \psi \rangle d\pi(x) \geq 0.$$

Let $\theta_k \in C_c\big(Q(a,\delta)\big)$ with $\theta_k \underset{k\to\infty}{\longrightarrow} \chi_{Q(a,\delta)}$. Then

$$\int_{Q(a,\delta)} \langle \lambda_x, \psi \rangle d\pi(x) \geq 0$$

for all $\psi \in \mathcal{D}$. By (15) inequality (16) follows.

3.2.2 **Sufficiency**

For the sufficiency we use the following lemma.

Lemma 4 *Let*

$$A := \{\mu \in \mathcal{M}^+(\overline{Q} \times S) : \ \langle \mu, \psi \rangle \geq 0 \ \ \textit{for all } \psi \in \mathcal{H}_p \ \textit{with } \psi_{\text{hom}}(0) = 0\}$$

and

$$H := \Big\{\mu \in \mathcal{M}^+(\overline{Q} \times S) : \ \langle \mu, \psi \rangle = \fint_{(0,T)^N} \psi(\langle y \rangle, \nabla u(y))\, dy \ \textit{ for all } \psi \in \mathcal{H}_p, \\ \textit{for some } T \in \mathbb{N} \ \textit{ and } \ \ u \in W_0^{1,p}((0,T)^N;\mathbb{R}^d)\Big\}.$$

Then H is a convex set. In addition $\overline{H} = A$ for the weak-topology.*

Proof We start by showing that H is convex. Let

$$\langle \mu_i, \psi \rangle = \fint_{(0,T_i)^N} \psi(\langle y \rangle, \nabla u_i(y))\, dy$$

for all $\psi \in \mathcal{H}_p$, for some $T_i \in \mathbb{N}$ and $u_i \in W_0^{1,p}((0,T_i)^n;\mathbb{R}^d)$ with $i = 1, 2$, and let $\theta \in [0,1]$. We want to see that

$$\theta \mu_1 + (1-\theta)\mu_2 \in H. \tag{17}$$

Without loss of generality we can assume that $T_2 \geq T_1$. Let $T := T_1 + T_2$ and let us define

$$u(y) := \theta^{1/p} \left(\frac{T}{T_1}\right)^{N/p} u_1(y) + (1-\theta)^{1/p} \left(\frac{T}{T_2}\right)^{N/p} u_2(y - x_0)$$

where $x_0 := (T_1, \cdots, T_1) \in \mathbb{N}^N$. Then $u \in W_0^{1,p}\big((0,T)^N;\mathbb{R}^N\big)$ and for all $\psi \in \mathcal{H}_p$ we obtain, by p-homogeneity and changing variables, that

$$\begin{aligned}\fint_{(0,T)^N} \psi(\langle y \rangle, \nabla u(y))\, dy &= \theta \fint_{(0,T_1)^N} \psi(\langle y \rangle, \nabla u_1(y))\, dy + (1-\theta) \fint_{x_0 + (0,T_2)^N} \psi(\langle y \rangle, \nabla u_2(y - x_0))\, dy \\ &= \theta \fint_{(0,T_1)^N} \psi(\langle y \rangle, \nabla u_1(y))\, dy + (1-\theta) \fint_{(0,T_2)^N} \psi(\langle y + x_0 \rangle, \nabla u_2(y))\, dy.\end{aligned}$$

By periodicity (17) follows since

$$\fint_{(0,T)^N} \psi(\langle y \rangle, \nabla u(y))\, dy = \theta \fint_{(0,T_1)^N} \psi(\langle y \rangle, \nabla u_1(y))\, dy + (1-\theta) \fint_{(0,T_2)^N} \psi(\langle y \rangle, \nabla u_2(y))\, dy.$$

To see that $\overline{H} = A$ we start by proving that $H \subset A$. Let $\mu \in H$. Then for all $\psi \in \mathcal{H}_p$

$$\langle \mu, \psi \rangle = \fint_{(0,T)^N} \psi(\langle y \rangle, \nabla u(y))\, dy$$

for some $T \in \mathbb{N}$ and $u \in W_0^{1,p}((0,T)^n;\mathbb{R}^d)$. By Example 1 the pair

$$\left(\delta_0, \mu \frac{T^N}{||\nabla u||_{L^p}^p} \otimes \delta_{x_0} \frac{||\nabla u||_{L^p}^p}{T^N}\right)$$

with $x_0 \in \Omega$ and $\mu \in H$ is a gradient two-scale Young measure for oscillations and concentrations. Thus using the necessary condition (see Subsection 3.2.1) we get that

$$\langle \mu, \psi \rangle \geq 0$$

for all $\psi \in \mathcal{H}_p$ with $\psi_{\mathrm{hom}}(0) = 0$, and so $H \subset A$. As A is a closed set then $\overline{H} \subset A$.

Now we prove the opposite inclusion by contradiction. Let $\mu \in A$ and assume that $\mu \notin \overline{H}$. As H is convex, by the Hahn-Banach Theorem (see also Proposition III.13 in Brezis [9]) there exist $\psi \in C(\overline{Q} \times S)$ such that for all $\nu \in \overline{H}$

$$\langle \nu, \psi \rangle \geq \alpha$$

and

$$\langle \mu, \psi \rangle < \alpha$$

with $\alpha \in \mathbb{R}$. Extending ψ by p-homogeneity then

$$\psi_{\mathrm{hom}}(0) \geq \alpha.$$

Thus $\psi_{\mathrm{hom}}(0) \neq -\infty$. As $\mu \in A$ then $\psi_{\mathrm{hom}}(0) \neq 0$, which is a contradiction (see Remark 1).

Remark 5 For any $R > 0$ it follows that

$$\overline{H} \cap \{||\mu|| \leq R\} = \overline{H \cap \{||\mu|| \leq R\}}.$$

Indeed, it is clear that $\overline{H} \cap \{||\mu|| \leq R\} = A \cap \{||\mu|| \leq R\} \subset \overline{H \cap \{||\mu|| \leq R\}}$. To see the other inclusion we consider $\mu \in \overline{H}$ with $||\mu|| \leq R$. Thus, there exist $\mu_n \in H$ such that $\mu_n \rightharpoonup \mu$ weakly-$*$. As

$$\limsup_{n\to\infty} ||\mu_n|| = \limsup_{n\to\infty} \mu_n(\overline{Q} \times S) \leq \mu(\overline{Q} \times S) = ||\mu|| \leq R$$

then $\mu \in \overline{H \cap \{||\mu|| \leq R\}}$.

Our goal now is to show that given $\nu \in L^\infty_w(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d\times N}))$ and $\Lambda \in \mathcal{M}^+(\overline{\Omega} \times \overline{Q} \times S)$ satisfying i)-iv) of Theorem 2 then (ν, Λ) is a two-scale gradient Young measure for oscillations and concentrations. By Lemma 3 it is enough to find sequences $\{v_n\}, \{z_n\} \subset W^{1,p}(\Omega; \mathbb{R}^d)$ such that $\{|\nabla v_n|^p\}$ is equi-integrable, $\{\nabla z_n\}$ generates (δ_0, Λ) and $\{\nabla v_n\}$ generates $(\nu, 0)$.

We start by constructing the sequence $\{v_n\}$. Using Theorem 3 and the Decomposition Lemma (see Lemma 1.2 in Fonseca, Müller and Pedregal [21]) we can find a bounded sequence $\{v_n\} \subset W^{1,p}(\Omega; \mathbb{R}^d)$ with $\{|\nabla v_n|\}^p$ equi-integrable and such that $\{\nabla v_n\}$ generates the two-scale Young measure ν. Then for $\theta \in C(\overline{\Omega})$ and $\psi \in \mathcal{H}_p$ we have

$$\lim_{n\to\infty} \int_\Omega \theta(x)\psi(\langle x/\varepsilon_n\rangle, \nabla v_n)\, dx = \int_\Omega \theta(x) \int_Q \int_{\mathbb{R}^{d\times N}} \psi\,(y, \xi)\, d\nu_{(x,y)}(\xi)\, dy\, dx$$

by the equi-integrability of $\{\theta(\cdot)\psi(\langle \cdot/\varepsilon_n\rangle, \nabla v_n)\}$. Thus $\{\nabla v_n\}$ generates $(\nu, 0)$.

Now we construct the sequence $\{z_n\}$. Let us assume first that

$$\Lambda = \lambda \otimes \delta_{x_0}$$

with $\lambda \in A$ and $x_0 \in \Omega$. As $\lambda \in A$ then by Lemma 4 and Remark 5 there exist $\{\lambda_k\} \subset H$ with $||\lambda_k|| \leq ||\lambda||$ such that $\lambda_k \rightharpoonup \lambda$ weakly-$*$. As $\lambda_k \in H$ there exists $T_k \in \mathbb{N}$ and $u_k \in W^{1,p}_0([0, T_k]); \mathbb{R}^d)$ such that

$$\langle \lambda_k, \psi \rangle = ⨍_{(0,T_k)^N} \psi(y, \nabla u_k(y))\, dy \tag{18}$$

for all $\psi \in \mathcal{H}_p$. Arguing as in the Example 1 the pair (δ_0, Λ_k), $\Lambda_k := \lambda_k \otimes \delta_{x_0}$, is a two-scale gradient Young measure for oscillations and concentrations generated by $\{\nabla {u_n}^k\}$ where

$${u_n}^k(x) = \frac{\varepsilon_n}{(\varepsilon_n T_k)^{N/p}} u_k\left(\frac{x - \varepsilon_n[x_0/\varepsilon_n]}{\varepsilon_n}\right).$$

Then for all $\theta \in C(\overline{\Omega})$ and $\varphi \in C_0(\overline{Q} \times \mathbb{R}^{d\times N})$

$$\lim_{n\to\infty} \int_\Omega \theta(x)\varphi(\langle x/\varepsilon_n\rangle, \nabla {u_n}^k)\, dx = \int_\Omega \theta(x) \int_Q \varphi(y, 0)\, dy\, dx$$

and moreover for all $\psi \in \mathcal{H}^p$

$$\lim_{n\to\infty} \int_\Omega \theta(x)\psi(\langle x/\varepsilon_n\rangle, \nabla {u_n}^k)\, dx = \langle \Lambda_k, \theta \otimes \psi \rangle.$$

Consequently

$$\lim_{k\to\infty} \lim_{n\to\infty} \int_\Omega \theta(x)\varphi(\langle x/\varepsilon_n\rangle, \nabla {u_n}^k)\, dx = \int_\Omega \theta(x) \int_Q \varphi(y, 0)\, dy\, dx \tag{19}$$

and

$$\lim_{k\to\infty} \lim_{n\to\infty} \int_\Omega \theta(x)\psi(\langle x/\varepsilon_n\rangle, \nabla {u_n}^k)\, dx = \langle \Lambda, \theta \otimes \psi \rangle. \tag{20}$$

On the other hand using $\psi(y,\xi) = |\xi|^p$ in (18) it follows that

$$\lim_{n\to\infty} \int_\Omega |\nabla {u_n}^k|^p \, dx = ||\lambda_k|| \le ||\lambda||$$

for each $k \in \mathbb{N}$. Hence

$$\limsup_{k\to\infty} \lim_{n\to\infty} \int_\Omega |\nabla {u_n}^k|^p \, dx \le ||\lambda||. \tag{21}$$

Therefore, by diagonalization in (19)-(21), we get a sequence $z_n = u_n^{k(n)}$ such that

$$\lim_{n\to\infty} \int_\Omega \theta(x)\varphi(\langle x/\varepsilon_n\rangle, \nabla z_n) \, dx = \int_\Omega \theta(x) \int_Q \varphi(y,0) \, dy \, dx$$

and

$$\lim_{n\to\infty} \int_\Omega \theta(x)\psi(\langle x/\varepsilon_n\rangle, \nabla z_n) \, dx = \langle \Lambda, \theta \otimes \psi \rangle$$

for all $\theta \in C(\overline{\Omega})$, $\varphi \in C_0(\overline{Q} \times \mathbb{R}^{d\times N})$ and $\psi \in \mathcal{H}^p$, and in addition

$$\sup_{n\in\mathbb{N}} ||\nabla z_n||^p_{L^p} \le ||\lambda||.$$

It follows that ∇z_n generates (δ_0, Λ).

We remark that the case

$$\Lambda = \sum_{i=1}^{I} c_i \lambda_i \otimes \delta_{x_i}$$

is similar, and the general case comes from an approximation argument as in Lemma 6.2 of Fonseca, Müller and Pedregal [21].

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