

3D-2D analysis of a thin film with periodic microstructure

Jean-François Babadjian and Margarida Baía

Abstract

The purpose of this article is to study the behavior of a heterogeneous thin film whose microstructure oscillates on a scale that is comparable to that of the thickness of the domain. The argument is based on a 3D-2D dimensional reduction through a Γ -convergence analysis, techniques of two-scale convergence and a decoupling procedure between the oscillating variable and the in-plane variable.

Keywords: dimension reduction, thin films, periodic integrands, Γ -convergence, two-scale convergence, quasiconvexity, equi-integrability.

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1 Introduction and main result

In this work we study the asymptotic behavior of a heterogeneous ε -thin domain with periodic microstructure of period ε , as $\varepsilon \rightarrow 0$, through a Γ -limit analysis. Techniques of two-scale convergence and a decoupling procedure between the microscopic oscillating variables and the macroscopic in-plane variables are used to derive the relaxed two-dimensional energy from its three-dimensional counterpart.

Let ω be an open and bounded subset of \mathbb{R}^2 . For each $0 < \varepsilon \ll 1$ define $\Omega_\varepsilon := \omega \times (-\varepsilon, \varepsilon)$. Consider a deformable thin body occupied by a hyperelastic material with a periodic microstructure of period ε whose reference configuration is given by the thin domain Ω_ε , and whose stored energy density $W(\varepsilon) : \Omega_\varepsilon \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ is assumed to be a Carathéodory function satisfying some p -growth and coercivity conditions ($1 < p < \infty$). We assume that the body is pinned on the lateral boundary, that is $v(x) = x$ on $\partial\omega \times (-\varepsilon, \varepsilon)$, for all its admissible deformations, and that it is subjected to the action of regular surface traction densities $g(\varepsilon)$ on $\Sigma_\varepsilon := \omega \times \{-\varepsilon, \varepsilon\}$, and regular dead loads $f(\varepsilon)$. The total energy of this body under the action of these forces is the difference between the elastic energy and the work of external forces. More precisely,

$$\mathcal{E}(\varepsilon)(v) := \int_{\Omega_\varepsilon} W(\varepsilon)(x; Dv) dx - \int_{\Omega_\varepsilon} f(\varepsilon) \cdot v dx - \int_{\Sigma_\varepsilon} g(\varepsilon) \cdot v d\mathcal{H}^2,$$

for $v \in \mathcal{V}(\varepsilon) := \{v \in W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3) : v(x) = x \text{ on } \partial\omega \times (-\varepsilon, \varepsilon)\}$, and where \mathcal{H}^2 stands for the two-dimensional Hausdorff measure. It may occur that the minimization problem associated with this energy admits no solution over the set of kinematically admissible fields $\mathcal{V}(\varepsilon)$. However, we can introduce the notion of *almost-minimizer* of $\mathcal{E}(\varepsilon)$, $v(\varepsilon) \in \mathcal{V}(\varepsilon)$, by requiring that

$$\mathcal{E}(\varepsilon)(v(\varepsilon)) \leq \inf_{v \in \mathcal{V}(\varepsilon)} \mathcal{E}(\varepsilon)(v) + \varepsilon h(\varepsilon),$$

where $h(\varepsilon) \searrow 0^+$ when $\varepsilon \rightarrow 0$. Note that if the minimization problem admits a solution – for instance if $W(\varepsilon)$ is quasiconvex in its second variable – then we can take $h \equiv 0$.

As usual, in order to study this problem as $\varepsilon \rightarrow 0$ we rescale the ε -thin body into a reference domain of unit thickness (see e.g. Acerbi, Buttazzo and Percivale [2], Anzellotti, Baldo and Percivale [4], Le Dret and Raoult

[18], Braides, Fonseca and Francfort [11]), so that the resulting energy will be defined on a fixed body, while the dependence on ε turns out to be explicit in the transverse derivative. For this, we consider the change of variables

$$\Omega_\varepsilon \rightarrow \Omega := \omega \times I, \quad (x_1, x_2, x_3) \mapsto \left(x_1, x_2, \frac{1}{\varepsilon} x_3 \right),$$

and define $u(x_\alpha, x_3/\varepsilon) = v(x_\alpha, x_3)$ on the rescaled cylinder Ω , where $I := (-1, 1)$ and $x_\alpha := (x_1, x_2)$ is the in-plane variable. It is well known that membrane theory arises at the order ε of a formal asymptotic expansion (see Fox, Raoult and Simo [17]), provided that the body forces are of order 1 and the surface loadings are of order ε . Since this energy is of order ε we divide the total energy by ε and, in addition we assume that

$$\begin{cases} f(\varepsilon)(x_\alpha, \varepsilon x_3) &= f(x_\alpha, x_3), \\ g(\varepsilon)(x_\alpha, \varepsilon x_3) &= \varepsilon g(x_\alpha, x_3), \end{cases}$$

where $f \in L^{p'}(\Omega; \mathbb{R}^3)$, $g \in L^{p'}(\Sigma; \mathbb{R}^3)$ ($1/p + 1/p' = 1$) and $\Sigma := \omega \times \{-1, 1\}$. If $W_\varepsilon(x_\alpha, x_3; \cdot) = W(\varepsilon)(x_\alpha, \varepsilon x_3; \cdot)$, for fixed ε minimizing $\mathcal{E}(\varepsilon)$ on $\mathcal{V}(\varepsilon)$ is equivalent to minimizing

$$\mathcal{E}_\varepsilon(u) := \frac{\mathcal{E}(\varepsilon)(v)}{\varepsilon} = \int_\Omega W_\varepsilon \left(x; D_\alpha u(x) \Big| \frac{1}{\varepsilon} D_3 u(x) \right) dx - \int_\Omega f \cdot u dx - \int_\Sigma g \cdot u d\mathcal{H}^2$$

on $\mathcal{V}_\varepsilon := \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : u(x) = (x_\alpha, \varepsilon x_3) \text{ on } \partial\omega \times I\}$. Denote by $D_i = \frac{\partial}{\partial x_i}$ for $i \in \{1, 2, 3\}$ and $D_\alpha = (D_1, D_2)$. In the sequel, we identify $\mathbb{R}^{d \times N}$ (resp. $\mathbb{Q}^{d \times N}$) with the space of real (resp. rational) $d \times N$ matrices. For all $\bar{\xi} = (z_1 | z_2) \in \mathbb{R}^{3 \times 2}$ and $z \in \mathbb{R}^3$, $(\bar{\xi} | z)$ is the matrix whose first two columns are z_1 and z_2 and whose last one is z . Denoting a almost-minimizer of the rescaled energy by $u_\varepsilon(x_\alpha, x_3) := v(\varepsilon)(x_\alpha, \varepsilon x_3)$, we obtain

$$\mathcal{E}_\varepsilon(u_\varepsilon) \leq \inf_{u \in \mathcal{V}_\varepsilon} \mathcal{E}_\varepsilon(u) + h(\varepsilon). \quad (1.1)$$

Our aim is to study the asymptotic behavior of the equilibrium problem (1.1) as $\varepsilon \rightarrow 0$ via a Γ -convergence method (we refer to Braides and Defranceschi [10], Braides [12] and Dal Maso [14] for a comprehensive treatment and bibliography on Γ -convergence).

The motivation for studying problem (1.1) comes from the work in Braides, Fonseca and Francfort [11] where the authors have established an abstract dimensional reduction variational convergence result in a general setting for a family of stored energies of the form $W_\varepsilon(x; \xi)$ and derived specific characterizations for particular cases. In Section 3 of [11] a heterogeneous nonlinear membrane model is derived by Γ -convergence, and heterogeneity in the transverse direction is considered. Precisely, the authors treat the case where the stored energy density is of the form $W(x_3; \xi)$, generalizing the previous work of Le Dret and Raoult in [18] who treated a homogeneous material, i.e. when W depends only in ξ . Later, Babadjian and Francfort [5] considered energies of the form $W(x; \xi)$ with a general heterogeneity. Furthermore in Section 4 of [11], a 3D-2D analysis coupled with a homogenization in the in-plane direction is studied in the case where $W_\varepsilon(x; \xi) = W(x_3, x_\alpha/\varepsilon; \xi)$. Shu [23] also investigated similar problems, in the framework of martensitic materials, with different length scales for the film thickness and the material microstructure.

Here we propose to establish a dimensional reduction and homogenization result, where both scales are identical, by adding in the stored energy density an explicit dependence on the macroscopic in-plane variable x_α . Namely, we assume that $W_\varepsilon(x_\alpha, x_3; \cdot) = W(x_\alpha, x_3, x_\alpha/\varepsilon; \cdot)$ for some function $W : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ whose hypotheses will be introduced later.

Two features differentiate our approach from what is available in most of the literature in the subject. The first one is the use of a two-scale convergence argument (see Nguetseng [21, 22] and Allaire [3] for the notion and properties of two-scale convergence). The same argument was used by Baía and Fonseca in [6] in a purely homogeneous context, i.e. without considering the dimensional reduction problem. The second feature is due to the definition of the homogenized stored energy in which two independent variables appear (see identity (1.4) below). To take into account this structure, we are led to decouple the macroscopic in-plane variable x_α from

the microscopic oscillating variable x_α/ε via an extension argument along the lines of Babadjian and Francfort [5].

For a comprehensive treatment on the homogenization of integral functionals via a Γ -limit approach, we refer to Braides and Defranceschi [10] and references therein. We will denote by \mathcal{L}^N the N -dimensional Lebesgue measure in \mathbb{R}^N (in the sequel N will be equal to 2 or 3).

For each $\varepsilon > 0$ we define $\mathcal{I}_\varepsilon : L^p(\Omega; \mathbb{R}^3) \rightarrow \overline{\mathbb{R}}$ by

$$\mathcal{I}_\varepsilon(u) := \begin{cases} \int_{\Omega} W\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon}; D_\alpha u(x) \Big| \frac{1}{\varepsilon} D_3 u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases} \quad (1.2)$$

with $1 < p < \infty$, where we assume that $W : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ satisfies the following hypotheses:

(H₁) $W(x, \cdot; \cdot)$ is continuous for a.e. $x \in \Omega$;

(H₂) $W(\cdot, y_\alpha; \xi)$ is measurable for all $(y_\alpha, \xi) \in \mathbb{R}^2 \times \mathbb{R}^{3 \times 3}$;

(H₃) there exists $0 < \beta < +\infty$ such that

$$\frac{1}{\beta} |\xi|^p - \beta \leq W(x, y_\alpha; \xi) \leq \beta(1 + |\xi|^p), \quad \text{for a.e. } x \in \Omega \text{ and for all } (y_\alpha, \xi) \in \mathbb{R}^2 \times \mathbb{R}^{3 \times 3};$$

(H₄) $W(x, \cdot; \xi)$ is Q' -periodic for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^{3 \times 3}$, where we denote by $Q' = (0, 1)^2$ the unit cube of \mathbb{R}^2 .

Remark 1.1. We remark that due to hypothesis (H₁) and (H₂) the function W is a Carathéodory integrand as $W(x, \cdot; \cdot)$ is continuous a.e. $x \in \Omega$ and $W(\cdot, y_\alpha; \xi)$ is measurable for all $y_\alpha \in \mathbb{R}^2$ and $\xi \in \mathbb{R}^{3 \times 3}$. This implies (see e.g. Proposition 3.3 in Braides and Defranceschi [10] or Proposition 1.1, Chapter VIII in Ekeland and Temam [15]) that W is equivalent to a Borel function, that is there exist a Borel function \tilde{W} such that $W(x, \cdot; \cdot) = \tilde{W}(x, \cdot; \cdot)$ for a.e. $x \in \Omega$. As a consequence the integral in (1.2) is well defined. As noted by Allaire in [3], Section 5, the measurability of W in the pair (x, y_α) does not let us conclude that, for fixed ξ , the function $x \mapsto W(x, x_\alpha/\varepsilon; \xi)$ is measurable. The continuity of $W(x, y_\alpha; \xi)$ in at least one of the variables x or y_α turns out to be sufficient to guarantee the measurability of this function. In the present paper, we decide to impose the continuity in the y_α variable. Note that we could also have considered W to be continuous in x and measurable in y_α but the proof of our main result does not hold anymore in this context.

As for notation, we will identify $W^{1,p}(\omega; \mathbb{R}^3)$ with the set of functions $u \in W^{1,p}(\Omega; \mathbb{R}^3)$ such that $D_3 u(x) = 0$ for a.e. $x \in \Omega$ and we will use the notation $\Gamma(L^p(\Omega))$ -limit whenever we refer to the Γ -convergence with respect to the usual metric in $L^p(\Omega; \mathbb{R}^3)$. We prove the following main result.

Theorem 1.2. *If W satisfies (H₁)-(H₄), then the family $\{\mathcal{I}_\varepsilon\}_{\varepsilon > 0}$ $\Gamma(L^p(\Omega))$ -converges to the functional $\mathcal{I}_{\text{hom}} : L^p(\Omega; \mathbb{R}^3) \rightarrow \overline{\mathbb{R}}$ defined by*

$$\mathcal{I}_{\text{hom}}(u) := \begin{cases} 2 \int_{\omega} W_{\text{hom}}(x_\alpha; D_\alpha u(x_\alpha)) dx_\alpha & \text{if } u \in W^{1,p}(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases} \quad (1.3)$$

where W_{hom} is given by

$$W_{\text{hom}}(x_\alpha; \bar{\xi}) := \liminf_{T \rightarrow +\infty} \inf_{\varphi} \left\{ \frac{1}{2T^2} \int_{(0,T)^2 \times I} W(x_\alpha, y_3, y_\alpha; \bar{\xi} + D_\alpha \varphi(y) | D_3 \varphi(y)) dy : \right. \\ \left. \varphi \in W^{1,p}((0,T)^2 \times I; \mathbb{R}^3), \varphi = 0 \text{ on } \partial(0,T)^2 \times I \right\} \quad (1.4)$$

for a.e. $x_\alpha \in \omega$ and all $\bar{\xi} \in \mathbb{R}^{3 \times 2}$.

As a consequence of Theorem 1.2 we deduce the usual convergence of (almost-)minimizers. More precisely, we have the following result.

Corollary 1.3. *Let $\{u_\varepsilon\} \subset \mathcal{V}_\varepsilon$ be a sequence of almost-minimizers for $\{\mathcal{I}_\varepsilon\}_{\varepsilon > 0}$ (see identity (1.1)). Then $\{u_\varepsilon\}$ is weakly relatively compact in $W^{1,p}(\Omega; \mathbb{R}^3)$. Furthermore, any limit point u of this sequence is a solution of the minimization problem*

$$\min_{v - (x_\alpha, 0) \in W_0^{1,p}(\omega; \mathbb{R}^3)} \left\{ 2 \int_\omega W_{\text{hom}}(x_\alpha; D_\alpha v(x_\alpha)) dx_\alpha - \int_\omega (\bar{f} + g^+ + g^-)(x_\alpha) \cdot v(x_\alpha) dx_\alpha \right\},$$

where $\bar{f} := \frac{1}{2} \int_I f(\cdot, x_3) dx_3$ and $g^\pm := g(\cdot, \pm 1)$.

This corollary departs from the classical result on the type of boundary conditions that have been considered (see e.g. Proposition 7.2 in Braides and Defranceschi [10]). This difficulty is overcome by the fact that we can prescribe the lateral boundary conditions of recovering sequences (see Remark 3.2). We do not include the proof of this corollary here because it is similar to that of Corollary 1.3 in Bouchitté, Fonseca and Mascarenhas [9].

The plan of this work is as follows: In Section 2 we will discuss some properties of W_{hom} , namely that it is well defined, proving that the limit on the right hand side of (1.4) exists, and that $W_{\text{hom}}(x_\alpha; \cdot)$ is continuous for a.e. $x_\alpha \in \omega$. Section 3 is devoted to the proof of our main result, Theorem 1.2. The starting point of our analysis is the Γ -limit integral representation result, Theorem 2.5, in Braides, Fonseca and Francfort [11]. Our objective is to identify the limit integrand, showing that it coincides (almost everywhere) with W_{hom} . We will use an argument of two-scale convergence to derive an upper bound for the limit integrand (Lemma 3.4). Since the problem at fixed ε and the asymptotic problem as $\varepsilon \rightarrow 0$ are of different nature (one is a full three-dimensional problem, the other a two-dimensional one), we will need to use a decoupling argument to prove the other inequality (Lemma 3.5). For this purpose it will be convenient to extend W to a function which is (separately) continuous everywhere. This is the aim of Lemma 4.1 (see Appendix in Section 4) which provides conditions under which a Carathéodory function such as W can be extended to a separately continuous function in the macroscopic in-plane variable x_α and the microscopic variable x_α/ε .

2 Preliminary results

In this section we will prove some properties of the stored energy W_{hom} that will be of use in the proof of Theorem 1.2.

Remark 2.1. To prove Theorem 1.2 we may assume, without loss of generality, that W is non negative. Indeed, in view of (H_3) it suffices to replace W by $W + \beta$.

We begin by showing that in the definition (1.4) of W_{hom} the limit as $T \rightarrow +\infty$ exists. The proof of this property is a direct consequence of a result due to Licht and Michaille [19], Theorem 3.1 (see also Lemma 4.3.6 in Bouchitté, Fonseca and Mascarenhas [8]).

Lemma 2.2. *If W satisfies (H_1) - (H_4) , then*

$$W_{\text{hom}}(x_\alpha; \bar{\xi}) = \lim_{T \rightarrow +\infty} \inf_{\varphi} \left\{ \frac{1}{2T^2} \int_{(0,T)^2 \times I} W(x_\alpha, y_3, y_\alpha; \bar{\xi} + D_\alpha \varphi(y) | D_3 \varphi(y)) dy : \right. \\ \left. \varphi \in W^{1,p}((0,T)^2 \times I; \mathbb{R}^3), \varphi = 0 \text{ on } \partial(0,T)^2 \times I \right\}$$

exists for a.e. $x_\alpha \in \omega$ and all $\bar{\xi} \in \mathbb{R}^{3 \times 2}$.

Proof. Let $x_\alpha \in \omega$ be such that (H_1) , (H_3) and (H_4) hold and let $\bar{\xi} \in \mathbb{R}^{3 \times 2}$. Define $\mu : \mathcal{A}(\mathbb{R}^2) \rightarrow \mathbb{R}^+$ by

$$\mu(A) := \inf_{\varphi} \left\{ \frac{1}{2} \int_{A \times I} W(x_\alpha, y_3, y_\alpha; \bar{\xi} + D_\alpha \varphi(y) | D_3 \varphi(y)) dy : \right. \\ \left. \varphi \in W^{1,p}(A \times I; \mathbb{R}^3), \varphi = 0 \text{ on } \partial A \times I \right\},$$

where $\mathcal{A}(\mathbb{R}^2)$ stands for the family of open subsets of \mathbb{R}^2 .

The function μ is well defined and, thanks to (H_3) , it is finite. Moreover this set function satisfies the assumptions of Theorem 3.1 in Licht and Michaille [19]. Indeed firstly, by (H_3) , $\mu(A) \leq \beta(1 + |\bar{\xi}|^p) \mathcal{L}^2(A)$ for all $A \in \mathcal{A}(\mathbb{R}^2)$. Secondly, μ is subadditive, that is $\mu(C) \leq \mu(A) + \mu(B)$ for all $A, B, C \in \mathcal{A}(\mathbb{R}^2)$ with $A \cap B \neq \emptyset$ and $\bar{C} = \bar{A} \cup \bar{B}$. Finally, by (H_4) , for any $\mathbf{i} \in \mathbb{Z}^2$, $\mu(A + \mathbf{i}) = \mu(A)$ for all $A \in \mathcal{A}(\mathbb{R}^2)$. As a consequence the limit

$$\lim_{T \rightarrow +\infty} \frac{\mu((0,T)^2)}{T^2} = W_{\text{hom}}(x_\alpha; \bar{\xi})$$

exists. ■

Remark 2.3. It can be proved that the limit as $T \rightarrow +\infty$ in (1.4) can be replaced by an infimum taken for every $T > 0$ (see Braides and Defranceschi [10] or Baía and Fonseca [6]).

Now that W_{hom} is well defined, we will show that $W_{\text{hom}}(x_\alpha; \cdot)$ is continuous for a.e. $x_\alpha \in \omega$ for later use in Theorem 1.2. To prove this property directly it seems that we need more than merely the continuity condition imposed on $W(x, y_\alpha; \cdot)$ (e.g. a p -Lipschitz condition). We remark that if $W(x, y_\alpha; \cdot)$ was quasiconvex, then by the p -growth condition (H_3) , $W(x, y_\alpha; \cdot)$ would satisfy a p -Lipschitz condition (see Lemma 2.6 below). Since we do not want to a priori restrict the stored energy density too much, in order to compensate for this lack of regularity we first prove in Lemma 2.5 that the value of W_{hom} does not change if we replace W by its quasiconvexification QW (see Remark 2.4 below).

Remark 2.4. For a.e. $x \in \Omega$, all $y_\alpha \in \mathbb{R}^2$ and all $\xi \in \mathbb{R}^{3 \times 3}$ define

$$QW(x, y_\alpha; \xi) := [QW(x, y_\alpha; \cdot)](\xi)$$

where $QW(x, y_\alpha; \cdot)$ stands for the usual quasiconvexification of $W(x, y_\alpha; \cdot)$. Then, the function $QW(x, y_\alpha; \cdot)$ is quasiconvex (see e.g. Dacorogna [13]) and satisfies (H_1) - (H_4) with the exception that $QW(x, \cdot; \xi)$ may only be upper semicontinuous for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^{3 \times 3}$ (as the infimum of continuous functions). By an argument similar to that of Lemma 2.2 we conclude that

$$(QW)_{\text{hom}}(x_\alpha; \bar{\xi}) = \lim_{T \rightarrow +\infty} \inf_{\varphi} \left\{ \frac{1}{2T^2} \int_{(0,T)^2 \times I} QW(x_\alpha, y_3, y_\alpha; \bar{\xi} + D_\alpha \varphi(y) | D_3 \varphi(y)) dy : \right. \\ \left. \varphi \in W^{1,p}((0,T)^2 \times I; \mathbb{R}^3), \varphi = 0 \text{ on } \partial(0,T)^2 \times I \right\}$$

exists for a.e. $x_\alpha \in \omega$ and all $\bar{\xi} \in \mathbb{R}^{3 \times 2}$.

Lemma 2.5. *If W satisfies (H_1) - (H_4) , then $(\mathcal{Q}W)_{\text{hom}}(x_\alpha; \bar{\xi}) = W_{\text{hom}}(x_\alpha; \bar{\xi})$ for a.e. $x_\alpha \in \omega$ and all $\bar{\xi} \in \mathbb{R}^{3 \times 2}$.*

Proof. Let $x_\alpha \in \omega$ be such that both $(\mathcal{Q}W)_{\text{hom}}(x_\alpha; \cdot)$ and $W_{\text{hom}}(x_\alpha; \cdot)$ are well defined. Since $W \geq \mathcal{Q}W$, we have $W_{\text{hom}}(x_\alpha; \bar{\xi}) \geq (\mathcal{Q}W)_{\text{hom}}(x_\alpha; \bar{\xi})$ for all $\bar{\xi} \in \mathbb{R}^{3 \times 2}$. Let us prove now the converse inequality. Let $\bar{\xi} \in \mathbb{R}^{3 \times 2}$. For each $n > 0$, let $T_n \in \mathbb{N}$ and $\varphi_n \in W^{1,\infty}((0, T_n)^2 \times I; \mathbb{R}^3)$ satisfying $\varphi_n = 0$ on $\partial(0, T_n)^2 \times I$, be such that

$$(\mathcal{Q}W)_{\text{hom}}(x_\alpha; \bar{\xi}) + \frac{1}{n} \geq \frac{1}{2T_n^2} \int_{(0, T_n)^2 \times I} \mathcal{Q}W(x_\alpha, y_3, y_\alpha; \bar{\xi} + D_\alpha \varphi_n(y) | D_3 \varphi_n(y)) dy.$$

The Lipschitz regularity of φ_n is ensured because of the density of $W^{1,\infty}((0, T_n)^2 \times I; \mathbb{R}^3)$ in $W^{1,p}((0, T_n)^2 \times I; \mathbb{R}^3)$ together with the p -growth condition (H_3) . Thus

$$(\mathcal{Q}W)_{\text{hom}}(x_\alpha; \bar{\xi}) \geq \limsup_{n \rightarrow +\infty} \frac{1}{2T_n^2} \int_{(0, T_n)^2 \times I} \mathcal{Q}W(x_\alpha, y_3, y_\alpha; \bar{\xi} + D_\alpha \varphi_n(y) | D_3 \varphi_n(y)) dy. \quad (2.1)$$

For each $n \in \mathbb{N}$ fixed, by Acerbi-Fusco Relaxation Theorem (see Lemma III.1 and Statement III.7 in [1]) and Remark 2.1, there exists a sequence $\{\varphi_{n,k}\}_k \subset W^{1,\infty}((0, T_n)^2 \times I; \mathbb{R}^3)$ satisfying $\varphi_{n,k} = \varphi_n$ on $\partial[(0, T_n)^2 \times I]$ with $\varphi_{n,k} \xrightarrow[k \rightarrow +\infty]{} \varphi_n$ and such that

$$\begin{aligned} & \frac{1}{2T_n^2} \int_{(0, T_n)^2 \times I} \mathcal{Q}W(x_\alpha, y_3, y_\alpha; \bar{\xi} + D_\alpha \varphi_n(y) | D_3 \varphi_n(y)) dy \\ &= \lim_{k \rightarrow +\infty} \frac{1}{2T_n^2} \int_{(0, T_n)^2 \times I} W(x_\alpha, y_3, y_\alpha; \bar{\xi} + D_\alpha \varphi_{n,k}(y) | D_3 \varphi_{n,k}(y)) dy. \end{aligned}$$

From (2.1) we have

$$\begin{aligned} (\mathcal{Q}W)_{\text{hom}}(x_\alpha; \bar{\xi}) &\geq \limsup_{n \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \frac{1}{2T_n^2} \int_{(0, T_n)^2 \times I} W(x_\alpha, y_3, y_\alpha; \bar{\xi} + D_\alpha \varphi_{n,k}(y) | D_3 \varphi_{n,k}(y)) dy \\ &\geq \limsup_{n \rightarrow \infty} \inf_{\varphi} \left\{ \frac{1}{2T_n^2} \int_{(0, T_n)^2 \times I} W(x_\alpha, y_3, y_\alpha; \bar{\xi} + D_\alpha \varphi(y) | D_3 \varphi(y)) dy : \right. \\ &\quad \left. \varphi \in W^{1,p}((0, T_n)^2 \times I; \mathbb{R}^3), \varphi = 0 \text{ on } \partial(0, T_n)^2 \times I \right\} \\ &= W_{\text{hom}}(x_\alpha; \bar{\xi}). \end{aligned}$$

■

We are now in position to prove the continuity of W_{hom} in its second variable :

Lemma 2.6. *Let W satisfying (H_1) - (H_4) , then $W_{\text{hom}}(x_\alpha; \cdot)$ is continuous on $\mathbb{R}^{3 \times 2}$ for a.e. $x_\alpha \in \omega$.*

Proof. We observe that by the p -growth condition in (H_3) and Remark 2.4, $\mathcal{Q}W$ satisfies a p -Lipschitz condition (see Marcellini [20]): There exists $\beta > 0$ such that for all $y_\alpha \in \mathbb{R}^2$ and a.e. $x \in \Omega$,

$$|\mathcal{Q}W(x, y_\alpha; \xi_1) - \mathcal{Q}W(x, y_\alpha; \xi_2)| \leq \beta(1 + |\xi_1|^{p-1} + |\xi_2|^{p-1})|\xi_1 - \xi_2|, \quad \xi_1, \xi_2 \in \mathbb{R}^{3 \times 3}. \quad (2.2)$$

Take $x_\alpha \in \omega$ such that both $(\mathcal{Q}W)_{\text{hom}}(x_\alpha; \cdot)$ and $W_{\text{hom}}(x_\alpha; \cdot)$ are well defined. By Lemma 2.5 we have $(\mathcal{Q}W)_{\text{hom}}(x_\alpha; \cdot) = W_{\text{hom}}(x_\alpha; \cdot)$. Given $\bar{\xi} \in \mathbb{R}^{3 \times 2}$ let $\bar{\xi}_n \rightarrow \bar{\xi}$ in $\mathbb{R}^{3 \times 2}$. From the definition of $W_{\text{hom}}(x_\alpha; \bar{\xi})$, for fixed $\delta > 0$ choose $T \in \mathbb{N}$ and $\varphi \in W^{1,p}((0, T)^2 \times I; \mathbb{R}^3)$, $\varphi = 0$ on $\partial(0, T)^2 \times I$, such that

$$W_{\text{hom}}(x_\alpha; \bar{\xi}) + \delta \geq \frac{1}{2T^2} \int_{(0,T)^2 \times I} W(x_\alpha, y_3, y_\alpha; \bar{\xi} + D_\alpha \varphi(y) | D_3 \varphi(y)) dy. \quad (2.3)$$

Therefore, Remark 2.3 yields

$$\begin{aligned} \limsup_{n \rightarrow +\infty} W_{\text{hom}}(x_\alpha; \bar{\xi}_n) &\leq \limsup_{n \rightarrow +\infty} \frac{1}{2T^2} \int_{(0,T)^2 \times I} W(x_\alpha, y_3, y_\alpha; \bar{\xi}_n + D_\alpha \varphi(y) | D_3 \varphi(y)) dy \\ &= \frac{1}{2T^2} \int_{(0,T)^2 \times I} W(x_\alpha, y_3, y_\alpha; \bar{\xi} + D_\alpha \varphi(y) | D_3 \varphi(y)) dy \end{aligned}$$

due to hypothesis (H_1) , the p -growth condition in (H_3) and Lebesgue's Dominated Convergence Theorem. So by (2.3) and letting $\delta \rightarrow 0$ we conclude that

$$\limsup_{n \rightarrow +\infty} W_{\text{hom}}(x_\alpha; \bar{\xi}_n) \leq W_{\text{hom}}(x_\alpha; \bar{\xi}). \quad (2.4)$$

Similarly, for each $n \in \mathbb{N}$ consider $T_n \in \mathbb{N}$ ($T_n \nearrow +\infty$) and $\varphi_n \in W^{1,p}((0, T_n)^2 \times I; \mathbb{R}^3)$, $\varphi_n = 0$ on $\partial(0, T_n)^2 \times I$, such that

$$\begin{aligned} W_{\text{hom}}(x_\alpha; \bar{\xi}_n) + \frac{1}{n} &\geq \frac{1}{2T_n^2} \int_{(0, T_n)^2 \times I} \mathcal{Q}W(x_\alpha, y_3, y_\alpha; \bar{\xi}_n + D_\alpha \varphi_n(y) | D_3 \varphi_n(y)) dy \\ &= \frac{1}{2} \int_{Q' \times I} \mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha; \bar{\xi}_n + D_\alpha \varphi_n(T_n y_\alpha, y_3) | D_3 \varphi_n(T_n y_\alpha, y_3)) dy \\ &= \frac{1}{2} \int_{Q' \times I} \mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha; \bar{\xi}_n + D_\alpha \psi_n(y) | T_n D_3 \psi_n(y)) dy, \end{aligned}$$

after a change of variables and where $\psi_n(y) := \frac{1}{T_n} \varphi_n(T_n y_\alpha, y_3)$. Clearly the function ψ_n belongs to $W^{1,p}(Q' \times I; \mathbb{R}^3)$ and $\psi_n = 0$ on $\partial Q' \times I$.

By the p -coercivity hypothesis in (H_3) and (2.4), the sequence $\{(D_\alpha \psi_n | T_n D_3 \psi_n)\}$ is bounded in $L^p(Q' \times I; \mathbb{R}^{3 \times 3})$ uniformly in n . We can write that

$$\begin{aligned} &\liminf_{n \rightarrow +\infty} \int_{Q' \times I} \mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha; \bar{\xi}_n + D_\alpha \psi_n(y) | T_n D_3 \psi_n(y)) dy \\ &\geq \liminf_{n \rightarrow +\infty} \int_{Q' \times I} [\mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha; \bar{\xi}_n + D_\alpha \psi_n(y) | T_n D_3 \psi_n(y)) \\ &\quad - \mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha; \bar{\xi} + D_\alpha \psi_n(y) | T_n D_3 \psi_n(y))] dy \\ &\quad + \liminf_{n \rightarrow +\infty} \int_{Q' \times I} \mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha; \bar{\xi} + D_\alpha \psi_n(y) | T_n D_3 \psi_n(y)) dy. \end{aligned}$$

Using (2.2), Hölder inequality, the fact that $\{\|(D_\alpha \psi_n | T_n D_3 \psi_n)\|_{L^p(Q' \times I; \mathbb{R}^{3 \times 3})}\}$ is bounded and $\bar{\xi}_n \rightarrow \bar{\xi}$, we obtain

$$\begin{aligned} &\liminf_{n \rightarrow +\infty} \int_{Q' \times I} [\mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha; \bar{\xi}_n + D_\alpha \psi_n(y) | T_n D_3 \psi_n(y)) \\ &\quad - \mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha; \bar{\xi} + D_\alpha \psi_n(y) | T_n D_3 \psi_n(y))] dy = 0, \end{aligned}$$

and consequently

$$\begin{aligned}
\liminf_{n \rightarrow +\infty} W_{\text{hom}}(x_\alpha; \bar{\xi}_n) &\geq \liminf_{n \rightarrow +\infty} \frac{1}{2} \int_{Q' \times I} \mathcal{Q}W(x_\alpha, y_3, T_n y_\alpha; \bar{\xi} + D_\alpha \psi_n(y) | T_n D_3 \psi_n(y)) dy \\
&= \liminf_{n \rightarrow +\infty} \frac{1}{2T_n^2} \int_{(0, T_n)^2 \times I} \mathcal{Q}W(x_\alpha, y_3, y_\alpha; \bar{\xi} + D_\alpha \varphi_n(y) | D_3 \varphi_n(y)) dy \\
&\geq (\mathcal{Q}W)_{\text{hom}}(x_\alpha; \bar{\xi}) \\
&= W_{\text{hom}}(x_\alpha; \bar{\xi}).
\end{aligned} \tag{2.5}$$

From (2.4) and (2.5), we conclude that $W_{\text{hom}}(x_\alpha; \cdot)$ is continuous at $\bar{\xi}$. \blacksquare

3 Proof of Theorem 1.2

We start by localizing our functionals. Representing by $\mathcal{A}(\omega)$ the class of all open subsets of ω , define $\mathcal{I}_\varepsilon : L^p(\Omega; \mathbb{R}^3) \times \mathcal{A}(\omega) \rightarrow \overline{\mathbb{R}}$ by

$$\mathcal{I}_\varepsilon(u; A) := \begin{cases} \int_{A \times I} W\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon}; D_\alpha u(x) \Big| \frac{1}{\varepsilon} D_3 u(x)\right) dx & \text{if } u \in W^{1,p}(A \times I; \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

We will prove that the family of functionals $\{\mathcal{I}_\varepsilon(\cdot; A)\}_{\varepsilon > 0}$ Γ -converges with respect to the $L^p(A \times I; \mathbb{R}^3)$ -topology to the functional $\mathcal{I}_{\text{hom}}(\cdot; A) : L^p(\Omega; \mathbb{R}^3) \rightarrow \overline{\mathbb{R}}$

$$\mathcal{I}_{\text{hom}}(u; A) := \begin{cases} 2 \int_A W_{\text{hom}}(x_\alpha; D_\alpha u(x_\alpha)) dx_\alpha & \text{if } u \in W^{1,p}(A; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.1}$$

for all $A \in \mathcal{A}(\omega)$. As a consequence, taking $A = \omega$ yields Theorem 1.2.

For any $A \in \mathcal{A}(\omega)$ and any sequence $\{\varepsilon_j\} \searrow 0^+$, consider the Γ -lower limit of the family $\{\mathcal{I}_{\varepsilon_j}(\cdot; A)\}_{j \in \mathbb{N}}$, $\mathcal{I}_{\{\varepsilon_j\}}(\cdot; A) : L^p(\Omega; \mathbb{R}^3) \rightarrow \overline{\mathbb{R}}$, given by

$$\mathcal{I}_{\{\varepsilon_j\}}(u; A) := \inf_{\{u_j\}} \left\{ \liminf_{j \rightarrow +\infty} \mathcal{I}_{\varepsilon_j}(u_j; A) : u_j \rightarrow u \text{ in } L^p(A \times I; \mathbb{R}^3) \right\}. \tag{3.2}$$

Remark 3.1. In view of the coercivity condition (H_4) , for all $A \in \mathcal{A}(\omega)$ we have that $\mathcal{I}_{\{\varepsilon_j\}}(u; A) = +\infty$ whenever $u \in L^p(\Omega; \mathbb{R}^3) \setminus W^{1,p}(A; \mathbb{R}^3)$, hence our objective is to characterize $\mathcal{I}_{\{\varepsilon_j\}}(u; A)$ for $u \in W^{1,p}(A; \mathbb{R}^3)$.

By virtue of Remark 3.1, together with Theorem 2.5 in Braides, Fonseca and Francfort [11], it follows that every sequence $\{\varepsilon_j\}$ admits a subsequence $\{\varepsilon_{j_n}\} \equiv \{\varepsilon_n\}$ such that $\mathcal{I}_{\{\varepsilon_n\}}(\cdot; A)$ defined in (3.2) is the $\Gamma(L^p(A \times I))$ -limit of $\{\mathcal{I}_{\varepsilon_n}(\cdot; A)\}_{n \in \mathbb{N}}$ for all $A \in \mathcal{A}(\omega)$. Further there exists a Carathéodory function $W_{\{\varepsilon_n\}} : \omega \times \mathbb{R}^{3 \times 2} \rightarrow \overline{\mathbb{R}}$ such that

$$\mathcal{I}_{\{\varepsilon_n\}}(u; A) = 2 \int_A W_{\{\varepsilon_n\}}(x_\alpha; D_\alpha u(x_\alpha)) dx_\alpha, \tag{3.3}$$

for all $A \in \mathcal{A}(\omega)$ and all $u \in W^{1,p}(A; \mathbb{R}^3)$. Our aim is to show that $\mathcal{I}_{\{\varepsilon_n\}}(\cdot; A) = \mathcal{I}_{\text{hom}}(\cdot; A)$ on $W^{1,p}(A; \mathbb{R}^3)$ for all $A \in \mathcal{A}(\omega)$. Given $A \in \mathcal{A}(\omega)$, in view of the integral representation (3.3) and (3.1), it is enough to show that $W_{\{\varepsilon_n\}}(x_\alpha; \bar{\xi}) = W_{\text{hom}}(x_\alpha; \bar{\xi})$ for a.e. $x_\alpha \in A$ and all $\bar{\xi} \in \mathbb{R}^{3 \times 2}$, and thus to work with affine functions instead of general Sobolev functions. We will prove that $W_{\{\varepsilon_n\}}(x_\alpha; \bar{\xi}) = W_{\text{hom}}(x_\alpha; \bar{\xi})$ for a.e. $x_\alpha \in \omega$ and all $\bar{\xi} \in \mathbb{R}^{3 \times 2}$.

Remark 3.2. Lemma 2.6 of Braides, Fonseca and Francfort [11] implies that $\mathcal{I}_{\{\varepsilon_j\}}(u; A)$ is unchanged if the approximating sequences $\{u_j\}$ are constrained to match the lateral boundary condition of their target, i.e. $u_j \equiv u$ on $\partial A \times I$.

From now onward, $\{\varepsilon_n\}$ will denote a subsequence of $\{\varepsilon_j\}$ for which the $\Gamma(L^p(A \times I))$ -limit of $\{\mathcal{I}_{\varepsilon_n}(\cdot; A)\}_{n \in \mathbb{N}}$ exists and coincides with $\mathcal{I}_{\{\varepsilon_n\}}(\cdot; A)$ for all $A \in \mathcal{A}(\omega)$.

For each $T > 0$ consider \mathcal{S}_T a countable set of functions in $\mathcal{C}^\infty([0, T]^2 \times [-1, 1]; \mathbb{R}^3)$ that is dense in

$$\mathcal{W}_T = \{\varphi \in W^{1,p}((0, T)^2 \times I; \mathbb{R}^3) : \varphi = 0 \text{ on } \partial(0, T)^2 \times I\}.$$

Definition 3.3. Let L be the set of Lebesgue points x_α^0 for all functions

$$W_{\{\varepsilon_n\}}(\cdot; \bar{\xi}), \quad W_{\text{hom}}(\cdot; \bar{\xi}) \quad (3.4)$$

and

$$x_\alpha \mapsto \int_{Q' \times I} W(x_\alpha, y_3, Ty_\alpha; \bar{\xi} + D_\alpha \varphi(Ty_\alpha, y_3) | D_3 \varphi(Ty_\alpha, y_3)) dy, \quad (3.5)$$

with $T \in \mathbb{N}$, $\varphi \in \mathcal{S}_T$ and $\bar{\xi} \in \mathbb{Q}^{3 \times 2}$, and for which $W_{\text{hom}}(x_\alpha^0; \cdot)$ is well defined.

We have that $\mathcal{L}^2(\omega \setminus L) = 0$. Given $x_\alpha^0 \in L$, we denote by $Q'(x_\alpha^0, \delta)$ the cube in \mathbb{R}^2 centered in x_α^0 and of side length $\delta > 0$ where δ is small enough so that $Q'(x_\alpha^0, \delta) \in \mathcal{A}(\omega)$.

To prove that $W_{\{\varepsilon_n\}}(x_\alpha; \bar{\xi}) = W_{\text{hom}}(x_\alpha; \bar{\xi})$ for a.e. $x_\alpha \in \omega$ and all $\bar{\xi} \in \mathbb{R}^{3 \times 2}$ we first show in Lemmas 3.4 and 3.5 below that both functions coincide on $L \times \mathbb{Q}^{3 \times 2}$. The general case will only be treated at the end of that section using the Carathéodory property of both integrands.

Fix $\bar{\xi} \in \mathbb{Q}^{3 \times 2}$ and set $v(x) := \bar{\xi} \cdot x_\alpha$. By (3.3) and (3.4)

$$\begin{aligned} W_{\{\varepsilon_n\}}(x_\alpha^0; \bar{\xi}) &= \lim_{\delta \rightarrow 0} \frac{1}{\delta^2} \int_{Q'(x_\alpha^0, \delta)} W_{\{\varepsilon_n\}}(x_\alpha; \bar{\xi}) dx_\alpha \\ &= \lim_{\delta \rightarrow 0} \frac{\mathcal{I}_{\{\varepsilon_n\}}(v; Q'(x_\alpha^0, \delta))}{2\delta^2}. \end{aligned} \quad (3.6)$$

Lemma 3.4. $W_{\{\varepsilon_n\}}(x_\alpha^0; \bar{\xi}) \leq W_{\text{hom}}(x_\alpha^0; \bar{\xi})$ for all $x_\alpha^0 \in L$ and all $\bar{\xi} \in \mathbb{Q}^{3 \times 2}$.

Proof. Given $k \in \mathbb{N}$, let $T_k \in \mathbb{N}$ and $\varphi_k \in \mathcal{S}_{T_k}$ with $\varphi_k = 0$ on $\partial(0, T_k)^2 \times I$, be such that

$$W_{\text{hom}}(x_\alpha^0; \bar{\xi}) + \frac{1}{k} \geq \frac{1}{2T_k^2} \int_{(0, T_k)^2 \times I} W(x_\alpha^0, y_3, y_\alpha; \bar{\xi} + D_\alpha \varphi_k(y) | D_3 \varphi_k(y)) dy.$$

This is possible because of the continuity properties (H_1) of W , the growth conditions (H_3) and the density of \mathcal{S}_{T_k} in \mathcal{W}_{T_k} . Extend φ_k periodically with period T_k to $\mathbb{R}^2 \times I$. For $x \in \mathbb{R}^2 \times I$, define $u_n^k(x) := \bar{\xi} \cdot x_\alpha + \varepsilon_n \varphi_k\left(\frac{x_\alpha}{\varepsilon_n}, x_3\right)$.

For fixed k , $u_n^k \rightarrow v$ in $L^p(Q'(x_\alpha^0, \delta) \times I; \mathbb{R}^3)$ as $n \rightarrow \infty$, hence, by (3.6)

$$\begin{aligned} W_{\{\varepsilon_n\}}(x_\alpha^0; \bar{\xi}) &\leq \liminf_{\delta \rightarrow 0} \liminf_{n \rightarrow +\infty} \frac{1}{2\delta^2} \int_{Q'(x_\alpha^0, \delta) \times I} W\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; D_\alpha u_n^k(x) \Big|_{\varepsilon_n} D_3 u_n^k(x)\right) dx \\ &= \liminf_{\delta \rightarrow 0} \liminf_{n \rightarrow +\infty} \frac{1}{2\delta^2} \int_{Q'(x_\alpha^0, \delta) \times I} W\left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha \varphi_k\left(\frac{x_\alpha}{\varepsilon_n}, x_3\right) \Big|_{\varepsilon_n} D_3 \varphi_k\left(\frac{x_\alpha}{\varepsilon_n}, x_3\right)\right) dx. \end{aligned}$$

Define

$$h_k(x_\alpha, y_\alpha) := \int_{-1}^1 W(x_\alpha, x_3, T_k y_\alpha; \bar{\xi} + D_\alpha \varphi_k(T_k y_\alpha, x_3) | D_3 \varphi_k(T_k y_\alpha, x_3)) dx_3,$$

for a.e. $x_\alpha \in \omega$ and $y_\alpha \in \mathbb{R}^2$. The continuity of W with respect to y_α , its measurability and periodicity properties, and the fact that $T_k \in \mathbb{N}$ lead us to conclude that the function $h_k \in L^1(Q'(x_\alpha^0, \delta); \mathcal{C}_{\text{per}}(Q'))$ for fixed $\delta > 0$, where $\mathcal{C}_{\text{per}}(Q')$ denotes the space of Q' -periodic and continuous functions defined on \mathbb{R}^2 (see Lemma 5.3 in Allaire [3]). Lemma 5.2 in [3] together with Fubini's Theorem yields to

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_{Q'(x_\alpha^0, \delta) \times I} W \left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha \varphi_k \left(\frac{x_\alpha}{\varepsilon_n}, x_3 \right) \middle| D_3 \varphi_k \left(\frac{x_\alpha}{\varepsilon_n}, x_3 \right) \right) dx \\ &= \lim_{n \rightarrow +\infty} \int_{Q'(x_\alpha^0, \delta)} h_k \left(x_\alpha, \frac{x_\alpha}{T_k \varepsilon_n} \right) dx_\alpha \\ &= \int_{Q'(x_\alpha^0, \delta)} \int_{Q'} h_k(x_\alpha, y_\alpha) dy_\alpha dx_\alpha \\ &= \int_{Q'(x_\alpha^0, \delta)} \int_{Q' \times I} W(x_\alpha, x_3, T_k y_\alpha; \bar{\xi} + D_\alpha \varphi_k(T_k y_\alpha, x_3) | D_3 \varphi_k(T_k y_\alpha, x_3)) dy_\alpha dx_3 dx_\alpha. \end{aligned} \quad (3.7)$$

Using (3.5) and passing to the limit in (3.7), as $\delta \rightarrow 0$, we have that

$$\begin{aligned} & W_{\{\varepsilon_n\}}(x_\alpha^0; \bar{\xi}) \\ & \leq \frac{1}{2} \int_{Q' \times I} W(x_\alpha^0, x_3, T_k y_\alpha; \bar{\xi} + D_\alpha \varphi_k(T_k y_\alpha, x_3) | D_3 \varphi_k(T_k y_\alpha, x_3)) dy_\alpha dx_3 \\ & \leq W_{\text{hom}}(x_\alpha^0; \bar{\xi}) + \frac{1}{k}. \end{aligned}$$

Letting $k \rightarrow +\infty$ we assert the claim. ■

Note that the same proof could be used to prove Lemma 2.5 in Babadjian and Francfort [5].

Lemma 3.5. $W_{\{\varepsilon_n\}}(x_\alpha^0; \bar{\xi}) \geq W_{\text{hom}}(x_\alpha^0; \bar{\xi})$ for all $x_\alpha^0 \in L$ and all $\bar{\xi} \in \mathbb{Q}^{3 \times 2}$.

Proof. Let $\{v_n\} \subset W^{1,p}(Q'(x_\alpha^0, \delta) \times I; \mathbb{R}^3)$ be a recovering sequence for the Γ -limit, i.e.

$$v_n \rightarrow 0 \text{ in } L^p(Q'(x_\alpha^0, \delta) \times I; \mathbb{R}^3)$$

and

$$\mathcal{I}_{\{\varepsilon_n\}}(v; Q'(x_\alpha^0, \delta)) = \lim_{n \rightarrow +\infty} \int_{Q'(x_\alpha^0, \delta) \times I} W \left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha v_n(x) \middle| \frac{1}{\varepsilon_n} D_3 v_n(x) \right) dx.$$

According to Theorem 1.1 in Bocea and Fonseca [7], there exists a subsequence of $\{\varepsilon_n\}$ (not relabelled) and a sequence $\{u_n\} \subset W^{1,p}(Q'(x_\alpha^0, \delta) \times I; \mathbb{R}^3)$ such that, upon setting $E_n := \{x \in Q'(x_\alpha^0, \delta) \times I : u_n(x) = v_n(x)\}$, we have that

$$\left\{ \begin{array}{l} u_n \rightarrow 0 \text{ in } L^p(Q'(x_\alpha^0, \delta) \times I; \mathbb{R}^3), \\ \left\{ \left| \left(D_\alpha u_n \middle| \frac{1}{\varepsilon_n} D_3 u_n \right) \right|^p \right\} \text{ is equi-integrable,} \\ \lim_{n \rightarrow +\infty} \mathcal{L}^3([Q'(x_\alpha^0, \delta) \times I] \setminus E_n) = 0. \end{array} \right. \quad (3.8)$$

Thus, in view of the p -growth condition (H_3) together with (3.8) and Remark 2.1,

$$\begin{aligned}
\mathcal{I}_{\{\varepsilon_n\}}(v; Q'(x_\alpha^0, \delta)) &\geq \limsup_{n \rightarrow +\infty} \int_{E_n} W \left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n \middle| \frac{1}{\varepsilon_n} D_3 u_n \right) dx \\
&= \limsup_{n \rightarrow +\infty} \int_{Q'(x_\alpha^0, \delta) \times I} W \left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n \middle| \frac{1}{\varepsilon_n} D_3 u_n \right) dx \\
&\quad - \limsup_{n \rightarrow +\infty} \int_{[Q'(x_\alpha^0, \delta) \times I] \setminus E_n} W \left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n \middle| \frac{1}{\varepsilon_n} D_3 u_n \right) dx \\
&\geq \limsup_{n \rightarrow +\infty} \int_{Q'(x_\alpha^0, \delta) \times I} W \left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n \middle| \frac{1}{\varepsilon_n} D_3 u_n \right) dx.
\end{aligned}$$

For any $h \in \mathbb{N}$, we split $Q'(x_\alpha^0, \delta)$ into h^2 disjoint cubes $Q'_{i,h}$ of side length δ/h so that $Q'(x_\alpha^0, \delta) = \bigcup_{i=1}^{h^2} Q'_{i,h}$ and

$$\mathcal{I}_{\{\varepsilon_n\}}(v; Q'(x_\alpha^0, \delta)) \geq \limsup_{h \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \int_{Q'_{i,h} \times I} W \left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n \middle| \frac{1}{\varepsilon_n} D_3 u_n \right) dx. \quad (3.9)$$

For every $\eta > 0$ and $\lambda > 0$, let $K_\eta \subset \Omega$ and $W^{\eta, \lambda}$ be given by Lemma 4.1 below (with $N = d = 3$, $m = 2$ and $f = W$). Then

$$\mathcal{L}^3(\Omega \setminus K_\eta) < \eta. \quad (3.10)$$

On the other hand, define

$$R_n^\lambda := \left\{ x \in Q'(x_\alpha^0, \delta) \times I : \left| \left(\bar{\xi} + D_\alpha u_n(x) \middle| \frac{1}{\varepsilon_n} D_3 u_n(x) \right) \right| \leq \lambda \right\}.$$

Chebyshev's inequality implies that there exists a constant $C > 0$ – which does not depend on n or λ – such that

$$\mathcal{L}^3([Q'(x_\alpha^0, \delta) \times I] \setminus R_n^\lambda) < \frac{C}{\lambda^p}. \quad (3.11)$$

In what follows we denote by $\limsup_{\lambda, \eta, h, n}$ the successive $\limsup_{\lambda \rightarrow +\infty} \limsup_{\eta \rightarrow 0} \limsup_{h \rightarrow +\infty} \limsup_{n \rightarrow +\infty}$. Since W and $W^{\eta, \lambda}$ coincide on $K_\eta \times \mathbb{R}^2 \times \bar{B}(0, \lambda)$, where in the sequel the set $\bar{B}(0, \lambda)$ stands for the closed ball $\{\xi \in \mathbb{R}^{3 \times 3} : |\xi| \leq \lambda\}$ of $\mathbb{R}^{3 \times 3}$, we get in view of (3.9)

$$\begin{aligned}
&\mathcal{I}_{\{\varepsilon_n\}}(v; Q'(x_\alpha^0, \delta)) \geq \\
&\limsup_{\lambda, \eta, h, n} \sum_{i=1}^{h^2} \int_{[Q'_{i,h} \times I] \cap R_n^\lambda \cap K_\eta} W^{\eta, \lambda} \left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n \middle| \frac{1}{\varepsilon_n} D_3 u_n \right) dx.
\end{aligned}$$

By virtue of (4.1) below and (3.10),

$$\sum_{i=1}^{h^2} \int_{([Q'_{i,h} \times I] \cap R_n^\lambda) \setminus K_\eta} W^{\eta, \lambda} \left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n \middle| \frac{1}{\varepsilon_n} D_3 u_n \right) dx \leq \beta(1 + \lambda^p) \eta \xrightarrow{\eta \rightarrow 0} 0,$$

uniformly in (n, h) , so that

$$\begin{aligned} & \mathcal{I}_{\{\varepsilon_n\}}(v; Q'(x_\alpha^0, \delta)) \geq \\ & \limsup_{\lambda, \eta, h, n} \sum_{i=1}^{h^2} \int_{[Q'_{i,h} \times I] \cap R_n^\lambda} W^{\eta, \lambda} \left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n \Big| \frac{1}{\varepsilon_n} D_3 u_n \right) dx. \end{aligned}$$

Fix $y_\alpha \in Q'$. Since $W^{\eta, \lambda}(\cdot, y_\alpha; \cdot)$ is continuous, it is uniformly continuous on $\bar{\Omega} \times \bar{B}(0, \lambda)$, and we define the modulus of continuity $\omega_{\eta, \lambda} : Q' \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by

$$\begin{aligned} \omega_{\eta, \lambda}(y_\alpha, t) := \sup \Big\{ & |W^{\eta, \lambda}(x, y_\alpha; \xi) - W^{\eta, \lambda}(x', y_\alpha; \xi')|, \text{ where} \\ & (x, \xi), (x', \xi') \in \bar{\Omega} \times \bar{B}(0, \lambda) \text{ and } |(x; \xi) - (x'; \xi')| \leq t \Big\}. \end{aligned}$$

Then

$$\begin{cases} \omega_{\eta, \lambda}(\cdot, t) \text{ is lower semicontinuous for all } t \in \mathbb{R}^+, \\ \omega_{\eta, \lambda}(y_\alpha, \cdot) \text{ is continuous and increasing for all } y_\alpha \in Q', \\ \omega_{\eta, \lambda}(y_\alpha, 0) = 0 \text{ for all } y_\alpha \in Q', \end{cases}$$

and

$$|W^{\eta, \lambda}(x, y_\alpha; \xi) - W^{\eta, \lambda}(x', y_\alpha; \xi')| \leq \omega_{\eta, \lambda}(y_\alpha, |x - x'| + |\xi - \xi'|) \quad (3.12)$$

for all $(x, \xi), (x', \xi') \in \bar{\Omega} \times \bar{B}(0, \lambda)$. The first property is a consequence of the fact that the supremum of continuous functions is lower semicontinuous, while the other ones are classical properties of moduli of continuity.

For all $t \in \mathbb{R}^+$, we extend $\omega_{\eta, \lambda}(\cdot, t)$ to \mathbb{R}^2 by Q' -periodicity. Since $W^{\eta, \lambda}(x, \cdot; \xi)$ is Q' -periodic, inequality (3.12) holds for all $y_\alpha \in \mathbb{R}^2$. Consequently, for every $(x_\alpha, x_3) \in [Q'_{i,h} \times I] \cap R_n^\lambda$ and every $x'_\alpha \in Q'_{i,h}$,

$$\begin{aligned} & \left| W^{\eta, \lambda} \left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n(x) \Big| \frac{1}{\varepsilon_n} D_3 u_n(x) \right) \right. \\ & \quad \left. - W^{\eta, \lambda} \left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n(x) \Big| \frac{1}{\varepsilon_n} D_3 u_n(x) \right) \right| \\ & \leq \omega_{\eta, \lambda} \left(\frac{x_\alpha}{\varepsilon_n}, |x_\alpha - x'_\alpha| \right) \\ & \leq \omega_{\eta, \lambda} \left(\frac{x_\alpha}{\varepsilon_n}, \frac{\sqrt{2}\delta}{h} \right). \end{aligned}$$

After integration in $(x_\alpha, x_3, x'_\alpha)$ and summation, we get

$$\begin{aligned} & \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \int_{Q'_{i,h}} \left\{ \int_{R_n^\lambda \cap [Q'_{i,h} \times I]} \left| W^{\eta, \lambda} \left(x_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n(x) \Big| \frac{1}{\varepsilon_n} D_3 u_n(x) \right) \right. \right. \\ & \quad \left. \left. - W^{\eta, \lambda} \left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n(x) \Big| \frac{1}{\varepsilon_n} D_3 u_n(x) \right) \right| dx \right\} dx'_\alpha \\ & \leq 2 \int_{Q'(x_\alpha^0, \delta)} \omega_{\eta, \lambda} \left(\frac{x_\alpha}{\varepsilon_n}, \frac{\sqrt{2}\delta}{h} \right) dx_\alpha. \end{aligned}$$

Riemann-Lebesgue's Lemma applied to the Q' -periodic function $\omega_{\eta,\lambda}(\cdot, \sqrt{2}\delta/h)$ yields,

$$\lim_{n \rightarrow +\infty} 2 \int_{Q'(x_\alpha^0, \delta)} \omega_{\eta,\lambda} \left(\frac{x_\alpha}{\varepsilon_n}, \frac{\sqrt{2}\delta}{h} \right) dx_\alpha = 2\delta^2 \int_{Q'} \omega_{\eta,\lambda} \left(x_\alpha, \frac{\sqrt{2}\delta}{h} \right) dx_\alpha,$$

and by Beppo-Levi's Monotone Convergence Theorem

$$\lim_{h \rightarrow +\infty} 2\delta^2 \int_{Q'} \omega_{\eta,\lambda} \left(x_\alpha, \frac{\sqrt{2}\delta}{h} \right) dx_\alpha = 0.$$

Hence

$$\begin{aligned} & \mathcal{I}_{\{\varepsilon_n\}}(v; Q'(x_\alpha^0, \delta)) \geq \\ & \limsup_{\lambda, \eta, h, n} \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \int_{Q'_{i,h}} \left\{ \int_{[Q'_{i,h} \times I] \cap R_n^\lambda} W^{\eta,\lambda} \left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n \Big| \frac{1}{\varepsilon_n} D_3 u_n \right) dx \right\} dx'_\alpha. \end{aligned}$$

Define the following sets which depend on all parameters (η, λ, i, h, n) :

$$\begin{cases} T := \{(x'_\alpha, x_\alpha, x_3) \in Q'_{i,h} \times Q'_{i,h} \times I : (x'_\alpha, x_3) \in K_\eta \text{ and } (x_\alpha, x_3) \in R_n^\lambda\}, \\ T_1 := \{(x'_\alpha, x_\alpha, x_3) \in Q'_{i,h} \times Q'_{i,h} \times I : (x'_\alpha, x_3) \notin K_\eta \text{ and } (x_\alpha, x_3) \in R_n^\lambda\}, \\ T_2 := \{(x'_\alpha, x_\alpha, x_3) \in Q'_{i,h} \times Q'_{i,h} \times I : (x_\alpha, x_3) \notin R_n^\lambda\}, \end{cases}$$

and note that $Q'_{i,h} \times Q'_{i,h} \times I = T \cup T_1 \cup T_2$. Since $W(\cdot, y_\alpha; \cdot)$ and $W^{\eta,\lambda}(\cdot, y_\alpha; \cdot)$ coincide on $K_\eta \times \bar{B}(0, \lambda)$, we have

$$\begin{aligned} & \mathcal{I}_{\{\varepsilon_n\}}(v; Q'(x_\alpha^0, \delta)) \\ & \geq \limsup_{\lambda, \eta, h, n} \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \int_T W^{\eta,\lambda} \left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n(x) \Big| \frac{1}{\varepsilon_n} D_3 u_n(x) \right) dx dx'_\alpha \\ & = \limsup_{\lambda, \eta, h, n} \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \int_T W \left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n(x) \Big| \frac{1}{\varepsilon_n} D_3 u_n(x) \right) dx dx'_\alpha. \end{aligned} \quad (3.13)$$

We will prove that the corresponding terms over T_1 and T_2 are zero. Indeed, in view of (3.10) and the p -growth condition (H_3) ,

$$\begin{aligned} & \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \int_{T_1} W \left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n(x) \Big| \frac{1}{\varepsilon_n} D_3 u_n(x) \right) dx dx'_\alpha \\ & \leq \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \mathcal{L}^2(Q'_{i,h}) \mathcal{L}^3([Q'_{i,h} \times I] \setminus K_\eta) \beta(1 + \lambda^p) \\ & < \beta(1 + \lambda^p) \eta \xrightarrow{\eta \rightarrow 0} 0, \end{aligned} \quad (3.14)$$

uniformly in (n, h) . The bound from above in (H_3) , the equi-integrability of $\left\{ \left| \left(D_\alpha u_n \Big|_{\frac{1}{\varepsilon_n}} D_3 u_n \right) \right|^p \right\}$ and (3.11) imply that

$$\begin{aligned}
& \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \int_{T_2} W \left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n(x) \Big|_{\frac{1}{\varepsilon_n}} D_3 u_n(x) \right) dx dx'_\alpha \\
& \leq \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \mathcal{L}^2(Q'_{i,h}) \beta \int_{[Q'_{i,h} \times I] \setminus R_n^\lambda} \left(1 + \left| \left(D_\alpha u_n(x) \Big|_{\frac{1}{\varepsilon_n}} D_3 u_n(x) \right) \right|^p \right) dx \\
& = \beta \int_{[Q'(x_\alpha^0, \delta) \times I] \setminus R_n^\lambda} \left(1 + \left| \left(D_\alpha u_n(x) \Big|_{\frac{1}{\varepsilon_n}} D_3 u_n(x) \right) \right|^p \right) dx \xrightarrow{\lambda \rightarrow +\infty} 0, \tag{3.15}
\end{aligned}$$

uniformly in (η, n, h) . Thus, in view of (3.13), (3.14), (3.15), Fatou's Lemma yields

$$\begin{aligned}
& \mathcal{I}_{\{\varepsilon_n\}}(v; Q'(x_\alpha^0, \delta)) \\
& \geq \limsup_{h \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \int_{Q'_{i,h}} \int_{Q'_{i,h} \times I} W \left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n \Big|_{\frac{1}{\varepsilon_n}} D_3 u_n \right) dx dx'_\alpha \\
& \geq \limsup_{h \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \int_{Q'_{i,h}} \liminf_{n \rightarrow +\infty} \int_{Q'_{i,h} \times I} W \left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n \Big|_{\frac{1}{\varepsilon_n}} D_3 u_n \right) dx dx'_\alpha.
\end{aligned}$$

Fix $x'_\alpha \in Q'_{i,h}$ such that $W_{\text{hom}}(x'_\alpha; \bar{\xi})$ is well defined and set $Z(x; \xi) := W(x'_\alpha, x_3, x_\alpha; \xi)$. It is easy to check that Z is a Carathéodory integrand hence, applying Theorem 4.2 of Braides, Fonseca and Francfort [11], we get since $u_n \rightarrow 0$ in $L^p(Q'(x_\alpha^0, \delta) \times I; \mathbb{R}^3)$,

$$2 \frac{\delta^2}{h^2} \bar{Z}(\bar{\xi}) \leq \liminf_{n \rightarrow +\infty} \int_{Q'(x_\alpha^0, \delta) \times I} Z \left(\frac{x_\alpha}{\varepsilon_n}, x_3; \bar{\xi} + D_\alpha u_n(x) \Big|_{\frac{1}{\varepsilon_n}} D_3 u_n(x) \right) dx,$$

where

$$\begin{aligned}
\bar{Z}(\bar{\xi}) & := \inf_{T>0, \varphi} \left\{ \int_{(0,T)^2 \times I} Z(x; \bar{\xi} + D_\alpha \varphi(x) | D_3 \varphi(x)) dx : \right. \\
& \quad \left. \varphi \in W^{1,p}((0,T)^2 \times I; \mathbb{R}^3), \quad \varphi = 0 \text{ on } \partial(0,T)^2 \times I \right\}.
\end{aligned}$$

In view of the previous formula together with (1.4) and Remark 2.3, we have that $\bar{Z}(\bar{\xi}) = W_{\text{hom}}(x'_\alpha; \bar{\xi})$. Then

$$\liminf_{n \rightarrow +\infty} \int_{Q'_{i,h} \times I} W \left(x'_\alpha, x_3, \frac{x_\alpha}{\varepsilon_n}; \bar{\xi} + D_\alpha u_n(x) \Big|_{\frac{1}{\varepsilon_n}} D_3 u_n(x) \right) dx \geq \frac{2\delta^2}{h^2} W_{\text{hom}}(x'_\alpha; \bar{\xi}),$$

and so

$$\begin{aligned}
\mathcal{I}_{\{\varepsilon_n\}}(v; Q'(x_\alpha^0, \delta)) & \geq \limsup_{h \rightarrow +\infty} \sum_{i=1}^{h^2} \frac{h^2}{\delta^2} \int_{Q'_{i,h}} \frac{2\delta^2}{h^2} W_{\text{hom}}(x'_\alpha; \bar{\xi}) dx'_\alpha \\
& = 2 \int_{Q'(x_\alpha^0, \delta)} W_{\text{hom}}(x'_\alpha; \bar{\xi}) dx'_\alpha.
\end{aligned}$$

Dividing both sides of the previous inequality by δ^2 and passing to the limit when $\delta \searrow 0^+$, we obtain by (3.4) and (3.6)

$$W_{\{\varepsilon_n\}}(x_\alpha^0; \bar{\xi}) \geq W_{\text{hom}}(x_\alpha^0; \bar{\xi}).$$

■

Proposition 3.6. $W_{\{\varepsilon_n\}}(x_\alpha; \bar{\xi}) = W_{\text{hom}}(x_\alpha; \bar{\xi})$ a.e. $x_\alpha \in \omega$ and all $\bar{\xi} \in \mathbb{R}^{3 \times 2}$.

Proof. Let E be the intersection of the set L (see Definition 3.3) with the subset of points $x_\alpha^0 \in \omega$ where $W_{\{\varepsilon_n\}}(x_\alpha^0; \cdot)$ and $W_{\text{hom}}(x_\alpha^0; \cdot)$ are continuous (see Lemma 2.6). Then $\mathcal{L}^2(\omega \setminus E) = 0$ and in view of Lemma 3.4 and 3.5, we have that for all $x_\alpha^0 \in E$ and for all $\bar{\xi} \in \mathbb{Q}^{3 \times 2}$,

$W_{\{\varepsilon_n\}}(x_\alpha^0; \bar{\xi}) = W_{\text{hom}}(x_\alpha^0; \bar{\xi})$. Since $W_{\{\varepsilon_n\}}(x_\alpha^0; \cdot)$ and $W_{\text{hom}}(x_\alpha^0; \cdot)$ are continuous for all $x_\alpha^0 \in E$, the equality $W_{\{\varepsilon_n\}}(x_\alpha^0; \bar{\xi}) = W_{\text{hom}}(x_\alpha^0; \bar{\xi})$ holds true for $x_\alpha^0 \in E$ and all $\bar{\xi} \in \mathbb{R}^{3 \times 2}$.

■

Corollary 3.7. $\Gamma(L^p(A \times I))$ - $\lim_{\varepsilon} \mathcal{I}_\varepsilon(\cdot; A) = \mathcal{I}_{\text{hom}}(\cdot; A)$ for all $A \in \mathcal{A}(\omega)$, where $\mathcal{I}_{\text{hom}}(\cdot; A)$ is the functional defined in (3.1).

Proof. From Proposition 3.6 we can conclude that $\mathcal{I}_{\text{hom}}(\cdot; A)$ is well defined and

$$\Gamma(L^p(A \times I))\text{-} \lim_n \mathcal{I}_{\varepsilon_n}(\cdot; A) = \mathcal{I}_{\text{hom}}(\cdot; A)$$

for all $A \in \mathcal{A}(\omega)$ (see Remark 3.1). Since this limit does not depend upon the extracted subsequence, in view of Proposition 7.11 in Braides and Defranceschi [10], the whole sequence $\{\mathcal{I}_\varepsilon(\cdot; A)\}_{\varepsilon > 0}$ $\Gamma(L^p(A \times I))$ -converges to $\mathcal{I}_{\text{hom}}(\cdot; A)$ for each $A \in \mathcal{A}(\omega)$.

■

The proof of Theorem 1.2 comes as a consequence of Corollary 3.7 taking $A = \omega$.

4 Appendix

We now prove a technical result on extension of Carathéodory functions that was useful in the proof of Lemma 3.5. The argument used is very close to that of Theorem 1, Section 1.2 in Evans and Gariepy [16].

Lemma 4.1. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{d \times N} \rightarrow \mathbb{R}$ a function such that

$$\left\{ \begin{array}{l} f(x, \cdot; \cdot) \text{ is continuous for a.e. } x \in \Omega; \\ f(\cdot; y; \xi) \text{ is } \mathcal{L}^N\text{-measurable for all } y \in \mathbb{R}^m \text{ and } \xi \in \mathbb{R}^{d \times N}; \\ f(x, \cdot; \xi) \text{ is } (0, 1)^m\text{-periodic for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^{d \times N}. \end{array} \right.$$

Assume also that there exists $\beta > 0$ and $1 \leq p < \infty$ such that

$$\frac{1}{\beta} |\xi|^p - \beta \leq f(x, y; \xi) \leq \beta(1 + |\xi|^p), \quad \text{for a.e. } x \in \Omega \text{ and all } (y, \xi) \in \mathbb{R}^m \times \mathbb{R}^{d \times N}.$$

Then for any $\eta > 0$ and $\lambda > 0$ there exist a compact set $K_\eta \subset \Omega$ and a function $f^{\eta, \lambda} : \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^{d \times N} \rightarrow \mathbb{R}$ such that

$$\left\{ \begin{array}{l} \mathcal{L}^N(\Omega \setminus K_\eta) < \eta, \\ f^{\eta,\lambda}(x, y; \xi) = f(x, y; \xi) \text{ for all } (x, y, \xi) \in K_\eta \times \mathbb{R}^m \times \overline{B}(0, \lambda), \\ f^{\eta,\lambda}(\cdot, y; \cdot) \text{ is continuous for all } y \in \mathbb{R}^m, \\ f^{\eta,\lambda}(x, \cdot; \xi) \text{ is continuous and } (0, 1)^m\text{-periodic for all } (x, \xi) \in \mathbb{R}^N \times \mathbb{R}^{d \times N}, \end{array} \right.$$

and

$$-\beta \leq f^{\eta,\lambda}(x, y; \xi) \leq \beta(1 + \lambda^p), \quad \text{for all } (x, y, \xi) \in \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^{d \times N}. \quad (4.1)$$

Proof. Since f is a Carathéodory function, by Scorza Dragoni's Theorem (see Ekeland and Teman [15]) for all $\eta > 0$ there exists a compact set $K_\eta \subset \Omega$ satisfying $\mathcal{L}^N(\Omega \setminus K_\eta) < \eta$ and such that f is continuous on $K_\eta \times \mathbb{R}^m \times \mathbb{R}^{d \times N}$. Let $C^{\eta,\lambda} := K_\eta \times \overline{B}(0, \lambda) \equiv C$ (to simplify notation) and $U^{\eta,\lambda} := (\mathbb{R}^N \times \mathbb{R}^{d \times N}) \setminus C^{\eta,\lambda} \equiv U$. Fix $(s, \gamma) \in C$, and for all $(x, \xi) \in U$ set

$$u_{(s,\gamma)}^{\eta,\lambda}(x, \xi) := \max \left\{ 2 - \frac{|(s, \gamma) - (x, \xi)|}{\text{dist}((x, \xi), C)}, 0 \right\} \equiv u_{(s,\gamma)}(x, \xi).$$

Clearly

$$\left\{ \begin{array}{l} u_{(s,\gamma)} \text{ is continuous on } U, \\ 0 \leq u_{(s,\gamma)} \leq 1, \\ u_{(s,\gamma)}(x, \xi) = 0 \text{ if and only if } |(s, \gamma) - (x, \xi)| \geq 2\text{dist}((x, \xi), C). \end{array} \right.$$

Let $\{s_j^\eta\}_{j \geq 1} \equiv \{s_j\}_{j \geq 1}$ and $\{\gamma_j^\lambda\}_{j \geq 1} \equiv \{\gamma_j\}_{j \geq 1}$ be countable dense families in K_η and $\overline{B}(0, \lambda)$, respectively. Define

$$\sigma^{\eta,\lambda}(x, \xi) := \sum_{j \geq 1} 2^{-j} u_{(s_j, \gamma_j)}(x, \xi) \equiv \sigma(x, \xi) \quad \text{for all } (x, \xi) \in U.$$

Since σ is the uniform limit of a sequence of continuous functions in U , then σ is continuous in U . Moreover, for all $(x, \xi) \in U$

$$0 < \sigma(x, \xi) \leq 1.$$

Indeed, assume that $\sigma(x, \xi) = 0$ for some $(x, \xi) \in U$. Then, for all $j \geq 1$, $u_{(s_j, \gamma_j)}(x, \xi) = 0$ and thus $|(s_j, \gamma_j) - (x, \xi)| \geq 2\text{dist}((x, \xi), C)$. The density of $\{(s_j, \gamma_j)\}$ in C yields that

$$|(s, \gamma) - (x, \xi)| \geq 2\text{dist}((x, \xi), C)$$

for all $(s, \gamma) \in C$. We obtain a contradiction if we choose (s, γ) to be those points of C such that $\text{dist}((x, \xi), C) = \text{dist}((x, \xi), (s, \gamma))$ so $\sigma(x, \xi) > 0$ for all $(x, \xi) \in U$. Consequently, the function

$$(x, \xi) \mapsto v_k(x, \xi) \equiv v_k^{\eta,\lambda}(x, \xi) := \frac{2^{-k} u_{(s_k, \gamma_k)}(x, \xi)}{\sigma(x, \xi)}$$

is well defined and continuous in U . Moreover it satisfies that

$$0 \leq v_k(x, \xi) \leq 1, \quad \sum_{k \geq 1} v_k(x, \xi) = 1 \quad \text{for all } (x, \xi) \in U.$$

Fix $y \in \mathbb{R}^m$ and define the continuous extension of $f(\cdot, y; \cdot)$ outside C as

$$f^{\eta, \lambda}(x, y; \xi) = \begin{cases} f(x, y, \xi) & \text{if } (x, \xi) \in C, \\ \sum_{k \geq 1} v_k(x, \xi) f(s_k, y; \gamma_k) & \text{if } (x, \xi) \in U. \end{cases}$$

Obviously, we have $f^{\eta, \lambda}(x, y; \xi) = f(x, y, \xi)$ for all $(x, y, \xi) \in K_\eta \times \mathbb{R}^m \times \overline{B}(0, \lambda)$. On the other hand, if (x, y, ξ) is such that $(x, \xi) \in U$, in view of the p -growth and the p -coercivity condition on f we get

$$-\beta \leq f^{\eta, \lambda}(x, y; \xi) \leq \sum_{k \geq 1} v_k(x, \xi) \beta (1 + |\gamma_k|^p) \leq \beta (1 + \lambda^p).$$

Since we have

$$\sup_{y \in \mathbb{R}^m, (x, \xi) \in U} \left[\sum_{k \geq n} |2^{-k} u_{(s_k, \gamma_k)}(x, \xi) f(s_k, y; \gamma_k)| \right] \leq \beta (1 + \lambda^p) \sum_{k \geq n} 2^{-k} \xrightarrow{n \rightarrow +\infty} 0, \quad (4.2)$$

then the function

$$(x, y, \xi) \mapsto \sum_{k \geq 1} 2^{-k} u_{(s_k, \gamma_k)}(x, \xi) f(s_k, y; \gamma_k)$$

is continuous on $\{(x, y, \xi) : (x, \xi) \in U, y \in \mathbb{R}^m\}$. In particular, for all $(x, \xi) \in \mathbb{R}^N \times \mathbb{R}^{d \times N}$ the function $f^{\eta, \lambda}(x, \cdot; \xi)$ is continuous. Furthermore, $f^{\eta, \lambda}(x, \cdot; \xi)$ is $(0, 1)^m$ -periodic because if $\mathbf{i} \in \mathbb{Z}^m$ then for $(x, \xi) \in U$

$$f^{\eta, \lambda}(x, y + \mathbf{i}; \xi) = \sum_{k \geq 1} v_k(x, \xi) f(s_k, y + \mathbf{i}; \gamma_k) = \sum_{k \geq 1} v_k(x, \xi) f(s_k, y; \gamma_k) = f^{\eta, \lambda}(x, y; \xi).$$

Finally we prove the continuity of $f^{\eta, \lambda}(\cdot, y; \cdot)$. By (4.2) it suffices to show that for all $(a, A) \in C$

$$\lim_{U \ni (x, \xi) \rightarrow (a, A)} f^{\eta, \lambda}(x, y; \xi) = f(a, y; A).$$

As $\{(s_j, \gamma_j)\}_{j \geq 1}$ is dense in C and $f(\cdot, y; \cdot)$ is continuous on C , for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(a, y; A) - f(s_j, y; \gamma_j)| < \varepsilon$ for all $j \geq 1$ with $|(a, A) - (s_j, \gamma_j)| < \delta$. Assume that $|(x, \xi) - (a, A)| < \delta/4$ and suppose that $j \geq 1$ is such that $|(a, A) - (s_j, \gamma_j)| \geq \delta$. Then

$$\delta \leq |(a, A) - (s_j, \gamma_j)| \leq |(a, A) - (x, \xi)| + |(x, \xi) - (s_j, \gamma_j)| \leq \frac{\delta}{4} + |(x, \xi) - (s_j, \gamma_j)|,$$

and thus

$$|(x, \xi) - (s_j, \gamma_j)| \geq \frac{3\delta}{4} > 2|(a, A) - (x, \xi)| \geq 2 \operatorname{dist}((x, \xi), C).$$

Consequently, $v_j(x, \xi) = 0$ if j is such that $|(a, A) - (s_j, \gamma_j)| \geq \delta$, and so

$$|f^{\eta, \lambda}(x, y; \xi) - f(a, y; A)| \leq \sum_{j \geq 1, |(a, A) - (s_j, \gamma_j)| < \delta} v_j(x, \xi) |f(s_j, y; \gamma_j) - f(a, y; A)| < \varepsilon,$$

because non zero terms of the sum are those which satisfy $|f(a, y; A) - f(s_j, y; \gamma_j)| < \varepsilon$. The continuity of $f^{\eta, \lambda}(\cdot, y; \cdot)$ now follows. \blacksquare

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Jean-François Babadjian
L.P.M.T.M., UNIVERSITÉ PARIS NORD, 93430, VILLETANEUSE, FRANCE
E-mail address: jfb@galilee.univ-paris13.fr

Margarida Baía
DEPARTAMENTO DE MATEMÁTICA, INSTITUTO SUPERIOR TÉCNICO,
1049-001 LISBOA, PORTUGAL
E-mail address: mbaia@math.ist.utl.pt