

CHARACTERIZATION OF TWO-SCALE GRADIENT YOUNG MEASURES AND APPLICATION TO HOMOGENIZATION

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ABSTRACT. This work is devoted to the study of two-scale gradient Young measures naturally arising in nonlinear elasticity homogenization problems. Precisely, a characterization of this class of measures is derived and an integral representation formula for homogenized energies, whose integrands satisfy very weak regularity assumptions, is obtained in terms of two-scale gradient Young measures.

KEYWORDS: Young measures, homogenization, Γ -convergence, two-scale convergence.

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1. INTRODUCTION

Young (or Parametrized) measures have been introduced in optimal control theory by L. C. Young [39] to study non convex variational problems for which there were no classical solution, and to provide an effective notion of generalized solution for problems in Calculus of Variations.

Starting with the works of Tartar [35] on hyperbolic conservation laws, Young measures have been an important tool for studying the asymptotic behavior of solutions of nonlinear partial differential equations (see also DiPerna [17]). A key feature of these measures is their capacity to capture the oscillations of minimizing sequences of non convex variational problems, and many applications arise *e.g.* in models of elastic crystals (see Chipot & Kinderlehrer [16] and Fonseca [19]), phase transition (see Ball & James [8]), optimal design (see Bonnetier & Conca [11], Maestre & Pedregal [24] and Pedregal [32]). The special properties of Young measures generated by sequences of gradients of Sobolev functions have been studied by Kinderlehrer & Pedregal [21, 22] and are relevant in the applications to nonlinear elasticity.

The lack of information on the spatial structure of oscillations presents an obstacle for the application of Young measures to homogenization problems. Two-scale Young measures have been introduced by E in [18] to study periodic homogenization of nonlinear transport equations. They contain some information on the amount of oscillations and extendNguetseng's notion of two-scale convergence (see [29] and Allaire [2]). Other (generalized) multiscale Young measures have been introduced in the works of Alberti & Müller [1] and Ambrosio & Frid [3].

From a variational point of view periodic homogenization of integral functionals rests on the study of the equilibrium states, or minimizers, of a family of functionals of the type

$$\mathcal{F}_\varepsilon(u) := \int_{\Omega} f\left(x, \left\langle \frac{x}{\varepsilon} \right\rangle, \nabla u(x)\right) dx, \quad (1.1)$$

as $\varepsilon \rightarrow 0$, under suitable boundary conditions. Here Ω (bounded open subset of \mathbb{R}^N) is the reference configuration of a nonlinear elastic body with periodic microstructure and whose heterogeneities scale like a small parameter $\varepsilon > 0$. The function $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ stands for a deformation and $f : \Omega \times Q \times \mathbb{R}^{d \times N} \rightarrow [0, +\infty)$, with $Q := (0, 1)^N$, is the stored energy density of this body that is assumed to satisfy standard p -coercivity and p -growth conditions, with $p > 1$. The presence of the term $\langle x/\varepsilon \rangle$ (fractional part of the vector x/ε componentwise) takes into account the periodic microstructure of

the body leading the integrand of (1.1) to be periodic with respect to that variable. The macroscopic (or averaged) description of this material may be understood by the Γ -limit of (1.1) with respect to the weak $W^{1,p}(\Omega; \mathbb{R}^d)$ -topology (or, equivalently, with respect to the $L^p(\Omega; \mathbb{R}^d)$ -topology if Ω is, for instance, Lipschitz) and it has already been studied by many authors in the Sobolev setting. Namely, under several regularity assumptions on f it has been proved that for all $u \in W^{1,p}(\Omega; \mathbb{R}^d)$,

$$\mathcal{F}_{\text{hom}}(u) := \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u) = \int_{\Omega} f_{\text{hom}}(x, \nabla u(x)) dx, \quad (1.2)$$

where for all $(x, \xi) \in \Omega \times \mathbb{R}^{d \times N}$

$$f_{\text{hom}}(x, \xi) = \inf_{T, \phi} \left\{ \int_{(0,T)^N} f(x, y, \xi + \nabla \phi(y)) dy : T \in \mathbb{N}, \phi \in W_0^{1,p}((0,T)^N; \mathbb{R}^d) \right\} \quad (1.3)$$

(see Baía & Fonseca [5, 6], Braides [12], Braides & Defranceschi [14], Marcellini [26] and Müller [27]).

We also refer to Anza Hafsa, Mandallena & Michaille [4] where a formula for the function f_{hom} has been given in terms of gradient Young measures. In the convex case, Barchiesi [9] and Pedregal [33] have derived the same Γ -limit result (1.2) with Young measures techniques. The main contribution in [9] is to weaken, as most as possible, the regularity of f that is assumed to be an “admissible integrand” in the sense of Valadier [38] (see Definition 4.1 below). Using the same kind of arguments, Pedregal has extended this result to the nonconvex case in [31].

Following these lines, we seek here to derive a representation of \mathcal{F}_{hom} in terms of two-scale gradient Young measures (see Definition 2.3 below). From a mathematical point of view, the main property of Young measures is their capability of describing the asymptotic behavior of integrals of the form

$$\int_{\Omega} f(v_\varepsilon(x)) dx,$$

where f is some nonlinear function and $\{v_\varepsilon\}$ is an oscillating sequence. To address the homogenization of (1.1) we consider Young measures generated by sequences of the type $\{(\cdot/\varepsilon, \nabla u_\varepsilon)\}$, which are, roughly speaking, what we will call *two-scale gradient Young measures*. From a physical point of view, we seek to capture microstructures – due to finer and finer oscillations of minimizing sequences that cannot reach an optimal state – at a given scale ε (period of the material heterogeneities). In this way, the minima of the limit problem captures two kinds of oscillations of the minimizing sequences: those due to the periodic heterogeneities of the material and those due to a possible multi-well structure. In fact this is the main advantage of deriving a characterization of \mathcal{F}_{hom} in terms of two-scale gradient Young measures, since solutions of

$$\min_{u=u_0 \text{ on } \partial\Omega} \int_{\Omega} f_{\text{hom}}(x, \nabla u(x)) dx$$

only give an average of the oscillations that minimizing sequences may develop. Precisely, we prove the following result.

Theorem 1.1. *Let Ω be a bounded open subset of \mathbb{R}^N with Lipschitz boundary and let $f : \Omega \times Q \times \mathbb{R}^{d \times N} \rightarrow [0, +\infty)$ be an admissible integrand. Assume that there exist constants $\alpha, \beta > 0$ and $p \in (1, +\infty)$ such that for all $(x, y, \xi) \in \Omega \times Q \times \mathbb{R}^{d \times N}$*

$$\alpha|\xi|^p \leq f(x, y, \xi) \leq \beta(1 + |\xi|^p). \quad (1.4)$$

Then the functional \mathcal{F}_ε Γ -converges with respect to the weak $W^{1,p}(\Omega; \mathbb{R}^d)$ -topology (or equivalently the strong $L^p(\Omega; \mathbb{R}^d)$ -topology) to the functional $\mathcal{F}_{\text{hom}} : W^{1,p}(\Omega; \mathbb{R}^d) \rightarrow [0, +\infty)$ given by

$$\mathcal{F}_{\text{hom}}(u) = \min_{\nu \in \mathcal{M}_u} \int_{\Omega} \int_Q \int_{\mathbb{R}^{d \times N}} f(x, y, \xi) d\nu_{(x,y)}(\xi) dy dx, \quad (1.5)$$

where

$$\mathcal{M}_u := \left\{ \nu \in L_w^\infty(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d \times N})) : \{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q} \text{ is a two-scale gradient Young measure such that } \right. \\ \left. \nabla u(x) = \int_Q \int_{\mathbb{R}^{d \times N}} \xi \, d\nu_{(x,y)}(\xi) \, dy \text{ for a.e. } x \in \Omega \right\} \quad (1.6)$$

for all $u \in W^{1,p}(\Omega; \mathbb{R}^d)$.

In order to prove Theorem 1.1 and make it more useful, we first derive an algebraic characterization of two-scale gradient Young measures in the spirit of Kinderlehrer & Pedregal [22]. This characterization is given in terms of a Jensen's inequality with test functions in the space \mathcal{E}_p of continuous functions $f : \overline{Q} \times \mathbb{R}^{d \times N} \rightarrow \mathbb{R}$ such that the limit

$$\lim_{|\xi| \rightarrow +\infty} \frac{f(y, \xi)}{1 + |\xi|^p}$$

exists uniformly with respect to $y \in \overline{Q}$. We prove the following result.

Theorem 1.2. *Let Ω be a bounded open subset of \mathbb{R}^N with Lipschitz boundary and let $\nu \in L_w^\infty(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d \times N}))$ be such that $\nu_{(x,y)} \in \mathcal{P}(\mathbb{R}^{d \times N})$ for a.e. $(x, y) \in \Omega \times Q$. The family $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ is a two-scale gradient Young measure if and only if the three conditions below hold:*

i) *there exist $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ and $u_1 \in L^p(\Omega; W_{per}^{1,p}(Q; \mathbb{R}^d))$ such that*

$$\int_{\mathbb{R}^{d \times N}} \xi \, d\nu_{(x,y)}(\xi) = \nabla u(x) + \nabla_y u_1(x, y) \quad \text{for a.e. } (x, y) \in \Omega \times Q; \quad (1.7)$$

ii) *for every $f \in \mathcal{E}_p$*

$$\int_Q \int_{\mathbb{R}^{d \times N}} f(y, \xi) \, d\nu_{(x,y)}(\xi) \, dy \geq f_{\text{hom}}(\nabla u(x)) \quad \text{for a.e. } x \in \Omega, \quad (1.8)$$

where

$$f_{\text{hom}}(\xi) := \inf_{T, \phi} \left\{ \int_{(0,T)^N} f(\langle y \rangle, \xi + \nabla \phi(y)) \, dy : T \in \mathbb{N}, \phi \in W_0^{1,p}((0,T)^N; \mathbb{R}^d) \right\}; \quad (1.9)$$

iii)

$$(x, y) \mapsto \int_{\mathbb{R}^{d \times N}} |\xi|^p \, d\nu_{(x,y)}(\xi) \in L^1(\Omega \times Q). \quad (1.10)$$

We note that \mathcal{E}_p is separable (see Section 3). Hence, condition (ii) above needs only to be checked for countably many test functions f .

The proof of Theorem 1.2 is similar to that of Kinderlehrer & Pedregal [22]. We first address the homogeneous case, that is, we consider two-scale gradient Young measures that are independent of the macroscopic variable $x \in \Omega$. This case rests on the Hahn-Banach Separation Theorem. The general case will be obtained by splitting Ω into suitable small subsets and by approximating these measures by two-scale Young measures that are piecewise constant with respect to the variable $x \in \Omega$.

The overall plan of this work in the ensuing sections will be as follows: Section 2 collects the main notations and results used throughout. Section 3 is devoted to the proof of Theorem 1.2, and in Section 4 we address the proof of the homogenization result Theorem 1.1.

2. SOME PRELIMINARIES

The purpose of this section is to give a brief overview of the concepts and results that are used in the sequel. Almost all these results are stated without proofs as they can be readily found in the references given below.

2.1. Notation. Throughout this work Ω is an open bounded subset of \mathbb{R}^N with Lipschitz boundary, $\mathcal{A}(\Omega)$ denotes the family of all open subsets of Ω , \mathcal{L}^N is the Lebesgue measure in \mathbb{R}^N , $\mathbb{R}^{d \times N}$ is identified with the set of real $d \times N$ matrices and $Q := (0, 1)^N$ is the unit cube in \mathbb{R}^N . The symbols $\langle \cdot \rangle$ and $[\cdot]$ stand, respectively, for the fractional and integer part of a number, or a vector, componentwise. The Dirac mass at a point $a \in \mathbb{R}^m$ is denoted by δ_a . The symbol f_A stands for the average $\mathcal{L}^N(A)^{-1} \int_A$.

Let U be an open subset of \mathbb{R}^m . Then:

- $\mathcal{C}_c(U)$ is the space of continuous functions $f : U \rightarrow \mathbb{R}$ with compact support.
- $\mathcal{C}_0(U)$ is the closure of $\mathcal{C}_c(U)$ for the uniform convergence; it coincides with the space of all continuous functions $f : U \rightarrow \mathbb{R}$ such that, for every $\eta > 0$, there exists a compact set $K_\eta \subset U$ with $|f| < \eta$ on $U \setminus K_\eta$.
- $\mathcal{M}(U)$ is the space of real-valued Radon measures with finite total variation. We recall that by the Riesz Representation Theorem $\mathcal{M}(U)$ can be identified with the dual space of $\mathcal{C}_0(U)$ through the duality

$$\langle \mu, \phi \rangle = \int_U \phi d\mu, \quad \mu \in \mathcal{M}(U), \quad \phi \in \mathcal{C}_0(U).$$

- $\mathcal{P}(U)$ denotes the space of probability measures on U , *i.e.* the space of all $\mu \in \mathcal{M}(U)$ such that $\mu \geq 0$ and $\mu(U) = 1$.
- $L^1(\Omega; \mathcal{C}_0(U))$ is the space of maps $\phi : \Omega \rightarrow \mathcal{C}_0(U)$ such that
 - ϕ is strongly measurable, *i.e.* there exists a sequence of simple functions $s_n : \Omega \rightarrow \mathcal{C}_0(U)$ such that $\|s_n(x) - \phi(x)\|_{\mathcal{C}_0(U)} \rightarrow 0$ for a.e. $x \in \Omega$;
 - $x \mapsto \|\phi(x)\|_{\mathcal{C}_0(U)} \in L^1(\Omega)$.

We recall that the linear space spanned by $\{\varphi \otimes \psi : \varphi \in L^1(\Omega) \text{ and } \psi \in \mathcal{C}_0(U)\}$ is dense in $L^1(\Omega; \mathcal{C}_0(U))$.

- $L_w^\infty(\Omega; \mathcal{M}(U))$ is the space of maps $\nu : \Omega \rightarrow \mathcal{M}(U)$ such that
 - ν is weak* measurable, *i.e.* $x \mapsto \langle \nu_x, \phi \rangle$ is measurable for every $\phi \in \mathcal{C}_0(U)$;
 - $x \mapsto \|\nu_x\|_{\mathcal{M}(U)} \in L^\infty(\Omega)$.

The space $L_w^\infty(\Omega; \mathcal{M}(U))$ can be identified with the dual space of $L^1(\Omega; \mathcal{C}_0(U))$ through the duality

$$\langle \mu, \phi \rangle = \int_\Omega \int_U \phi(x, \xi) d\mu_x(\xi) dx, \quad \mu \in L_w^\infty(\Omega; \mathcal{M}(U)), \quad \phi \in L^1(\Omega; \mathcal{C}_0(U)),$$

where $\phi(x, \xi) := \phi(x)(\xi)$ for all $(x, \xi) \in \Omega \times U$. Hence it can be endowed with the weak* topology (see *e.g.* Theorem 2.11 in Málek, Nečas, Rokyta & Růžička [25]).

- The space $W_{\text{per}}^{1,p}(Q; \mathbb{R}^d)$ stands for the $W^{1,p}$ -closure of all functions $f \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R}^d)$ which are Q -periodic.

2.2. Young measures. We recall here the notion of Young measure and some of its basic properties. We refer the reader to Braides [13], Müller [28], Pedregal [30], Roubíček [34], Valadier [37] and references therein for a detailed description on the subject.

Definition 2.1. (Young measure) *Let $\nu \in L_w^\infty(\Omega; \mathcal{M}(\mathbb{R}^m))$ and let $z_n : \Omega \rightarrow \mathbb{R}^m$ be a sequence of measurable functions. The family of measures $\{\nu_x\}_{x \in \Omega}$ is said to be the Young measure generated by $\{z_n\}$ provided $\nu_x \in \mathcal{P}(\mathbb{R}^m)$ for a.e. $x \in \Omega$ and*

$$\delta_{z_n} \xrightarrow{*} \nu \text{ in } L_w^\infty(\Omega; \mathcal{M}(\mathbb{R}^m)),$$

i.e. for all $\psi \in L^1(\Omega; \mathcal{C}_0(\mathbb{R}^m))$

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \psi(x, z_n(x)) dx = \int_{\Omega} \int_{\mathbb{R}^m} \psi(x, \xi) d\nu_x(\xi) dx.$$

The family $\{\nu_x\}_{x \in \Omega}$ is said to be a *homogeneous* Young measure if the map $x \mapsto \nu_x$ is independent of x . In this case the family $\{\nu_x\}_{x \in \Omega}$ is identified with a single element ν of $\mathcal{M}(\mathbb{R}^m)$.

The following result asserts the existence of Young measures (see Ball [7]).

Theorem 2.2. *Let $\{z_n\}$ be a sequence of measurable functions $z_n : \Omega \rightarrow \mathbb{R}^m$. Then there exists a subsequence $\{z_{n_k}\}$ and $\nu \in L_w^\infty(\Omega; \mathcal{M}(\mathbb{R}^m))$ with $\nu_x \geq 0$ for a.e. $x \in \Omega$, such that $\delta_{z_{n_k}} \xrightarrow{*} \nu$ in $L_w^\infty(\Omega; \mathcal{M}(\mathbb{R}^m))$ and the following properties hold:*

- (i) $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} = \nu_x(\mathbb{R}^m) \leq 1$ for a.e. $x \in \Omega$;
- (ii) if $\text{dist}(z_{n_k}, K) \rightarrow 0$ in measure for some closed set $K \subset \mathbb{R}^m$, then $\text{Supp}(\nu_x) \subset K$ for a.e. $x \in \Omega$;
- (iii) $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ if and only if there exists a Borel function $g : \mathbb{R}^m \rightarrow [0, +\infty]$ such that

$$\lim_{|\xi| \rightarrow +\infty} g(\xi) = +\infty \quad \text{and} \quad \sup_{k \in \mathbb{N}} \int_{\Omega} g(z_{n_k}(x)) dx < +\infty;$$

- (iv) if $f : \Omega \times \mathbb{R}^m \rightarrow [0, +\infty]$ is a normal integrand, then

$$\liminf_{k \rightarrow +\infty} \int_{\Omega} f(x, z_{n_k}(x)) dx \geq \int_{\Omega} \int_{\mathbb{R}^m} f(x, \xi) d\nu_x(\xi) dx;$$

- (v) if (iii) holds and if $f : \Omega \times \mathbb{R}^m \rightarrow [0, +\infty]$ is a Carathéodory integrand such that the sequence $\{f(\cdot, z_{n_k})\}$ is equi-integrable then

$$\lim_{k \rightarrow +\infty} \int_{\Omega} f(x, z_{n_k}(x)) dx = \int_{\Omega} \int_{\mathbb{R}^m} f(x, \xi) d\nu_x(\xi) dx.$$

2.3. Two-scale gradient Young measures. As remarked by Pedregal [33], regular Young measures do not always provide enough information on the oscillations of a certain sequence $\{v_\varepsilon\}$. To better understand oscillations that occur at a given length scale ε we may study the Young measure generated by the pair $\{(\cdot/\varepsilon), v_\varepsilon\}$. In this paper we are interested in the case where $v_\varepsilon = \nabla u_\varepsilon$, for some sequence $\{u_\varepsilon\} \subset W^{1,p}(\Omega; \mathbb{R}^d)$, with $1 < p < \infty$.

Let $\mu \in L_w^\infty(\Omega; \mathcal{M}(\mathbb{R}^N \times \mathbb{R}^{d \times N}))$ and $\{u_\varepsilon\} \subset W^{1,p}(\Omega; \mathbb{R}^d)$ be such that the pair $\{(\cdot/\varepsilon), \nabla u_\varepsilon\}$ generates the Young measure $\{\mu_x\}_{x \in \Omega}$. By an application of the Generalized Riemann-Lebesgue Lemma (see e.g. Lemma 5.2 in Allaire [2] or Theorem 3 in Lukkassen, Nguetseng & Wall [23]) the sequence $\{(\cdot/\varepsilon)\}$ generates the homogeneous Young measure $dy := \mathcal{L}^N \llcorner Q$ (restriction of the Lebesgue measure to Q). Then by the Disintegration Theorem (see Valadier [36]) there exists a map $\nu \in L_w^\infty(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d \times N}))$ with $\nu_{(x,y)} \in \mathcal{P}(\mathbb{R}^{d \times N})$ for a.e. $(x, y) \in \Omega \times Q$ and such that $\mu_x = \nu_{(x,y)} \otimes dy$ for a.e. $x \in \Omega$, i.e.

$$\int_{\mathbb{R}^N \times \mathbb{R}^{d \times N}} \phi(y, \xi) d\mu_x(y, \xi) = \int_Q \int_{\mathbb{R}^{d \times N}} \phi(y, \xi) d\nu_{(x,y)}(\xi) dy \quad \text{for every } \phi \in \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^{d \times N}).$$

The family $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ is referred in [33] as the *two-scale (gradient) Young measure* associated to the sequence $\{\nabla u_\varepsilon\}$ at scale ε . More precisely, we give the following definition.

Definition 2.3. *Let $\nu \in L_w^\infty(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d \times N}))$. The family $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ is said to be a two-scale gradient Young measure if $\nu_{(x,y)} \in \mathcal{P}(\mathbb{R}^{d \times N})$ for a.e. $(x, y) \in \Omega \times Q$ and if for every sequence*

$\{\varepsilon_n\} \rightarrow 0$ there exists a bounded sequence $\{u_n\}$ in $W^{1,p}(\Omega; \mathbb{R}^d)$ such that $\{(\langle \cdot / \varepsilon_n \rangle, \nabla u_n)\}$ generates the Young measure $\{\nu_{(x,y)} \otimes dy\}_{x \in \Omega}$ i.e. for every $z \in L^1(\Omega)$ and $\varphi \in \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^{d \times N})$,

$$\lim_{n \rightarrow +\infty} \int_{\Omega} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x) \right) dx = \int_{\Omega} \int_Q \int_{\mathbb{R}^{d \times N}} z(x) \varphi(y, \xi) d\nu_{(x,y)}(\xi) dy dx.$$

In this case $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ is also called the two-scale Young measure associated to $\{\nabla u_n\}$.

Example 2.4. Let $\{\varepsilon_n\} \rightarrow 0$, and let $u : \Omega \rightarrow \mathbb{R}^d$ and $u_1 : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^d$ be smooth functions such that $u_1(x, \cdot)$ is Q -periodic for all $x \in \Omega$. Define

$$u_n(x) := u(x) + \varepsilon_n u_1 \left(x, \frac{x}{\varepsilon_n} \right).$$

The two-scale gradient Young measure associated to $\{\nabla u_n\}$ is given by $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ where

$$\nu_{(x,y)} := \delta_{\nabla u(x) + \nabla_y u_1(x,y)} \text{ for all } (x,y) \in \Omega \times Q.$$

Indeed, let us show that $\{(\langle \cdot / \varepsilon_n \rangle, \nabla u_n)\}$ generates the Young measure $\{\nu_{(x,y)} \otimes dy\}_{x \in \Omega}$. First we note that $\nabla u_n(x) = \nabla u(x) + \varepsilon_n \nabla_x u_1(x, x/\varepsilon_n) + \nabla_y u_1(x, x/\varepsilon_n)$ for every $x \in \Omega$. As $\varepsilon_n \nabla_x u_1(\cdot, \cdot / \varepsilon_n) \rightarrow 0$ strongly in $L^p(\Omega; \mathbb{R}^{d \times N})$ because $x \mapsto \nabla_x u_1(x, x/\varepsilon)$ is weakly convergent in $L^p(\Omega; \mathbb{R}^{d \times N})$ (see e.g. Example 3 in Lukkassen, Nguetseng & Wall [23]), then in particular

$$(\langle \cdot / \varepsilon_n \rangle, \nabla u_n(\cdot)) - (\langle \cdot / \varepsilon_n \rangle, \nabla u(\cdot) + \nabla_y u_1(\cdot, \cdot / \varepsilon_n)) = (0, \varepsilon_n \nabla_x u_1(\cdot, \cdot / \varepsilon_n)) \rightarrow 0 \text{ in measure.}$$

Thus from Lemma 6.3 in Pedregal [30] the sequences

$$\{(\langle \cdot / \varepsilon_n \rangle, \nabla u_n)\} \text{ and } \{(\langle \cdot / \varepsilon_n \rangle, \nabla u(\cdot) + \nabla_y u_1(\cdot, \cdot / \varepsilon_n))\}$$

generate the same Young measure. By Riemann-Lebesgue's Lemma we have for every $\psi \in L^1(\Omega)$ and every $\varphi \in \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^{d \times N})$

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_{\Omega} \psi(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u(x) + \nabla_y u_1 \left(x, \frac{x}{\varepsilon_n} \right) \right) dx \\ &= \int_{\Omega} \int_Q \psi(x) \varphi(y, \nabla u(x) + \nabla_y u_1(x, y)) dy dx \end{aligned}$$

which proves the claim.

Example 2.5. Let $\{\varepsilon_n\} \rightarrow 0$, and let $u : \Omega \rightarrow \mathbb{R}^d$ and $u_2 : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^d$ be smooth functions such that $u_2(x, \cdot, \cdot)$ is separately Q -periodic with respect to its second and third variable, for all $x \in \Omega$. Define

$$v_n(x) := u(x) + \varepsilon_n^2 u_2 \left(x, \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2} \right).$$

Arguing as previously, both sequences

$$\{(\langle \cdot / \varepsilon_n \rangle, \nabla v_n)\} \text{ and } \{(\langle \cdot / \varepsilon_n \rangle, \nabla u(\cdot) + \nabla_z u_2(\cdot, \cdot / \varepsilon_n, \cdot / \varepsilon_n^2))\}$$

generate the same Young measure. Using once more the Riemann-Lebesgue Lemma we have that for every $\psi \in L^1(\Omega)$ and every $\varphi \in \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^{d \times N})$

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_{\Omega} \psi(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u(x) + \nabla_z u_2 \left(x, \frac{x}{\varepsilon_n}, \frac{x}{\varepsilon_n^2} \right) \right) dx \\ &= \int_{\Omega} \int_Q \int_Q \psi(x) \varphi(y, \nabla u(x) + \nabla_z u_2(x, y, z)) dz dy dx. \end{aligned}$$

Hence, the two-scale Young measure associated to $\{\nabla v_n\}$ is $\nu_{(x,y)} := \int_Q \delta_{\nabla u(x) + \nabla_z u_2(x,y,z)} dz$, for a.e. $(x,y) \in \Omega \times Q$, i.e.

$$\langle \nu_{(x,y)}, \phi \rangle = \int_Q \phi(\nabla u(x) + \nabla_z u_2(x, y, z)) dz \quad \text{for all } \phi \in \mathcal{C}_0(\mathbb{R}^{d \times N}).$$

Note that in this example we do not get a Dirac mass because there are oscillations occurring at different scales than ε_n , namely at scale ε_n^2 , that the two-scale Young measure misses (see Valadier in [38] for more details).

Remark 2.6. Let $\{\varepsilon_n\}$, $\{u_n\}$ and ν be as in Definition 2.3. Since ∇u_n do not change if we add or remove a constant, there is no loss of generality to assume that all the functions u_n have zero average. Moreover, let $\{u_{n_k}\}$ be a subsequence of $\{u_n\}$. Then there exists a subsequence $\{u_{n_{k_j}}\}$ and $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ (with zero average) such that $u_{n_{k_j}} \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$ and

$$\nabla u(x) = \int_Q \int_{\mathbb{R}^{d \times N}} \xi \, d\nu_{(x,y)}(\xi) \, dy \quad \text{a.e. in } \Omega$$

(see e.g. the proof of Lemma 3.1 below). It follows that u is uniquely defined because if v is the weak $W^{1,p}(\Omega; \mathbb{R}^d)$ -limit of another subsequence of u_{n_k} then

$$\nabla u(x) = \int_Q \int_{\mathbb{R}^{d \times N}} \xi \, d\nu_{(x,y)}(\xi) \, dy = \nabla v(x) \quad \text{a.e. in } \Omega,$$

which implies that $u = v$ since they both have zero average. As a consequence $u_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$ and we can show in a similar way that u is also independent of the sequence $\{\varepsilon_n\}$. The function u is called *the underlying deformation* of $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$.

In the following lemma, we show that there is no loss of generality to assume that sequences of generators in Definition 2.3 match the boundary condition of the underlying deformation.

Lemma 2.7. *Let $\{\varepsilon_n\} \rightarrow 0$ and $\{u_n\} \subset W^{1,p}(\Omega; \mathbb{R}^d)$ be such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$ for some $u \in W^{1,p}(\Omega; \mathbb{R}^d)$. Suppose that $\{(\cdot/\varepsilon_n, \nabla u_n)\}$ generates the Young measure $\{\nu_{(x,y)} \otimes dy\}_{x \in \Omega}$. Then there exists a sequence $\{v_n\} \subset W^{1,p}(\Omega; \mathbb{R}^d)$ such that $v_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$, $v_n = u$ on a neighborhood of $\partial\Omega$ and $\{(\cdot/\varepsilon_n, \nabla v_n)\}$ also generates $\{\nu_{(x,y)} \otimes dy\}_{x \in \Omega}$.*

Proof. For any $k \in \mathbb{N}$ let $\Omega_k := \{x \in \Omega : \text{dist}(x, \partial\Omega) > 1/k\}$ and let $\Phi_k \in C_c^\infty(\Omega; [0, 1])$ be a cut-off function such that

$$\Phi_k := \begin{cases} 1 & \text{if } x \in \Omega_k, \\ 0 & \text{if } x \in \Omega \setminus \overline{\Omega}_{k+1} \end{cases}$$

and $|\nabla \Phi_k| \leq Ck$, for some constant $C > 0$. Let $w_{n,k} \in W^{1,p}(\Omega; \mathbb{R}^d)$ be given by

$$w_{n,k} := (1 - \Phi_k)u + \Phi_k u_n,$$

from where

$$\nabla w_{n,k} = (1 - \Phi_k)\nabla u + \Phi_k \nabla u_n + (u_n - u) \otimes \nabla \Phi_k.$$

Since $u_n \rightarrow u$ strongly in $L^p(\Omega; \mathbb{R}^d)$ then

$$\lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \|w_{n,k} - u\|_{L^p(\Omega; \mathbb{R}^d)} = 0 \quad (2.1)$$

and, as a consequence of

$$\lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \|(u_n - u) \otimes \nabla \Phi_k\|_{L^p(\Omega; \mathbb{R}^{d \times N})} = 0,$$

it follows that

$$\sup_{n,k \in \mathbb{N}} \|\nabla w_{n,k}\|_{L^p(\Omega; \mathbb{R}^{d \times N})} < +\infty. \quad (2.2)$$

Let z and φ be in a countable dense subset of $L^1(\Omega)$ and $\mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^{d \times N})$, respectively. Then

$$\begin{aligned}
& \lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{\Omega} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla w_{n,k}(x) \right) dx \\
&= \lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{\Omega_k} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x) \right) dx \\
&\quad + \lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{\Omega \setminus \Omega_k} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla w_{n,k}(x) \right) dx \\
&= \int_{\Omega} z(x) \int_Q \int_{\mathbb{R}^{d \times N}} \varphi(y, \xi) d\nu_{(x,y)} dy dx. \tag{2.3}
\end{aligned}$$

By a diagonalization argument (see *e.g.* Lemma 7.2 in Braides, Fonseca & Francfort [15]) and taking into account (2.1)-(2.3), we can find a sequence $\{k(n)\} \nearrow +\infty$ such that, upon setting $v_n := w_{n,k(n)}$, we have $v_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$, $v_n = u$ on a neighborhood of $\partial\Omega$ and for every z and φ in a countable dense subset of $L^1(\Omega)$ and $\mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^{d \times N})$, respectively,

$$\lim_{n \rightarrow +\infty} \int_{\Omega} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla v_n(x) \right) dx = \int_{\Omega} z(x) \int_Q \int_{\mathbb{R}^{d \times N}} \varphi(y, \xi) d\nu_{(x,y)}(\xi) dy dx.$$

□

A two-scale gradient Young measure $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ is said to be *homogeneous* if the map $(x, y) \mapsto \nu_{(x,y)}$ is independent of x . In this case, ν can be identified with an element of $L_w^\infty(Q; \mathcal{M}(\mathbb{R}^{d \times N}))$ and we write $\{\nu_y\}_{y \in Q} \equiv \{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$.

We next define the average of a map $\nu \in L_w^\infty(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d \times N}))$ for which $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ is a two-scale gradient Young measure. This notion, useful for the analysis developed on Section 3.2.1, will provide an important example of homogeneous two-scale gradient Young measure.

Definition 2.8. *Let $\nu \in L_w^\infty(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d \times N}))$ be such that $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ is a two-scale gradient Young measure. The average of $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ (with respect to the variable x) is the family $\{\bar{\nu}_y\}_{y \in Q}$ defined by*

$$\langle \bar{\nu}_y, \varphi \rangle := \int_{\Omega} \int_{\mathbb{R}^{d \times N}} \varphi(\xi) d\nu_{(x,y)} dx$$

for every $\varphi \in \mathcal{C}_0(\mathbb{R}^{d \times N})$.

If $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ is a two-scale gradient Young measure, then it can be seen that $\bar{\mu} := \bar{\nu}_y \otimes dy$ is the average of $\{\mu_x\}_{x \in \Omega}$ with $\mu_x := \nu_{(x,y)} \otimes dy$ and $\mu \in L_w^\infty(\Omega; \mathcal{M}(\mathbb{R}^N \times \mathbb{R}^{d \times N}))$. Thus, $\bar{\mu}$ is a homogeneous Young measure by Definition 2.3 and Theorem 7.1 in Pedregal [30].

In the following Lemma we prove that $\{\bar{\nu}_y\}_{y \in Q}$ is actually a homogeneous two-scale gradient Young measure. We will use the same kind of blow up argument as in the proof of Theorem 7.1 in Pedregal [30], splitting Q into suitable subsets. However, contrary to [30] we will not use Vitali's Covering Theorem because the radii of this sets (which may vary from one to another) may interact with the length scale of our problem, ε , in a way we are unable to control. We will construct a covering consisting of subsets of fixed radius. It is enough for our purposes to consider the case where the underlying deformation is affine and $\Omega = Q$.

Lemma 2.9. *Let $\nu \in L_w^\infty(Q \times Q; \mathcal{M}(\mathbb{R}^{d \times N}))$ be such that $\{\nu_{(x,y)}\}_{(x,y) \in Q \times Q}$ is a two-scale gradient Young measure with underlying deformation $F \cdot$, for $F \in \mathbb{R}^{d \times N}$. Then $\{\bar{\nu}_y\}_{y \in Q}$ is a homogeneous two-scale gradient Young measure with the same underlying deformation.*

Proof. Note that by Definition 2.8 and Fubini's Theorem, it follows that $y \mapsto \bar{\nu}_y$ is weakly*-measurable and thus $\bar{\nu} \in L_w^\infty(Q; \mathcal{M}(\mathbb{R}^{d \times N}))$. We have to show that for every sequence $\{\varepsilon_n\} \rightarrow 0$, there exists $\{v_n\} \subset W^{1,p}(Q; \mathbb{R}^d)$ such that $\{(\langle \cdot / \varepsilon_n \rangle, \nabla v_n)\}$ generates $\bar{\mu} := \bar{\nu}_y \otimes dy$ and $v_n \rightharpoonup F \cdot$ in $W^{1,p}(Q; \mathbb{R}^d)$.

Let $\{u_n\} \subset W^{1,p}(Q; \mathbb{R}^d)$ be such that $\{(\langle n \cdot \rangle, \nabla u_n)\}$ generates $\{\nu_{(x,y)} \otimes dy\}_{x \in \Omega}$ and $u_n \rightharpoonup F \cdot$ in $W^{1,p}(Q; \mathbb{R}^d)$ (see Remark 2.6). Without loss of generality we may assume that $u_n(x) = Fx$ on a neighborhood of ∂Q (see Lemma 2.7).

Let $\{\varepsilon_n\} \rightarrow 0$ and for each n define $\rho_n := \varepsilon_n [1/\sqrt{\varepsilon_n}]$. Then there exist $m_n \in \mathbb{N}$, $a_i^n \in \rho_n \mathbb{Z}^N \cap Q$ and a measurable set $E_n \subset Q$ with $\mathcal{L}^N(E_n) \rightarrow 0$ such that

$$Q = \bigcup_{i=1}^{m_n} (a_i^n + \rho_n Q) \cup E_n.$$

Define

$$v_n(x) := \begin{cases} \rho_n u_{\rho_n/\varepsilon_n} \left(\frac{x - a_i^n}{\rho_n} \right) + F a_i^n & \text{if } x \in a_i^n + \rho_n Q \text{ and } i \in \{1, \dots, m_n\}, \\ Fx & \text{otherwise.} \end{cases}$$

Note that the previous definition makes sense since $\rho_n/\varepsilon_n \in \mathbb{N}$. We remark that $\{v_n\} \subset W^{1,p}(Q; \mathbb{R}^d)$ and $v_n \rightharpoonup F \cdot$ in $W^{1,p}(Q; \mathbb{R}^d)$ since $u_n \rightharpoonup F \cdot$ in this space. Let $z \in \mathcal{C}_c(Q)$ and $\varphi \in \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^{d \times N})$. Then we have

$$\begin{aligned} & \int_Q z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla v_n(x) \right) dx \\ &= \sum_{i=1}^{m_n} \int_{a_i^n + \rho_n Q} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_{\rho_n/\varepsilon_n} \left(\frac{x - a_i^n}{\rho_n} \right) \right) dx \\ & \quad + \int_{E_n} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, F \right) dx \\ &= \sum_{i=1}^{m_n} z(a_i^n) \int_{a_i^n + \rho_n Q} \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_{\rho_n/\varepsilon_n} \left(\frac{x - a_i^n}{\rho_n} \right) \right) dx \\ & \quad + \sum_{i=1}^{m_n} \int_{a_i^n + \rho_n Q} (z(x) - z(a_i^n)) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_{\rho_n/\varepsilon_n} \left(\frac{x - a_i^n}{\rho_n} \right) \right) dx \\ & \quad + \int_{E_n} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, F \right) dx, \end{aligned}$$

and, consequently,

$$\begin{aligned} & \int_Q z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla v_n(x) \right) dx \\ &= \sum_{i=1}^{m_n} \rho_n^N z(a_i^n) \int_Q \varphi \left(\left\langle \frac{a_i^n + \rho_n x}{\varepsilon_n} \right\rangle, \nabla u_{\rho_n/\varepsilon_n}(x) \right) dx \\ & \quad + o(1), \text{ as } n \rightarrow +\infty \end{aligned} \tag{2.4}$$

by changing variables, using the uniform continuity of z and the fact that $\mathcal{L}^N(E_n) \rightarrow 0$. Hence, as $a_i^n/\varepsilon_n \in \mathbb{Z}^N$, it follows that

$$\begin{aligned} & \int_Q z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla v_n(x) \right) dx \\ &= \sum_{i=1}^{m_n} \rho_n^N z(a_i^n) \int_Q \varphi \left(\left\langle \frac{x}{\varepsilon_n/\rho_n} \right\rangle, \nabla u_{\rho_n/\varepsilon_n}(x) \right) dx \\ & \quad + o(1), \text{ as } n \rightarrow +\infty, \end{aligned} \tag{2.5}$$

and passing to the limit in (2.5) and using Definition 2.8, we conclude that

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_Q z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla v_n(x) \right) dx \\ &= \int_Q z(x) dx \int_Q \int_{\mathbb{R}^{d \times N}} \varphi(y, \xi) d\nu_{(x,y)}(\xi) dy dx \\ &= \langle \bar{\nu}, \varphi \rangle \int_Q z(x) dx. \end{aligned}$$

Since by density the previous equality holds for every $z \in L^1(\Omega)$, then $\{(\cdot/\varepsilon_n), \nabla v_n\}$ generates the homogeneous Young measure $\bar{\nu}_y \otimes dy$ and, consequently, $\{\bar{\nu}_y\}_{y \in Q}$ is a homogeneous two-scale gradient Young measure. \square

3. PROOF OF THEOREM 1.2

The aim of this section is to prove Theorem 1.2. We start by introducing the space \mathcal{E}_p of all continuous functions $f : \bar{Q} \times \mathbb{R}^{d \times N} \rightarrow \mathbb{R}$ such that the limit

$$\lim_{|\xi| \rightarrow +\infty} \frac{f(y, \xi)}{1 + |\xi|^p}$$

exists uniformly with respect to $y \in \bar{Q}$. As an example, the function $(y, \xi) \mapsto a(y)|\xi|^p$, where $a \in \mathcal{C}(\bar{Q})$, is in \mathcal{E}_p .

It can be checked that \mathcal{E}_p is a Banach space under the norm

$$\|f\|_{\mathcal{E}_p} := \sup_{y \in \bar{Q}, \xi \in \mathbb{R}^{d \times N}} \frac{|f(y, \xi)|}{1 + |\xi|^p}.$$

In addition, \mathcal{E}_p is isomorphic to the space $\mathcal{C}(\bar{Q} \times (\mathbb{R}^{d \times N} \cup \{\infty\}))$ under the map

$$\begin{aligned} \mathcal{E}_p & \longrightarrow \mathcal{C}(\bar{Q} \times (\mathbb{R}^{d \times N} \cup \{\infty\})) \\ f & \longmapsto (y, \xi) \mapsto \begin{cases} \frac{f(y, \xi)}{1 + |\xi|^p} & \text{if } (y, \xi) \in \bar{Q} \times \mathbb{R}^{d \times N} \\ \lim_{|\xi| \rightarrow +\infty} \frac{f(y, \xi)}{1 + |\xi|^p} & \text{if } |\xi| = +\infty, \end{cases} \end{aligned}$$

where $\mathbb{R}^{d \times N} \cup \{\infty\}$ denotes the one-point compactification of $\mathbb{R}^{d \times N}$, and, consequently, it is separable. Furthermore, there exists a constant $c > 0$ such that for all $f \in \mathcal{E}_p$

$$|f(y, \xi)| \leq c(1 + |\xi|^p), \quad \text{for all } (y, \xi) \in \bar{Q} \times \mathbb{R}^{d \times N}. \tag{3.1}$$

We denote by $(\mathcal{E}_p)'$ the dual space of \mathcal{E}_p and the brackets $\langle \cdot, \cdot \rangle_{(\mathcal{E}_p)', \mathcal{E}_p}$ stand for the duality product between $(\mathcal{E}_p)'$ and \mathcal{E}_p .

3.1. Necessity. We start by showing that conditions i)-iii) in (1.7)-(1.10) are necessary. Precisely we prove the following result.

Lemma 3.1. *Let $\nu \in L_w^\infty(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d \times N}))$ be such that $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ is a two-scale gradient Young measure. Then*

i) *there exist $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ and $u_1 \in L^p(\Omega; W_{per}^{1,p}(Q; \mathbb{R}^d))$ such that*

$$\int_{\mathbb{R}^{d \times N}} \xi \, d\nu_{(x,y)}(\xi) = \nabla u(x) + \nabla_y u_1(x, y) \quad \text{for a.e. } (x, y) \in \Omega \times Q;$$

ii) *for every $f \in \mathcal{E}_p$ we have that*

$$\int_Q \int_{\mathbb{R}^{d \times N}} f(y, \xi) \, d\nu_{(x,y)}(\xi) \, dy \geq f_{\text{hom}}(\nabla u(x)) \quad \text{for a.e. } x \in \Omega,$$

where f_{hom} is given by (1.9);

iii) $(x, y) \mapsto \int_{\mathbb{R}^{d \times N}} |\xi|^p \, d\nu_{(x,y)}(\xi) \in L^1(\Omega \times Q)$.

Proof. Let $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ be a two-scale gradient Young measure.

We start by proving that i) holds. By Definition 2.3 and Remark 2.6 there exists $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ such that for every sequence $\{\varepsilon_n\} \rightarrow 0$ one can find $\{u_n\} \subset W^{1,p}(\Omega; \mathbb{R}^d)$ such that $\{(\cdot/\varepsilon_n, \nabla u_n)\}$ generates the Young measure $\{\nu_{(x,y)} \otimes dy\}_{x \in \Omega}$ and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$. Up to a subsequence (still denoted by u_n), we can also assume that $\{|\nabla u_n|^p\}$ is equi-integrable (see the Decomposition Lemma in Fonseca, Müller & Pedregal [20]) and that there exists a function $u_1 \in L^p(\Omega; W_{per}^{1,p}(Q; \mathbb{R}^d))$ such that the sequence $\{\nabla u_n\}$ two-scale converges to $\nabla u + \nabla_y u_1$ (see *e.g.* Theorem 13 in Lukkassen, Nguetseng & Wall [23]; see also Allaire [2] or Nguetseng [29]). Consequently, for all $\phi \in \mathcal{C}_c^\infty(\Omega \times Q; \mathbb{R}^{d \times N})$ we have that

$$\lim_{n \rightarrow +\infty} \int_\Omega \nabla u_n(x) \cdot \phi \left(x, \left\langle \frac{x}{\varepsilon_n} \right\rangle \right) dx = \int_\Omega \int_Q (\nabla u(x) + \nabla_y u_1(x, y)) \cdot \phi(x, y) \, dy \, dx. \quad (3.2)$$

Set $f(x, y, \xi) = \xi \cdot \phi(x, y)$ for $(x, y, \xi) \in \Omega \times Q \times \mathbb{R}^{d \times N}$. As f is a Carathéodory integrand (measurable in x and continuous in (y, ξ)) and the sequence $\{f(\cdot, \cdot/\varepsilon_n, \nabla u_n(\cdot))\}$ is equi-integrable, by Theorem 2.2 v) we get that

$$\lim_{n \rightarrow +\infty} \int_\Omega \int_Q \int_{\mathbb{R}^{d \times N}} \nabla u_n(x) \cdot \phi \left(x, \left\langle \frac{x}{\varepsilon_n} \right\rangle \right) dx = \int_\Omega \int_Q \int_{\mathbb{R}^{d \times N}} \xi \cdot \phi(x, y) \, d\nu_{(x,y)}(\xi) \, dy \, dx. \quad (3.3)$$

Consequently, from (3.2) and (3.3) we get for a.e. $(x, y) \in \Omega \times Q$

$$\int_{\mathbb{R}^{d \times N}} \xi \, d\nu_{(x,y)}(\xi) = \nabla u(x) + \nabla_y u_1(x, y),$$

which proves i).

Let us see now that iii) is satisfied. As $\{\nabla u_n\}$ is p -equi-integrable then by Theorem 2.2 v) we get that

$$\int_\Omega \int_Q \int_{\mathbb{R}^{d \times N}} |\xi|^p \, d\nu_{(x,y)}(\xi) \, dy \, dx = \lim_{n \rightarrow +\infty} \int_\Omega |\nabla u_n|^p \, dx < +\infty,$$

which completes the proof of iii).

Finally, let us see that condition ii) holds by application of the classical Γ -convergence result for the homogenization of integral functionals (see Braides [12] or Müller [27]). Let $f \in \mathcal{E}_p$. In particular f satisfies the p -growth condition (3.1) but it is not necessarily p -coercive. For every $\alpha > 0$ and $M > 0$, define $f_{M,\alpha}(y, \xi) := f_M(y, \xi) + \alpha|\xi|^p$ where $f_M(y, \xi) = \max\{-M, f(y, \xi)\}$. Then

$$\alpha|\xi|^p - M \leq f_{M,\alpha}(y, \xi) \leq (c + \alpha)(1 + |\xi|^p), \quad \text{for all } (y, \xi) \in \overline{Q} \times \mathbb{R}^{d \times N}.$$

Hence, by e.g. Theorem 14.5 in Braides [12] (Γ -lim inf inequality) and since $f_{M,\alpha} \geq f$, we get that for every $A \in \mathcal{A}(\Omega)$

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \int_A f_{M,\alpha} \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x) \right) dx &\geq \int_A (f_{M,\alpha})_{\text{hom}}(\nabla u(x)) dx \\ &\geq \int_A f_{\text{hom}}(\nabla u(x)) dx \end{aligned} \quad (3.4)$$

where f_{hom} is defined in (1.9). On the other hand,

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \int_A f_{M,\alpha} \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x) \right) dx &\leq \liminf_{n \rightarrow +\infty} \int_A f_M \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x) \right) dx \\ &\quad + \alpha \sup_{n \in \mathbb{N}} \int_A |\nabla u_n|^p dx. \end{aligned} \quad (3.5)$$

Gathering (3.4) and (3.5), and passing to the limit as $\alpha \rightarrow 0$, we obtain that

$$\liminf_{n \rightarrow +\infty} \int_A f_M \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x) \right) dx \geq \int_A f_{\text{hom}}(\nabla u(x)) dx. \quad (3.6)$$

Define the set

$$A_n^M := \left\{ x \in A : f \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x) \right) \leq -M \right\}$$

and notice that by Chebyshev's Inequality

$$\mathcal{L}^N(A_n^M) \leq c/M,$$

for some constant $c > 0$ independent of n and M . Then

$$\begin{aligned} \int_A f_M \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x) \right) dx &= -M \mathcal{L}^N(A_n^M) \\ &\quad + \int_{A \setminus A_n^M} f \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x) \right) dx \\ &\leq \int_{A \setminus A_n^M} f \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x) \right) dx. \end{aligned} \quad (3.7)$$

As $\{|\nabla u_n|^p\}$ is equi-integrable, by the p -growth condition (3.1), the sequence $\{f(\langle \cdot / \varepsilon_n \rangle, \nabla u_n)\}$ is also equi-integrable. Thus

$$\int_{A_n^M} f \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x) \right) dx \xrightarrow{M \rightarrow +\infty} 0 \quad (3.8)$$

uniformly with respect to $n \in \mathbb{N}$. By (3.6), (3.7) and (3.8) we get that

$$\liminf_{n \rightarrow +\infty} \int_A f \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x) \right) dx \geq \int_A f_{\text{hom}}(\nabla u(x)) dx. \quad (3.9)$$

Finally, since $\{f(\langle \cdot / \varepsilon_n \rangle, \nabla u_n)\}$ is equi-integrable, by Theorem 2.2 v) we have that

$$\lim_{n \rightarrow +\infty} \int_A f \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x) \right) dx = \int_A \int_Q \int_{\mathbb{R}^{d \times N}} f(y, \xi) d\nu_{(x,y)}(\xi) dy dx \quad (3.10)$$

and we conclude the proof of ii) thanks to (3.9) and (3.10) together with a localization argument. \square

3.2. Sufficiency. We show here that these conditions are also sufficient to characterize two-scale gradient Young measures. Following the lines of Kinderlehrer & Pedregal [22], we first study the homogeneous case. The non-homogeneous one will be obtained through a suitable approximation of two-scale gradient Young measures by piecewise constant ones.

3.2.1. *Homogeneous case.* Our aim here is to prove the following result.

Lemma 3.2. *Let $F \in \mathbb{R}^{d \times N}$ and $\nu \in L_w^\infty(Q; \mathcal{M}(\mathbb{R}^{d \times N}))$ be such that $\nu_y \in \mathcal{P}(\mathbb{R}^{d \times N})$ for a.e. $y \in Q$. Assume that*

$$F = \int_Q \int_{\mathbb{R}^{d \times N}} \xi \, d\nu_y(\xi) \, dy, \quad (3.11)$$

$$f_{\text{hom}}(F) \leq \int_Q \int_{\mathbb{R}^{d \times N}} f(y, \xi) \, d\nu_y(\xi) \, dy \quad (3.12)$$

for every $f \in \mathcal{E}_p$, and that

$$\int_Q \int_{\mathbb{R}^{d \times N}} |\xi|^p \, d\nu_y(\xi) \, dy < +\infty. \quad (3.13)$$

Then $\{\nu_y\}_{y \in Q}$ is a homogeneous two-scale gradient Young measure.

As Kinderlehrer & Pedregal [22], the argument in this case will rest on the Hahn-Banach Separation Theorem that implies any element $\nu \in L_w^\infty(Q; \mathcal{M}(\mathbb{R}^{d \times N}))$, for which the hypothesis of Theorem 1.2 are satisfied, to be in a suitable convex and weak* closed subset of homogeneous two-scale gradient Young measures. To prove Lemma 3.2 we start by giving some notations and auxiliary lemmas.

For every $F \in \mathbb{R}^{d \times N}$ let

$$M_F := \left\{ \nu \in L_w^\infty(Q; \mathcal{M}(\mathbb{R}^{d \times N})) : \{\nu_y\}_{y \in Q} \text{ is a homogeneous two-scale gradient Young measure and } \int_Q \int_{\mathbb{R}^{d \times N}} \xi \, d\nu_y(\xi) \, dy = F \right\}. \quad (3.14)$$

Remark 3.3. The set M_F is independent of Ω , i.e. if $\nu \in M_F$ and $\Omega' \subset \mathbb{R}^N$ is another domain, then for all $\{\varepsilon_n\} \rightarrow 0$ there exist a sequence $\{v_n\} \in W^{1,p}(\Omega'; \mathbb{R}^d)$ such that $\{(\cdot/\varepsilon_n), \nabla v_n\}$ generates $\nu_y \otimes dy$. Indeed, let $r > 0$ such that $\Omega' \subset r\Omega$. Fix an arbitrary sequence $\{\varepsilon_n\} \rightarrow 0$ and define $\delta_n = \varepsilon_n/r$. Then there exists a sequence $\{u_n\} \subset W^{1,p}(\Omega; \mathbb{R}^d)$ such that $\{(\cdot/\delta_n), \nabla u_n\}$ generates the homogeneous Young measure $\nu_y \otimes dy$. Define now $v_n(x) = r u_n(x/r)$ so that v_n belongs to $W^{1,p}(r\Omega; \mathbb{R}^d)$ and thus *a fortiori* to $W^{1,p}(\Omega'; \mathbb{R}^d)$. A simple change of variable shows that the sequence $\{(\cdot/\varepsilon_n), \nabla v_n\}$ generates the homogeneous Young measure $\nu_y \otimes dy$ as well.

The next technical result allows us to construct two-scale gradient Young measures from measures of this class that are defined on disjoint subsets of Ω . It will be of use in Lemma 3.5 below to prove the convexity of the set M_F .

Lemma 3.4. *Let D be an open subset of Ω with Lipschitz boundary, and let $\mu, \nu \in L_w^\infty(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d \times N}))$ be such that $\{\mu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ and $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ are two-scale gradient Young measures with same underlying deformation $u \in W^{1,p}(\Omega; \mathbb{R}^d)$. Let*

$$\sigma_{(x,y)} := \begin{cases} \mu_{(x,y)} & \text{if } (x,y) \in D \times Q \\ \nu_{(x,y)} & \text{if } (x,y) \in (\Omega \setminus D) \times Q. \end{cases}$$

Then $\sigma \in L_w^\infty(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d \times N}))$ and $\{\sigma_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ is a two-scale gradient Young measure with underlying deformation $u \in W^{1,p}(\Omega; \mathbb{R}^d)$.

Proof. We have to show that for every sequence $\{\varepsilon_n\} \rightarrow 0$ there exists $\{w_n\} \subset W^{1,p}(\Omega; \mathbb{R}^d)$ such that $w_n \rightarrow u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$ and $\{(\cdot/\varepsilon_n), \nabla w_n\}$ generates the Young measure $\{\sigma_{(x,y)} \otimes dy\}_{x \in \Omega}$.

By Lemma 2.7, there exist sequences $\{u_n\} \subset W^{1,p}(D; \mathbb{R}^d)$ and $\{v_n\} \subset W^{1,p}(\Omega \setminus \overline{D}; \mathbb{R}^d)$ such that $u_n \rightarrow u$ in $W^{1,p}(D; \mathbb{R}^d)$, $v_n \rightarrow u$ in $W^{1,p}(\Omega \setminus \overline{D}; \mathbb{R}^d)$, $u_n = v_n = u$ on ∂D and such that $\{(\cdot/\varepsilon_n), \nabla u_n\}$ and $\{(\cdot/\varepsilon_n), \nabla v_n\}$ generate, respectively, the Young measures $\{\mu_{(x,y)} \otimes dy\}_{x \in D}$ and $\{\nu_{(x,y)} \otimes dy\}_{x \in \Omega \setminus \overline{D}}$.

Define

$$w_n := \begin{cases} u_n & \text{if } x \in D, \\ v_n & \text{if } x \in \Omega \setminus \overline{D} \end{cases}$$

Then $\{w_n\} \subset W^{1,p}(\Omega; \mathbb{R}^d)$, $w_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$ and given $z \in L^1(\Omega)$ and $\varphi \in \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^{d \times N})$ we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{\Omega} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla w_n(x) \right) dx &= \lim_{n \rightarrow +\infty} \int_D z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x) \right) dx \\ &\quad + \lim_{n \rightarrow +\infty} \int_{\Omega \setminus D} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla v_n(x) \right) dx \\ &= \int_{\Omega} z(x) \int_Q \int_{\mathbb{R}^{d \times N}} \varphi(x, \xi) d\sigma_{(x,y)}(\xi) dy dx, \end{aligned}$$

which concludes the proof. \square

As a consequence of Remark 3.3 there is no loss of generality to assume hereafter that $\Omega = Q$. We can now prove the following result.

Lemma 3.5. *M_F is a convex and weak*-closed subset of $(\mathcal{E}_p)'$.*

Proof. We identify every element $\nu \in M_F$ with a homogeneous Young measure $\nu_y \otimes dy$.

We start by showing that M_F is a subset of $(\mathcal{E}_p)'$. For this purpose let $\nu \in M_F$. Arguing exactly as in the proof of Lemma 3.1 one can show that

$$K := \int_Q \int_{\mathbb{R}^{d \times N}} |\xi|^p d\nu_y(\xi) dy < +\infty.$$

Hence, using the fact that ν_y are probability measures for a.e. $y \in Q$, for every $f \in \mathcal{E}_p$ we have that

$$\begin{aligned} \int_Q \int_{\mathbb{R}^{d \times N}} f(y, \xi) d\nu_y(\xi) dy &\leq \|f\|_{\mathcal{E}_p} \int_Q \int_{\mathbb{R}^{d \times N}} (1 + |\xi|^p) d\nu_y(\xi) dy \\ &= (1 + K) \|f\|_{\mathcal{E}_p}. \end{aligned}$$

As a consequence, $M_F \subset (\mathcal{E}_p)'$.

Let us now prove that M_F is closed for the weak*-topology of $(\mathcal{E}_p)'$. Denoting by $\overline{M_F}$ the closure of M_F for the weak*-topology of $(\mathcal{E}_p)'$ it is enough show that $\overline{M_F} \subset M_F$. Since \mathcal{E}_p is separable, the weak*-topology of $(\mathcal{E}_p)'$ is locally metrizable and thus, if $\nu \in \overline{M_F}$, there exists a sequence $\{\nu^k\} \subset M_F$ such that $\nu^k \xrightarrow{*} \nu$ in $(\mathcal{E}_p)'$. Hence, since the map $(y, \xi) \mapsto \xi_{ij}$ is in \mathcal{E}_p (where $1 \leq i \leq d$ and $1 \leq j \leq N$), we get, from the definition of weak*-convergence in $(\mathcal{E}_p)'$, that

$$\int_Q \int_{\mathbb{R}^{d \times N}} \xi d\nu_y(\xi) dy = \lim_{k \rightarrow +\infty} \int_Q \int_{\mathbb{R}^{d \times N}} \xi d\nu_y^k(\xi) dy = F. \quad (3.15)$$

It remains to show that $\{\nu_y\}_{y \in Q}$ is a homogeneous two-scale Young measure. By definition, given $\{\varepsilon_n\} \rightarrow 0$, for each $k \in \mathbb{N}$ there exist sequences $\{u_n^k\}_{n \in \mathbb{N}} \subset W^{1,p}(Q; \mathbb{R}^d)$ such that $\{(\cdot/\varepsilon_n), \nabla u_n^k\}_{n \in \mathbb{N}}$ generate the homogeneous Young measures $\nu_y^k \otimes dy$. For every (z, φ) in countable dense subsets of $L^1(Q) \times \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^{d \times N})$ we have that

$$\begin{aligned} \lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_Q z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n^k(x) \right) dx \\ &= \lim_{k \rightarrow +\infty} \int_Q \int_Q \int_{\mathbb{R}^{d \times N}} z(x) \varphi(y, \xi) d\nu_y^k(\xi) dy dx \\ &= \int_Q z(x) dx \int_Q \int_{\mathbb{R}^{d \times N}} \varphi(y, \xi) d\nu_y(\xi) dy, \end{aligned}$$

where we have used the fact that $\mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^{d \times N}) \subset \mathcal{E}_p$ in the second equality. By a diagonalization argument we can find a sequence $\{k(n)\} \nearrow +\infty$ such that, setting $v_n := u_n^{k(n)}$, we have that

$$\lim_{n \rightarrow +\infty} \int_Q z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla v_n(x) \right) dx = \int_\Omega z(x) dx \int_Q \int_{\mathbb{R}^{d \times N}} \varphi(y, \xi) d\nu_y(\xi) dy.$$

Thus, $\{\nu_y\}_{y \in Q}$ is a homogeneous two-scale Young measure, which together with (3.15) implies that $\nu \in M_F$.

Next we show that M_F is convex. Given $\mu, \nu \in M_F$ and $t \in (0, 1)$ we have to show that $t\mu + (1-t)\nu \in M_F$. Let $D = (0, t) \times (0, 1)^{N-1} \subset Q$ and define

$$\sigma_{(x,y)} := \begin{cases} \mu_y & \text{if } (x, y) \in D \times Q \\ \nu_y & \text{if } (x, y) \in (Q \setminus D) \times Q. \end{cases}$$

By Lemma 3.4 we have that $\{\sigma_{(x,y)}\}_{(x,y) \in Q \times Q}$ is a two-scale gradient Young measure and from Lemma 2.9 its average $\{\bar{\sigma}_y\}_{y \in Q}$ is a homogeneous two-scale gradient Young measure. We claim that $\bar{\sigma} = t\mu + (1-t)\nu$. Indeed, for every $\varphi \in L^1(Q; \mathcal{C}_0(\mathbb{R}^{d \times N}))$

$$\begin{aligned} \int_Q \int_{\mathbb{R}^{d \times N}} \varphi(y, \xi) d\bar{\sigma}_y(\xi) dy &= \int_Q \int_Q \int_{\mathbb{R}^{d \times N}} \varphi(y, \xi) d\sigma_{(x,y)}(\xi) dy dx \\ &= t \int_Q \int_{\mathbb{R}^{d \times N}} \varphi(y, \xi) d\mu_y(\xi) dy \\ &\quad + (1-t) \int_Q \int_{\mathbb{R}^{d \times N}} \varphi(y, \xi) d\nu_y(\xi) dy. \end{aligned}$$

In particular,

$$\int_Q \int_{\mathbb{R}^{d \times N}} \xi d\bar{\sigma}_y(\xi) dy = t \int_Q \int_{\mathbb{R}^{d \times N}} \xi d\mu_y(\xi) dy + (1-t) \int_Q \int_{\mathbb{R}^{d \times N}} \xi d\nu_y(\xi) dy = F,$$

and thus $\bar{\sigma} = t\mu + (1-t)\nu \in M_F$. \square

We are now in position to show the sufficiency of conditions i)-iii) in (1.7)-(1.10) in the homogeneous case.

Proof of Lemma 3.2. Let $F \in \mathbb{R}^{d \times N}$ and $\nu \in L_w^\infty(Q; \mathcal{M}(\mathbb{R}^{d \times N}))$ be such that $\nu_y \in \mathcal{P}(\mathbb{R}^{d \times N})$ for a.e. $y \in Q$, and (3.11)-(3.13) hold. We will proceed by contradiction using the Hahn-Banach Separation Theorem. Assume that $\{\nu_y\}_{y \in Q}$ is not a homogeneous two-scale Young measure.

By Lemma 3.5, M_F is a convex and weak* closed subset of $(\mathcal{E}_p)'$. Moreover, by (3.13) and the fact that $\{\nu_y\}_{y \in Q}$ is a family of probability measures, we get that $\nu \in (\mathcal{E}_p)'$ as well (see *e.g.* the first part of the proof of Lemma 3.5). As $\nu \notin M_F$, according to the Hahn-Banach Separation Theorem, we can separate ν from M_F *i.e.* there exist a linear weak* continuous map $L : (\mathcal{E}_p)' \rightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that $\langle L, \nu \rangle_{(\mathcal{E}_p)', \mathcal{E}_p} < \alpha$ and $\langle L, \mu \rangle_{(\mathcal{E}_p)', \mathcal{E}_p} \geq \alpha$ for all $\mu \in M_F$. Let $f \in \mathcal{E}_p$ be such that

$$\alpha \leq \langle L, \mu \rangle_{(\mathcal{E}_p)'', (\mathcal{E}_p)'} = \langle \mu, f \rangle_{(\mathcal{E}_p)', \mathcal{E}_p} = \int_Q \int_{\mathbb{R}^{d \times N}} f(y, \xi) d\mu_y(\xi) dy \quad \text{for all } \mu \in M_F, \quad (3.16)$$

and

$$\alpha > \langle L, \nu \rangle_{(\mathcal{E}_p)'', (\mathcal{E}_p)'} = \langle \nu, f \rangle_{(\mathcal{E}_p)', \mathcal{E}_p} = \int_Q \int_{\mathbb{R}^{d \times N}} f(y, \xi) d\nu_y(\xi) dy \geq f_{\text{hom}}(F). \quad (3.17)$$

Let

$$f_{\text{H}}(F) := \inf_{\mu \in M_F} \int_Q \int_{\mathbb{R}^{d \times N}} f(y, \xi) d\mu_y(\xi) dy, \quad F \in \mathbb{R}^{d \times N}.$$

Then, by (3.16), we have that $\alpha \leq f_{\text{H}}(F)$. We are going to show that

$$f_{\text{H}}(F) \leq f_{\text{hom}}(F), \quad (3.18)$$

which is a contradiction with (3.17) and asserts the conclusion of this lemma.

To prove (3.18), let $T \in \mathbb{N}$ and $\phi \in W_0^{1,p}((0, T)^N; \mathbb{R}^d)$. Extend ϕ to \mathbb{R}^N by $(0, T)^N$ -periodicity and consider the sequence

$$\phi_n(x) = Fx + \varepsilon_n \phi \left(\frac{x}{\varepsilon_n} \right),$$

where $\{\varepsilon_n\} \rightarrow 0$ is an arbitrary sequence. Let $\varphi \in \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^{d \times N})$ and $z \in L^1(Q)$. Then, since $T \in \mathbb{N}$, the function $y \mapsto \varphi(\langle y \rangle, F + \nabla \phi(y))$ is $(0, T)^N$ -periodic and according to the Riemann-Lebesgue Lemma, we get that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_Q z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla \phi_n(x) \right) dx &= \lim_{n \rightarrow +\infty} \int_Q z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, F + \nabla \phi \left(\frac{x}{\varepsilon_n} \right) \right) dx \\ &= \int_Q z(x) dx \int_{(0, T)^N} \varphi(\langle y \rangle, F + \nabla \phi(y)) dy. \end{aligned} \quad (3.19)$$

Observe that

$$\begin{aligned} &\int_{(0, T)^N} \varphi(\langle y \rangle, F + \nabla \phi(y)) dy \\ &= \frac{1}{T^N} \sum_{a_i \in \mathbb{Z}^N \cap [0, T)^N} \int_{a_i + Q} \varphi(\langle y \rangle, F + \nabla \phi(y)) dy \\ &= \frac{1}{T^N} \sum_{a_i \in \mathbb{Z}^N \cap [0, T)^N} \int_Q \varphi(\langle a_i + y \rangle, F + \nabla \phi(a_i + y)) dy \\ &= \frac{1}{T^N} \sum_{a_i \in \mathbb{Z}^N \cap [0, T)^N} \int_Q \varphi(y, F + \nabla \phi(a_i + y)) dy. \end{aligned} \quad (3.20)$$

Thus, from (3.19) and (3.20), the pair $\{(\langle \cdot / \varepsilon_n \rangle, \nabla \phi_n)\}$ generates the homogeneous Young Measure

$$\mu := \sum_{a_i \in \mathbb{Z}^N \cap [0, T)^N} \frac{1}{T^N} \delta_{F + \nabla \phi(a_i + y)} \otimes dy.$$

Then

$$\int_Q \int_{\mathbb{R}^{d \times N}} \xi d\nu_y(\xi) dy = F,$$

which implies that $\mu \in M_F$. In addition

$$\int_{(0, T)^N} f(\langle y \rangle, F + \nabla \phi(y)) dy = \int_Q \int_{\mathbb{R}^{d \times N}} f(y, \xi) d\mu_y(\xi) dy,$$

and then

$$\int_{(0, T)^N} f(\langle y \rangle, F + \nabla \phi(y)) dy \geq f_H(F).$$

As a consequence, taking the infimum over all $\phi \in W_0^{1,p}((0, T)^N; \mathbb{R}^d)$ and all $T \in \mathbb{N}$ we get that $f_{\text{hom}}(F) \geq f_H(F)$ which proves (3.18). \square

Let us conclude this section by stating a localization result which allows us to construct homogeneous two-scale gradient Young measures starting from any kind of them.

Proposition 3.6. *Let $\nu \in L_w^\infty(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d \times N}))$ be such that $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ is a two-scale gradient Young measure. Then for a.e. $a \in \Omega$, $\{\nu_{(a,y)}\}_{y \in Q}$ is a homogeneous two-scale gradient Young measure.*

Proof. Since $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ is a two-scale gradient Young measure, from Lemma 3.1 it satisfies properties (1.7), (1.8) and (1.10) above. Since $u_1(x, \cdot)$ is Q -periodic for a.e. $x \in \Omega$, integrating (1.7) with respect to $y \in Q$, it follows that

$$\int_Q \int_{\mathbb{R}^{d \times N}} \xi d\nu_{(x,y)}(\xi) dy = \nabla u(x) \quad \text{for a.e. } x \in \Omega. \quad (3.21)$$

Furthermore, (1.10) implies that

$$\int_Q \int_{\mathbb{R}^{d \times N}} |\xi|^p d\nu_{(x,y)}(\xi) dy < +\infty, \quad \text{for a.e. } x \in \Omega. \quad (3.22)$$

Let $E \subset \Omega$ be a set of Lebesgue measure zero such that (3.21), (1.8) and (3.22) do not hold. Then for every $a \in \Omega \setminus E$

$$\begin{aligned} \int_Q \int_{\mathbb{R}^{d \times N}} \xi d\nu_{(a,y)}(\xi) dy &= \nabla u(a), \\ \int_Q \int_{\mathbb{R}^{d \times N}} |\xi|^p d\nu_{(a,y)}(\xi) dy &< +\infty, \end{aligned}$$

and

$$\int_Q \int_{\mathbb{R}^{d \times N}} f(y, \xi) d\nu_{(a,y)}(\xi) dy \geq f_{\text{hom}}(\nabla u(a))$$

for every $f \in \mathcal{E}_p$.

As a consequence of Lemma 3.2, for every $a \in \Omega \setminus E$, the family $\{\nu_{(a,y)}\}_{y \in Q}$ is a homogeneous two-scale gradient Young measure. \square

3.2.2. The nonhomogeneous case. We treat now the general case whose proof is based on Proposition 3.6 and a suitable decomposition of the domain Ω . We use (a variant of) Vitali's covering Theorem and an approximation of two-scale gradient Young measures by measures of this class that are piecewise constant with respect to x .

Lemma 3.7. *Let Ω be a bounded and open subset of \mathbb{R}^N with Lipschitz boundary. Let $\nu \in L_w^\infty(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d \times N}))$ be such that $\nu_{(x,y)} \in \mathcal{P}(\mathbb{R}^{d \times N})$ for a.e. $(x, y) \in \Omega \times Q$. Suppose that*

- (i) *there exist $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ and $u_1 \in L^p(\Omega; W_{\text{per}}^{1,p}(Q; \mathbb{R}^d))$ satisfying*

$$\int_{\mathbb{R}^{d \times N}} \xi d\nu_{(x,y)}(\xi) = \nabla u(x) + \nabla_y u_1(x, y) \quad \text{for a.e. } (x, y) \in \Omega \times Q; \quad (3.23)$$

- (ii) *for every $f \in \mathcal{E}_p$,*

$$f_{\text{hom}}(\nabla u(x)) \leq \int_Q \int_{\mathbb{R}^{d \times N}} f(y, \xi) d\nu_{(x,y)}(\xi) dy \quad \text{for a.e. } x \in \Omega; \quad (3.24)$$

- (iii) $(x, y) \mapsto \int_{\mathbb{R}^{d \times N}} |\xi|^p d\nu_{(x,y)}(\xi) \in L^1(\Omega \times Q)$.

Then $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ is a two-scale gradient Young measure with underlying deformation u .

Proof. In a first step, we address the case where the underlying deformation is zero, while the general case is treated afterwards.

Step 1. Assume $u = 0$ and let (φ, z) be in a countable dense subset of $\mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^{d \times N}) \times L^1(\Omega)$. Set

$$\overline{\varphi}(x) := \int_Q \int_{\mathbb{R}^{d \times N}} \varphi(y, \xi) d\nu_{(x,y)}(\xi) dy.$$

Let $k \in \mathbb{N}$ and let $E \subset \Omega$ be the set of Lebesgue measure zero given by Proposition 3.6. According to Lemma 7.9 in Pedregal [30], there exist points $a_i^k \in \Omega \setminus E$ and positive numbers $\rho_i^k \leq 1/k$ such that $\{a_i^k + \rho_i^k \bar{\Omega}\}$ are pairwise disjoint for each k ,

$$\bar{\Omega} = \bigcup_{i \geq 1} (a_i^k + \rho_i^k \bar{\Omega}) \cup E_k, \quad \mathcal{L}^N(E_k) = 0$$

and

$$\int_{\Omega} z(x) \bar{\varphi}(x) dx = \lim_{k \rightarrow +\infty} \sum_{i \geq 1} \bar{\varphi}(a_i^k) \int_{a_i^k + \rho_i^k \Omega} z(x) dx. \quad (3.25)$$

For each $k \in \mathbb{N}$, let $m_k \in \mathbb{N}$ large enough so that

$$\left| \sum_{i=1}^{m_k} \bar{\varphi}(a_i^k) \int_{a_i^k + \rho_i^k \Omega} z(x) dx - \sum_{i \geq 1} \bar{\varphi}(a_i^k) \int_{a_i^k + \rho_i^k \Omega} z(x) dx \right| < \frac{1}{k}. \quad (3.26)$$

For fixed i and k , by the choice of a_i^k and Proposition 3.6 the family $\{\nu_{(a_i^k, y)}\}_{y \in Q}$ is a homogeneous two-scale gradient Young measure. Hence by Remark 3.3 and Lemma 2.7, for every sequence $\{\varepsilon_n\} \rightarrow 0$, there exist sequences $\{u_n^{i,k}\}_{n \in \mathbb{N}} \subset W_0^{1,p}(a_i^k + \rho_i^k \Omega; \mathbb{R}^d)$ such that

$$\lim_{n \rightarrow +\infty} \int_{a_i^k + \rho_i^k \Omega} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n^{i,k}(x) \right) dx = \bar{\varphi}(a_i^k) \int_{a_i^k + \rho_i^k \Omega} z(x) dx.$$

Summing up

$$\lim_{n \rightarrow +\infty} \sum_{i=1}^{m_k} \int_{a_i^k + \rho_i^k \Omega} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n^{i,k}(x) \right) dx = \sum_{i=1}^{m_k} \bar{\varphi}(a_i^k) \int_{a_i^k + \rho_i^k \Omega} z(x) dx. \quad (3.27)$$

Let us define

$$u_n^k(x) := \begin{cases} u_n^{i,k}(x) & \text{if } x \in a_i^k + \rho_i^k \Omega, \\ 0 & \text{otherwise} \end{cases}$$

and remark that $u_n^k \in W_0^{1,p}(\Omega; \mathbb{R}^d)$. Since the sets $a_i^k + \rho_i^k \Omega$ are pairwise disjoint for each k we have that

$$\begin{aligned} & \int_{\Omega} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n^k(x) \right) dx \\ &= \sum_{i \geq 1} \int_{a_i^k + \rho_i^k \Omega} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n^{i,k}(x) \right) dx \\ &= \sum_{i=1}^{m_k} \int_{a_i^k + \rho_i^k \Omega} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n^{i,k}(x) \right) dx \\ & \quad + \int_{\Omega \cap \bigcup_{i > m_k} (a_i^k + \rho_i^k \Omega)} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n^k(x) \right) dx. \end{aligned} \quad (3.28)$$

But as $z \in L^1(\Omega)$ and $\mathcal{L}^N(\Omega \cap \bigcup_{i > m_k} (a_i^k + \rho_i^k \Omega)) \rightarrow 0$, as $k \rightarrow +\infty$, it follows that

$$\begin{aligned} & \lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \left| \int_{\Omega \cap \bigcup_{i > m_k} (a_i^k + \rho_i^k \Omega)} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n^k(x) \right) dx \right| \\ & \leq \|\varphi\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^d \times \mathbb{N})} \lim_{k \rightarrow +\infty} \int_{\Omega \cap \bigcup_{i > m_k} (a_i^k + \rho_i^k \Omega)} |z(x)| dx = 0. \end{aligned} \quad (3.29)$$

Then, gathering (3.25)-(3.29) we obtain that

$$\begin{aligned}
& \lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{\Omega} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n^k(x) \right) dx \\
&= \lim_{k \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sum_{i=1}^{m_k} \int_{a_i^k + \rho_i^k \Omega} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n^{i,k}(x) \right) dx \\
&= \lim_{k \rightarrow +\infty} \sum_{i=1}^{m_k} \overline{\varphi}(a_i^k) \int_{a_i^k + \rho_i^k \Omega} z(x) dx \\
&= \lim_{k \rightarrow +\infty} \sum_{i \geq 1} \overline{\varphi}(a_i^k) \int_{a_i^k + \rho_i^k \Omega} z(x) dx \\
&= \int_{\Omega} z(x) \overline{\varphi}(x) dx.
\end{aligned}$$

A diagonalization argument implies the existence of a sequence $\{k(n)\} \nearrow +\infty$, as $n \rightarrow +\infty$, such that upon setting $u_n := u_n^{k(n)}$, then

$$\lim_{n \rightarrow +\infty} \int_{\Omega} z(x) \varphi \left(\left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x) \right) dx = \int_{\Omega} z(x) \overline{\varphi}(x) dx \quad \text{and } u_n \rightharpoonup 0 \text{ in } W^{1,p}(\Omega; \mathbb{R}^d),$$

which completes the proof whenever $u = 0$.

Step 2. Consider now a general $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ and ν satisfying properties (i)-(iii). We define $\tilde{\nu} \in L_w^\infty(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d \times N}))$ by

$$\langle \tilde{\nu}, \varphi \rangle := \int_{\Omega} \int_Q \int_{\mathbb{R}^{d \times N}} \varphi(x, y, \xi - \nabla u(x)) d\nu_{(x,y)}(\xi) dy dx, \quad (3.30)$$

for every $\varphi \in L^1(\Omega \times Q; \mathcal{C}_0(\mathbb{R}^{d \times N}))$. We can easily check that $\tilde{\nu}$ satisfies the analogue of properties (i)-(iii) with $\tilde{u} = 0$. Hence, applying Step 1, for every sequence $\{\varepsilon_n\} \rightarrow 0$ we may find a sequence $\{\tilde{u}_n\} \subset W^{1,p}(\Omega; \mathbb{R}^d)$ such that $\{(\cdot/\varepsilon_n), \nabla \tilde{u}_n\}$ generates the Young measure $\{\tilde{\nu}_{(x,y)} \otimes dy\}_{x \in \Omega}$. Defining $u_n := \tilde{u}_n + u$, we claim that $\{(\cdot/\varepsilon_n), \nabla u_n\}$ generates $\{\nu_{(x,y)} \otimes dy\}_{x \in \Omega}$. Indeed let $\psi \in L^1(\Omega; \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^{d \times N}))$ and define the $\tilde{\psi}(x, y, \xi) := \psi(x, y, \xi + \nabla u(x))$ where $\tilde{\psi} \in L^1(\Omega; \mathcal{C}_0(\mathbb{R}^N \times \mathbb{R}^{d \times N}))$ as well. Then by (3.30),

$$\begin{aligned}
\lim_{n \rightarrow +\infty} \int_{\Omega} \psi \left(x, \left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla u_n(x) \right) dx &= \lim_{n \rightarrow +\infty} \int_{\Omega} \tilde{\psi} \left(x, \left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla \tilde{u}_n(x) \right) dx \\
&= \int_{\Omega} \int_Q \int_{\mathbb{R}^{d \times N}} \tilde{\psi}(x, y, \xi) d\tilde{\nu}_{(x,y)}(\xi) dy dx \\
&= \int_{\Omega} \int_Q \int_{\mathbb{R}^{d \times N}} \psi(x, y, \xi) d\nu_{(x,y)}(\xi) dy dx
\end{aligned}$$

which completes the proof. \square

The next corollary asserts the independence of the sequence in Definition 2.3.

Corollary 3.8. *Let $\{u_n\}$ be a bounded sequence in $W^{1,p}(\Omega; \mathbb{R}^d)$. Assume that there exists a sequence $\{\varepsilon_n\} \rightarrow 0$ such that the pair $\{(\cdot/\varepsilon_n), \nabla u_n\}$ generates a Young measure $\{\nu_{(x,y)} \otimes dy\}_{x \in \Omega}$. Then the family $\{\nu_{(x,y)}\}_{(x,y) \in \Omega \times Q}$ is a two-scale gradient Young measure.*

4. PROOF OF THEOREM 1.1

Before proving Theorem 1.1 we start by recalling Valadier's notion of *admissible integrand* (see [38]).

Definition 4.1. *A function $f : \Omega \times Q \times \mathbb{R}^{d \times N} \rightarrow [0, +\infty)$ is said to be an admissible integrand if for any $\eta > 0$, there exist compact sets $K_\eta \subset \Omega$ and $Y_\eta \subset Q$, with $\mathcal{L}^N(\Omega \setminus K_\eta) < \eta$ and $\mathcal{L}^N(Q \setminus Y_\eta) < \eta$, and such that $f|_{K_\eta \times Y_\eta \times \mathbb{R}^{d \times N}}$ is continuous.*

We observe that from Lemma 4.11 in Barchiesi [9], if f is an admissible integrand then, for fixed $\varepsilon > 0$, the function $(x, \xi) \mapsto f(x, \langle x/\varepsilon \rangle, \xi)$ is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^{d \times N})$ -measurable, where $\mathcal{L}(\Omega)$ and $\mathcal{B}(\mathbb{R}^{d \times N})$ denote, respectively, the σ -algebra of Lebesgue measurable subsets of Ω and Borel subsets of $\mathbb{R}^{d \times N}$. In particular, the functional (1.1) is well defined in $W^{1,p}(\Omega; \mathbb{R}^d)$.

Proof of Theorem 1.1. Let $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ and let $\{\varepsilon_n\} \rightarrow 0$. We start by showing that

$$\Gamma\text{-}\limsup_{n \rightarrow +\infty} \mathcal{F}_{\varepsilon_n}(u) \leq \inf_{\nu \in \mathcal{M}_u} \int_{\Omega} \int_Q \int_{\mathbb{R}^{d \times N}} f(x, y, \xi) d\nu_{(x,y)}(\xi) dy dx. \quad (4.1)$$

where \mathcal{M}_u is the set defined in (1.6). Let $\nu \in \mathcal{M}_u$, by Remark 2.6 there exists a sequence $\{u_n\} \subset W^{1,p}(\Omega; \mathbb{R}^d)$ such that $\{(\langle \cdot/\varepsilon_n \rangle, \nabla u_n)\}$ generates the Young measure $\{\nu_{(x,y)} \otimes dy\}_{x \in \Omega}$ and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$. Extract a subsequence $\{\varepsilon_{n_k}\} \subset \{\varepsilon_n\}$ such that

$$\limsup_{n \rightarrow +\infty} \mathcal{F}_{\varepsilon_n}(u_n) = \lim_{k \rightarrow +\infty} \mathcal{F}_{\varepsilon_{n_k}}(u_{n_k})$$

and that $\{|\nabla u_{n_k}|^p\}$ is equi-integrable, which is always possible by the Decomposition Lemma (see Lemma 1.2 in Fonseca, Müller & Pedregal [20]). In particular, due to the p -growth condition (1.4), the sequence $\{f(\cdot, \langle \cdot/\varepsilon_{n_k} \rangle, \nabla u_{n_k})\}$ is equi-integrable as well and applying Theorem 2.8 (ii) in Barchiesi [10] we get that

$$\Gamma\text{-}\limsup_{n \rightarrow +\infty} \mathcal{F}_{\varepsilon_n}(u) \leq \lim_{k \rightarrow +\infty} \int_{\Omega} f\left(x, \left\langle \frac{x}{\varepsilon_{n_k}} \right\rangle, \nabla u_{n_k}(x)\right) dx \quad (4.2)$$

$$= \int_{\Omega} \int_Q \int_{\mathbb{R}^{d \times N}} f(x, y, \xi) d\nu_{(x,y)}(\xi) dy dx. \quad (4.3)$$

Taking the infimum over all $\nu \in \mathcal{M}_u$ in the right hand side of the (4.2) yields to (4.1).

Let us prove now that

$$\Gamma\text{-}\liminf_{n \rightarrow +\infty} \mathcal{F}_{\varepsilon_n}(u) \geq \inf_{\nu \in \mathcal{M}_u} \int_{\Omega} \int_Q \int_{\mathbb{R}^{d \times N}} f(x, y, \xi) d\nu_{(x,y)}(\xi) dy dx. \quad (4.4)$$

Let $\eta > 0$ and $\{u_n\} \subset W^{1,p}(\Omega; \mathbb{R}^d)$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$ and

$$\liminf_{n \rightarrow +\infty} \mathcal{F}_{\varepsilon_n}(u_n) \leq \Gamma\text{-}\liminf_{n \rightarrow +\infty} \mathcal{F}_{\varepsilon_n}(u) + \eta. \quad (4.5)$$

For a subsequence $\{n_k\}$, we can assume that there exists $\nu \in L_w^\infty(\Omega \times Q; \mathcal{M}(\mathbb{R}^{d \times N}))$ such that $\{(\langle \cdot/\varepsilon_{n_k} \rangle, \nabla u_{n_k})\}$ generates a Young measure $\{\nu_{(x,y)} \otimes dy\}_{x \in \Omega}$ and

$$\lim_{k \rightarrow +\infty} \mathcal{F}_{\varepsilon_{n_k}}(u_{n_k}) = \liminf_{n \rightarrow +\infty} \mathcal{F}_{\varepsilon_n}(u_n). \quad (4.6)$$

We remark that $\{\nabla u_{n_k}\}$ is equi-integrable since it is bounded in $L^p(\Omega; \mathbb{R}^{d \times N})$ and $p > 1$. Thus, by Theorem 2.2 (v) we get that for every $A \in \mathcal{A}(\Omega)$,

$$\int_A \nabla u(x) dx = \lim_{k \rightarrow +\infty} \int_A \nabla u_{n_k}(x) dx = \int_A \int_Q \int_{\mathbb{R}^{d \times N}} \xi d\nu_{(x,y)}(\xi) dy dx.$$

By the arbitrariness of the set A , it follows that

$$\nabla u(x) = \int_Q \int_{\mathbb{R}^{d \times N}} \xi d\nu_{(x,y)}(\xi) dy \quad \text{a.e. in } \Omega. \quad (4.7)$$

As a consequence of Corollary 3.8 $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$ is a two-scale gradient Young measure and, by (4.7), we also have that $\nu \in \mathcal{M}_u$. Applying now Theorem 2.8 (i) in Barchiesi [10] we get that

$$\begin{aligned} \lim_{k\rightarrow+\infty} \int_{\Omega} f\left(x, \left\langle \frac{x}{\varepsilon_{n_k}} \right\rangle, \nabla u_{n_k}(x)\right) dx \\ \geq \int_{\Omega} \int_Q \int_{\mathbb{R}^{d\times N}} f(x, y, \xi) d\nu_{(x,y)}(\xi) dy dx \\ \geq \inf_{\nu \in \mathcal{M}_u} \int_{\Omega} \int_Q \int_{\mathbb{R}^{d\times N}} f(x, y, \xi) d\nu_{(x,y)}(\xi) dy dx. \end{aligned}$$

Hence by (4.5), (4.6) and the arbitrariness of η we get the desired result. Gathering (4.1) and (4.4), we obtain that

$$\Gamma\text{-}\lim_{n\rightarrow+\infty} \mathcal{F}_{\varepsilon_n}(u) = \inf_{\nu \in \mathcal{M}_u} \int_{\Omega} \int_Q \int_{\mathbb{R}^{d\times N}} f(x, y, \xi) d\nu_{(x,y)}(\xi) dy dx.$$

It remains to prove that the minimum is attained. To this aim, consider a recovering sequence $\{\bar{u}_n\} \subset W^{1,p}(\Omega; \mathbb{R}^d)$. Arguing exactly as before we can assume that (a subsequence of) $\{\nabla \bar{u}_n\}$ generates a two-scale gradient Young measure $\{\nu_{(x,y)}\}_{(x,y)\in\Omega\times Q}$, that $\nu \in \mathcal{M}_u$ and $\{f(\cdot, \langle \cdot/\varepsilon_n \rangle, \nabla \bar{u}_n)\}$ is equi-integrable. According to Theorem 2.8 (ii) in Barchiesi [10] and using the fact that $\{\bar{u}_n\}$ is a recovering sequence,

$$\begin{aligned} \Gamma\text{-}\lim_{n\rightarrow+\infty} \mathcal{F}_{\varepsilon_n}(u) &= \lim_{n\rightarrow+\infty} \int_{\Omega} f\left(x, \left\langle \frac{x}{\varepsilon_n} \right\rangle, \nabla \bar{u}_n(x)\right) dx \\ &= \int_{\Omega} \int_Q \int_{\mathbb{R}^{d\times N}} f(x, y, \xi) d\nu_{(x,y)}(\xi) dy dx \end{aligned}$$

which completes the proof. \square

Let us conclude by stating a Corollary which provides an alternative formula to derive the homogenized energy density f_{hom} in (1.9).

Corollary 4.2. *If $f : Q \times \mathbb{R}^{d\times N} \rightarrow [0, +\infty)$ is a Carathéodory integrand (independent of x) and satisfying (1.4), then for every $u \in W^{1,p}(\Omega; \mathbb{R}^d)$,*

$$\mathcal{F}_{\text{hom}}(u) = \int_{\Omega} f_{\text{hom}}(\nabla u(x)) dx,$$

where for every $F \in \mathbb{R}^{d\times N}$,

$$f_{\text{hom}}(F) = \min_{\nu \in M_F} \int_Q \int_{\mathbb{R}^{d\times N}} f(y, \xi) d\nu_y(\xi) dy$$

and M_F is defined in (3.14).

Proof. It known from e.g. Theorem 14.5 in Braides & Defranceschi [14] that

$$\mathcal{F}_{\text{hom}}(u) = \int_{\Omega} f_{\text{hom}}(\nabla u(x)) dx$$

where f_{hom} is defined in (1.9). By Theorem 1.1 with $\Omega = Q$ and $u(x) = Fx$, we get that

$$f_{\text{hom}}(F) = \min_{\nu \in \mathcal{M}_u} \int_Q \int_Q \int_{\mathbb{R}^{d\times N}} f(x, y, \xi) d\nu_{(x,y)}(\xi) dy dx.$$

The thesis follows from Lemma 2.9. \square

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