OPEN BOOK DECOMPOSITIONS AND CONTACT GEOMETRY

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1. Open book decompositions and differential topology

1.1. **Introduction and definitions.** In the sequel M^m will be a smooth compact oriented manifold, with empty boundary unless otherwise stated. All functions and tensors to appear are smooth,

We find useful to give decompositions/presentations of M. Examples of them are:

- Handlebody decompositions: this is a diffeomeorphism $M \simeq H$, where H - a handlebody- is the result of attaching handles
$$h_1^0 \cup \cdots \cup h_{i_0}^0 \cup h_1^1 \cup \cdots \cup h_{i_m}^m$$
(Recall that the result of attaching a handle to a manifold with boundary is a manifold with corners which inherits a canonical smooth structure up to diffeomorphism). Handlebody decompositions can be associated to Morse functions in M. The link with function theory is very useful. In particular results of Cerf [6] tell us how to relate two different handlebody decompositions associated to the same manifold M. There might be changes in the isotopy class of the gluing maps, in the order in which handles are attached, and more importantly births/deaths of canceling pairs of handles can occur. Applications include computation of $H_*(M)$, definition of geometric structures by extending them across the gluing (for example some problems are seen to abide an h-principle in that way [11]), Kirby calculus for 4-manifolds (this with an additional result that allows to neglect 3 and 4 handles up to diffeomorphism [15]).
- Round handle decompositions: this time a k-handle is $S^1 \times D^k \times D^{m-1-k}$ with core $S^1 \times S^{k-1} \times \{0\}$. Asimov proves that if m>3 round handle decompositions for M exist iff $\chi(M) = 0$. These decompositions are applied in dynamical systems to prove the existence of non-singular Morse-Smale flows [2], in foliation theory to prove the existence of codimension 1 foliations [20], and to study Engel structures [22].
- Lefschetz pencils for closed projective varieties: it amounts to present $M \subset \mathbb{CP}^N$ as a $\mathbb{CP}^1$ family of complex hypersurfaces. This is done slicing M with an appropriate pencil of complex

hyperplanes $\lambda\mathbb{H}_0 + \mu\mathbb{H}_1$ to get

$$\begin{array}{rcl} M\backslash B & \longrightarrow & \mathbb{CP}^1 \\ x & \longmapsto & [\lambda(x):\mu(x)],\ x \in \lambda\mathbb{H}_0 + \mu\mathbb{H}_1,\ B = X \cap \mathbb{H}_0 \cap \mathbb{H}_1 \end{array}$$

where a finite number of fibers present quadratic singularities. Lefschetz pencils always exist, perhaps having to consider degree 2 hypersurfaces rather than hyperplanes [13].

- Open book decompositions $\overset{1:1}{\longleftrightarrow}$ relative mapping torus.

Recall that for F a closed manifold a mapping torus P is a surjective submersion $P \to S^1$ with F the fibre over 1 say. Let θ be the periodic coordinate in S^1. Any lift of $\frac{\partial}{\partial\theta}$ is a vector field transverse to the fibers. We cut open along F and use it to trivialize the resulting manifold with boundary as $F \times [0,1]$. To recover P we must identify $(0,x) \sim (1,\phi(x))$, ϕ the (inverse of the) return map associated to the vector field. Changing the vector field amounts to isotoping the return map. Therefore, a mapping torus with fiber F is given by a pair (F,ϕ), $\phi \in \mathrm{Diff}(F)/\overset{\text{isot}}{\sim}$.

Definition 1. *Let F be a compact manifold with boundary. A relative mapping torus P with page F is a surjective submersion which restricted to the boundary is a relative surjective submersion with trivial monodromy. In other words, we can choose a lift of $\frac{\partial}{\partial\theta}$ in P whose return map is the identity near ∂P. Therefore, a relative mapping torus with fiber F is given by a pair (F,ϕ), $\phi \in \mathrm{Diff}^{\mathrm{comp}}(F,\partial F)/\overset{\text{isot}}{\sim}$.*

A relative mapping torus is a manifold with boundary. There is a canonical way of turning it into a closed manifold: it is the boundary connected sum of P and $\partial F \times D^2$. The boundary of the former is identified with $\partial F \times S^1$, and we glue via the identity map.

Definition 2 (OBI)**.** *An open book decomposition $\mathcal{B}$ on M is a diffeomorphism from M to the closed manifold associated to a relative mapping torus.*

The page of the open book is the image of the fiber F. The binding is the image of ∂F.

Since our manifold is oriented, the canonical orientation of S^1 coorients the page, and by the outward/positive normal first rule, it orients the page. For the same reason the binding inherits an orientation from $\mathcal{B}$. Conversely, if the page is abstractly oriented, we choose the rotation direction so that it becomes the induced orientation.

One can give an alternative definition:

Definition 3 (OBII)**.** *An open book decomposition is given by a codimension 2 submanifold $B \subset M$, the binding, with trivial normal bundle, and a fibration $M\backslash B \to S^1$ such that for a trivialization $\nu(B) = B \times D^2$ it coincides with the angular coordinate in $\nu(B)\backslash B$.*

There is a third very useful definition, because it brings function theory.

In $\mathbb{R}^2 = \mathbb{C}$ the standard open book decomposition $\mathcal{B}_{\mathrm{std}}$ has the origin as binding and the half lines as pages.

Definition 4 (OBIII)**.** *An open book decomposition on M is given by a function $f\colon M \to \mathbb{C}$ which is transverse to $\mathcal{B}_{\mathrm{std}}$ and whose image contains the origin. The binding and pages are the inverse images of the binding and pages respectively of $\mathcal{B}_{\mathrm{std}}$.*

Remark 1. *If M has boundary an open book decomposition must have binding and pages transverse to ∂M, so it induces an open book decomposition there.*

Remark 2. *Definition 4 is very useful in the following sense: often geometric structures are given by sections of bundles associated to M. If we want to define some sort of compatibility of an open book w.r.t. it, it might possible that the compatibility might be expressed by choosing suitable classes of functions.*

Example 1 (Odd Spheres)**.** *Look at $S^{2n+1} \subset \mathbb{C}^{n+1} \subset \mathbb{CP}^{n+1}$. Take $\mathbb{H}$ any projective hyperplane intersecting the sphere. $\mathbb{H} \cap S^{2n+1}$ is the binding. Pages are intersections with each of the real half hyperplanes through $\mathbb{H}$. They are diffeomorphic to $\mathbb{C}^n$. A nice way of seeing this is using the projection $\pi\colon \mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{CP}^n$. A complex line through the origin, if not in $\mathbb{H}$, cuts a fixed real half hyperplane in a half line, therefore the sphere in a point which is in the corresponding page. Thus each page is identified with the complement of $\mathbb{H}$ in the projective space at infinity. One can check easily the associated function by using a complex line transverse to $\mathbb{H}$ in $\mathbb{C}^{n+1}$. For $S^3 \subset \mathbb{C}^2$ with coordinates (z_0, z_1), take $f(z) = z_0$.*

If n=2, then stereographic projection sends the open book to a pencil of real planes.

Definition 5. *Given an annulus A with coordinates θ, t, $t \in [-1, 1]$, a Dehn twist is the compactly supported isotopy class of maps defined by $\tau(\theta, t) = (\beta(t) + \theta, t)$, $\beta\colon [-1, 1] \to [0, 2\pi]$ a step function sending -1 to 0 and 1 to 2π. Otherwise said it is one of the generators of $\mathrm{Diff}(A, \partial A)/ \overset{\text{isot}}{\sim} \simeq \mathbb{Z}$.*

Example 2 (Hopf Bands)**.** *For S^3 as above take the function $z_0 z_1$ (resp. $z_0 \bar{z}_1$). The monodromy is a Dehn twist. The resulting open book is denoted $\mathcal{B}_{\pm H}$. (see http://sketchesoftopology.wordpress.com/2008/09/04/a-product-disk-for-the-hopf-band/).*

Lemma 1. *If $(M, \mathcal{B})$, $(M', \mathcal{B}')$ are open books, then the connected sum (for the identity map in the sphere) carries a canonical open book decomposition $\mathcal{B}\#\mathcal{B}'$. One just needs to drill out the m-ball centered at the binding and written as $D^m = D^{m-2} \times D^2 \subset B \times D^2 = \nu(B)$. The* *page of $\mathcal{B}\#\mathcal{B}'$ is the boundary connected sum $F\#F'$ and the monodromy $\phi\#\phi'$.*

1.2. **Applications.**

- Foliation theory: any 3-dimensional closed manifold admits a codimension 1 foliation if it carries an open book decomposition. One needs to introduce a Reeb component on a neighborhood of each component of the link. More generally, for and open book with binding B so that $B \times D^2$ admits a codimension 1 foliation tangent to the boundary, one can construct codimension 1 foliations [14].
- Twisted doubles and cobordism: A twisted double is a generalization of a Heegard splitting of a 3-manifold. M is presented as the result of gluing two copies of the same manifold W by a diffeomorphism of its boundary. If a manifold admits an open book decomposition then it admits a twisted double. Using that one concludes
  1. A manifold admitting an open book decomposition is cobordant to a manifold that fibers over the circle.
  2. For a given homotopy sphere Σ one can construct M such that M is diffeomorphic to $M\#\Sigma$ [23].

There is a long list of applications in appendix A5 in [18].

1.3. **Existence.**

Theorem 1. [1] *Any closed oriented 3-manifold admits an open book decomposition. The binding=link is also called a fibred link.*

In particular this gives another proof of Lickorish' result on the existence of codimension 1 foliations on surfaces.

How about uniqueness of open book decompositions, or relations among them?. Recall that for Heegard decompositions/ handlebody structures we know that two of then differ by isotopy and introduction or elimination of pairs of cancelling 1 and 2 handles [6].

By lemma 1 $(M^3, \mathcal{B})\#(S^3, \mathcal{B}_{\pm H})$ is an open book in $M\#S^3 = M^3$. More generally one can modify $(M^3, \mathcal{B}) = (F, \phi)$ as follows: add a 1-handle to the page, and compose the resulting monodromy ϕ with Dehn twist about a curve γ cutting once the core of the 1-handle. That produces an open book on a manifold M'. But M' is cobordant to M and the cobordism W amounts to add a 1-handle, and then a 2-handle with attaching circle γ and certain framing. Hence the two handles are a canceling pair and the cobordism is trivial; this is known as the Murasugi sum.

Remark 3. *One can actually perform the Murasugi sum inside M. Otherwise said the diffeomorphism between the ends of W can be fixed up to isotopy. To isotope the two dynamical systems relative to M one* *would have trouble near critical points due to the absence in general of smooth normal forms (Grobman-Hartman), so they have to be imposed by asking the metrics giving rise to the gradients to be standard in Morse coordinates; then one isotopes the cores of the cancelling pair; this is possible because they fit do fit a disk, whose intersection with* M *is* γ. *The same can be done for the union of unstable manifolds; then one can isotope the dymanical systems in a neighborhood of the union of stable and unstable manifolds (because of the aforementioned imposed normal forms); the complement has a couple of non-singular flows matching in the boundary, and one can isotope so that the corresponding Morse functions to coincide, that giving an isotopy between the diffeomorphisms induced by both gradient flows.*

Theorem 2. [10] *Homologous open books in* M^3 *(with oriented page) are isotopic up to stabilizations.*

Any open book carries an associated oriented plane field. Away from the binding if determined by the foliation, or by $d\theta$; in $B \times D^2$ with coordinates b, r, θ, take the 1-form $(1-\rho(r))db+\rho(r)d\theta$, where $\rho(r)$ is a bump function that vanishes near zero and equals 1 near the boundary of the tubular neighborhood. Plane fields are homologous if their graph in the sphere bundle $S(TM)$ define the same homology class.

Theorem 3. [17] *Let* M *be a compact orientable of dimension* ≥ 6, *and let* $\mathcal{B}_{\partial M}$ *an open book decomposition in the boundary. Then*

(1) *If* M *has odd dimension the open book decomposition extends to* $\mathcal{B}$ *an open book decomposition in* M.
(2) *If* M *has even dimension 2n then* $\mathcal{B}_{\partial M}$ *defines an element*

$$i(M) \in W(Z[\pi_1])$$

This is roughly associated to an intersection form on $H_n(M, F)$, F *the page in* $\partial\mathcal{B}$, *with values on* $Z[\pi_1]$. *The vanishing of the obstruction is equivalent to the possibility of extending the open book.*
(3) *If* M *is odd dimensional of dimension* ≥ 5, *then for any* $i \in W(Z[\pi_1])$ *and any open book decomposition* $\mathcal{B}$, *one can find another book decomposition such that the induced one on* $\partial(M \times [0,1])$ *has invariant* i.

Moreover, the open books above can be chosen to have almost canonical page F, *that is*

$$\pi_j(M, F) = 0, j < [m/2], j > [m/2] \tag{1}$$

Sketch of the proof. We mention the key parts of Quinn's proof in the closed case. It is about “guessing” a submanifold F that can be the page. It is done as follows: if F is indeed a page of an open book then

$$W := M \backslash (F \times (0, 1/2))$$

is a manifold with boundary

$$\partial W = (F \times \{0\}) \cup (F \times \{1\}) \cup ([0,1] \times \partial F)$$

That is to say that W is a relative cobordism. By hypothesis it is trivial. Quinn's idea is find F such that W can be checked to be trivial by applying the h-cobordism or s-cobordism theorem.

In the odd dimensional case m=2k+1 one picks the k-1 skeleton, by gluing product/ribbon handles (a product d-handle is $D^d \times D^{m-(d+1)} \times [0,1/2]$), so the result is $F' \times [0,1/2]$, F' a hypersurface with boundary. We assume for simplicity $\pi_1(M) = 0$. The complement W will have no homotopy (general position) and also boundary component (diffeomorphic to F') simply connected. By construction $(F', \partial F')$ is k-connected, so up to degree k the relative homology of the inclusion $(F', \partial F') \hookrightarrow (W, F')$ is $H_*(W, F')$, which by excision is $H_*(M, F')$. Therefore it vanishes up to degree k-1. By Poincaré duality it would be enough to kill the k-th homology. So one possibility might be removing some k-handles from W, which is to say adding them to $F' \times [0,1/2]$. Of course, they have to be product k-handles, so that the product structure $F' \times [0,1/2]$ is extended; because $(F', \partial F')$ was k-connected there is no trouble is sliding the attaching sphere to $\partial F'$, so the gluing can be assumed w.l.o.g. to extend the product. Here it is important that we are in odd dimensions, so general position grants that by removing k-handles we do not change the j-th homotopy, j<k, and that the k-handles to be removed can be chosen to be disjoint. Therefore this gives open books with almost canonical pages.

In the even dimensional case 2k, to extend open books from the boundary one uses the same approach. This time the product k-handles may intersect. Those intersections, arranged in a suitable way determine a pairing of $H_k(M, F)$ with values in $W(Z[\pi_1 M])$. In the simply connected case it is easy to understand geometrically: By Poincaré duality $H_k(M, F) \simeq H^k(M, \partial M \backslash F)$, and pushing by the flow of the monodromy we get $H^k(M, \partial M \backslash F) \simeq H^k(M, F)$. In other words, given two relative classes, we use the return map to "double" one of them and compute the pairing. If M has empty boundary, then we are just speaking of the intersection pairing in the middle homology. In this case the obstruction is exactly the signature. □

Regarding uniqueness and construction of open books (in odd dimensional manifolds) using functions: the analogy with handlebody decompositions and Morse functions does not go through. Let

$$\mathcal{M} = \{f \in C^\infty(M), \mid df \pitchfork \mathbf{0} \subset T^*M\}$$

$$\mathcal{B}(M) = \{f \in C^\infty(M, \mathbb{C}), \mid f \pitchfork \mathcal{B}_{\text{std}}\}$$

The inclusion of Morse functions

$$\mathcal{M} \hookrightarrow C^\infty(M)$$

is dense in the C^1-topology. So we can construct a handlebody decomposition by picking a function and perturbing it a bit. Even more, we know how we can relate two different handlebody decompositions [6].

If M is even dimensional and with non-zero signature then $\mathcal{B}(M) = \emptyset$, so there is no way to perturb any given function to one giving a book. Even when there are open book decompositions for the manifold, $\mathcal{B}(M) \subset C^\infty(M, \mathbb{C})$ need not be dense.

Lemma 2. *Let M be a oriented manifold and $B \subset M$ a null-homologous codimension 2 submanifold which is known not to be the binding of an open book decomposition of M (there are knots known to be not fibered). Then for any $f\colon M \to \mathbb{C}$, $f \pitchfork 0$, $f^{-1}(0) = B$, we have $f \notin \overline{\mathcal{B}(M)}$.*

Proof. Because B is null-homologous we know we have functions $f\colon M \to \mathbb{C}$ transverse to zero with zero subset M. Any small enough perturbation will have zero subset isotopic to B. But by hypothesis it cannot give an open book decomposition. □

A natural way to compare structures is to call for a bordism relation. Open books $\mathcal{B}, \mathcal{B}'$ in M are bordant if the induced open book in the boundary of $M \times [0,1]$ extends. This a substitute for a "reasonable" path joining open book decompositions. Of course, for each $t \in [0,1]$ the induced slice need not carry an open book, but we would like the situation to be as generic as possible.

For open books Quinn proves that any open book is bordant to one with almost canonical page. Recall that if for to such books $\mathcal{B}, \mathcal{B}'$ the induced invariant $i(M \times I)$ does not vanish, they are not bordant. Even for bordant almost canonical open books, it is not clear to me whether they can be related by a sequence of elementary surgeries. One such possibility is stabilization, which is defined pretty much as in the 3-dimensional case:

Definition 6. *Given S^n a sphere, a generalized Dehn twist is a compactly supported class of diffeomorphisms of T^*S^n with a representative defined by taking all planes through the origin and perform on each intersection - which is T^*S^1- a Dehn twist in a compatible way. In a different way, fix the round metric and the pick the normalized geodesic flow for time π near the 0-section and decreasing to the identity as the distance=norm goes far away*

To stabilize an open book with page F^n we attach an n-1-handle to ∂F. Next we need to complete the core n-1-disk to a n-1-sphere in F such that $T_F S^{n-1} \simeq T^* S^{n-1}$. Next we compose ϕ with a generalized Dehn twist along the sphere. In that way we obtain M' with an open book decomposition. Again M' is cobordant to M and the cobordism amounts to adding a cancelling pair of n-1 and n-handle.

Remark 4. *One should also check how to put an open book decomposition on the cobordism associated to the stabilization.*

2. Open book decompositions in contact geometry

It is often the case that when studying the space of certain structures we may obtain information by understanding the way in which they can degenerate. This can go for example by embedding our space as a subspace of a topological space, an interpret convergence to points in the boundary (or perhaps just to some of them) in a geometric sense (e.g. Thurston compactification of Teichmuller space [21]); degenerations into geometries on other associated manifolds are also considered (tropical geometry [16], degenerations of metrics with some curvature properties).

2.1. **Open book decompositions in 3-dimensional contact geometry.** For a three dimensional manifold M we let $\Omega^1_{\text{cont}}(M) \subset \Omega^1(M)$ be the space of (positive) contact 1-forms with the C^1-topology. The so called Giroux correspondence [9] can be interpreted as follows:

(1) An open book $\mathcal{B}$ gives rise to $\theta_{\mathcal{B}} \subset \partial\Omega^1_{\text{cont}}(M)$ (as a matter of fact there are choices involved but the corresponding subset of 1-forms is convex). More precisely it gives rise to $\alpha_{\mathcal{B}} \in \Omega^1_{\text{cont}}(M)$ (again there are choices but the corresponding subset is path connected) such that

$$\lambda\theta_{\mathcal{B}} + (1-\lambda)\alpha_{\mathcal{B}} \in \Omega^1_{\text{cont}}(M) \text{ iff } \lambda \in [0,1)$$

(2) There is a precise test for $\alpha \in \Omega^1_{\text{cont}}(M)$ to be such that $\lambda\theta_{\mathcal{B}} + (1-\lambda)\alpha \in \Omega^1_{\text{cont}}(M)$, $\lambda \in [0,1)$ (for some of the $\theta_{\mathcal{B}}$). The contact form has to be adapted to $\mathcal{B}$ (see definition in the statement of theorem 4).

(3) The space of contact forms adapted to $\mathcal{B}$ and close enough to $\theta_{\mathcal{B}}$ is convex. That is for any two α_0, α_1 adapted to $\mathcal{B}$ (plus some extra condition near the binding) and for all λ close enough to 1 (depending on α_0, α_1)

$$t(\lambda\theta_{\mathcal{B}} + (1-\lambda)\alpha_0) + (1-t)(\lambda\theta_{\mathcal{B}} + (1-\lambda)\alpha_1) \in \Omega^1_{\text{cont}}(M),\ t \in [0,1]$$

In particular (because the condition near the binding can be attained by isotopy) the isotopy class of contact structures associated to contact forms adapted to $\mathcal{B}$ is unique.

Seen from $\mathcal{B}$ it means that for any $\theta_{\mathcal{B}}$ there exist a subset of $\Omega^1_{\text{cont}}(M) \subset T_{\theta_{\mathcal{B}}}\Omega^1(M)$ whose intersection with a ball of small enough radius is convex (if the two directions fulfilled the aforementioned condition near the binding; in general it is just path connected). For all θ_B associated to $\mathcal{B}$ the union of this directions is exactly the contact forms adapted to $\mathcal{B}$.

(4) For any $\alpha \in \Omega^1_{\text{cont}}(M)$ there exist $\mathcal{B}$ so that α is adapted to $\mathcal{B}$. That is to say there is a direction in $T_\alpha\Omega^1_{\text{cont}}(M)$ (actually the directions defined by many of the $\theta_{\mathcal{B}}$ as in point 1) so that the

segment in that direction hits $\partial\Omega^1_{\text{cont}}(M)$ in the corresponding $\theta_{\mathcal{B}}$.

(5) If α is adapted to $\mathcal{B}$ and $\mathcal{B}'$, then the two open books are the same up to positive stabilization.

(6) In short, there is a one to one correspondence between isotopy classes of contact structures and open books up to positive stabilization, and that correspondence is proven by understanding a subset of $\partial\Omega^1_{\text{cont}}(M)$.

Remark 5. *Other interesting 1-forms in $\partial\Omega^1_{\text{cont}}(M)$ are those defining taut foliations* [7].

We start to give details of the correspondence by constructing the contact 1-forms α_β. This in particular reproves the result of Martinet on the existence of contact structures on closed oriented 3-manifolds.

Theorem 4. [19] *Any open book $\mathcal{B}$ in an oriented 3-manifold induces a co-oriented contact structure $\xi_{\mathcal{B}}$ on M. A positive contact form $\alpha_{\mathcal{B}}$ (or just α to ease the notation a bit) can be chosen so that:*

(1) *The binding is a contact submanifold ($\xi_{\mathcal{B}} \pitchfork B$) and $d\alpha$ makes the pages into symplectic manifolds; that is to say the Reeb vector field of α is transverse to the pages (in $M \backslash B$).*

(2) *$d\alpha$ induces the given orientation in the page and α is positive in the binding.*

We say that $\mathcal{B}$ carries $\xi_{\mathcal{B}}$ and that α is adapted to $\mathcal{B}$.

Proof. Fix ω a symplectic form in F.

1. Normal form for a potential for ω: Parametrize $\nu(\partial F)$ with $b \in \partial F$ coordinate and $r \in [1,2]$ inwards pointing transverse coordinate so that $\omega = -dr \wedge db$. Because F has no second cohomology, $w = d\beta$. Writing $w = \omega_c + \varphi(r)dr \wedge db$ and applying Poincaré's lemma to compactly supported forms, one can assume

$$\beta_{|\nu(\partial F)} = (a - r)db,\ a \in \mathbb{R} \tag{2}$$

2. Contact structure in the relative mapping torus: In $F \times \mathbb{R}$, with real coordinate θ, consider the 1-form

$$\alpha = \beta + d\theta$$

This is clearly a contact form, since

$$\alpha \wedge d\alpha = d\theta \wedge \omega$$

Assume for the moment that $\phi^*\beta = \beta$. Define

$$\begin{aligned} \Phi\colon F \times [0,1] &\longrightarrow F \times [0,1] \\ (x,\theta) &\longmapsto (\phi(x), \theta + 1) \end{aligned} \tag{3}$$

By construction

$$\Phi^*\alpha = \alpha,$$

and thus the quotient -which is the relative mapping torus- inherits a positive contact form α (its associated volume form defines the fixed orientation) whose Reeb vector field is transverse to the pages, and with positive symplectic pages.

3. Extending the contact form to the open book: We parametrize $\nu(B)$ using the coordinates r, θ, b as above, but with $r \in [0,2]$. For $r \geq 1$ we have

$$\alpha = (a - r)db + d\theta \tag{4}$$

We are going to extend it to a 1-form

$$\alpha = f(r, \theta, b)db + g(r, \theta, b)d\theta,$$

The conditions

$$\alpha_{|B} > 0,\ \alpha \wedge d\alpha_{|B} > 0,$$

are equivalent to

$$f_{|B} > 0,\ g = r^2 G,\ G_{|B} > 0 \tag{5}$$

On $\nu(B)\backslash B$ the conditions to be positive contact and restrict to a positive symplectic form are

$$f\frac{\partial g}{\partial r} - \frac{\partial f}{\partial r}g > 0,\ \frac{\partial f}{\partial r} < 0 \tag{6}$$

We can extend $f = a - r > 0$, $r \geq 1$, to a function $f(r)$ with $f' < 0$ and $f = 1 - r^2$ near zero; we can extend $g = 1$, $r \geq 1$, to a function $g(r)$ with $g' > 0$ and $g = r^2$ near zero. By equations 5 and 6 α is a contact form adapted to the open book.

4. Avoiding the assumption $\phi^*\beta = \beta$: By Moser's lemma within the compactly supported isotopy class of the monodromy we can always choose a representative so that $\phi^*\omega = \omega$. That implies $\phi^*\beta - \beta$ is closed. If it is exact choose a function h so that

$$dh = \beta - \phi^*\beta,$$

h positive, $h = 1$ near ∂F.

Then just modify the map in 3 into

$$\Phi(x, \theta) = (\phi(x), \theta + h(x))$$

Then α descends to $F \times \mathbb{R}/ \sim \Phi$, which is diffeomorphic to the relative mapping torus of (F, ϕ), inducing a contact form which makes the pages symplectic. Notice that because $h = 1$ near ∂F step 3 can be carried with no change.

We can isotope ϕ so that $\gamma = \phi^*\beta - \beta$ is exact: the result has to be $\phi \circ \psi_s$, $s \in [0,1]$, ψ_s the flow of a vector field X.

Take C any loop (1-chain). We want

$$\langle \phi \circ \psi_1^* \beta, C \rangle = \langle \beta, C \rangle$$

If A is the annulus $\psi_s(C)$, we can always compare

$$\langle \phi \circ \psi_1^* \beta - \phi^* \beta, C \rangle = \langle \omega, A \rangle = \int_0^1 \langle i_X \omega, \phi \circ \psi_{s*} C \rangle$$

The form $\mu := i_X \omega$ can be chosen at will, and if closed we get

$$[\phi \circ \psi_1^* \beta - \phi^* \beta] = [\phi^* \mu]$$

By choosing $\mu = \phi_* \beta - \beta$ we are done. □

Remark 6. *The effect of moving ϕ within the symplectic isotopy class for a fixed open book $(M^3, \mathcal{B})$ produces a family of contact forms α_t, $t \in [0,1]$. All the contact forms are adapted to $\mathcal{B}$ and are such that $d\alpha_t = d\alpha_0$ restricted to any page. Similarly, changing the function Φ that defines the mapping cone can be seen as an produces another contact from α_1; they can be connected by a family α_t of contact forms with exactly the same properties as above.*

Definition 7. *Let r, θ, b, $r \in [0,1]$ be coordinates in $\nu(B)$ so that the pages are the fiber of θ. Let $\rho(r,\theta) \geq 0$ which is constant for r near 1, vanishes exactly at $r = 0$, and it is monotone along radial segment. Define the 1-form*

$$\theta_{\mathcal{B}} = \rho(r,\theta) d\theta \tag{7}$$

Observe that

$$\theta_{\mathcal{B}} \wedge d\theta_{\beta} = 0$$

Notice that for fixed coordinates the choices gives rise to a convex subset of 1-forms. Changing the b, r coordinates does not affect the definition (we assume the θ coordinate to be fixed). Typically on the fixed coordinates we take ρ to be independent of θ.

Remark 7. *One could also allow functions ρ vanishing near the binding. We made our choice because θ vanishes exactly along B, and defines the pages away from the binding.*

Observe that ρ is not a 1-form defining a classical structure on a manifold; the reasonable limits are the non-continuous "closed" 1-forms

$$\begin{cases} a d\theta, \; r > 0 \\ 0, \; r = 0, \end{cases} \tag{8}$$

$a > 0$. *They can be made smooth at the expense of modifying the manifold. One possibility is blowing up B along normal directions, but this produces a non-orientable manifold. The second is capping off each page with a disk, or equivalently performing Dehn surgery along B with the framing defined by the page. This is a different way of saying that an open book decomposition on M^3 canonically defines an oriented $\hat{M}^3$ which fibers over S^1 (and so that we have $M \backslash B \hookrightarrow \hat{M}$ and the fibration restricts to the fibration coming from the open book decomposition); note that M and $\hat{M}$ are cobordant.*

We next state a lemma which explains the geometric meaning/consequence of $\alpha_{|B} > 0$.

Lemma 3. *Let α be a contact form adapted to $\mathcal{B}$. Then $(1-\lambda)\alpha+\lambda\theta_{\mathcal{B}}$, $\lambda \in [0,1)$, is a contact form adapted to $\mathcal{B}$ provided ρ in 7 has support closed enough to the binding.*

Proof. Fix coordinates r, θ, b so that the pages are given by θ. By sliding along the b coordinates we can arrange in $\nu(B)$

$$\alpha = f(r,\theta,b)db + g(r,\theta,b)d\theta$$

with the properties in equations 5 and 6. By reducing the domain of the r coordinate if necessary we may assume $f > 0$. We then take $\theta_{\mathcal{B}}$ so that the support of ρ is contained in the subset for which $f > 0$. Then the segment

$$(1-\lambda)\alpha + \lambda\theta_{\mathcal{B}}$$

is by contact forms except for $\lambda = 1$. Away from the tubular neighborhood the wedging with the exterior derivative we get

$$(1-\lambda)^2\alpha \wedge d\alpha + \lambda(1-\lambda)\theta_{\mathcal{B}} \wedge d\alpha > 0, \lambda < 1$$

Also the restriction to the pages of the exterior derivative is $(1-\lambda)d\alpha$, therefore positive. In the tubular neighborhood we have

$$(1-\lambda)f db + ((1-\lambda)g + \lambda\rho)d\theta$$

Because convex combination preserves signs equations 5 and 6 are fulfilled. □

Remark 8. *In the coordinates r, θ, b, and along Legendrian segments in the radial direction the contact distribution rotates from the b to the θ coordinates (actually towards the $-b$ direction); by moving towards $\theta_{\mathcal{B}}$ so that for very large values of ρ the contact distribution becomes very close to the foliation by the pages. That the result is contact depends heavily on $\alpha_{|B} > 0$ (equivalently $f > 0$). This implies that the Reeb vector field is not in the sector (in the b, θ planes) whose area decreases as the the kernel of $(1-\lambda)\alpha + \lambda\theta_{\mathcal{B}}$ approaches the pages.*

In particular this proves assertions (1), (2) at the beginning of this section. Point (3) is the following:

Theorem 5. [9] *If α_0, α_1 are carried by $\mathcal{B}$ and B has a normal bundle whose disks have Legendrian rays for both contact structures, then for suitable $\theta_{\mathcal{B}}$ and $\epsilon > 0$, the segment*

$$t((1-\lambda)\alpha_0 + \lambda\theta_{\mathcal{B}}) + (1-t)((1-\lambda)\alpha_1 + \lambda\theta_{\mathcal{B}}),\ t \in [0,1] \tag{9}$$

is by contact forms adapted to $\mathcal{B}$ for all $\lambda \in (1-\epsilon, 1)$. In particular co-oriented contact structures carried by the same book are isotopic.

Proof. By our assumptions we have coordinates r, θ, b of $\nu(B)$ such that

$$\alpha_i = f_i db + g_i d\theta,\ i = 0, 1$$

with equations 5 and 6 fulfilled and $f_i > 0$. We take $\theta_{\mathcal{B}}$ defined by ρ supported where the coordinates are defined. Then in that neighborhood the forms in 9 are contact and adapted to $\mathcal{B}$ for all $t \in [0, 1]$, $\lambda \in [0, 1)$. Away from the neighborhood the restriction of the exterior derivative to each page is the convex combination of $(1 - \lambda)d\alpha_0), (1 - \lambda)d\alpha_1$, therefore symplectic. Regarding the 3-form we obtain

$$A(t, \lambda) + \lambda(1 - \lambda)\theta_{\mathcal{B}} \wedge (td\alpha_0 + (1 - t)d\alpha_1),$$

and as λ approaches 1 the second summand becomes dominant.

For fixed coordinates r, θ, b by just sliding along the b coordinate we can arrange the radial segments to be isotropic. That means that for two given contact forms adapted to $\mathcal{B}$ we can change one of them (by ambient isotopy) so that the hypothesis in the statement are fulfilled. □

Points (4) and (5) will not be discussed here. A proof can be found in [8].

2.2. **Open book decompositions and Liouville structures.** Given an open book $\mathcal{B}$ on M^{2n+1}, we need some extra structure to produce a contact form as in theorem 4.

Definition 8. *A Liouville structure on F is a pair (ω, X) where ω is a symplectic form and X a Liouville vector field $L_X\omega = \omega$ transverse to ∂F. It is often given as just the 1-form β where $\omega = d\beta$, $\beta = i_X\omega$. We will consider Liouville structures making the boundary convex, i.e. with outwards point Liouville vector field.*

The 1-form $\gamma := \beta_{|\partial F}$ is contact. For $r \in [1, 1 + \delta]$ an appropriate inwards pointing transverse coordinate (the result of a suitable orientation reversing change of coordinates of the coordinate associated to X), $r = 1$ defining the boundary, then

$$\beta = (2 - r)\gamma \tag{10}$$

Definition 9. *The completion of F is the result of gluing to F the symplectization of $(\partial F, \gamma)$, which can be done in a unique way. The result is a complete Liouville manifold convex at infinity. This is the same as saying that we allow $-r$ to go to ∞.*

Once more if we have an open book with oriented page, we assume that the manifold is oriented so that the open book is positively co-oriented.

Theorem 6. [9] *Given a Liouville manifold (F, β, X) and a compactly supported symplectomorphism ϕ, the open book M -which is oriented* *since the page is oriented by the Liouville volume form- carries a canonical positive contact structure $\xi_{\mathcal{B}}$ co-oriented. More precisely one can build α a positive contact form for which*

(1) *The binding is a contact submanifold ($\xi_{\mathcal{B}} \pitchfork B$) and $d\alpha$ makes the pages into symplectic manifolds; that is to say the Reeb vector field of α is transverse to the pages (in $M \backslash B$).*
(2) *The orientations on the pages induced by $d\alpha$ are positive (and the Liouville structure in the page is strictly pseudoconvex).*

The contact 1-form is said to be adapted to $\mathcal{B}$, and $\mathcal{B}$ carries $\xi_{\mathcal{B}}$.

Proof. We just need to repeat the four steps in theorem 4. For surfaces a nice normal form for β near ∂F followed for free. Here we have to use the Liouville structure to write as in equation 10

$$\beta_{|\nu(\partial F)} = (2-r)\gamma,$$

γ the induced contact form in ∂F.

Step 2 is exactly the same using $\alpha = \beta + d\theta$.

In step 3 one again looks at 1-forms in $\nu(B)$

$$\alpha = f(r,\theta,b)\beta + g(r,\theta,b)d\theta,\ b \in B$$

with $g = r^2 G$, $G(0,b) > 0$ for the contact condition to hold along B, $f_{|B} > 0$ (γ is a contact form on the binding).

The good news is that the contact condition and compatibility with B in $\nu(B)\backslash B$ are exactly the same equations

$$f\frac{\partial g}{\partial r} - \frac{\partial f}{\partial r} g > 0,\ \frac{\partial f}{\partial r} < 0,$$

so one can extend the contact form α defined in $\nu(\partial F)$ exactly as we did in the 3-dimensional case.

Step 4 again is the same, because once we have $\phi^*\omega = \omega$, we only use the symplectic nature of the page to make the necessary adjustment. □

Remark 9. *One of the complications in higher dimensions is the need of asking for a Liouville structure which makes ∂F strictly convex. That gives the right sign for $\frac{\partial f}{\partial r}$ near ∂F so it can be extended to the open book. With a concave structure one does not know how to do it. This difference between concave and convex does not appear in the 3-dimensional case. The reason -apart from simply noting that the normal form procedure gives the right answer- is that there is no difference between convex and concave ends in dimension 2 because we can send the punctured disk (annulus) inside out symplectically. The only result is the reversal of the orientation of the boundary.*

Theorem 7. [9] *Let ξ be a co-oriented contact form carried by an open book $\mathcal{B}$. Then if a is a positive form representing ξ and carried by $\mathcal{B}$, then ξ is isotopic to $\xi_{\mathcal{B}}$ as in theorem 6 for a suitable Liouville structure on the page F.*

Proof.

1. Normal form near the binding: We fix coordinates r, θ, b, $b \in B$, $r \in [0,1]$ adapted to $\mathcal{B}$. Let γ denote $\alpha_{|B}$. Recall that without altering the polar coordinates we can arrange

$$\alpha = f(r,\theta,b)\gamma + g(r,\theta,b)d\theta$$

Indeed, by Gray's theorem we can obtain the restriction of ξ to any fiber over a point of the disk (r,θ) to match $\ker\gamma$. We further need to make sure that the radial segments are isotropic. We choose as new normal disks the only ones which are tangent to the symplectic orthogonal to $\ker\gamma$ (at B) and are made of isotropic rays. By construction the transformation on each fiber is a contactomorphism (follows from the choices together with Cartan's formula).

By making the radial coordinate smaller if necessary we can assume

$$\frac{\partial g}{\partial r} > 0,\ f > 0$$

2. Recognizing the Liouville structure in the open page: The restriction of α defines a non-complete Liouville structure $(F \backslash B, \beta)$. In our previous coordinates, the Liouville vector field is

$$X = \frac{f}{\frac{\partial f}{\partial r}} \frac{\partial}{\partial r} \tag{11}$$

Assume for the moment that f only depends on r, so the Liouville vector field has the same property. Because the latter goes to infinity the Liouville structure is not complete, but it defines a complete unique structure up to completion: we just take parallel copies of B for small values of the radial coordinate. In that way we get a Liouville structure in a compact manifold, as in definition 8. Of course, different values give different structures, but with diffeomorphic completion. It is clear that for any two choices we can apply theorem 6 to produce a one parameter family of contact forms adapted to $\mathcal{B}$.

In general we can always modify a given f to $\hat{f}$ which only depends on r near the binding. Then β and $\hat{\beta} := \hat{f}\gamma$ again define the same Liouville structure up to completion and the resulting contact 1-forms can be joined by a 1-parameter family of contact 1-forms adapted to $\mathcal{B}$ (this is easily seen by taking $\hat{f}$ so that $\hat{f}/\frac{\partial \hat{f}}{\partial r} \leq f/\frac{\partial f}{\partial r}$)

It is also clear that the choice of different radial coordinates (different tubular neighborhoods of B) has the effect of producing diffeomorphic completions; just take a function f which only depends on the initial radial coordinates. Isotope one radial coordinate into the other, and use the isotopy φ_s, $s \in [0,1]$ to push the function giving rise to $f_s\gamma$ a family of Liouville structures. Clearly $\varphi_s^* f_s \gamma = f\gamma$, $s \in [0,1]$. Obviously the isotopy extends to a diffeomorphism of the completions.

3. Modifying the contact form to obtain a suitable Liouville structure and returm map: we start by modifying f to $\hat{f}$ which only depends on

r if $r \leq 1/2$ and coincides with f if $r \geq 1$. and consider

$$\hat{\alpha} := \hat{f}\gamma + g d\theta$$

Notice that $(1-t)\alpha + t\hat{\alpha}$ are all contact forms adapted to $\mathcal{B}$. We will use the Liouville structure $(F, \hat{\beta})$ for which the radial coordinate is a multiple of the coordinate associated to $-X$.

The monodromy associated to $\hat{\alpha}$ is generated by

$$R_\gamma - \tfrac{f'}{g'}\tfrac{\partial}{\partial\theta}$$

and therefore not compactly supported. To correct it we modify g into $\check{g}$ which is monotone on r but constant for $r \leq 1/2$. That produces $\check{\alpha}$ a contact form away from B but still adapted to $\mathcal{B}$. In any case the restriction to each page is still $\hat{\beta}$ so this amounts to no change in the Liouville structure.

The Reeb vector field of $\check{\alpha}$ is transverse to the pages, and by construction has for $r \leq 1/2$ closed orbits bounding the normal disks associated to the coordinates r, θ, b. Let ϕ denote the return map which is a symplectomorphism, and let α_ϕ the contact 1-form adapted to $(M, \mathcal{B})$ constructed as in theorem 6 out of $(F, \hat{\beta}, \phi)$, that we now see as a contact 1-form on M. We claim that $\hat{\alpha}$ and α_ϕ can be connected by a path of 1-forms adapted to $\mathcal{B}$.

As before, we want to use the segments

$$t(\lambda\theta_{\mathcal{B}} + (1-\lambda)\hat{\alpha}) + (1-t)(\lambda\theta_{\mathcal{B}} + (1-\lambda)\alpha_\phi), \tag{12}$$

where ρ as in remark 8 but now constant for $\rho \geq 1/2$. Recall that by construction

$$\alpha_{\phi|r\leq 1/2} = F\gamma + G d\theta$$

with F strictly decreasing, G increasing monotone until it becomes constant for $r \geq 1/2$. Convexity considerations imply that for all $t \in [0,1]$, $\lambda \in [0,1)$, the forms in equation 12 are contact and adapted to $\mathcal{B}$ if $r \leq 1/2$. If $r \geq 1/2$ we recall by construction and remark 6 on each page

$$d\hat{\alpha} = d\alpha_\phi$$

Then exactly the same argument used in the 3-dimensional case shows that for some $\epsilon > 0$ and $\lambda \in (1-\epsilon, 1)$, $t \in [0,1]$, the corresponding 1-form is contact and adapted to $\mathcal{B}$. □

Remark 10. *What we saw is that there we have two ways of producing a complete Liouville manifold starting with data in a compact manifold F: one possibility is a Liouville structure as in definition 8. Another is a one form β which is contact in the boundary and produces a Liouville structure in the interior.*

3. Applications and examples of open book carrying contact structures

An interesting application of the existence of open book is the following

Theorem 8. [3] *If M admits a co-oriented contact structure then so $M \times \Sigma$ does, where Σ is any closed orientable surface different from the sphere. In particular $\mathbb{T}^{2n+1}$ admits a contact structure.*

Proof. One first proves that $\Sigma = \mathbb{T}^2$ admits a contact form making all the copies of M contact submanifold. Then a argument using a ramified cover $\Sigma \to \mathbb{T}^2$ yields the desired result.

Let β be a contact form on M and fix $f\colon M \to \mathbb{C}^2$ an open book decomposition adapted to it. In coordinates (r, θ, b) of $\nu(B)$ we assume $f = (h(r), \theta)$ where h is the identity near the binding and constant near the boundary. We let ϑ_1, ϑ_2 be the angular coordinates of the torus. Then for a small enough domain of the radial coordinate the 1-form

$$\alpha := f_1\vartheta_1 + f_2\vartheta_2 + \beta$$

is contact, where $f = (f_1, f_2)$ in the coordinates of the plane $\mathbb{R}^2 = \mathbb{C}$. $\square$

Remark 11. *Unfortunately α is not adapted to any of the natural open books that we can put in $M \times \mathbb{T}^2$ once f has been fixed (For $g\colon N \to S^1$ a mapping torus, we have open book decompositions in $M \times N$ by composing $f \times g$ with either the projection of the multiplication $\mathbb{C} \times S^1 \to \mathbb{C}$).*

3.1. Open book decompositions for Boothby-Wang contact structures.

We go back for a second to open book decompositions, with no reference to contact structures.

Let M be a closed manifold endowed with the action of a (compact) group G. We would like to construct -if possible- G-invariant open books. Here the point of view of functions (OBIII) seems the most appropriate. Assume G acts in $\mathbb{C}$ preserving $\mathcal{B}_{\text{std}}$ (fixing the binding and sending pages to pages). Then we look for G-equivariant functions providing open books $f \in \mathcal{B}(M)^G$. The case that gives nice results is $G = S^1$ acting on $\mathbb{C}^*$ by complex multiplication. As a matter of fact we find useful to consider the full $\mathbb{C}^*$-action. On the side of the manifold we "extend" it to its cone $C(M)$, and observe that once we remove the apex we get $C(M)^*$ a manifold with a $\mathbb{C}^*$-action.

The following to elementary lemmas are quite useful:

Lemma 4. *Let M be a manifold endowed with a locally free S^1-action, so its cone carries a locally free $\mathbb{C}^*$-action. Let $F\colon C(M) \to \mathbb{C}$ be a $\mathbb{C}^*$-equivariant function, which can always be written as the cone of $f\colon M \to \mathbb{C}$ an S^1-equivariant function. Then the following statements are equivalent:*

(1) *F is transverse to $\mathcal{B}_{\mathrm{std}}$.*
(2) *F is transverse to $\mathbf{0}$.*
(3) *f is transverse to $\mathbf{0}$.*
(4) *f is transverse to $\mathcal{B}_{\mathrm{std}}$.*

Lemma 5. *Let M be a principal S^1-bundle over X, and let L be the associated complex line bundle. Then there is a one to one correspondence between $\mathbb{C}^*$-equivariant functions of L^*-and sections of $L \to X$, and this correspondence takes functions transverse to $\mathbf{0}$ to sections transverse to the zero section.*

Thus we obtain

Corollary 1. *Given M a closed principal S^1-bundle with associated complex line bundle $p\colon L \to X$, any section s of L^* transverse to the zero section provides open book decomposition for M. The binding is the restriction of the principal bundle to the zero set $Z(s)$, the projection p sends any page diffeomorphically into $X \backslash Z(s)$, and the free S^1-action interchanges pages.*

Recall that we have a diffeomorphism of fibrations $L^* \backslash \mathbf{0} \simeq L \backslash \mathbf{0}$ which restricts to circle bundle for a hermitian metric and its dual. Therefore using this diffeomorphism a section s of L also defines an equivariant open book decomposition on M (for the S^1-action in the opposite direction). One checks that

(1) The binding is as before $M_{|Z(s)}$.
(2) The positive real page is

$$F = \{\tfrac{s(x)}{|s(x)|} \,|\, x \in X \backslash Z(s)\}$$

(3) The return map is a family Dehn twist. That is, the end of F is $\nu(Z(s)) \backslash Z(s)$. The circle bundle intersects each normal annulus, which is oriented by pulling back the orientation of the complex fiber. Then the circle is oriented using the outward normal first convention (outwards w.r.t. $X \backslash Z(s)$). Then one performs a Dehn twist in these family of curves.

Let M be a principal S^1-bundle with base X. A Boothby-Wang contact form is a connection 1-form $A \in \Omega^1(X)^{S^1}$ which is contact, or equivalently whose curvature is a symplectic form on X. Let $p\colon L \to X$ be the complex line bundle associated to M: we have the relation

$$-\frac{dA}{2\pi} = p^* c_1(L)$$

For a fixed integral symplectic form ω on X the previous process can be reversed. Because any two Boothby-Wang connection forms differ by the action of the Gauge group, we simply speak of the Boothby-Wang structure associated to the symplectic base (X, ω).

Proposition 1. *Let M be a closed manifold and let A be a Boothby-Wang contact form with associated complex line bundle $L \to (X, \omega)$. Then any section $s\colon X \to L^*$ (i) transverse to the zero section, (ii) for which $Z(s)$ is a symplectic submanifold of (X, ω), and (iii) for which $\nabla s\colon \nu(Z(s)) \to L^*$ sends the symplectic orientation of normal disks to the complex orientation of fibers, defines an open book decomposition adapted to A.*

Proof. By corollary 1 s defines an open book. Because $Z(s)$ is symplectic the binding $M_{|Z(s)}$ is contact. Because the pages are transverse to the infinitesimal generator of the S^1-action, and this equals the Reeb vector field, the pages are symplectic. The orientation condition (iii) in the statement of proposition 1 is equivalent to the orientation condition required for an open book to be adapted to a contact form. □

Remark 12. *Following theorem 7 the page admits a Liouville structure and the return map is a symplectomorphism. In these case the (non-complete) Liouville structure is the one in the complement of $Z(s)$ with potential p_*A. In suitable coordinates about $Z(s)$ the circle bundles of $\nu(Z(s))$ associated to the radial coordinates inherit contact forms of Boothby-Wang type which up to a multiplicative constant $c(r)$ they are $A_{|Z(s)}$ (this is an standard fact for the neighborhood of any symplectic divisor Y for which $c_1(\nu(Y))$ lifts $[\omega_{|Y}]$). Up to isotopy of the section near its zero set this is exactly the Liouville structure in the open end of the hyperplane section induced by p_*A. A symplectic representative for the family Dehn twist is chosen by starting to rotate along the Reeb vector fields as the radial coordinate approaches zero until a full turn on each normal disk has been completed).*

Remark 13. *If (X, w) is a Hodge manifold and s is a holomorphic section of L^*, then proposition 1 is a special case of the construction of Milnor open books in* [5]*. The complex manifold with isolated singularity is the result of collapsing the zero section of L to a point. Because we only allow $\mathbb{C}^*$-equivariant holomorphic functions the proof becomes much simpler.*

Remark 14. *One may try to find invariant open books for contact toric manifolds by restricting to the Boothby-Wang case. For that one restricts to contact toric actions of Reeb type which are regular* [4]*. The quotient is a symplectic toric manifold. The existence of an invariant open book implies that the binding is itself a toric contact manifold, and therefore fibers over a union of facets of the symplectic toric base with non-empty intersection. Those facets should represent the Poincaré dual of the toric symplectic form. This might also be a sufficient condition (for the family Dehn twist appears to be equivariant w.r.t. toric action). Projective spaces are example of such symplectic toric varieties, but we do not know how frequent these symplectic toric varieties are (for example among Hirzebruch surfaces $\mathbb{CP}^2$ is the only one).*

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