Γ-convergence of functionals with periodic integrands via 2-scale convergence

M. Baía and I. Fonseca

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Abstract

Γ-convergence techniques combined with techniques of 2-scale convergence are used to give a characterization of the behavior as ε goes to zero of a family of integral functionals defined on $L^p(\Omega;\mathbb{R}^d)$ by

$$\mathcal{I}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\left(x, \frac{x}{\varepsilon}, \nabla u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases}$$

under periodicity (and nonconvexity) hypothesis, standard p-coercivity and p-growth conditions with $p > 1$. Precisely it is assumed that the integrand $f = f(x, y, \xi)$ is measurable in x, continuous with respect to the pair (y, ξ), and Q-periodic as a function of the variable y. Uniform continuity with respect to the x variable, as it is customary in the existing literature, is not required.

Keywords: integral functionals, periodic integrands, Γ-convergence, 2-scale convergence, quasiconvexity, equi-integrability

MSC 2000 ($\mathcal{AMS}$): $35E99, 35M10, 49J45, 74G65$

1 Introduction and main result

The asymptotic analysis of ordinary or partial differential (systems of) equations with oscillating and periodic coefficients was first started using asymptotic expansions (see Bensoussan, Lions and Papanicolau [8], Jikov, Kozlov and Oleinik [38] and Sanchez-Palencia [50]), and it evolved toward more general situations through the concepts of G-convergence introduced by Spagnolo (see [51]), of H-convergence due to Murat and Tartar (see [47] and [52]), of Γ-convergence due to De Giorgi (see [26] and [28]), and of 2-scale convergence introduced by Nguetseng (see [41], [48] and [49]), further developed by Allaire and Briane (see [2] and [3]) and generalized by many other authors.

From a variational point of view the asymptotic analysis or homogenization of integral functionals, as it is referred in the literature, may rest on the study of the equilibrium states, or minimizers, of a family of functionals of the type

$$\mathcal{I}_\varepsilon(u) = \int_\Omega f_\varepsilon(x, \nabla u(x))\, dx,$$

where the functions f_ε are increasingly oscillating in the first variable as the parameter ε goes to zero, Ω is an open bounded set in $\mathbb{R}^N$ with $N \geqslant 1$, and u is a scalar or vector-valued function in some Sobolev space. The homogenization of these functionals leads to a "homogenized" functional $\mathcal{I}_{\text{hom}}$ such that a

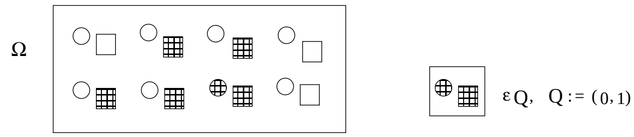


Figure 1: Typical example

sequence of minimizers u_ε of $\mathcal{I}_\varepsilon$ converges, as ε goes to zero, to a limit u, which in turn, is a minimizer of the functional $\mathcal{I}_{\text{hom}}$. Hence $\mathcal{I}_{\text{hom}}$ captures the limiting behavior of equilibria, and usually the main goal of these techniques is to obtain explicit characterizations of this functional, in particular to reach an integral representation formula of the type

$$\mathcal{I}_{\text{hom}} = \int_\Omega f_{\text{hom}}(x, \nabla u(x))\, dx,$$

where the effective energy density f_{hom} is to be determined.

Our aim here is to characterize the behavior as ε tends to zero of the family of functionals defined on $L^p(\Omega;\mathbb{R}^d)$ by

$$\mathcal{I}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\left(x, \frac{x}{\varepsilon}, \nabla u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise}, \end{cases} \tag{1.1}$$

under periodicity (and nonconvexity) hypothesis and p-coercivity and p-growth conditions with $p > 1$. The term $\mathcal{I}_\varepsilon(u)$ can be interpreted as the energy under a deformation u of an elastic body whose microstructure is periodic of period ε (Figure. 1). We seek to approximate in a Γ-convergence sense the microscopic behavior of this kind of material by a macroscopic, or average, description. We combine a Γ-limit argument with techniques of 2-scale convergence.

In the sequel given Ω an open bounded set in $\mathbb{R}^N$ with $N \geqslant 1$, we define $\mathcal{A}(\Omega)$ as the family of all its open subsets, and the notation $\Gamma(L^p(\Omega))$-limit stands for the Γ-convergence with respect to the usual metric in $L^p(\Omega;\mathbb{R}^d)$ $(d \geqslant 1)$. The space $\mathbb{R}^{d\times N}$ will be identified with the set of real $d \times N$ matrices and $Q := (0,1)^N$ stands for the unit cube in $\mathbb{R}^N$. Given $c \in \mathbb{R}$, we denote its integer part by $|[c]|$ and throughout the text C and β will represent generic constants.

Functionals of the type (1.1) have been already studied by many authors within the Sobolev and BV settings. In a Sobolev framework the main references for functionals of the form

$$\int_\Omega f\Big(\frac{x}{\varepsilon}, \nabla u(x)\Big)\, dx$$

are Marcellini [43] and Carbone-Sbordone [20] for the scalar case, where it is assumed that $f = f(y,\xi)$ is measurable, Q-periodic with respect to y, and convex with respect to ξ (see also Cioranescu, Damlamian and De Arcangelis [22]). In the vectorial (and nonconvex) case we refer the reader to Müller [46] and Braides [10]. In [9] Braides studied functionals of the form

$$\int_\Omega f\Big(\, x, \frac{x}{\varepsilon}, u(x), \nabla u(x)\,\Big)\, dx,$$

for scalar-valued u under the assumptions that the integrand $f = f(x,y,s,\xi)$ is convex in ξ, and that there exist a locally integrable function b and a continuous positive real function ω, with $\omega(0) = 0$, such that

$$|f(x,y,s,\xi) - f(x',y,s',\xi)| \leqslant \omega(|x-x'| + |s-s'|)(b(y) + f(x,y,s,\xi)) \tag{1.2}$$

for all x, x', $y \in \mathbb{R}^N$ and s, $s' \in \mathbb{R}^d$. In addition f is assumed to be measurable, Q-periodic with respect to y, and continuous with respect to the variables x and s.

A sketch of the proof of an analogous result in the vectorial setting for $f = f(x,y,\xi)$ can be found in Braides and Defranceschi [13] (convex and nonconvex case) and also in Braides and Lukkassen [15] (convex case).

The great advantage of using 2-scale convergence techniques is that it allows us to substantially weaken the continuity hypothesis (1.2) required in the previous works. We treat the nonconvex case assuming that the integrand $f = f(x,y,\xi)$ is continuous with respect to the pair (y,ξ), measurable in x, and Q-periodic as a function of y. We note that the improvement in (1.2) is done at the expense of requiring continuity in the variable y, as opposed to only measurability as in [9], [13] and [15]. Precisely, we prove in Section 4 the following result.

Theorem 1.1. *Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ be a function such that*

(H_1) *$f(x,\cdot,\cdot)$ is continuous a.e. $x \in \Omega$;*

(H_2) *$f(\cdot,y,\xi)$ is measurable for all $y \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{d\times N}$;*

(H_3) *$f(x,\cdot,\xi)$ is Q-periodic for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^{d\times N}$;*

(H_4) *there exist a real number $p > 1$ and a constant $\alpha > 0$ such that*

$$\frac{|\xi|^p}{\alpha} - \alpha \leqslant f(x,y,\xi) \leqslant \alpha(1 + |\xi|^p),$$

for a.e. $x \in \Omega$, all $y \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{d\times N}$.

For each $\varepsilon > 0$ define the functional $\mathcal{I}_\varepsilon : L^p(\Omega;\mathbb{R}^d) \to [0,\infty]$ by

$$\mathcal{I}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\left(x, \frac{x}{\varepsilon}, \nabla u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.3}$$

If $u \in L^p(\Omega;\mathbb{R}^d)$ then

$$\mathcal{I}_{\text{hom}}(u) := \Gamma(L^p(\Omega))\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega f_{\text{hom}}(x, \nabla u(x))\, dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.4}$$

where the integrand f_{hom} is given by

$$f_{\text{hom}}(x,\xi) := \lim_{T\to+\infty} \inf_\phi \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(x,y,\xi + \nabla\phi(y))\, dy,\ \phi \in W_0^{1,p}((0,T)^N;\mathbb{R}^d) \right\} \tag{1.5}$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^{d\times N}$. It turns out that f_{hom} is (equivalent to) a Carathéodory function and satisfies similar p-coercivity and p-growth conditions than those of f. Moreover $f_{\text{hom}}(x,\cdot)$ is quasiconvex for a.e. $x \in \Omega$.

The proof of Theorem 1.1 follows by the direct method of Γ-convergence. The existence of Γ-converging (sub)sequences to an abstract integral functional follows by the integral representation theorem of Buttazzo and Dal Maso (Theorem 2.10). Arguments of two-scale convergence are used to derive an upper bound for the integrand of this functional. To get the other bound we use the fact that, under hypotheses (H_1)-(H_4), the integrand f is "uniformly continuous up to a small error". Indeed, since f is a Carathéodory integrand, Scorza-Dragoni's Theorem (Theorem 2.8) implies that the restriction of f to $K \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$ is continuous, for some compact set $K \subset \Omega$ whose complement has arbitrarily small Lebesgue measure. Then the periodicity of f with respect to its second variable leads f to be uniformly continuous on $K \times \mathbb{R}^N \times \overline{B}$, for some closed ball $\overline{B}$ of $\mathbb{R}^{d\times N}$ of sufficiently large radius. Finally, to ensure that the energy remains arbitrarily small on the complement of K and on the set of x's such that the gradient of the deformation does not belong to $\overline{B}$, we use the Decomposition Lemma (Theorem 2.19) which allows us to select minimizing sequences with p^{th}-equi-integrable gradients. Thus, in view of the p-growth character of the integrand, the energy over sets of arbitrarily small Lebesgue measure tends to zero.

Like for quasiconvex envelopes, there are very few explicit examples of homogenized densities in the literature. A classical explicit derivation of the function $f_{\hom}$ for elliptic operators in the homogeneous case, that is, for integrands f that do not depend on the variable x, can be found in De Giorgi and Spagnolo [30]. Other classical examples in this homogeneous case can be found in the book of Dal Maso [24] (see references therein).

Remark 1.2. By hypothesis (H_1) and (H_2) the integrand f is of Carathéodory-type and this establishes that $f(x,\cdot,\cdot)$ is a Borel function for a.e. $x \in \Omega$ (see Proposition 2.9; in particular the integral in (1.3) is well defined). Moreover, by hypothesis (H_4) replacing f by $f+\alpha$ we may assume that f is nonnegative almost everywhere. As a consequence of this two remarks, in the sequel without loss of generality we may assume that f is a positive Borel function such that hypothesis (H_1), (H_3) and (H_4) hold for every $(x,y,\xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$.

As noted by Allaire (see Section 5 in [2]) the continuity of f in at least one of the variables x or y turns out to be sufficient to guarantee the measurability of $x \to f(x, \frac{x}{\varepsilon}; \xi)$ for fixed $\xi \in \mathbb{R}^{d\times N}$. From the applications point of view it will be interesting to prove an analogous of Theorem 1.1 for functions that are continuous with respect to the first variable and measurable with respect to the others ones (problem of mixtures), this case being addressed in a future report.

For $p = 1$ our argument fails to characterize this Γ-limit for $u \in W^{1,1}(\Omega;\mathbb{R}^d)$ either with the strong L^1-topology (see Step 2 of the proof of Proposition 4.6) or with the weak $W^{1,p}$-topology (see the proof of Lemma 4.2 and Step 2 of the proof of Proposition 4.6).

Two more remarks are worthy of note (see [46] and [13]; see also Lemma 3.2 and Lemma 3.8 below). First, it can be seen that for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^{d\times N}$

$$f_{\hom}(x,\xi) = \inf_{T\in\mathbb{N}} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(x,y,\xi+\nabla\phi(y))\, dy,\ \phi \in W_0^{1,p}((0,T)^N;\mathbb{R}^d) \right\} \tag{1.6}$$

and

$$f_{\hom}(x,\xi) = \inf_{T\in\mathbb{N}} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(x,y,\xi+\nabla\phi(y))\, dy,\ \phi \in W_{\mathrm{per}}^{1,p}((0,T)^N;\mathbb{R}^d) \right\}. \tag{1.7}$$

Secondly, we observe that under the additional hypothesis that $f(x,y,\cdot)$ is convex for a.e. x and all y (in which case (H_1) is equivalent to requiring that $f(x,\cdot,\xi)$ is continuous for a.e. x and all ξ), equalities (1.6) and (1.7) simplify to read, respectively,

$$f_{\hom}(x,\xi) = \inf_{\phi} \left\{ \int_Q f(x,y,\xi+\nabla\phi(y))\, dy,\ \phi \in W_0^{1,p}((0,1)^N;\mathbb{R}^d) \right\} \tag{1.8}$$

and

$$f_{\hom}(x,\xi) = \inf_{\phi} \left\{ \int_Q f(x,y,\xi + \nabla\phi(y))\,dy,\ \phi \in W^{1,p}_{\mathrm{per}}((0,1)^N;\mathbb{R}^d) \right\}. \tag{1.9}$$

Moreover, by hypothesis (H_1), the p-growth condition in (H_4) and by a variant of the Dominated Convergence Theorem (see e.g. Theorem 4 in section 1.3 of Evans and Gariepy [33]) equalities (1.5)-(1.9) hold if the admissible test functions are taken in any smooth dense subset of $W^{1,p}_0((0,T)^N;\mathbb{R}^d)$ and $W^{1,p}_{\mathrm{per}}((0,T)^N;\mathbb{R}^d)$, respectively. Identities (1.8) and (1.9) show that for convex integrands f it is sufficient to consider variations which are periodic in one cell Q, while for nonconvex integrands f it is necessary to consider variations which are periodic over an infinite ensemble of cells. As noted by Müller, (1.8) or, equivalently, (1.9) hold for scalar u without assuming any convexity hypothesis on f, and do not hold in the general vector-valued nonconvex case (see Section 4 in [46]).

The fundamental property of Γ-convergence and its main link to other homogenization techniques, is that it implies (under certain compactness properties) the convergence of approximate minimizers of $\mathcal{I}_\varepsilon$ to a minimizer of the limiting functional $\mathcal{I}_{\hom}$. This is made precise in the following corollary of Theorem 1.1.

Corollary 1.3. *The functional* $\mathcal{I}_{\hom}$ *has a minimum on* $V_\varphi = \left\{ u \in W^{1,p}(\Omega;\mathbb{R}^d) :\ u - \varphi \in W^{1,p}_0(\Omega;\mathbb{R}^d) \right\}$ *and*

$$\min_{u\in V_\varphi} \mathcal{I}_{\hom}(u) = \lim_{\varepsilon\to 0} \inf_{u\in V_\varphi} \mathcal{I}_\varepsilon(u).$$

Moreover, given $\varepsilon_n \downarrow 0$ *and* $\{u_n\}_n \subset V_\varphi$ *s.t.*

$$\lim_{n\to\infty} \mathcal{I}_{\varepsilon_n}(u_n) = \min_{u\in V_\varphi} \mathcal{I}_{\hom}(u),$$

then (up to a subsequence) $\{u_n\}_n$ $W^{1,p}$*-weakly converges to a minimum of* $\mathcal{I}_{\hom}$ *on* V_φ

The homogenization of functionals of linear growth in the framework of Γ-convergence and in the space of (special)bounded variation functions has been considered , among others, by Bouchitté [16], Braides and Chiadò Piat [12] and Carbone, Cioranescu, De Arcangelis and Gaudiello (see [21] and references there in) in the convex case; in the nonconvex case it has been treated by De Arcangelis and Gargiulo [5] and Bouchitté, Fonseca and Mascarenhas [17]. We finally refer to Lukkassen [40], Braides and Lukkassen [15], Braides and Defranceschi [13], Fonseca and Zappale [37], and Babadjian and Baía [6] for other multiscale problems in the Γ-convergence setting.

The overall plan of this work in the ensuing sections will be as follows. *Section 2* collects the main definitions and auxiliary results used in the proof of Theorem 1.1. In *Section 3* we record the properties of $f_{\hom}$ for later use in the proof of this theorem. These properties can be deduced from previous works (see references below) but we present here alternative proofs for the sake of completeness. We will come finally to the proof of Theorem 1.1 in *Section 4*.

2 Preliminaries

The purpose of this section is to give a survey of the concepts and results that are used throughout this work. All these results are stated without proofs as they can be readily found in the references given below.

2.1 Γ-convergence of a family of functionals

We start by recalling De Giorgi's notion of Γ-convergence and some of its basic properties (see De Giorgi and dal Maso [26] and De Giorgi and Franzoni [28]). We refer to Braides [11] and Dal Maso [24] for a comprehensive treatment and bibliography on the subject.

Throughout this part (X,d) denotes a metric space.

Definition 2.1. (Γ-convergence of a sequence of functionals) *Let $\{\mathcal{I}_n\}_n$ be a sequence of functionals defined on X with values on $\overline{\mathbb{R}}$. The functional $\mathcal{I} : X \to \overline{\mathbb{R}}$ is said to be the* Γ-lim inf *(resp.* Γ-lim sup*) of $\{\mathcal{I}_n\}_n$ with respect to the metric d if for every $u \in X$*

$$\mathcal{I}(u) = \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} \mathcal{I}_n(u_n) : \ u_n \in X,\, u_n \to u \text{ in } X \right\} \ (\text{resp.} \quad \limsup_{n\to\infty}).$$

In this case we write

$$\mathcal{I} = \Gamma\text{-}\liminf_{n\to\infty} \mathcal{I}_n \qquad \left(\text{resp.} \quad \mathcal{I} = \Gamma\text{-}\limsup_{n\to\infty} \mathcal{I}_n\right).$$

Moreover, the functional $\mathcal{I}$ is said to be the Γ-lim *of $\{\mathcal{I}_n\}_n$ if*

$$\mathcal{I} = \Gamma\text{-}\liminf_{n\to\infty} \mathcal{I}_n = \Gamma\text{-}\limsup_{n\to\infty} \mathcal{I}_n,$$

and in this case we write

$$\mathcal{I} = \Gamma\text{-}\lim_{n\to\infty} \mathcal{I}_n.$$

For every $\varepsilon > 0$ let $\mathcal{I}_\varepsilon$ be a functional over X with values on $\overline{\mathbb{R}}$, $\mathcal{I}_\varepsilon : X \to \overline{\mathbb{R}}$.

Definition 2.2. (Γ-convergence of a family of functionals)

A functional $\mathcal{I} : X \to \overline{\mathbb{R}}$ is said to be the Γ-lim inf *(resp.* Γ-lim sup *or* Γ-lim*) of $\{\mathcal{I}_\varepsilon\}_\varepsilon$ with respect to the metric d, as $\varepsilon \to 0$, if for every sequence $\varepsilon_n \downarrow 0$,*

$$\mathcal{I} = \Gamma\text{-}\liminf_{n\to\infty} \mathcal{I}_{\varepsilon_n} \qquad \left(\text{resp.} \quad \mathcal{I} = \Gamma\text{-}\limsup_{n\to\infty} \mathcal{I}_{\varepsilon_n} \ \text{ or } \mathcal{I} = \Gamma\text{-}\lim_{n\to\infty} \mathcal{I}_{\varepsilon_n}\right),$$

and we write

$$\mathcal{I} = \Gamma\text{-}\liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon \qquad \left(\text{resp.} \quad \mathcal{I} = \Gamma\text{-}\limsup_{\varepsilon\to 0} \mathcal{I}_\varepsilon \ \text{ or } \mathcal{I} = \Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon\right).$$

The expressions "Γ-converge to $\mathcal{I}$" or "is the Γ-lim of $\mathcal{I}_\varepsilon$" are used interchangeably.

The next result states that a metric space (X,d) satisfies the Urysohn property with respect to Γ-convergence.

Proposition 2.3. *Given $\mathcal{I} : X \to \overline{\mathbb{R}}$ and $\varepsilon_n \downarrow 0$, $\mathcal{I} = \Gamma\text{-}\lim_{n\to\infty} \mathcal{I}_{\varepsilon_n}$ if and only if for every subsequence $\{\varepsilon_{n_j}\}_j$ there exists a further subsequence $\{\varepsilon_{j_k}\}_k$ such that $\{\mathcal{I}_{\varepsilon_{j_k}}\}_k$ Γ-converges to $\mathcal{I}$.*

If in addition (X,d) is a separable metric space then the following compactness property hold.

Theorem 2.4. *Each sequence $\varepsilon_n \downarrow 0$ has a subsequence $\{\varepsilon_{n_j}\}_j \equiv \{\varepsilon_j\}_j$ such that $\Gamma\text{-}\lim_{j\to\infty} \mathcal{I}_{\varepsilon_j}$ exists.*

Proposition 2.5. *If $\mathcal{I} = \Gamma\text{-}\liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon$ (or $\Gamma\text{-}\limsup_{\varepsilon\to 0}$) then $\mathcal{I}$ is lower semicontinuous (with respect to the metric d). Consequently, if $\mathcal{I} = \Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon$ then $\mathcal{I}$ is lower semicontinuous.*

Definition 2.6. *A family of functionals $\{\mathcal{I}_\varepsilon\}_\varepsilon$ is said to be equi-coercive if for every real number λ there exists a compact set K_λ in X such that for each sequence $\varepsilon_n \downarrow 0$,*

$$\{u \in X : \ \mathcal{I}_{\varepsilon_n}(u) \leqslant \lambda\} \subseteq K_\lambda \text{ for every } n \in \mathbb{N}.$$

As mentioned before, one of the most important properties of Γ-convergence, and the reason why this kind of convergence is so important in the asymptotic analysis of variational problems, is that under appropriate compactness properties it implies the convergence of (almost) minimizers of a family of equi-coercive functionals to the minimum of the limiting functional. More precisely, we have the following result.

Theorem 2.7. (Fundamental Theorem of Γ-convergence) *If $\{\mathcal{I}_\varepsilon\}_\varepsilon$ is a family of equi-coercive functionals on X and if*

$$\mathcal{I} = \Gamma\text{-}\lim_{\varepsilon \to 0} \mathcal{I}_\varepsilon,$$

then the functional $\mathcal{I}$ has a minimum on X and

$$\min_{u \in X} \mathcal{I}(u) = \lim_{\varepsilon \to 0} \ \inf_{u \in X} \mathcal{I}_\varepsilon(u).$$

Moreover, given $\varepsilon_n \downarrow 0$ and $\{u_n\}_n$ a converging sequence such that

$$\lim_{n \to \infty} \mathcal{I}_{\varepsilon_n}(u_n) = \lim_{n \to \infty} \ \inf_{u \in X} \mathcal{I}_{\varepsilon_n}(u), \tag{2.1}$$

then its limit is a minimum point for $\mathcal{I}$ on X.

If (2.1) holds, then $\{u_n\}_n$ is said to be a sequence of almost-minimizers for $\mathcal{I}$.

2.2 Quasiconvex functions

We recall that a Borel measurable function $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ is said to be *quasiconvex at a point* $x \in \mathbb{R}^{d\times N}$ if

$$f(x) \leqslant \int_\Omega f(x + \nabla\phi(y))\, dy$$

for every $\phi \in W_0^{1,\infty}(\Omega;\mathbb{R}^d)$, and for every open bounded set $\Omega \subset \mathbb{R}^N$ with $\mathcal{L}^N(\partial\Omega) = 0$. The function f is said to be *quasiconvex* if it is quasiconvex at any $x \in \mathbb{R}^{d\times N}$ (see Morrey [45]). As it is known, if $1 \leqslant p < \infty$ and if $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ is quasiconvex and there exists $C > 0$ such that

$$0 \leqslant f(\xi) \leqslant C(1 + |\xi|^p)$$

for all $\xi \in \mathbb{R}^{d\times N}$, then the p-Lipschitz condition

$$|f(x) - f(y)| \leqslant C(1 + |x|^{p-1} + |y|^{p-1})|x - y| \tag{2.2}$$

holds for all x, $y \in \mathbb{R}^{d\times N}$, $C > 0$ (see Marcellini [44]).

2.3 Carathéodory functions

We also recall that given Ω an open subset of $\mathbb{R}^N$ and B a Borel set of $\mathbb{R}^l$, with $l \geqslant 1$, a function $f : \Omega \times B \to \overline{\mathbb{R}}$ is said to be a *Carathéodory integrand* if

i) $x \to f(x,\xi)$ is measurable for every $\xi \in B$,

ii) $\xi \to f(x,\xi)$ is continuous for almost all $x \in \Omega$.

In this work we deal with Carathéodory integrands $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}$ so that $l = N + (d \times N)$. We will use the following characterization.

Theorem 2.8. (Scorza-Dragoni Theorem) (see Ekeland and Teman [32]) *Let $\Omega \subset \mathbb{R}^N$ with $N \geqslant 1$ be an open set. A function $f : \Omega \times \mathbb{R}^l \to \mathbb{R}$, with $l \geqslant 1$, is Carathéodory if and only if given a compact set $K \subset \Omega$ and a positive number ε, there exists a compact set $K_\varepsilon \subset K$ such that $\mathcal{L}^N(K \setminus K_\varepsilon) \leqslant \varepsilon$ and the restriction of f to $K_\varepsilon \times \mathbb{R}^l$ is continuous.*

The following result shows that every Carathéodory integrand is (equivalent to) a Borel function.

Proposition 2.9. (see Proposition 3.3 in Braides and Defranceschi [13] or Ekeland and Teman [32]) *Let $\Omega \subset \mathbb{R}^N$ with $N \geqslant 1$ be an open set, and let B be a Borel set of $\mathbb{R}^l$ with $l \geqslant 1$. Every Carathéodory integrand $f : \Omega \times B \to \mathbb{R}$ is (equivalent to) a Borel function, that is there exists a Borel function $g : \Omega \times B \to \mathbb{R}$ such that $f(x,\cdot) = g(x,\cdot)$ for a.e. x.*

2.4 An integral representation theorem for functionals defined over Sobolev spaces

In this section we recall an integral representation theorem for local functionals depending on Sobolev functions and on open sets, that is useful in the sequel. This result was obtained by Buttazzo and Dal Maso and gives abstract conditions under which a local functional $\mathcal{I}$ admits an integral representation of the form

$$\mathcal{I}(u,A) = \int_A f(x, \nabla u(x))\, dx$$

for some Carathéodory integrand f (see Theorem 1.1 in [18] and references therein).

Theorem 2.10. *Let Ω be an open subset of $\mathbb{R}^N$ with $N \geqslant 1$. Let $\mathcal{I} : W^{1,p}(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to \mathbb{R}$, with $d \geqslant 1$ and $1 \leqslant p \leqslant \infty$, satisfies the following properties*

i) $\mathcal{I}$ is local on $\mathcal{A}(\Omega)$, i.e. $\mathcal{I}(u,A) = \mathcal{I}(v,A)$ whenever $A \in \mathcal{A}(\Omega)$, $u,v \in W^{1,p}(\Omega;\mathbb{R}^d)$ and $u = v$ a.e. on A;

ii) $\mathcal{I}$ is a measure on $\mathcal{A}(\Omega)$, i.e. for every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ the set function $\mathcal{I}(u,.)$ is the restriction to $\mathcal{A}(\Omega)$ of a finite Radon measure;

iii) $\mathcal{I}$ satisfies a growth condition of order p, i.e. when $p < \infty$ there exist $a \in L^1(\Omega)$ and $b \geqslant 0$ such that for every $A \in \mathcal{A}(\Omega)$ and every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$,

$$|\mathcal{I}(u,A)| \leqslant \int_A [a(x) + b|\nabla u|^p]\, dx,$$

and when $p = \infty$ for every $r \geqslant 0$ there exist $a_r \in L^1(\Omega)$ such that

$$|\mathcal{I}(u,A)| \leqslant \int_A a_r(x)\, dx,$$

for every $A \in \mathcal{A}(\Omega)$ and every $u \in W^{1,\infty}(\Omega;\mathbb{R}^d)$ with $|\nabla u| \leqslant r$ a.e. in A;

iv) $\mathcal{I}$ is translation invariant, i.e. for every $A \in \mathcal{A}(\Omega)$, $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, $c \in \mathbb{R}^d$,

$$\mathcal{I}(u+c,A) = \mathcal{I}(u,A);$$

v) for every $A \in \mathcal{A}(\Omega)$, the function $\mathcal{I}(\cdot, A)$ is s.w.l.s.c on $W^{1,p}$ (s.w⋆.l.s.c if $p = \infty$).

Then, there exists a function $f : \Omega \times \mathbb{R}^{d\times N} \to \mathbb{R}$ such that

a) *for every $A \in \mathcal{A}(\Omega)$ and every $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ the integral representation formula holds*

$$\mathcal{I}(u, A) = \int_A f(x, \nabla u(x))\, dx;$$

b) *f is a Carathéodory integrand;*

c) *$f(x, z)$ satisfies a growth condition of order p, that is, when $p < \infty$ there exist $a \in L^1(\Omega)$ and $b \geqslant 0$ such that*

$$|f(x, z)| \leqslant a(x) + b|z|^p,$$

for a.e. $x \in \Omega$ and for all $z \in \mathbb{R}^{d\times N}$, and when $p = \infty$ for every $r \geqslant 0$ there exists $a_r \in L^1(\Omega)$ such that

$$|f(x, z)| \leqslant a_r(x)$$

for a.e. $x \in \Omega$ and for all $z \in \mathbb{R}^{d\times N}$ with $|z| \leqslant r$.

Remark 2.11.

- *Conditions a), b), c) imply i), ii), iii), iv) but not v). Nevertheless, the integral representation theorem does not hold if we drop hypothesis v) (see examples in Buttazzo and Dal Maso [18]).*
- *Conditions a), b), c) and v) imply that for a.e. $x \in \Omega$ the function $\xi \to f(x, \xi)$ is quasiconvex (see Statment II.5 in Acerbi and Fusco [1]).*

2.5 Sufficient conditions for a mapping to be the trace of a Radon measure

The following lemma provides sufficient conditions for a set function $\Pi : \mathcal{A}(X) \to [0, \infty)$ to be the restriction of a Radon measure on $\mathcal{A}(X)$. It is close in spirit to De Giorgi-Letta's criterion (see [29]) and it is of importance to apply the Direct Method of Γ-convergence as well as for the use of relaxation methods that strongly rely on the structure of Radon measures.

Lemma 2.12. (see Fonseca and Malý [35]; also Fonseca and Leoni [34]) *Let X be a locally compact Hausdorff space, let $\Pi : \mathcal{A}(X) \to [0, \infty)$, and let μ be a finite Radon measure μ on X satisfying*

i) *(nested-subadditivity) $\Pi(D) \leq \Pi(D\backslash\overline{B}) + \Pi(C)$ for all $B, C, D \in \mathcal{A}(X)$ with $B \subset\subset C \subset D$;*

ii) *Given $D \in \mathcal{A}(X)$, for all $\epsilon > 0$ there exists $D_\varepsilon \in \mathcal{A}(X)$ such that $D_\varepsilon \subset\subset D$ and $\Pi(D\backslash\overline{D}_\varepsilon) \leq \varepsilon$;*

iii) $\Pi(X) \geq \mu(X)$;

iv) $\Pi(D) \leq \mu(\overline{D})$ *for all $D \in \mathcal{A}(X)$.*

Then $\Pi = \mu|_{\mathcal{A}(X)}$.

2.6 The notion of 2-scale convergence

The aim of this section is to present in a schematic way the main properties of two-scale convergence introduced by Nguetseng [48] (see [41] and also [49]) and further developed by Allaire and Briane (see [3] and Lukkassen, Nguetseng and Wall [41]).

Definition 2.13. (Periodic function) *A function $f : \mathbb{R}^N \to \mathbb{R}$, with $N \geqslant 1$, is*

i) Q- periodic if $f(\cdot) = f(\cdot + le_i)$ for all $l \in \mathbb{Z}$, where $\{e_1, ..., e_N\}$ is the canonical basis of $\mathbb{R}^N$;

ii) kQ- periodic (or k- periodic), with $k \in \mathbb{N}$, if $f(k\,\cdot)$ is Q-periodic.

As for notation, we denote by $C_{\rm per}(Q)$ the Banach space of all Q-periodic continuous functions defined on $\mathbb{R}^N$ with values in $\mathbb{R}$ endowed with the supremum norm, and by $W^{1,p}_{\rm per}(kQ)$ the $W^{1,p}$-closure of all kQ-periodic and C^1-functions defined on $\mathbb{R}^N$ with values in $\mathbb{R}$ endowed with the $W^{1,p}$-norm.

Given Ω an open bounded subset of $\mathbb{R}^N$ and $1 \leqslant p < \infty$, we denote by $L^p(\Omega; C_{\rm per}(Q))$ (resp. $L^p(\Omega; W^{1,p}_{\rm per}(kQ))$) the space of all measurable functions $f : \Omega \to C_{\rm per}(Q)$ ($f : \Omega \to W^{1,p}_{\rm per}(kQ)$) such that

$$\int_\Omega ||f(x)||^p_{C_{\rm per}(Q)}\, dx < \infty, \quad \Big(\text{resp. } \int_\Omega ||f(x)||^p_{W^{1,p}_{\rm per}(kQ)}\, dx < \infty\Big)$$

where

$$||f(x)||_{C_{\rm per}(Q)} := \sup_{y \in Q} |f(x,y)|$$

$$\Big(\text{resp. } ||f(x)||^p_{W^{1,p}_{\rm per}(kQ)} = \int_{kQ} |f(x,y)|^p\, dy + \int_{kQ} |\nabla_y f(x,y)|^p\, dy < \infty\Big).$$

Clearly a function $f \in L^p(\Omega; C_{\rm per}(Q))$ (resp. $L^p(\Omega; W^{1,p}_{\rm per}(kQ))$) may be identified with the function defined on $\Omega \times \mathbb{R}^N$ via $f(x,y) := f(x)(y)$ ($\nabla_y f$ denotes its derivative with respect to the second argument y).

A generalized version of the Riemann-Lebesgue Lemma holds for functions in $L^p(\Omega; C_{\rm per}(Q))$.

Lemma 2.14. (see Lemma 5.2 in Allaire [2]; see also Bensoussan, Lions and Papanicolaou [8] and Donato [31]) *Let $f \in L^p(\Omega; C_{\rm per}(Q))$ and let $\{\varepsilon_n\}_n$ be a fixed sequence of positive real numbers converging to zero. Then, for every $n \in \mathbb{N}$, the function $f(\cdot, \frac{\cdot}{\varepsilon_n})$ is measurable in Ω,*

$$\left|\left|f\left(\cdot, \frac{\cdot}{\varepsilon_n}\right)\right|\right|_{L^p(\Omega)} \leqslant ||f||_{L^p(\Omega; C_{\rm per}(Q))} := \left(\int_\Omega ||f(x)||^p_{C_{\rm per}(\Omega)}\, dx\right)^{1/p}$$

and

$$\lim_n \int_\Omega \left|f\left(x, \frac{x}{\varepsilon_n}\right)\right|^p dx = \int_\Omega \int_Q |f(x,y)|^p\, dx dy.$$

A useful characterization of functions in $L^1(\Omega; C_{\rm per}(Q))$ is given by the following result.

Lemma 2.15. (*see Lemma 5.3 in Allaire [2]*) *A function f belongs to $L^1(\Omega; C_{\rm per}(Q))$ if and only if there exists a subset $E \subset \Omega$ of measure zero such that*

i) *the function $y \to f(x,y)$ is continuous and Q- periodic for any $x \in \Omega \setminus E$;*

ii) *the function $x \to f(x,y)$ is measurable for any $y \in Q$;*

iii) $x \to \sup_{y \in Q} |f(x,y)|$ *has finite $L^1(\Omega)$-norm.*

Let p and q be real numbers such that $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, and let $\{\varepsilon_n\}_n$ be a sequence of positive numbers converging to zero.

Definition 2.16. (Two-scale convergence) *A sequence of functions* $\{f_n\}_n$ *in* $L^p(\Omega)$ *is said to* two-scale converge *to a limit* $f \in L^p(\Omega \times Q)$*, and we will write* $f_n \overset{2s}{\rightharpoonup} f$*, if*

$$\int_\Omega f_n(x)\phi\Big(x, \frac{x}{\varepsilon_n}\Big)\, dx \to \int_\Omega \int_Q f(x,y)\phi(x,y)\, dy\, dx$$

for all $\phi \in L^q(\Omega; C_{\text{per}}(Q))$.

Lemma 2.17. (see e.g. Lukkassen, Nguetseng and Wall [41]) *For each sequence* $\{f_n\}_n$ *bounded in* $L^p(\Omega)$ *there exists a subsequence (still denoted by* $\{f_n\}_n$*) and* $f \in L^p(\Omega \times Q)$ *such that* $f_n \overset{2s}{\rightharpoonup} f$.

For sequences weakly convergent in $W^{1,p}(\Omega)$ the following compactness result holds.

Theorem 2.18. (*see Allaire [2] or Nguetseng [48]*) *Assume that* $\{f_n\}_n$ *weakly converges to a function* f *in* $W^{1,p}(\Omega)$*. Then* $f_n \overset{2s}{\rightharpoonup} f$*, and there exist a subsequence* (*still denoted by* $\{f_n\}_n$) *and* $f_1 \in L^p(\Omega; W^{1,p}_{\text{per}}(Q))$ *such that*

$$\nabla f_n \overset{2s}{\rightharpoonup} \nabla f + \nabla_y f_1.$$

2.7 The Decomposition Lemma

We recall that a sequence of functions $\{u_n\} \subset L^1(\Omega)$ is said *equi-integrable* if for all $\varepsilon > 0$ there exist $\delta > 0$ such that

$$\sup_{n\in\mathbb{N}} \int_A |u_n|\, dx < \varepsilon$$

whenever $A \subset \Omega$ with $\mathcal{L}^N(A) < \delta$.

As a consequence of next theorem each sequence with bounded gradients in L^p, for $1 < p < \infty$, admits a subsequence that can be decomposed as a sum of a sequence with p-equi-integrable gradients and a remainder that converges to zero in measure. This property turns out to be an important tool for the asymptotic analysis of integral functionals relying on localization arguments.

Theorem 2.19. (Decomposition Lemma) (see Fonseca and Leoni [34]; see also Fonseca, Müller and Pedregal [36] and Kristensen [42]) *Let* $1 < p < \infty$ *and assume that* $\partial\Omega$ *is Lipschitz, and that* $u_n \rightharpoonup v_0$ *in* $W^{1,p}(\Omega; \mathbb{R}^d)$*. Then, there exists a subsequence* $\{u_{n_k}\}_k$ *of* $\{u_n\}_n$ *and a sequence* $\{v_k\}_k \subset W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^d)$ *such that*

i) $v_k \rightharpoonup v_0$ *in* $W^{1,p}(\Omega; \mathbb{R}^d)$,

ii) $v_k = v_0$ *in a neighborhood of* $\partial\Omega$,

iii) $\{\nabla v_k\}_k$ *is* p*-equi-integrable,*

iv) $\lim_{k\to\infty} \mathcal{L}^N(\{x \in \Omega : v_k(x) \neq u_{n_k}(x)\}) = 0.$

3 Properties of f_{hom}

In this section we turn our attention to the main properties of the function f_{hom} defined in (1.5). By Remark 1.2 we restrict our analysis to the case where f is a positive Borel function such that hypotheses (H_1), (H_3) and (H_4) hold for every $(x, y, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$.

We start by showing that the limit in (1.5) is well defined. The argument is analogous to that used in Bouchitté, Fonseca and Mascarenhas [17] and relies on Lemma 5.1 presented on Appendix.

Lemma 3.1. *For all $(x, \xi) \in \Omega \times \mathbb{R}^{d\times N}$ there exists*

$$\lim_{T\to+\infty} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(x, y, \xi + \nabla\phi(y))\, dy,\ \phi \in W_0^{1,p}((0,T)^N; \mathbb{R}^d) \right\}. \tag{3.1}$$

Proof. (see also Braides and Defranceschi [13]) Let $(x, \xi) \in \Omega \times \mathbb{R}^{d\times N}$ and let

$$S(A) := \inf_{\phi} \left\{ \int_A f(x, y, \xi + \nabla\phi(y))\, dy :\ \phi \in W_0^{1,p}(A; \mathbb{R}^d) \right\}$$

for $A \in \mathcal{A}(\mathbb{R}^N)$. Under the assumptions on f the function $S : \mathcal{A}(\mathbb{R}^N) \to \mathbb{R}^+$ is well defined, and it can be seen that satisfies the hypotheses of Lemma 5.1 with $T = \mathbb{Z}^N$ and $M = 1$. Hence we conclude that the limit

$$\lim_{T\to+\infty} \frac{S((0,T)^N)}{T^N},$$

or, equivalently, (3.1) exists. ■

We now want to show that f_{hom} is a Carathéodory integrand so that the functional $\mathcal{I}_{\text{hom}}$ in (1.4) is well defined. We start by noting the following result.

Lemma 3.2. *Let $\overline{f}_{\text{hom}}, \hat{f}_{\text{hom}} : \Omega \times \mathbb{R}^{d\times N} \to [0, \infty)$ be defined, respectively, by*

$$\overline{f}_{\text{hom}}(x, \xi) := \inf_{T\in\mathbb{N}} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(x, y, \xi + \nabla\phi(y))\, dy,\ \phi \in W_0^{1,p}((0,T)^N; \mathbb{R}^d) \right\}$$

and

$$\hat{f}_{\text{hom}}(x, \xi) := \inf_{T\in\mathbb{N}} \inf_{\phi} \left\{ \frac{1}{T^N} \int_{(0,T)^N} f(x, y, \xi + \nabla\phi(y))\, dy,\ \phi \in W_{\text{per}}^{1,p}((0,T)^N; \mathbb{R}^d) \right\},$$

for all $(x, \xi) \in \Omega \times \mathbb{R}^{d\times N}$. Then the relations $f_{\text{hom}} = \overline{f}_{\text{hom}} = \hat{f}_{\text{hom}}$ hold.

Proof. Let $(x, \xi) \in \Omega \times \mathbb{R}^{d\times N}$. We first show that $f_{\text{hom}}(x, \xi) = \overline{f}_{\text{hom}}(x, \xi)$. It is clear that $f_{\text{hom}}(x, \xi) \geqslant \overline{f}_{\text{hom}}(x, \xi)$. To prove the other inequality, fixed $\delta > 0$ let $S \in \mathbb{N}$ and $\varphi \in W_0^{1,p}((0,S)^N; \mathbb{R}^d)$ be such that

$$\overline{f}_{\text{hom}}(x, \xi) + \delta \geqslant \frac{1}{S^N} \int_{(0,S)^N} f(x, y, \xi + \nabla\varphi(y))\, dy. \tag{3.2}$$

Extend φ periodically to $\mathbb{R}^N$ with period S. Using Riemann-Lebesgue's Lemma

$$\begin{aligned} \frac{1}{S^N} \int_{(0,S)^N} f(x, y, \xi + \nabla\varphi(y))\, dy &= \lim_{\varepsilon\to 0} \frac{1}{S^N} \int_{(0,S)^N} f\left(x, \frac{y}{\varepsilon}, \xi + \nabla\varphi\left(\frac{y}{\varepsilon}\right)\right) dy \\ &= \lim_{\varepsilon\to 0} \frac{\varepsilon^N}{S^N} \int_{\left(0,\frac{S}{\varepsilon}\right)^N} f(x, z, \xi + \nabla\theta_\varepsilon(z))\, dz, \end{aligned} \tag{3.3}$$

where $\theta_\varepsilon(z) := \frac{1}{\varepsilon}\varphi(\varepsilon z) \in W_0^{1,p}\Big(\big(0,\frac{S}{\varepsilon}\big)^N;\mathbb{R}^d\Big)$. Therefore from (3.2) and (3.3)

$$\begin{aligned}\overline{f}_{\text{hom}}(x,\xi)+\delta &\geqslant \lim_{\varepsilon\to 0}\ \inf_{\theta}\left\{\tfrac{\varepsilon^N}{S^N}\int_{(0,\frac{S}{\varepsilon})^N} f(x,z,\xi+\nabla\theta(z))\,dz,\ \theta\in W_0^{1,p}\Big(\Big(0,\frac{S}{\varepsilon}\Big)^N;\mathbb{R}^d\Big)\right\}\\ &= f_{\text{hom}}(x,\xi).\end{aligned}$$

Letting $\delta\to 0$ we conclude that

$$\overline{f}_{\text{hom}}(x,\xi)\geqslant f_{\text{hom}}(x,\xi).$$

Finally we show that $\overline{f}_{\text{hom}}(x,\xi)=\hat{f}_{\text{hom}}(x,\xi)$ (see also Braides [13] and Müller [46] for an alternative justification). It is clear that $\overline{f}_{\text{hom}}(x,\xi)\geqslant\hat{f}_{\text{hom}}(x,\xi)$. To verify the opposite inequality, fix $\delta>0$ and take $S\in\mathbb{N}$ and a smooth function $\varphi\in W^{1,p}_{\text{per}}\big((0,S)^N;\mathbb{R}^d\big)$ such that

$$\hat{f}_{\text{hom}}(x,\xi)+\delta\geqslant\frac{1}{S^N}\int_{(0,S)^N} f(x,y,\xi+\nabla\varphi(y))\,dy. \tag{3.4}$$

By hypothesis (H_3) the function $f(x,\cdot,\xi+\nabla\varphi(\cdot))$ is $(0,S)^N$-periodic, and thus

$$\begin{aligned}\frac{1}{S^N}\int_{(0,S)^N} f(x,y,\xi+\nabla\varphi(y))\,dy &= \lim_{\varepsilon\to 0}\int_Q f\Big(x,\frac{y}{\varepsilon},\xi+\nabla\varphi\Big(\frac{y}{\varepsilon}\Big)\Big)\,dy\\ &= \lim_{\varepsilon\to 0}\int_Q f\Big(x,\frac{y}{\varepsilon},\xi+\nabla\psi_\varepsilon(y)\Big)\,dy,\end{aligned} \tag{3.5}$$

where $\psi_\varepsilon(y) := \varepsilon\varphi\Big(\frac{y}{\varepsilon}\Big)$. For each $\varepsilon>0$ define

$$Q_\varepsilon := \Big\{y\in Q:\ \operatorname{dist}(y,\partial Q)\geqslant\varepsilon\Big\}.$$

Take $\theta_\varepsilon\in C_c^\infty(Q)$ with $\theta_\varepsilon\in[0,1]$ such that $\theta_\varepsilon=1$ on Q_ε and $||\nabla\theta_\varepsilon||_{L^\infty}\leqslant C\varepsilon^{-1}$. Then

$$\lim_{\varepsilon\to 0}\int_Q f\Big(x,\frac{y}{\varepsilon},\xi+\nabla(\theta_\varepsilon\psi_\varepsilon)(y)\Big)\,dy = \lim_{\varepsilon\to 0}\int_{Q_\varepsilon} f\Big(x,\frac{y}{\varepsilon},\xi+\nabla\psi_\varepsilon(y)\Big)\,dy, \tag{3.6}$$

since by the p-growth condition in (H_4) we have

$$\begin{aligned}&\int_{Q\setminus Q_\varepsilon} f\Big(x,\frac{y}{\varepsilon},\xi+\nabla(\theta_\varepsilon\psi_\varepsilon)(y)\Big)\,dy\\ \leqslant\ & C\int_{Q\setminus Q_\varepsilon}\left(1+|\xi|^p+|\nabla\psi_\varepsilon(y)|^p+\varepsilon^{-p}|\psi_\varepsilon(y)|^p\right)\,dy\\ =\ & C\Big[|Q\setminus Q_\varepsilon|+\int_{Q\setminus Q_\varepsilon}\Big|\nabla\varphi\Big(\frac{y}{\varepsilon}\Big)\Big|^p dy+\int_{Q\setminus Q_\varepsilon}\Big|\varphi\Big(\frac{y}{\varepsilon}\Big)\Big|^p dy\Big]\to 0.\end{aligned}$$

Hence by (3.4)-(3.6), defining $\phi_\varepsilon(y) := \frac{1}{\varepsilon}(\theta_\varepsilon\psi_\varepsilon)(\varepsilon y)\in W_0^{1,p}\Big(\Big(0,\frac{1}{\varepsilon}\Big)^N;\mathbb{R}^N\Big)$, we obtain

$$\begin{aligned}\hat{f}_{\text{hom}}(x,\xi)+\delta &\geqslant \lim_{\varepsilon\to 0}\int_Q f\Big(x,\frac{y}{\varepsilon},\xi+\nabla(\theta_\varepsilon\psi_\varepsilon)(y)\Big)\,dy\\ &\geqslant \lim_{\varepsilon\to 0}\varepsilon^N\int_{\left(0,\frac{1}{\varepsilon}\right)^N} f\Big(x,y,\xi+\nabla\phi_\varepsilon(y)\Big)\,dy\\ &\geqslant \overline{f}_{\text{hom}}(x,\xi).\end{aligned}$$

Letting $\delta \to 0$ we get $f_{\text{hom}}(x,\xi) \geqslant \overline{f}_{\text{hom}}(x,\xi)$.

■

The measurability of f_{hom} follows as a consequence.

Lemma 3.3. *The function $f_{\text{hom}}(\cdot,\xi)$ is measurable for all $\xi \in \mathbb{R}^{d\times N}$.*

Proof. Let $\xi \in \mathbb{R}^{d\times N}$. By Lemma 3.2 we can write

$$f_{\text{hom}}(x,\xi) = \inf_{T\in\mathbb{N}} \inf_{\phi\in\mathcal{S}_T} f_{T,\phi}(x)$$

where

$$f_{T,\phi}(x) := \int_T f(x,y,\xi+\nabla\phi(z))\,dz$$

for $x \in \Omega$, and $\mathcal{S}_T$ is a countable subset of $C_c^\infty((0,T)^N;\mathbb{R}^d)$ dense in $W_0^{1,p}((0,T)^N;\mathbb{R}^d)$. By Tonelli's Theorem the functions $f_{T,\phi}$ are measurable, and so is $f_{\text{hom}}(\cdot,\xi)$ as the infimum of a countable family of measurable functions.

■

Our next objective is to show that $f_{\text{hom}}(x,\cdot)$ is continuous for all $x \in \Omega$. We are not able to prove it directly unless f would satisfy a p-Lipschitz condition of the form (2.2). As quasiconvex functions satisfies inequality (2.2), the first step will be to show that $f_{\text{hom}} = (\mathcal{Q}f)_{\text{hom}}$ where $\mathcal{Q}f : \Omega\times\mathbb{R}^N\times\mathbb{R}^{d\times N} \to \mathbb{R}$ denote the usual quasiconvexification of f with respect to the last variable ξ, that we know to be quasiconvex in this last variable (see Dacorogna [23]). We remark that

$$\mathcal{Q}f(x,y,\xi) = \inf_\phi \left\{ \int_Q f(x,y,\xi+\nabla\phi(z))\,dz : \ \phi \in W_0^{1,p}(Q;\mathbb{R}^d) \right\} \tag{3.7}$$

for all $(x,y,\xi) \in \Omega\times\mathbb{R}^N\times\mathbb{R}^{d\times N}$ (see Dacorogna [23] and Ball and Murat [7]) and that consequently $\mathcal{Q}f$ satisfies conditions (H_3) and (H_4). The following properties of $\mathcal{Q}f$ are of interest for the argument that follows.

Lemma 3.4. *We have that*

i) *$\mathcal{Q}f(x,\cdot,\cdot)$ is continuous for all $x \in \Omega$;*

ii) *$\mathcal{Q}f(\cdot,y,\xi)$ is measurable for all $(y,\xi) \in \mathbb{R}^N\times\mathbb{R}^{d\times N}$;*

iii) $$(\mathcal{Q}f)_{\text{hom}}(x,\xi) := \liminf_{T\to\infty} \inf\left\{ \frac{1}{T^N}\int_{(0,T)^N} \mathcal{Q}f(x,y,\xi+\nabla\phi(y))\,dy,\ \phi \in W_0^{1,p}((0,T)^N;\mathbb{R}^d) \right\}$$
$$= \lim_{T\to\infty} \inf\left\{ \frac{1}{T^N}\int_{(0,T)^N} \mathcal{Q}f(x,y,\xi+\nabla\phi(y))\,dy,\ \phi \in W_0^{1,p}((0,T)^N;\mathbb{R}^d) \right\}$$
for all $(x,\xi) \in \Omega\times\mathbb{R}^N\times\mathbb{R}^{d\times N}$.

iv) *$(\mathcal{Q}f)_{\text{hom}}(x,\xi) = f_{\text{hom}}(x,\xi)$ for all $(x,\xi) \in \Omega\times\mathbb{R}^{d\times N}$.*

Proof. i) The upper semicontinuity of $\mathcal{Q}f(x,\cdot,\cdot)$ for $x \in \mathbb{R}^N$ follows from equality (3.7) and hypothesis (H_1), while its lower semicontinuity can be obtained by an argument analogous to that of Lemma **??**.
ii) The proof is identical to that of Lemma 3.3 above.
iii) Is a consequence of identities i) and ii), the coerciveness, growth and periodicity properties of $\mathcal{Q}f$, and of Lemma 3.1.

iv) Let $(x,\xi) \in \Omega \times \mathbb{R}^{d\times N}$. Obviously $f_{\rm hom}(x,\xi) \geqslant (\mathcal{Q}f)_{\rm hom}(x,\xi)$. Let us prove the converse inequality. Let $n \in \mathbb{N}$ and let $T_n \in \mathbb{N}$ and $\phi_n \in W_0^{1,p}((0,T_n)^N;\mathbb{R}^d)$ be such that

$$(\mathcal{Q}f)_{\rm hom}(x,\xi) + \frac{1}{n} \geqslant \frac{1}{T_n^N} \int_{(0,T_n)^N} \mathcal{Q}f(x,y;\xi + \nabla\phi_n(y))\,dy.$$

Thus

$$(\mathcal{Q}f)_{\rm hom}(x,\xi) \geqslant \limsup_{n\to\infty} \frac{1}{T_n^N} \int_{(0,T_n)^N} \mathcal{Q}f(x,y;\xi + \nabla\phi_n(y))\,dy. \tag{3.8}$$

To compare (3.8) with $f_{\rm hom}(x,\xi)$ we apply the Acerbi and Fusco Relaxation Theorem (Theorem **??**) and the Decomposition Lemma (Lemma 2.19). As a consequence of Theorem **??**, for every n fixed there exists a sequence $\{\phi_{n,k}\}_k \subset W^{1,p}((0,T_n)^N;\mathbb{R}^d)$ such that $\phi_{n,k} \underset{k}{\rightharpoonup} \phi_n$ in $W^{1,p}((0,T_n)^N;\mathbb{R}^d)$ and

$$\frac{1}{T_n^N} \int_{(0,T_n)^N} \mathcal{Q}f(x,y;\xi + \nabla\phi_n(y))\,dy = \lim_{k\to\infty} \frac{1}{T_n^N} \int_{(0,T_n)^N} f(x,y;\xi + \nabla\phi_{n,k}(y))\,dy. \tag{3.9}$$

By Lemma 2.19 we can now find a subsequence (still denoted by $\{\phi_{n,k}\}_k$) and a sequence $\{\psi_{n,k}\}_k \subset W_0^{1,\infty}(\mathbb{R}^N;\mathbb{R}^d)$ such that $\psi_{n,k} \rightharpoonup \phi_n$ in $W^{1,p}((0,T_n)^N;\mathbb{R}^d)$ with

$$\{|\nabla\psi_{n,k}|^p\} \quad \text{equi-integrable} \tag{3.10}$$

and

$$\mathcal{L}^N\{y \in (0,T_n)^N : \ \psi_{n,k}(y) \neq \phi_{n,k}(y)\} \underset{k\to\infty}{\longrightarrow} 0. \tag{3.11}$$

As f is nonnegative, by (4.29) and (4.30)

$$\begin{aligned}
&\lim_{k\to\infty} \frac{1}{T_n^N} \int_{(0,T_n)^N} f(x,y;\xi + \nabla\phi_{n,k}(y))\,dy \\
&\geqslant \limsup_{k\to\infty} \frac{1}{T_n^N} \int_{\{y\in(0,T_n)^N :\ \psi_{n,k}(y)=\phi_{n,k}(y)\}} f(x,y;\xi + \nabla\psi_{n,k}(y))\,dy \\
&= \limsup_{k\to\infty} \frac{1}{T_n^N} \int_{(0,T_n)^N} f(x,y;\xi + \nabla\psi_{n,k}(y))\,dy.
\end{aligned} \tag{3.12}$$

Thus from (3.8), (3.9) and (3.12)

$$(\mathcal{Q}f)_{\rm hom}(x,\xi) \geqslant \limsup_{n\to\infty}\ \limsup_{k\to\infty} \frac{1}{T_n^N} \int_{(0,T_n)^N} f(x,y,\xi + \nabla\psi_{n,k}(y))\,dy \geqslant f_{\rm hom}(x,\xi).$$

■

We are now in a position to prove the continuity property of $f_{\rm hom}$.

Lemma 3.5. *The function $f_{\rm hom}(x,\cdot)$ (or equivalently $(\mathcal{Q}f)_{\rm hom}(x,\cdot)$) is continuous for all $x \in \Omega$.*

Proof. (see also Braides [13]) Fix $x \in \Omega$. Let $\xi \in \mathbb{R}^{d\times N}$ and $\xi_n \to \xi$ in $\mathbb{R}^{d\times N}$. We wish to show that

$$f_{\rm hom}(x,\xi) = \lim_{n\to\infty} f_{\rm hom}(x,\xi_n).$$

We first establish that (upper semicontinuity)

$$f_{\rm hom}(x,\xi) \geqslant \limsup_{n\to\infty} f_{\rm hom}(x,\xi_n). \tag{3.13}$$

Fixed $\delta > 0$, choose $S \in \mathbb{N}$ and by density a function $\varphi \in C_0^\infty((0,S)^N;\mathbb{R}^d)$ such that

$$\begin{aligned} f_{\mathrm{hom}}(x,\xi)+\delta &\geqslant \frac{1}{S^N}\int_{(0,S)^N} f(x,y,\xi+\nabla\varphi(y))\,dy \\ &= \frac{1}{S^N}\lim_{n\to\infty}\int_{(0,S)^N} f(x,y,\xi_n+\nabla\varphi(y))\,dy \\ &\geqslant \limsup_{n\to\infty} f_{\mathrm{hom}}(x,\xi_n), \end{aligned}$$

as a consequence of a variant of the Dominated Convergence Theorem and Lemma 3.2. Letting $\delta \to 0$ we get (3.13).

We show now the converse inequality (lower semicontinuity), i.e.

$$f_{\mathrm{hom}}(x,\xi) \leqslant \liminf_{n\to\infty} f_{\mathrm{hom}}(x,\xi_n). \tag{3.14}$$

We start by remarking that $f_{\mathrm{hom}}(x,\xi_n) = (\mathcal{Q}f)_{\mathrm{hom}}(x,\xi_n)$ for all $n \in \mathbb{N}$ (property iv) in Lemma 3.4). Given $n \in \mathbb{N}$, consider $T_n \in \mathbb{N}$ ($T_n \nearrow \infty$) and $\phi_n \in W_0^{1,p}((0,T_n)^N;\mathbb{R}^d)$ such that

$$\begin{aligned} f_{\mathrm{hom}}(x,\xi_n)+\frac{1}{n} &\geqslant \frac{1}{{T_n}^N}\int_{(0,T_n)^N} \mathcal{Q}f(x,y,\xi_n+\nabla\phi_n(y))\,dy \\ &= \int_{(0,1)^N} \mathcal{Q}f(x,T_n y,\xi_n+\nabla\phi_n(T_n y))\,dy \\ &= \int_{(0,1)^N} \mathcal{Q}f(x,T_n y,\xi_n+\nabla\psi_n(y))\,dy, \end{aligned} \tag{3.15}$$

where $\psi_n(y) := \frac{1}{T_n}\phi_n(T_n y)$, $\psi_n \in W_0^{1,p}((0,1)^N;\mathbb{R}^d)$. By the p-coervivity condition of $(\mathcal{Q}f)$ the sequence $\{||\nabla\psi_n||_{L^p((0,1)^N;\mathbb{R}^d)}\}$ is bounded. We write

$$\begin{aligned} &\int_{(0,1)^N} \mathcal{Q}f(x,T_n y,\xi_n+\nabla\psi_n(y))\,dy \\ =\ &\int_{(0,1)^N} \mathcal{Q}f(x,T_n y,\xi_n+\nabla\psi_n(y)) - \mathcal{Q}f(x,T_n y,\xi+\nabla\psi_n(y))\,dy \\ +\ &\int_{(0,1)^N} \mathcal{Q}f(x,T_n y,\xi+\nabla\psi_n(y))\,dy. \end{aligned} \tag{3.16}$$

Our task now is to show that the term (3.16) goes to zero as n goes to infinity. As $\xi_n \to \xi$ and the sequence $\{||\nabla\psi_n||_{L^p((0,1)^N;\mathbb{R}^d)}\}$ is bounded, using the p-Lipschitz condition (2.2) and Hölder Inequality we have

$$\begin{aligned} &\limsup_{n\to\infty}\int_{(0,1)^N} |\mathcal{Q}f(x,T_n y,\xi_n+\nabla\psi_n(y)) - \mathcal{Q}f(x,T_n y,\xi+\nabla\psi_n(y))|\ dy \\ &\leqslant C\ \limsup_{n\to\infty}\int_{(0,1)^N} \left(1+|\xi_n+\nabla\psi_n(y)|^{p-1}+|\xi+\nabla\psi_n(y)|^{p-1}\right)|\xi_n-\xi|\,dy \\ &\leqslant C\ \limsup_{n\to\infty}\int_{(0,1)^N} \left(1+|\nabla\psi_n(y)|^{p-1}\right)|\xi_n-\xi|\,dy \\ &\leqslant C\ \lim_{n\to\infty}|\xi_n-\xi| = 0, \end{aligned}$$

which, together with (3.15), leads to

$$f_{\mathrm{hom}}(x,\xi_n)+\frac{1}{n} \geqslant \limsup_{n\to\infty}\int_{(0,1)^N} \mathcal{Q}f(x,T_n y,\xi+\nabla\psi_n(y))\,dy \geqslant (\mathcal{Q}f)_{\mathrm{hom}}(x,\xi) = f_{\mathrm{hom}}(x,\xi).$$

Inequality (3.14) holds letting $n \to \infty$.

■

By Lemmas 3.3 and 3.5, we conclude that f_{hom} is a Carathéodory integrand and thus $\mathcal{I}_{\text{hom}}$ is well defined.

Remark 3.6. We note that f_{hom} satisfies analog growth and coercivity conditions to the ones of f, which together with the continuity properties of f_{hom} implies by standard arguments (approximation of $W^{1,p}$ by piecewise affine functions together with the invariance of the domain of f_{hom}) that this function is quasiconvex with respect to the last variable.

We will show next that in the convex case it is enough to consider one cell period for the definition of f_{hom} (1.5) (see also Braides and Defranceschi [13] or Müller [46]).

Remark 3.7. We recall that if $f : \mathbb{R}^N \to \mathbb{R}^+$ is convex and $f(x) \leqslant C(1+|x|^p)$ holds for all $x \in \mathbb{R}^N$, $1 \leqslant p < \infty$, then $|f(x)-f(y)| \leqslant C(1+|x|^{p-1}+|y|^{p-1})|x-y|$ for all x and y in $\mathbb{R}^N$. Moreover f is differentiable almost everywhere (see e.g section 3.1.2 in Evans and Gariepy [33]) and $f(y) \geqslant f(x) + \nabla f(x) \cdot (y-x)$ for all points x where f is differentiable.

We define for all $(x,\xi) \in \Omega \times \mathbb{R}^{d\times N}$

$$f^{\star}_{\text{hom}}(x,\xi) = \inf_{\phi} \left\{ \int_Q f(x,y,\xi+\nabla\phi(y))\,dy, \quad \phi \in W^{1,p}_0((0,1)^N;\mathbb{R}^d) \right\}$$

and

$$f^{\star\star}_{\text{hom}}(x,\xi) = \inf_{\phi} \left\{ \int_Q f(x,y,\xi+\nabla\phi(y))\,dy, \quad \phi \in W^{1,p}_{\text{per}}((0,1)^N;\mathbb{R}^d) \right\}.$$

Lemma 3.8. *Assume in addition to the hypotheses on f that $f(x,y,\cdot)$ is convex for all $(x,y) \in \Omega \times \mathbb{R}^N$. Then*

$$f^{\star}_{\text{hom}} = f^{\star\star}_{\text{hom}} = f_{\text{hom}}. \tag{3.17}$$

Proof. (see [13] and [46] for an alternative proof). Equality $f^{\star}_{\text{hom}} = f^{\star\star}_{\text{hom}}$ is proven by an argument analog to that of the proof of Lemma 3.2.

We show that $f^{\star}_{\text{hom}} = f_{\text{hom}}$. Let $(x,\xi) \in \Omega \times \mathbb{R}^{d\times N}$. By definition $f^{\star}_{\text{hom}}(x,\xi) \geqslant f_{\text{hom}}(x,\xi)$. To prove the opposite inequality, for each $n \in \mathbb{N}$ take $T_n \in \mathbb{N}$ and a function $\phi_n \in W^{1,p}_0((0,T_n)^N;\mathbb{R}^d)$ such that

$$\begin{aligned} f_{\text{hom}}(x,\xi) + \tfrac{1}{n} &\geqslant \tfrac{1}{T_n^N} \int_{(0,T_n)^N} f(x,y,\xi+\nabla\phi_n(y))\,dy \\ &= \int_Q f(x,T_n y,\xi+\nabla\psi_n(y))\,dy, \end{aligned} \tag{3.18}$$

where $\psi_n(y) := \frac{1}{T_n}\phi_n(T_n y)$, $\psi_n \in W^{1,p}_0(Q;\mathbb{R}^d)$. We note that by the p-growth condition in (H_4) the sequence $\{||\nabla\psi_n||_{L^p(Q;\mathbb{R}^d)}\}_n$ is bounded, and so is $\{||\psi_n||_{W^{1,p}(Q;\mathbb{R}^d)}\}_n$ by Poincaré Inequality. Hence, there exists a subsequence (still denoted by $\{\psi_n\}_n$) such that

$$\psi_n \overset{W^{1,p}}{\rightharpoonup} \psi$$

for some $\psi = \psi(y) \in W^{1,p}_0(Q;\mathbb{R}^d)$. As a consequence, by Theorem 2.18 and up to a subsequence

$$\psi_n \overset{2s}{\rightharpoonup} \psi$$

and

$$\nabla\psi_n \overset{2s}{\rightharpoonup} \nabla\psi + \nabla_z\overline{\psi}$$

for some $\overline{\psi} = \overline{\psi}(y,z) \in L^p\big(Q; W^{1,p}_{\mathrm{per}}(Q;\mathbb{R}^d)\big)$. We divide the rest of the proof in two steps.
Step 1. We follow an argument of Allaire [2] assuming in addition that

(H_5) $\frac{\partial f}{\partial \eta}(x,y,\eta)$ exists for all $(x,y,\eta) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$.

Let $\varphi = \varphi(y,z) \in C_c^\infty\big(Q; C^\infty_{\mathrm{per}}(Q;\mathbb{R}^{d\times N})\big)$. Since f is convex in the last variable then

$$\begin{aligned}\int_Q f\big(x, T_n y, \xi + \nabla\psi_n(y)\big)\,dy \geqslant &\int_Q f\big(x, T_n y, \xi + \varphi(y, T_n y)\big)\,dy \\ &+ \int_Q \frac{\partial f}{\partial \eta}\big(x, T_n y, \varphi(y, T_n y)\big) \cdot (\nabla\psi_n(y) - \varphi(y, T_n y))\,dy,\end{aligned} \tag{3.19}$$

for each $n \in \mathbb{N}$. By Lemma 2.14 and as $\nabla\psi_n \overset{2s}{\rightharpoonup} \nabla\psi + \nabla_z\overline{\psi}$ (see Definition 2.16) we have

$$\lim_{n\to\infty} \int_Q f\big(x, T_n y, \xi + \varphi(y, T_n y)\big)\,dy = \int_Q \left[\int_Q f\big(x, z, \xi + \varphi(y,z)\big)\,dz\right] dy; \tag{3.20}$$

$$\begin{aligned}&\lim_{n\to\infty} \int_Q \frac{\partial f}{\partial \eta}\Big(x, T_n y, \varphi(y, T_n y)\Big) \cdot (\nabla\psi_n(y) - \varphi(y, T_n y))\,dy \\ &= \int_Q \left[\int_Q \frac{\partial f}{\partial \eta}\big(x, z, \varphi(y,z)\big) \cdot (\nabla\psi(y) + \nabla_z\overline{\psi}(y,z) - \varphi(y,z))\,dz\right] dy.\end{aligned} \tag{3.21}$$

Therefore from (3.19)-(3.21) we get

$$\begin{aligned}\lim_{n\to\infty} \int_Q f\big(x, T_n y, \xi + \nabla\psi_n(y)\big)\,dy \geqslant &\int_Q \left[\int_Q f\big(x, z, \xi + \varphi(y,z)\big)\,dz\right] dy \\ &+ \int_Q \left[\int_Q \frac{\partial f}{\partial \eta}\big(x, z, \varphi(y,z)\big) \cdot (\nabla\psi(y) + \nabla_z\overline{\psi}(y,z) - \varphi(y,z))\,dz\right] dy,\end{aligned} \tag{3.22}$$

for all $\varphi = \varphi(y,z) \in C_c^\infty\big(Q; C^\infty_{\mathrm{per}}(Q;\mathbb{R}^{d\times N})\big)$. Let now $\{\varphi_k\}_k \subset C_c^\infty\big(Q; C_c^\infty(Q;\mathbb{R}^{d\times N})\big)$ be a convergent sequence to $\nabla\psi + \nabla_z\overline{\psi}$ in $L^p\big(Q\times Q;\mathbb{R}^{d\times N}\big)$. In particular from (3.22)

$$\begin{aligned}\lim_{n\to\infty} \int_Q f\big(x, T_n y, \xi + \nabla\psi_n(y)\big)\,dy \geqslant &\int_Q \left[\int_Q f\big(x, z, \xi + \varphi_k(y,z)\big)\,dz\right] dy \\ &+ \int_Q \left[\int_Q \frac{\partial f}{\partial \eta}\big(x, z, \varphi_k(y,z)\big) \cdot (\nabla\psi(y) + \nabla_z\overline{\psi}(y,z) - \varphi_k(y,z))\,dz\right] dy,\end{aligned} \tag{3.23}$$

for every $k \in \mathbb{N}$. By Remark 3.7 and the p-growth condition in (H_4) there exists a constant $C > 0$ such that

$$\left|\frac{\partial f}{\partial \eta}(x,y,\eta)\right| \leqslant C\big[1 + |\eta|^{p-1}\big] \tag{3.24}$$

for all $(x,y,\eta) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$. Inequality (3.24) and the growth conditions on f and $\frac{\partial f}{\partial \eta}$ imply that we can pass to the limit in (3.23) and get

$$\liminf_{n\to\infty} \int_Q f\big(x, T_n y, \xi + \nabla\psi_n(y)\big)\, dy \geqslant \int_Q \left[\int_Q f\big(x, z, \xi + \nabla\psi(y) + \nabla_z \overline{\psi}(y,z)\big)\, dz \right] dy. \tag{3.25}$$

By Jensen's Inequality and Fubini's Theorem, for each $z \in Q$

$$\begin{aligned} \int_Q f\big(x, z, \xi + \nabla\psi(y) + \nabla_z\overline{\psi}(y,z)\big)\, dy \ &\geqslant\ f\Big(x, z, \int_Q \big[\, \xi + \nabla\psi(y) + \nabla_z \overline{\psi}(y,z)\,\big]\, dy\Big) \\ &=\ f\Big(x, z, \xi + \nabla_z\Big(\int_Q \overline{\psi}(y,z)\, dy\Big)\Big). \end{aligned} \tag{3.26}$$

Thus by (3.18), (3.25), (3.26) and once more Fubini's Theorem

$$f_{\text{hom}}(\xi) \geqslant \int_Q f\Big(z, \xi + \nabla_z\Big(\int_Q \overline{\psi}(y,z)\, dy\Big)\Big)\, dz \geqslant f^{\star\star}_{\text{hom}}(\xi) = f^{\star}_{\text{hom}}(\xi).$$

Step 2. We show now the general case. For each $\varepsilon > 0$ set $\zeta_\varepsilon(\eta) := \frac{1}{\varepsilon^N}\zeta\Big(\frac{\eta}{\varepsilon}\Big)$ where $\zeta \in C^\infty(\mathbb{R}^{d\times N})$ denotes the standard mollifier, that is,

$$\zeta(\eta) := \begin{cases} C\exp\Big(\frac{1}{|\eta|^2 - 1}\Big) & \text{if } |\eta| < 1, \\ 0 & \text{if } |\eta| \geqslant 1, \end{cases}$$

and the constant C is selected so that $\int_{\mathbb{R}^{d\times N}} \zeta(\eta)\, d\eta = 1$. Let

$$f_\varepsilon(x,y,\xi) := \int_{B(0,\varepsilon)} \zeta_\varepsilon(\eta) f(x,y,\xi - \eta)\, d\eta$$

for all $\varepsilon > 0$ and all $(x,y,\xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$. It is straightforward to show that f_ε satisfies conditions $(H_1) - (H_5)$. Fixed $\delta > 0$, by density let $S \in \mathbb{N}$ and $\psi \in W^{1,\infty}((0,S)^N;\mathbb{R}^d)$ be such that

$$f_{\text{hom}}(x,\xi) + \delta \geqslant \frac{1}{S^N}\int_{(0,S)^N} f(x,y,\xi + \nabla\psi(y))\, dy.$$

Then

$$\begin{aligned} f_{\text{hom}}(x,\xi) + \delta \ &\geqslant\ \lim_{\varepsilon\to 0} \frac{1}{S^N}\int_{(0,S)^N} f_\varepsilon(x,y,\xi + \nabla\psi(y))\, dy \\ &\geqslant\ \limsup_{\varepsilon\to 0}\, (f_\varepsilon)_{\text{hom}}(x,\xi) \\ &=\ \limsup_{\varepsilon\to 0}\, (f_\varepsilon)^{\star\star}_{\text{hom}}(x,\xi) \\ &\geqslant\ f^{\star\star}_{\text{hom}}(x,\xi) \end{aligned}$$

since f_ε is convex and $f_\varepsilon \geqslant f$ for all $\varepsilon > 0$. Indeed it is easy to show that f_ε is convex and the last assertion is a consequence of the convexity of f and Jensen's Inequality in Banach spaces (see e.g. Lemma 23.2 in Dal Maso [24]).

■

We finish this section by remarking that in the scalar case, that is when $d = 1$, the identity (3.17) still holds independent of any convexity assumption on f (see Müller [46]).

4 Proof of Theorem 1.1

As in the previous section, by Remark 1.2, we may suppose that f is a positive Borel function satisfying hypotheses (H_1), (H_3) and (H_4) for every $(x, y, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$. Due to the p-coercivity condition in (H_4) to prove Theorem 1.1 it suffices to show that

$$\Gamma(L^p(\Omega))\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u) = \int_\Omega f_{\text{hom}}(x, u(x), \nabla u(x))\, dx, \tag{4.1}$$

for all $u \in W^{1,p}(\Omega; \mathbb{R}^d)$, where f_{hom} is the function defined in (1.5), since

$$\Gamma(L^p(\Omega))\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u) = \infty$$

for all $u \in L^p(\Omega; \mathbb{R}^N) \setminus W^{1,p}(\Omega; \mathbb{R}^d)$. The idea behind the proof of identity (4.1) is to use the direct method of Γ- convergence, first outlined by De Giorgi (see [27]; see also Dal Maso [24] and De Giorgi and Dal Maso [26]) and later used by many other authors. Accordingly, we start by localizing the functionals $\mathcal{I}_\varepsilon$ in order to highlight their dependence on the domain of integration, that is, we consider a family of functionals $\mathcal{I}_\varepsilon : L^p(\Omega; \mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0, \infty]$ defined by

$$\mathcal{I}_\varepsilon(u; A) := \begin{cases} \int_A f\left(x, \frac{x}{\varepsilon}, \nabla u(x)\right)\, dx & \text{if } u \in W^{1,p}(A; \mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases}$$

Our goal is to show that

$$\Gamma(L^p(\Omega))\text{-}\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u, A) = \int_A f_{\text{hom}}(x, \nabla u(x))\, dx, \tag{4.2}$$

for all $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$. In particular (4.1) will follow by taking $A = \Omega$.

The next step toward the proof of Theorem 1.1 is to establish a compactness property that ensures the existence of Γ-converging subsequences of $\mathcal{I}_\varepsilon$. Namely, the following result holds.

Proposition 4.1. *For every sequence $\{\varepsilon_n\}_n$ of positive real numbers converging to zero there exists a further subsequence $\{\varepsilon_{n_j}\}_j \equiv \{\varepsilon_j\}_j$ such that*

$$\Gamma(L^p(A))\text{-}\lim_{j\to\infty} \mathcal{I}_{\varepsilon_j}(\cdot; A)(u) =: \mathcal{I}_{\{\varepsilon_j\}}(u; A) \tag{4.3}$$

exists for all $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ and all $A \in \mathcal{A}(\Omega)$.

The proof of this proposition follows an argument analog to the one used in Braides, Fonseca and Francfort [14], but for completeness we present it here.

Let $\mathcal{C}$ be a countable collection of subsets of Ω such that, for any $\delta > 0$ and any $A \in \mathcal{A}(\Omega)$, there exists a finite union C_A of disjoint elements of $\mathcal{C}$ satisfying

$$\begin{cases} \overline{C_A} \subset A, \\ \mathcal{L}^N(A) \leqslant \mathcal{L}^N(C_A) + \delta. \end{cases}$$

We may take $\mathcal{C}$ as the set of open squares with faces parallel to the axes, centered at points $x \in \Omega \cap \mathbb{Q}^N$ and with rational edge lengths. We denote by $\mathcal{R}$ the countable collection of all finite unions of elements of $\mathcal{C}$, i.e,

$$\mathcal{R} = \left\{ \bigcup_{i=1}^k C_i : \ k \in \mathbb{N},\, C_i \in \mathcal{C} \right\}.$$

The next lemma is the start point for the proof of Proposition 4.1.

Lemma 4.2. *For every sequence $\{\varepsilon_n\}_n$ of positive real numbers converging to zero there exists a further subsequence $\{\varepsilon_{n_j}\}_j$ (depending in $\mathcal{R}$) such that the Γ-limit*

$$\Gamma(L^p(C))\text{-}\lim_{j\to\infty} \mathcal{I}_{\varepsilon_{n_j}}(\cdot;C)(u) =: \mathcal{I}^{\{\varepsilon_{n_j}\}}(u;C) \tag{4.4}$$

exists for all $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ and all $C \in \mathcal{R}$.

Proof. Let $C \in \mathcal{R}$. From Proposition 2.4 and as $L^p(\Omega;\mathbb{R}^d)$ is separable there exist a subsequence $\{\varepsilon_{n_j}\}_j$ (depending on C) such that the $\Gamma(L^p(C))$-limit of $\mathcal{I}_{\varepsilon_{n_j}}(\cdot;C)$ exists for all $u \in W^{1,p}(\Omega, \mathbb{R}^d)$. But then by a diagonalization procedure we can find a subsequence $\{\varepsilon_{n_j}\}_j$ (depending on $\mathcal{R}$) such that (4.4) holds. ■

Let now $\{\varepsilon_n\}_n$ be a fixed sequence of positive real numbers converging to zero and $\{\varepsilon_j\}_j$ a subsequence for which (4.4) holds.

In order to prove that the Γ-limit in (4.3) exists for all $A \in \mathcal{A}(\Omega)$, it is crucial to establish the existence of recovering sequences for the Γ-limit in (4.4) with identical values on the boundaries of the elements of $\mathcal{R}$. Precisely, it is important to establish the following result.

Lemma 4.3. *Given $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and $C \in \mathcal{R}$ there exists a sequence $\{w_j\} \subset W^{1,p}(C;\mathbb{R}^d)$ such that*

i) *$w_j \to u$ strongly in $L^p(C;\mathbb{R}^d)$;*

ii) $\Gamma(L^p(C))\text{-}\lim_{j\to\infty} \mathcal{I}_{\varepsilon_j}(\cdot;C)(u) = \lim_{j\to\infty} \int_C f\left(x, \frac{x}{\varepsilon_j}, \nabla w_j(x)\right) dx;$

iii) *$w_j = u$ in $C \setminus \mathcal{K}_j$ for some compact subset $\mathcal{K}_j \subset C$ with $|C \setminus \mathcal{K}_j| \to 0$, for every $j \in \mathbb{N}$.*

Proof. The proof relies on De Giorgi's slicing argument introduced in De Giorgi [27]. Let $C \in \mathcal{R}$ and let $\{v_j\} \subset W^{1,p}(\Omega;\mathbb{R}^d)$ be a sequence given by Lemma 4.2 such that $||u - v_j||_{L^p(C)} \to 0$ and

$$\mathcal{I}^{\{\varepsilon_j\}}(u;C) = \lim_{j\to\infty} \mathcal{I}_{\varepsilon_j}(v_j;C).$$

Set

$$\beta_0 := \sup_j \int_C (1 + |Dv_j|^p)\, dx < \infty \text{ (by } (H_4)),$$

and define for $j \in \mathbb{N}$

$$K_j := \left|\left[\frac{1}{||v_j - u||^{\frac{1}{2}}{}_{L^p(C)}}\right]\right|,$$

$$M_j := \left|\left[\sqrt{K_j}\,\right]\right|,$$

and finally

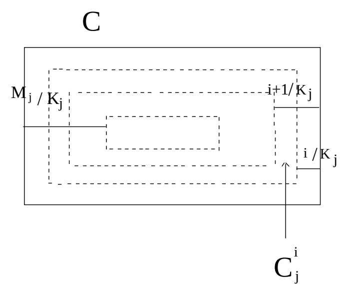
$$C_j := \left\{x \in C : \ \operatorname{dist}(x, \partial C) < \frac{M_j}{K_j}\right\}.$$

We observe that by definition $K_j \uparrow \infty$ and $\mathcal{L}^N(C_j) \downarrow 0$ as $j \to \infty$. For each j subdivide C_j into M_j disjoint subsets

$$C_j^i := \left\{x \in C_j : \ \operatorname{dist}(x, \partial C) \in \left[\frac{i}{K_j}, \frac{i+1}{K_j}\right]\right\}, \quad i = 0, ..., M_j - 1,$$

and choose $i_j \in \{0, ..., M_j - 1\}$ such that

$$\int_{C_j^{i_j}} (1 + |Dv_j|^p)\, dx \leqslant \int_{C_j^i} (1 + |Dv_j|^p)\, dx$$



for all $i = 0, ..., M_j - 1$. Then

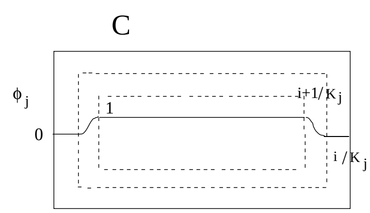
$$M_j \int_{C_j^{i_j}} (1 + |Dv_j|^p)\, dx \leqslant \int_{C_j} (1 + |Dv_j|^p)\, dx = \sum_{i=0}^{M_j - 1} \int_{C_j^i} (1 + |Dv_j|^p)\, dx \leqslant \int_C (1 + |Dv_j|^p)\, dx \leqslant \beta_0,$$

or, equivalently,

$$\int_{C_j^{i_j}} (1 + |Dv_j|^p)\, dx \leqslant \frac{\beta_0}{M_j}.$$

Let $\phi_j \in C_0^\infty(C)$ be such that $0 \leqslant \phi_j \leqslant 1$, $||D\phi_j||_{L^\infty} \leqslant K_j$,

$$\phi_j := \begin{cases} 1 & \text{if } \operatorname{dist}(x, \partial C) \geqslant \frac{i_j + 1}{K_j}, \\ \\ 0 & \text{if } \operatorname{dist}(x, \partial C) \leqslant \frac{i_j}{K_j}, \end{cases}$$



and define $w_j := \phi_j v_j + (1 - \phi_j) u \in W^{1,p}(C; \mathbb{R}^d)$. Clearly $w_j \to u$ strongly in $L^p(C; \mathbb{R}^d)$, $w_j = u$ in $C \backslash \mathcal{K}_j$, with $\mathcal{K}_j := \left\{ x \in C : \ \operatorname{dist}(x, \partial C) \geqslant \frac{i_j}{K_j} \right\}$ and $|C \backslash \mathcal{K}_j| \to 0$. Moreover

$$\begin{aligned} I^{\{\varepsilon_j\}}(u; C) &= \lim_{j \to \infty} I_{\varepsilon_j}(v_j; C) \\ &\geqslant \limsup_{j \to \infty} \int_{C \cap \left\{ x:\ \operatorname{dist}(x, \partial C) \geqslant \frac{i_j + 1}{K_j} \right\}} f\left(x, \frac{x}{\varepsilon_j}, Dw_j \right) dx. \end{aligned}$$

And consequently

$$
\begin{aligned}
\mathcal{I}^{\{\varepsilon_j\}}(u;C) \quad &\geqslant \limsup_{j\to\infty} \int_C f\left(x, \frac{x}{\varepsilon_j}, Dw_j\right) - \alpha \ \liminf_{j\to\infty} \int_{C_j \cap \left\{x:\ \mathrm{dist}(x,\partial C) < \frac{i_j}{K_j}\right\}} (1+|Du|^p)\, dx \\
&\quad -\alpha\beta \ \liminf_{j\to\infty} \textstyle\int_{C_j^{i_j}} (1+|Dv_j|^p)\, dx - \alpha\beta \ \liminf_{j\to\infty} |K_j|^p \displaystyle\int_{C_j^{i_j}} |v_j - u|^p\, dx \\
&\quad -\alpha\beta \ \liminf_{j\to\infty} \textstyle\int_{C_j^{i_j}} |Du|^p \\
&\geqslant \limsup_{j\to\infty} \int_C f\left(x, \frac{x}{\varepsilon_j}, Dw_j\right) - \alpha\beta\beta_0 \ \liminf_{j\to\infty} \frac{1}{M_j} - \alpha\beta \ \liminf_{j\to\infty} ||v_j - u||^{\frac{p}{2}}_{L^p(C)} \\
&= \limsup_{j\to\infty} I_{\varepsilon_j}(w_j; C),
\end{aligned}
$$

where α is the constant given in hypothesis (H_4) and β is some positive constant. Then

$$
\begin{aligned}
\mathcal{I}^{\{\varepsilon_j\}}(u;C) \quad &\geqslant \limsup_{j\to\infty} \int_\Omega f\left(x, \frac{x}{\varepsilon_j}, Dw_j\right) - \alpha\beta\beta_0 \ \liminf_{j\to\infty} \frac{1}{M_j} - \alpha\beta \ \liminf_{j\to\infty} ||v_j - u||^{\frac{p}{2}}_{L^p(\Omega)} \\
&= \limsup_{n\to\infty} \mathcal{I}_{\varepsilon_j}(w_j, C),
\end{aligned}
$$

Accordingly

$$
\Gamma(L^p(C))\text{-} \ \lim_{j\to\infty} \mathcal{I}_{\varepsilon_j}(\cdot; C)(u) = \lim_{j\to\infty} \int_C f\left(x, \frac{x}{\varepsilon_j}, \nabla w_j(x)\right)\, dx.
$$

■

Proof of Proposition 4.1. We wish to show that for all $A \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(\Omega, \mathbb{R}^d)$

$$
\begin{aligned}
&\inf\left\{ \liminf_{j\to\infty} \int_A f\Big(x, \frac{x}{\varepsilon_j}, \nabla u_j(x)\Big)\, dx :\ u_j \in W^{1,p}(A, \mathbb{R}^d), u_j \overset{L^p(A;\mathbb{R}^d)}{\longrightarrow} u \right\} \\
&= \inf\left\{ \limsup_{j\to\infty} \int_A f\Big(x, \frac{x}{\varepsilon_j}, \nabla u_j(x)\Big)\, dx :\ u_j \in W^{1,p}(A, \mathbb{R}^d), u_j \overset{L^p(A;\mathbb{R}^d)}{\longrightarrow} u \right\}.
\end{aligned} \tag{4.5}
$$

Fixed $A \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(\Omega, \mathbb{R}^d)$, to prove (4.5) it suffices to find a sequence $\{v_j\}_j \subset W^{1,p}(A, \mathbb{R}^d)$ with $v_j \overset{L^p(A;\mathbb{R}^d)}{\longrightarrow} u$ and such that

$$
\begin{aligned}
&\inf\left\{ \liminf_{j\to\infty} \int_A f\left(x, \frac{x}{\varepsilon_j}, \nabla u_j(x)\right)\, dx :\ u_j \in W^{1,p}(A, \mathbb{R}^d), u_j \overset{L^p(A;\mathbb{R}^d)}{\longrightarrow} u \right\} \\
&\geqslant \limsup_{j\to\infty} \int_A f\left(x, \frac{x}{\varepsilon_j}, \nabla v_j(x)\right)\, dx.
\end{aligned} \tag{4.6}
$$

Fix $\delta > 0$ and choose $C_\delta \in \mathcal{R}$ with $\overline{C_\delta} \subset A$ and $\mathcal{L}^N(A \setminus C_\delta) << 1$, so that

$$\int_{A\setminus C^\delta}(1+|\nabla u|^p)\,dx \leqslant \frac{\delta}{\alpha}, \tag{4.7}$$

where α is the constant in (H_4). By Lemma 4.3 consider a sequence $\{w_j^\delta\}\in W^{1,p}(C^\delta,\mathbb{R}^d)$ such that

$$w_j^\delta \overset{L^p(C^\delta;\mathbb{R}^d)}{\longrightarrow} u,$$

$$\Gamma(L^p(C^\delta))\text{-}\lim_{j\to\infty}\mathcal{I}_{\varepsilon_j}(\cdot\,;C^\delta)(u)=\lim_{j\to\infty}\int_{C^\delta} f\left(x,\frac{x}{\varepsilon_j},Dw_j^\delta(x)\right)dx, \tag{4.8}$$

and $w_j^\delta = u$ outside a compact subset $\mathcal{K}_j^\delta$ of C^δ with $|C^\delta\setminus\mathcal{K}_j^\delta|\to 0$ (we could also have used Proposition 11.7 in Braides and Defranceschi [13]). Extending w_j^δ by u outside C^δ (still denoted by w_j^δ) it follows that $w_j^\delta \overset{L^p(A;\mathbb{R}^d)}{\longrightarrow} u$, and in view of (H_3), (4.7) and (4.8) we obtain that

$$\begin{aligned}
&\limsup_{\delta\to 0}\ \limsup_{j\to\infty}\int_A f\left(x,\frac{x}{\varepsilon_j},\nabla w_j^\delta(x)\right)dx \qquad (4.9)\\
\leqslant\ &\limsup_{\delta\to 0}\ \lim_{j\to\infty}\int_{C^\delta} f\left(x,\frac{x}{\varepsilon_j},\nabla w_j^\delta(x)\right)dx+\alpha\ \limsup_{\delta\to 0}\int_{A\setminus C^\delta} C(1+|\nabla u|^p)\,dx\\
=\ &\limsup_{\delta\to 0}\Gamma(L^p(C^\delta))\text{-}\lim_{j\to\infty}\mathcal{I}_{\varepsilon_j}(\cdot;C^\delta)(u)\\
=\ &\limsup_{\delta\to 0}\ \inf\left\{\liminf_{j\to\infty}\int_{C_\delta} f\left(x,\frac{x}{\varepsilon_j},\nabla u_j(x)\right)dx:\ u_j\in W^{1,p}(C_\delta,\mathbb{R}^d),\ u_j \overset{L^p(C_\delta;\mathbb{R}^d)}{\longrightarrow} u\right\}\\
=\ &\inf\left\{\liminf_{j\to\infty}\int_A f\left(x,\frac{x}{\varepsilon_j},\nabla u_j(x)\right)dx:\ u_j\in W^{1,p}(A,\mathbb{R}^d),\ u_j \overset{L^p(A;\mathbb{R}^d)}{\longrightarrow} u\right\}\ \text{(by 4.9)}\\
\leqslant\ &\liminf_{\delta\to 0}\ \liminf_{j\to\infty}\int_A f\left(x,\frac{x}{\varepsilon_j},\nabla w_j^\delta(x)\right)dx.
\end{aligned}$$

By Lemma 5.2 there exists a decreasing sequence $\{\delta(\varepsilon_j)\}\searrow 0$ such that

$$\begin{aligned}
&\lim_{j\to\infty}\textstyle\int_A f\left(x,\frac{x}{\varepsilon_j},\nabla w_j^{\delta(\varepsilon_j)}(x)\right)dx\\
&=\inf\left\{\liminf_{j\to\infty}\textstyle\int_A f(x,\frac{x}{\varepsilon_j},\nabla u_j(x))\,dx:\ u_j\in W^{1,p}(A,\mathbb{R}^d),\ u_j \overset{L^p(A;\mathbb{R}^d)}{\longrightarrow} u\right\},
\end{aligned}$$

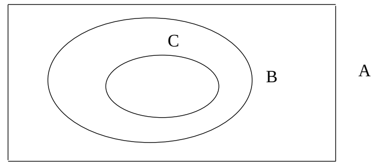
for $v_j := w_j^{\delta(\varepsilon_j)}$, and this implies (4.6).

■

We now seek to ensure that $\mathcal{I}^{\{\varepsilon_j\}}$, regarded both as a functional on $W^{1,p}(\Omega,\mathbb{R}^d)$ and as a set function, admits an integral representation of the form

$$\mathcal{I}^{\{\varepsilon_j\}}(u;A)=\int_A g^{\{\varepsilon_j\}}(x,\nabla u(x))\,dx.$$

We will verify that the hypotheses of Theorem 2.10 hold. Using Lemma 2.12 and the conditions imposed on f, it is possible to show that $\mathcal{I}^{\{\varepsilon_j\}}(u;.)$ is a measure, precisely we prove the following result.



Lemma 4.4. *For each $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, $\mathcal{I}^{\{\varepsilon_j\}}(u;.)$ is the trace of a finite, positive Radon measure restricted to $\mathcal{A}(\Omega)$.*

Proof. Let $u \in W^{1,p}(\Omega;\mathbb{R}^d)$. In view of Lemma 4.1 let $\{u_j\} \subset W^{1,p}(\Omega;\mathbb{R}^d)$ be a sequence such that

$$I^{\{\varepsilon_j\}}(u;\Omega) = \lim_{j\to\infty} \int_\Omega f\left(x, \frac{x}{\varepsilon_j}, \nabla u_j(x)\right) dx,$$

and consider $\mu_j := f(\cdot, \frac{\cdot}{\varepsilon_j}, \nabla u_j)\chi_\Omega(\cdot)\mathcal{L}^N$. By (H_4), and up to a subsequence (still denoted by μ_j), there exists a finite positive Radon measure on $\mathbb{R}^N$ such that

$$\mu_j \overset{\star}{\rightharpoonup} \mu.$$

We claim that $\mathcal{I}^{\{\varepsilon_j\}}(u;.)\lfloor\mathcal{A}(\Omega) = \mu$, that is $\mathcal{I}^{\{\varepsilon_j\}}(u;A) = \mu(A)$ for all $A \in \mathcal{A}(\Omega)$. We apply Lemma 2.12 with $\Pi(\cdot) = \mathcal{I}^{\{\varepsilon_j\}}(u;.)$.

We start by proving that condition i) in Lemma 2.12 holds, that is $\mathcal{I}^{\{\varepsilon_j\}}(u;.)$ is nested-subadditive. Given A, B, $C \in \mathcal{A}(\Omega)$ with $C \subset\subset B \subset A$ we have to show that

$$\mathcal{I}^{\{\varepsilon_j\}}(u;A) \leqslant \mathcal{I}^{\{\varepsilon_j\}}(u;B) + \mathcal{I}^{\{\varepsilon_j\}}(u;A \setminus \overline{C}).$$

Choose C^δ and $D^\delta \in \mathcal{R}$ with $C^\delta \subset C$ and $D^\delta \subset A \setminus \overline{C}$ such that

$$\int_{C\setminus C^\delta} (1 + |\nabla u|^p)\, dx < \delta \quad \text{and} \quad \int_{(A\setminus\overline{C})\setminus D^\delta} (1 + |\nabla u|^p)\, dx < \delta.$$

By Lemma 4.3 there exist two sequences $\{v^{C^\delta}\} \subset W^{1,p}(C^\delta;\mathbb{R}^d)$ and $\{v_j^{D^\delta}\} \subset W^{1,p}(D^\delta;\mathbb{R}^d)$ satisfying

$$||v_j^{D^\delta} - u||_{L^p(D^\delta;\mathbb{R}^d)} \to 0, \quad ||v_j^{C^\delta} - u||_{L^p(C^\delta;\mathbb{R}^d)} \to 0,$$

$$\mathcal{I}^{\{\varepsilon_j\}}(u;D^\delta) = \lim I_{\varepsilon_j}(v_j^{D^\delta};D^\delta), \quad \mathcal{I}^{\{\varepsilon_j\}}(u;C^\delta) = \lim I_{\varepsilon_j}(v_j^{C^\delta};C^\delta),$$

$$v_j^{D^\delta} = u \quad \text{on} \quad \partial D^\delta \quad \text{and} \quad v_j^{C^\delta} = u \quad \text{on} \quad \partial C^\delta.$$

Extend $v_j^{C^\delta}$ and $v_j^{D^\delta}$ by u to all A and set

$$w_j^\delta := \begin{cases} v_j^{D^\delta} & \text{if} \quad x \in A \setminus \overline{C} \\ v_j^{C^\delta} & \text{if} \quad x \in C. \end{cases}$$

Clearly $||w_j^\delta - u||_{L^p(A;\mathbb{R}^d)} \to 0$ and we have

$$
\begin{aligned}
I^{\{\varepsilon_j\}}(u;A) &\leqslant \liminf_{\delta\to 0}\ \liminf_{j\to\infty} I_{\varepsilon_j}(w_j^\delta;A) \\
&\leqslant I^{\{\varepsilon_j\}}(u;D^\delta) + I^{\{\varepsilon_j\}}(u;C^\delta) + \lim_{\delta\to 0}\int_{(A\setminus\overline{C})\setminus D^\delta\cup(C\setminus C^\delta)} (1+|\nabla u|^p)\,dx \\
&= I^{\{\varepsilon_j\}}(u;D^\delta) + I^{\{\varepsilon_j\}}(u;C^\delta).
\end{aligned}
$$

To establish condition ii) in Lemma 2.12: Given $A \in \mathcal{A}(\Omega)$ and $\varepsilon > 0$, consider $A_\varepsilon \in \mathcal{A}(\Omega)$ such that $\bar{A}_\varepsilon \subset A$ and

$$
\alpha(1+|\nabla u|^p)\mathcal{L}^N\lfloor(\Omega)(A\setminus\bar{A}_\varepsilon) < \varepsilon.
$$

Due to the growth conditions (H_4)

$$
\begin{aligned}
\mathcal{I}^{\{\varepsilon_j\}}(u;A\setminus\bar{A}_\varepsilon) &\leqslant \liminf_{j\to\infty} \int_{A\setminus\bar{A}_\varepsilon} f(x,\tfrac{x}{\varepsilon_j},\nabla u(x))\,dx \\
&\leqslant \alpha\int_{A\setminus\bar{A}_\varepsilon}(1+|\nabla u(x)|^p)\,dx \\
&\leqslant \varepsilon.
\end{aligned}
$$

To show iv) fix $A \in \mathcal{A}(\Omega)$. We have

$$
\begin{aligned}
\mathcal{I}^{\{\varepsilon_j\}}(u;A) &\leqslant \liminf_{j\to\infty} \int_A f(x,\tfrac{x}{\varepsilon_j},\nabla u_j(x))\,dx \\
&= \liminf_{j\to\infty} \mu_j(A) \\
&\leqslant \mu(\overline{A}).
\end{aligned}
$$

Finally, to establish iii) take $\Omega' \subset \mathbb{R}^N$ such that $\Omega \subset\subset \Omega'$. As $\{\mu_j\}_j$ weakly converge to μ

$$
\mu(\Omega') = \lim_{j\to\infty}\mu_j(\Omega') = \lim_{j\to\infty}\int_\Omega f\Big(x,\frac{x}{\varepsilon_j},\nabla u_j(x)\Big)\,dx = \mathcal{I}^{\varepsilon_j}(u;\Omega).
$$

Therefore

$$
\mu(\Omega') \leqslant \mathcal{I}^{\{\varepsilon_j\}}(u;\Omega).
$$

for all such Ω'. Hence $\mathcal{I}^{\{\varepsilon_j\}}(u;\Omega) \geqslant \mu(\mathbb{R}^N)$, and as a consequence of Lemma 2.12 we conclude that $\mathcal{I}^{\{\varepsilon_j\}}(u;A) = \mu(A)$ for all $A \in \mathcal{A}(\Omega)$.

■

By Proposition 2.5 the functional $\mathcal{I}^{\{\varepsilon_j\}}(.,A)$ is lower semicontinuous with respect to the L^p- topology for all $A \in \mathcal{A}(\Omega)$, hence it is sequentially lower semicontinuous with respect to the weak topology in $W^{1,p}$. As a consequence of the integral representation Theorem 2.10 and Remark 2.11 we derive an integral representation formula for $\mathcal{I}^{\{\varepsilon_j\}}$.

Lemma 4.5. *There exist a Carathéodory function*

$$
g^{\{\varepsilon_j\}} : \Omega\times\mathbb{R}^{d\times N} \to [0,\infty)
$$

quasiconvex with respect to its second variable for a.e. $x \in \Omega$ satisfying the same growth conditions than f does, and such that

$$\mathcal{I}^{\{\varepsilon_j\}}(u,A) = \int_A g^{\{\varepsilon_j\}}(x,\nabla u(x))\,dx$$

for all $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ *and* $A \in \mathcal{A}(\Omega)$.

To conclude that (4.2) holds, by Proposition 2.3 it suffices to prove that the function $g^{\{\varepsilon_j\}}$ is independent on this particular (sub)sequence, so that each Γ-convergent (sub)sequence has the same limit. The remaining of this section is devoted to showing that

$$g^{\{\varepsilon_j\}}(x,\xi) = f_{\mathrm{hom}}(x,\xi) \tag{4.10}$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^{d\times N}$. To start, let $T \in \mathbb{N}$ and let $\mathcal{S}_T$ denote a countable set of $C_c^\infty((0,T)^N;\mathbb{R}^d)$-functions dense in $W_0^{1,p}((0,T)^N;\mathbb{R}^d)$. Let L be the set of Lebesgue points x_0 for all functions

$$g^{\{\varepsilon_j\}}(\cdot,\eta) \tag{4.11}$$

and

$$x \to \int_Q f(x,Ty,\eta + \nabla\phi(Ty))\,dy, \tag{4.12}$$

with $\phi \in \mathcal{S}_T$, $\eta \in \mathbb{Q}^{d\times N}$ and $T \in \mathbb{N}$.

Since the function $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,\infty)$ is of Carathéodory-type, by Scorza-Dragoni Theorem (see Ekeland and Temam [32]) there exists a non decreasing sequence of compact subsets $\{K_m\}_{m\in\mathbb{N}} \subset \Omega$ with $|\Omega\backslash K_m| \leqslant \frac{1}{m}$ such that $f : K_m \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,\infty)$ is continuous for all $m \in \mathbb{N}$. Let $K_m^\star$ be the set of Lebesgue points for χ_{K_m} with $m \in \mathbb{N}$, and define

$$W := \bigcup_{m=0}^{\infty} (K_m \cap K_m^\star) \quad \text{and} \quad E := L \cap W.$$

We have $|\Omega \setminus L| = 0$ and $|\Omega\backslash W| \leqslant |\Omega\backslash K_m| \leqslant \frac{1}{m}$ for each $m \in \mathbb{N}$. Consequently $|\Omega \setminus W| = 0$ and $|\Omega \setminus E| = 0$. We are ready now to show equality (4.10).

Proposition 4.6. $g^{\{\varepsilon_j\}}(x_0,\xi) = f_{\mathrm{hom}}(x_0,\xi)$ *for all* $x_0 \in E$ *and* $\xi \in \mathbb{Q}^{d\times N}$.

Proof. Consider $x_0 \in E$ and $\xi \in \mathbb{Q}^{d\times N}$. We denote by $Q(x_0,\delta)$ the cube in $\mathbb{R}^3$ centered in x_0 and of radius $\delta > 0$. By (4.11) we have

$$\begin{aligned} g^{\{\varepsilon_j\}}(x_0,\xi) &= \lim_{\delta\to 0} \tfrac{1}{\delta^N} \textstyle\int_{Q(x_0,\delta)} g^{\{\varepsilon_j\}}(x,\xi)\,dx \\ &= \lim_{\delta\to 0} \tfrac{\mathcal{I}^{\{\varepsilon_j\}}(\xi\,\cdot\,;Q(x_0,\delta))}{\delta^N}. \end{aligned} \tag{4.13}$$

Step 1. We first establish the upper bound inequality for the Γ-limit of $\{\mathcal{I}_{\varepsilon_j}\}$, i.e.

$$g^{\{\varepsilon_j\}}(x_0,\xi) \leqslant f_{\mathrm{hom}}(x_0,\xi).$$

Given $n \in \mathbb{N}$, let $T_n \in \mathbb{N}$ and $\overline{\phi}_n \in W_0^{1,p}((0,T_n)^N;\mathbb{R}^d)$ such that

$$f_{\mathrm{hom}}(x_0,\xi) + \frac{1}{2n} \geqslant \frac{1}{{T_n}^N} \int_{(0,T_n)^N} f(x_0,y,\xi + \nabla\overline{\phi}_n(y))\,dy.$$

By conditions (H_1) and (H_4), and by the density of $\mathcal{S}_{T_n}$ in $W_0^{1,p}((0,T_n)^N;\mathbb{R}^d)$ we may take $\phi_n \in \mathcal{S}_{T_n}$ with

$$f_{\mathrm{hom}}(x_0,\xi)+\frac{1}{n}\geqslant \frac{1}{{T_n}^N}\int_{(0,T_n)^N} f(x_0,y,\xi+\nabla\phi_n(y))\,dy.$$

Extend ϕ_n periodically with period T_n to $\mathbb{R}^N$ (still denoted by $\{\phi_n\}_n$). For $x\in\mathbb{R}^N$ define

$$u_j^n(x):=\xi\cdot x+\varepsilon_j\phi_n\Big(\frac{x}{\varepsilon_j}\Big)$$

and let $\delta>0$ be small enough so that $Q(x_0,\delta)\in\mathcal{A}(\Omega)$. As $\{\phi_n\}_n$ is bounded in $W^{1,p}$, for fixed n we have $\lim_{j\to\infty} u_j^n=v$ in $L^p(Q(x_0;\delta);\mathbb{R}^d)$ where $v(x)=\xi\cdot x$. Hence by equality (4.13)

$$g^{\{\varepsilon_j\}}(x_0,\xi)\leqslant \liminf_{\delta\to 0}\ \liminf_{j\to\infty}\frac{1}{\delta^N}\int_{Q(x_0;\delta)} f\left(x,\frac{x}{\varepsilon_j},\xi+\nabla\phi_n\left(\frac{x}{\varepsilon_j}\right)\right)dx. \tag{4.14}$$

Define now $h_n(x,y):=f(x,T_ny,\xi+\nabla\phi_n(T_ny))$ for all $x\in\Omega$ and $y\in\mathbb{R}^N$, and for all $n\in\mathbb{N}$. Clearly $h_n\in L^1(Q(x_0;\delta);C_{\mathrm{per}}(Q;\mathbb{R}^d))$ for $n\in\mathbb{N}$ (see Lemma 2.15) and then by Lemma 2.14

$$\begin{aligned}
&\lim_{j\to\infty}\int_{Q(x_0;\delta)} f\left(x,\frac{x}{\varepsilon_j},\xi+\nabla\phi_n\left(\frac{x}{\varepsilon_j}\right)\right)dx\\
&=\lim_{j\to\infty}\int_{Q(x_0;\delta)} f\left(x,\frac{T_nx}{T_n\varepsilon_j},\xi+\nabla\phi_n\left(\frac{T_nx}{T_n\varepsilon_j}\right)\right)dx\\
&=\lim_{j\to\infty}\int_{Q(x_0;\delta)} h_n\left(x,\frac{x}{T_n\varepsilon_j}\right)dx\\
&=\int_{Q(x_0;\delta)}\int_Q h_n(x,y)\,dy\,dx\\
&=\int_{Q(x_0;\delta)}\int_Q f(x,T_ny,\xi+\nabla\phi_n(T_ny))\,dy\,dx
\end{aligned}\tag{4.15}$$

(Allaire [2] shows with counterexamples that this convergence does not hold if the continuity in one of the variables of f is not assumed). Therefore by identities (4.12), (4.14) and (4.15)

$$\begin{aligned}
g^{\{\varepsilon_j\}}(x_0,\xi)\ &\leqslant \liminf_{\delta\to 0}\frac{1}{\delta^N}\int_{Q(x_0;\delta)}\int_Q f(x,T_ny,\xi+\nabla\phi_n(T_ny))\,dy\,dx\\
&=\int_Q f(x_0,T_ny,\xi+\nabla\phi_n(T_ny))\,dy\\
&=\frac{1}{{T_n}^N}\int_{(0,T_n)^N} f(x_0,y,\xi+\nabla\phi_n(y))\,dy\\
&\leqslant f_{\mathrm{hom}}(x_0,\xi)+\frac{1}{n}.
\end{aligned}$$

Letting $n\to\infty$ we deduce that

$$g^{\{\varepsilon_j\}}(x_0,\xi)\leqslant f_{\mathrm{hom}}(x_0,\xi). \tag{4.16}$$

Step 2. We now show that the converse inequality holds, that is, we show that the upper bound inequality for the Γ-limit of $\{\mathcal{I}_{\varepsilon_j}\}$

$$g^{\{\varepsilon_j\}}(x_0,\xi)\geqslant f_{\mathrm{hom}}(x_0,\xi)$$

holds. Fix $\delta > 0$ small enough so that $Q(x_0;\delta) \in \mathcal{A}(\Omega)$, and consider $\{u_j^\delta\}_j \subset W^{1,p}(Q(x_0;\delta);\mathbb{R}^d)$ with $\lim_{j\to\infty} u_j^\delta = 0$ in $L^p(Q(x_0;\delta);\mathbb{R}^d)$ and

$$I^{\{\varepsilon_j\}}(\xi\,\cdot\,; Q(x_0;\delta)) = \lim_{j\to\infty} \int_{Q(x_0;\delta)} f\left(x, \frac{x}{\varepsilon_j}, \xi + \nabla u_j^\delta(x)\right)\, dx.$$

By identity (4.13)

$$\begin{aligned} g^{\{\varepsilon_j\}}(x_0,\xi) &= \lim_{\delta\to 0}\ \lim_{j\to\infty} \tfrac{1}{\delta^N} \int_{Q(x_0;\delta)} f\left(x, \tfrac{x}{\varepsilon_j}, \xi + \nabla u_j^\delta(x)\right)\, dx \\ &= \lim_{\delta\to 0} \tfrac{1}{\delta^N} \int_{Q(x_0;\delta)} f\left(x, \tfrac{x}{\varepsilon_{j(\delta)}}, \xi + \nabla u_{j(\delta)}^\delta(x)\right)\, dx \\ &= \lim_{\delta\to 0} \int_Q f\left(x_0 + \delta y, \tfrac{x_0+\delta y}{\varepsilon_{j(\delta)}}, \xi + \nabla v_{j(\delta)}^\delta(y)\right)\, dy, \end{aligned}$$

where $v_{j(\delta)}^\delta(y) := \frac{1}{\delta} u_{j(\delta)}^\delta(x_0+\delta y) \in W^{1,p}(Q;\mathbb{R}^d)$, and where using a diagonalization argument, we choose the sequence $\{j(\delta)\}$ in such a way that

$$\frac{\delta}{\varepsilon_{j(\delta)}} > \frac{1}{\delta} \quad \text{and} \quad \lim_{\delta\to 0} ||v_{j(\delta)}^\delta||_{L^p(Q;\mathbb{R}^d)} = 0. \tag{4.17}$$

For simplicity denote $j(\delta) \equiv \delta$, $v_{j(\delta)}^\delta \equiv v_\delta$ and $u_{j(\delta)}^\delta \equiv u_\delta$. By the Decomposition Lemma (Theorem 2.19) there exists a subsequence of $\{v_\delta\}$ (still denoted by $\{v_\delta\}$) and a sequence $\{w_\delta\} \subset W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^d)$ such that

$$w_\delta \rightharpoonup 0 \text{ in } W^{1,p}, \quad w_\delta = 0 \text{ in a neighborhood of } \partial Q,$$

$$\{|\ \nabla w_\delta\ |^p\} \text{ is equi-integrable} \tag{4.18}$$

and

$$|\ \{y \in Q:\ v_\delta(y) \neq w_\delta(y)\}\ | \to 0. \tag{4.19}$$

As by assumption f is nonnegative

$$\begin{aligned} g^{\{\varepsilon_j\}}(x_0,\xi) &\geqslant \liminf_{\delta\to 0} \int_{\{y\in Q:\ v_\delta(y)=w_\delta(y)\}} f\left(x_0+\delta y, \frac{x_0+\delta y}{\varepsilon_\delta}, \xi + \nabla w_\delta(y)\right)\, dy \\ &= \liminf_{\delta\to 0} \int_Q f\left(x_0+\delta y, \frac{x_0+\delta y}{\varepsilon_\delta}, \xi + \nabla w_\delta(y)\right)\, dy \end{aligned}$$

from (4.18), (4.19) and (H_4). Since $x_0 \in W$, there exists $m_0 \in \mathbb{N}$ such that $x_0 \in K_{m_0} \cap K_{m_0}^\star$ and then we can write

$$g^{\{\varepsilon_j\}}(x_0,\xi) \geqslant \liminf_m\ \liminf_\delta \int_{Q_{m,\delta}} f\left(x_0+\delta y, \frac{x_0+\delta y}{\varepsilon_\delta}, \xi + \nabla w_\delta(y)\right)\, dy, \tag{4.20}$$

where the set

$$Q_{m,\delta} := \{y \in Q: \quad x_0 + \delta y \in K_{m_0} \text{ and } |\nabla w_\delta(y)| \leqslant m\}$$

is such that

$$\lim_m \lim_\delta |Q \setminus Q_{m,\delta}| = 0. \tag{4.21}$$

Indeed this set has measure zero because on one hand as

$$|\{y \in Q : \ x_0 + \delta y \notin K_{m_0}\}| \quad \leqslant \quad 1 - \tfrac{1}{\delta^N} \textstyle\int_{Q(x_0;\delta)} \chi_{K_{m_0}}(y)\, dy$$

and as $x_0 \in K^\star_{m_0}$ it follows

$$\lim_m \lim_\delta |\{y \in Q : \ x_0 + \delta y \notin K_{m_0}\}| = 0,$$

and on the other hand, by Chebyshev Inequality and the fact that $\{\nabla w_\delta\}$ is bounded in $L^p(Q;\mathbb{R}^d)$ we have

$$\limsup_m \ \limsup_\delta |\{y \in Q : \ |\nabla w_\delta(y)| > m\}| = 0.$$

After Euclidean division we can write

$$\frac{x_0}{\varepsilon_\delta} = m_\delta + s_\delta$$

with $m_\delta \in \mathbb{Z}^N$ and $s_\delta \in [0,1)^N$; we define

$$x_\delta := \frac{-\varepsilon_\delta}{\delta} s_\delta.$$

Note that by (4.17) $x_\delta \to 0$ as $\delta \to 0$. After changing variables once more,

$$\begin{aligned}
&\textstyle\int_{Q_{m,\delta}} f\left(x_0 + \delta y, \frac{x_0+\delta y}{\varepsilon_\delta}, \xi + \nabla w_\delta(y)\right)\, dy \\
&= \textstyle\int_{Q_{m,\delta}-\{x_\delta\}} f\left(x_0 + \delta(y+x_\delta), \frac{x_0+\delta(y+x_\delta)}{\varepsilon_\delta}, \xi + \nabla w_\delta(y+x_\delta)\right)\, dy \\
&= \textstyle\int_{Q_{m,\delta}-\{x_\delta\}} f\left(x_0 + \delta(y+x_\delta), \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y+x_\delta)\right)\, dy,
\end{aligned}$$

by the periodicity hypothesis (H_3) and the fact that $\frac{x_0+\delta x_\delta}{\varepsilon_\delta} = m_\delta \in \mathbb{Z}^N$. Then by inequality (4.20) it comes

$$g^{\{\varepsilon_j\}}(x_0,\xi) \geqslant \liminf_m \ \liminf_\delta \int_{Q_{m,\delta}-\{x_\delta\}} f\left(x_0 + \delta(y+x_\delta), \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y+x_\delta)\right)\, dy. \tag{4.22}$$

We now write

$$\begin{aligned}
&\textstyle\int_{Q_{m,\delta}-\{x_\delta\}} f\left(x_0 + \delta(y+x_\delta), \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y+x_\delta)\right)\, dy \\
&= \textstyle\int_{Q_{m,\delta}-\{x_\delta\}} \left[f\left(x_0 + \delta(y+x_\delta), \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y+x_\delta)\right) - f\left(x_0, \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y+x_\delta)\right)\right] dy \\
&\qquad + \textstyle\int_{Q_{m,\delta}-\{x_\delta\}} f\left(x_0, \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y+x_\delta)\right)\, dy.
\end{aligned} \tag{4.23}$$

By hypothesis (H_3) and the continuity of f with respect to y, given $m \in \mathbb{N}$ the restriction of f to the set $K_{m_0} \times \mathbb{R}^N \times \overline{B(\xi,m)}$, where $B(\xi,m) := \{x \in \mathbb{R}^N : |\xi - x| < m\}$, is uniformly continuous - recall that if

$g : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function kQ-periodic with respect to the first variable, then for each compact $K \subset \mathbb{R}^N$, $g : \mathbb{R}^N \times K \to \mathbb{R}$ is uniformly continuous. Hence there exist $\rho_m \in (0,1)$ such that

$$|f(x,y,\phi) - f(\overline{x},\overline{y},\overline{\phi})| < \frac{1}{m}$$

for all $x, \overline{x} \in K_{m_0}$, $y, \overline{y} \in \mathbb{R}^N$, and $\phi, \overline{\phi} \in \overline{B(\xi, m)}$ satisfying $|x - \overline{x}| + |y - \overline{y}| + |\phi - \overline{\phi}| < \rho_m$. As a result

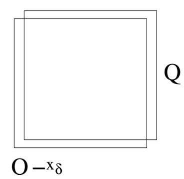
$$\lim_m \lim_\delta \int_{Q_{m,\delta}-\{x_\delta\}} \left| f\Big(x_0 + \delta(y + x_\delta), \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y + x_\delta)\Big) - f\Big(x_0, \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y + x_\delta)\Big) \right| dy = 0,$$

and then by inequality (4.22) and identity (4.23) we get

$$g^{\{\varepsilon_j\}}(x_0,\xi) \quad \geqslant \liminf_m \; \liminf_{\delta \to 0} \int_{Q_{m,\delta}-\{x_\delta\}} f\left(x_0, \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y + x_\delta)\right) dy.$$

Consequently

$$g^{\{\varepsilon_j\}}(x_0,\xi) \geqslant \liminf_{\delta \to 0} \int_{Q-\{x_\delta\}} f\left(x_0, \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y + x_\delta)\right) dy. \tag{4.24}$$



Indeed, first we note that

$$\begin{aligned} &\int_{(Q-\{x_\delta\})\setminus(Q_{m,\delta}-\{x_\delta\})} f\left(x_0, \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y + x_\delta)\right) dy \\ &\quad \leqslant \alpha \int_{(Q-\{x_\delta\})\setminus(Q_{m,\delta}-\{x_\delta\})} (1 + |\xi + \nabla w_\delta(y + x_\delta)|^p)\, dy \\ &\qquad = \alpha \int_{Q\setminus Q_{m,\delta}} (1 + |\xi + \nabla w_\delta(y)|^p)\, dy. \end{aligned} \tag{4.25}$$

As $\{|\nabla w_\delta|^p\}$ is equi-integrable, by inequality (4.25) and condition (4.21) we have

$$\limsup_m \; \limsup_\delta \int_{(Q-\{x_\delta\})\setminus(Q_{m,\delta}-\{x_\delta\})} f\left(x_0, \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y + x_\delta)\right) dy = 0,$$

which in turn implies inequality (4.24). In addition since

$$\begin{aligned} \textstyle\int_{Q\setminus Q-\{x_\delta\}} f\left(x_0, \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y + x_\delta)\right) dy \quad &\leqslant \alpha \textstyle\int_{Q\setminus Q-\{x_\delta\}} (1 + |\xi + \nabla w_\delta(y + x_\delta)|^p) dy \\ &= \alpha \textstyle\int_{Q + x_\delta \setminus Q} (1 + |\xi + \nabla w_\delta(z)|^p) dz \to 0 \end{aligned}$$

as $\delta \to 0$ (once again because $\{|\nabla w_\delta|^p\}$ is equi-integrable and $|(Q+x_\delta)\setminus Q| \to 0$ as $\delta \to 0$), we get from inequality (4.24)

$$g^{\{\varepsilon_j\}}(x_0,\xi) \quad \geqslant \liminf_{\delta\to 0} \int_Q f\left(x_0, \tfrac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y+x_\delta)\right)\, dy. \tag{4.26}$$

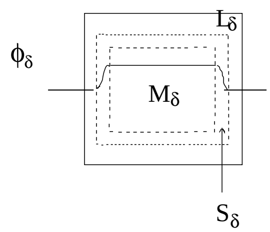
In order to compare $g^{\{\varepsilon_j\}}(x_0,\xi)$ with $f_{\rm hom}(x_0,\xi)$ we need to modify $w_\delta(\cdot + x_\alpha)$ close to the boundary of Q in order to be admissible for $f_{\rm hom}$. For this purpose define the sets

$$L_\delta := \{y \in Q : \ \mathrm{dist(y,\partial Q)} \leqslant |x_\delta|\}, \quad M_\delta = \{y \in Q : \ \mathrm{dist(y,\partial Q)} \geqslant 2|x_\delta|\},$$

and

$$S_\delta = \{y \in Q : \ \mathrm{dist(y,\partial Q)} \in (|x_\delta|, 2|x_\delta|)\}.$$

Consider a function $\phi_\delta \in C_c^\infty(Q;\mathbb{R})$ with $||\nabla \phi_\delta||_\infty \leqslant \frac{C}{|x_\delta|}$ such that



$$\phi_\delta(y) = \begin{cases} 1 & \text{if } y \in \ M_\delta, \\ 0 & \text{if } y \in \ L_\delta \end{cases}$$

and finally set $v_\delta(y) := \phi_\delta(y) w_\delta(y+x_\delta) + (1-\phi_\delta(y)) w_\delta(y) \in W_0^{1,\infty}(Q;\mathbb{R}^d)$. We claim that

$$g^{\{\varepsilon_j\}}(x_0,\xi) \quad \geqslant \quad \liminf_{\delta\to 0} \int_Q f\Big(x_0, \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla v_\delta(y)\Big)\, dy \tag{4.27}$$

which implies

$$g^{\{\varepsilon_j\}}(x_0,\xi) \quad \geqslant \quad \liminf_{\delta\to 0} \Big(\frac{\varepsilon_\delta}{\delta}\Big)^N \int_{\frac{\delta}{\varepsilon_\delta} Q} f\Big(x_0, y, \xi + \nabla \phi_\delta(y)\Big)\, dy$$

with $\phi_\delta(y) := \frac{\delta}{\varepsilon_\delta} v_\delta(\frac{\varepsilon_\delta}{\delta} y) \in W_0^{1,\infty}(\frac{\delta}{\varepsilon_\delta} Q;\mathbb{R}^d)$, and, consequently,

$$g^{\{\varepsilon_j\}}(x_0,\xi) \geqslant f_{\rm hom}(x_0,\xi).$$

In order to prove inequality (4.27) we note that

$$\begin{aligned}
\liminf_{\delta\to 0} \int_Q f\left(x_0, \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla v_\delta(y)\right)\, dy \leqslant \quad & \liminf_{\delta\to 0} \int_Q f\left(x_0, \frac{\delta}{\varepsilon_\delta} y, \xi + \nabla w_\delta(y+x_\delta)\right)\, dy \\
& + \ \limsup_{\delta\to 0} \alpha \int_{L_\delta} (1 + |\nabla w_\delta(x)|^p)\, dx \\
& + \ \limsup_{\delta\to 0} \alpha \int_{S_\delta} (|\nabla w_\delta(x+x_\delta)|^p + |\nabla w_\delta(x)|^p)\, dx \\
& + \ \limsup_{\delta\to 0} \alpha ||\nabla \phi_\delta||_\infty^p \int_{S_\delta} |w_\delta(x+x_\delta) - w_\delta(x)|^p\, dx.
\end{aligned} \tag{4.28}$$

But we have

$$\limsup_{\delta\to 0}\int_{L_\delta}(1+|\nabla w_\delta(x)|^p)\,dx = 0 = \limsup_{\delta\to 0}\int_{S_\delta}(|\nabla w_\delta(x+x_\delta)|^p+|\nabla w_\delta(x)|^p)\,dx, \tag{4.29}$$

because of the integrability property of $\{|\nabla w_\delta|^p\}_\delta$. This property also implies that

$$\limsup_{\delta\to 0}||\nabla\phi_\delta||_\infty^p\int_{S_\delta}|w_\delta(x+x_\delta)-w_\delta(x)|^p\,dx = 0 \tag{4.30}$$

in view of

$$\begin{aligned}
||\nabla\phi_\delta||_\infty^p\int_{S_\delta}|w_\delta(x+x_\delta)-w_\delta(x)|^p\,dx &\leqslant \frac{C}{|x_\delta|^p}\int_{S_\delta}\left|\int_0^1\frac{dw_\delta(x+tx_\delta)}{dt}\,dt\right|^p dx\\
&\leqslant \frac{C}{|x_\delta|^p}\int_{S_\delta}\int_0^1|\nabla w_\delta(x+tx_\delta)|^p\,.\,|x_\delta|^p\,dt\,dx\\
&\leqslant C\int_{S_\delta}\int_0^1|\nabla w_\delta(x+tx_\delta)|^p\,dt\,dx\\
&= C\int_0^1\int_{S_\delta-tx_\delta}|\nabla w_\delta(y)|^p\,dy\,dt\\
&\leqslant C\int_{N_\delta}|\nabla w_\delta(y)|^p\,dy,
\end{aligned}$$

where

$$N_\delta = \{x\in Q:\ \mathrm{dist(x,\partial Q)}\leqslant 3|x_\delta|\}.$$

Hence inequality (4.27) holds as a consequence of inequalities (4.26), (4.28), (4.29) and (4.30).

■

As a consequence we derive the equality (a.e.) of the density functions $g^{\{\varepsilon_j\}}$ and $f_{\rm hom}$.

Corollary 4.7. *$g^{\{\varepsilon_j\}}(x,\xi) = f_{\rm hom}(x,\xi)$ for a.e. $x\in\Omega$ and for all $\xi\in\mathbb{R}^{d\times N}$.*

Proof. In view of Proposition 4.6 we have that for a.e. $x\in\Omega$ and for all $\xi\in\mathbb{Q}^{d\times N}$, $g^{\{\varepsilon_j\}}(x;\xi) = f_{\rm hom}(x;\xi)$. Since $g^{\{\varepsilon_j\}}(x;\cdot)$ and $f_{\rm hom}(x;\cdot)$ are continuous, the equality $g^{\{\varepsilon_j\}}(x;\xi) = f_{\rm hom}(x;\xi)$ holds true for all $x\in\Omega$ and for all $\xi\in\mathbb{R}^{d\times N}$.

■

The proof of our main result is now straightforward.

Proof of Theorem 1.1 As a consequence of Corollary 4.7 the functional $\mathcal{I}_{\rm hom}$ is well defined and $\Gamma(L^p(A))$-limit of $\mathcal{I}_{\varepsilon_j}(\cdot\,;A)$ is equal to $\mathcal{I}_{\rm hom}(\cdot\,,A)$. In particular, since it does not depend upon the extracted subsequence, in view of Proposition 2.3, the whole sequence $\mathcal{I}_\varepsilon(\cdot\,;A)$ $\Gamma(L^p(A))$-converges to $\mathcal{I}_{\rm hom}(\cdot\,;A)$. Taking $A=\Omega$ we conclude the proof of Theorem 1.1. ■

5 Appendix

In this section we recall two auxiliary lemmas of use for the proof of Theorem 1.1. First we invoke a lemma by Licht and Michaille [39] that allowed us to justify that the function $f_{\rm hom}$ given in (1.5) is well defined (see Lemma 3.1 above).

Lemma 5.1. *Let $N \in \mathbb{N}$ with $N \geqslant 1$ and let $S : \mathcal{A}(\mathbb{R}^N) \to \mathbb{R}^+$ be such that*

i) *$S(A) \leqslant \beta \mathcal{L}^N(A)$, for all $A \in \mathcal{A}(\mathbb{R}^N)$, where β is a positive constant,*

ii) *$S(C) \leqslant S(A) + S(B)$ for all $A, B, C \in \mathcal{A}(\mathbb{R}^N)$, with $A \cap B \neq \emptyset$, $\overline{C} = \overline{A} \cup \overline{B}$,*

iii) *there exists $T \subset \mathbb{R}^N$ and $M > 0$ such that $T + [0, M)^N = \mathbb{R}^N$ and $S(A + \tau) = S(A)$ for all $A \in \mathcal{A}(\mathbb{R}^N)$ and $\tau \in T$.*

Then, for any cube A of the form $[a, b)^N$ there exists the limit of the sequence $\left\{ \frac{S(sA)}{\mathcal{L}^N(sA)} \right\}$ as $s \to +\infty$ and

$$\lim_{s \to +\infty} \frac{S(sA)}{\mathcal{L}^N(sA)} = \lim_{s \to +\infty} \frac{S([0, s)^N)}{s^N}.$$

Furthermore, if $\{S_L\}_L$ is a family of set functions satisfying $i), ii), iii)$ for C, T and M independent of L, then the above limits are attained uniformly with respect to L.

Next, we recall an auxiliary lemma that was stated in [14] and it is useful when diagonalization arguments are required (see Lemma 4.1).

Lemma 5.2. *Let $a_{k,j}$ be a doubly indexed sequence of real numbers $(k, j \nearrow +\infty)$. If*

$$\lim_{k \to \infty} \lim_{j \to \infty} a_{k,j} = L,$$

then there exists a subsequence $\{k(j)\}_j \nearrow +\infty$ such that

$$\lim_{j \to \infty} a_{k(j),j} = L.$$

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Department of Mathematical Sciences,
Carnegie Mellon Universtiy,
Pittsburgh, PA, 15213, USA,
mbaia@andrew.cmu.edu,
fonseca@andrew.cmu.edu.