ORBIFOLDS VIA LIE GROUPOIDS

OLIVIER BRAHIC

These are notes for a survey talk, to be held in the meeting on Poisson geometry in Porto on May 23, 2009. I will briefly review the classical definition of orbifold by gluing local charts. Then I'll explain the groupoid approach, following the lines of [6], and try to explain some advantages.

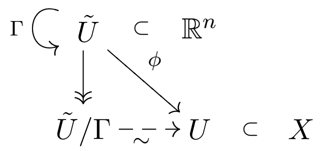
1. Local approach to orbifolds

Orbifolds were introduced in the 50's by Satake [9, 10], under the name of V-manifolds. They were thought as a generalization of manifolds, locally modelled as *quotients* of open balls by linear actions of finite groups.

The original definition carried a few problems, in particular it was not so clear how to construct maps between orbifolds, and if their composition was well-defined. Also there were problem with pull-backs of vector orbi-bundles. These definitions then have been modified in order to sort these problems [2, 3]. Here, we will follow the terminology of Thurston.

The first approach to orbifold is a *local* one, by means of atlases of compatible orbifold charts.

Let X be a topological space (Haussdorf, paracompact) and $n \in \mathbb{N}$ fixed (it will be the dimension of the orbifold).

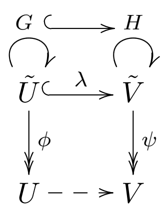


Def. An **orbifold chart** on X is a triple $(\tilde{U}, \Gamma, \phi)$ where:

- $\tilde{U} \subset \mathbb{R}^n$ is a connected, open subset;
- Γ is a finite group acting *effectively* on $\tilde{U}$;
- $\phi : \tilde{U} \to X$ is G-invariant;

and such that ϕ induces a homeomorphism on $\tilde{U}/\Gamma \to U \subset X$.

Remark 1.1. There a slightly more general definition that allows *non-effective* actions (this will matter later on when discussing orbifold groupoids).



Def. An **embedding** $\lambda : (\tilde{U}, \Gamma, \phi) \to (\tilde{V}, H, \psi)$ between two orbifolds charts on X is given by:

- an injection $G \subset H$
- an embedding $\tilde{U} \hookrightarrow \tilde{V}$ such that $\phi \circ \lambda = \psi$.

Note that in the case U and V have same dimension, the injection $G \hookrightarrow H$ is in fact determined by λ.

Remark 1.2. If one wants to allow non-effective actions, the group morphism shall be required to induce an isomorphism $\ker \rho_1 \simeq \ker \rho_2$.

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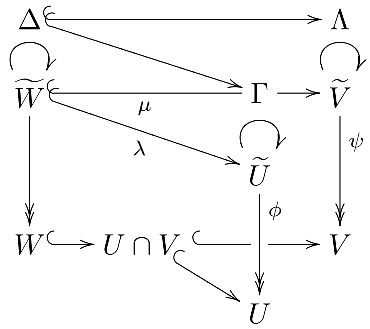
Def. Two charts $(\tilde{U}, \Gamma, \phi), (\tilde{V}, \Lambda, \psi)$ are **locally compatible** if for any $x \in U \cap V$, there exists:

- an open subset $W \subset U \cap V$ containing x;
- an orbifold chart $(\tilde{W}, \Delta, \eta)$;
- orbifold embeddings $\lambda : (\tilde{W}, \Delta, \eta) \to (\tilde{U}, \Gamma, \phi), \mu : (\tilde{W}, \Delta, \eta) \to (\tilde{V}, \Lambda, \psi)$.



Def. An **orbifold atlas** on X is a collection $\mathcal{U}$ of pairwise locally compatible orbifold charts.

Def. An atlas $\mathcal{U}$ is said to **refine** another atlas $\mathcal{V}$ if every chart of $\mathcal{U}$ has an injection into some chart of $\mathcal{V}$. Two orbifold atlases are said to be **equivalent** if they admit a common refinement

Def. A (say *classical*) **orbifold structure** on X is equivalence class of orbifold atlases on X.

Given an orbifold structure on X, there is a notion of tangent bundle TX. It is itself an orbifold, that can be obtained by differentiating the above structure maps. This is important since it means one can do differential geometry on orbifolds.

Remark 1.3. An orbifold structure does not only carry information on the topology of X. There is also, for any $x \in X$, some attached group action. This can be seen as follows:

i) by applying the slice theorem for compact group actions, one sees that any orbifold admits an atlas consisting of **linear charts**, *i.e.* $\tilde{U} \subset \mathbb{R}^n$ on which G acts linearly, by means of orthogonal trabsformations. By effectiveness, one can thus always see G as a finite subgroup of O_n.
ii) given an arbitrary point $x \in X$, one may choose such a linear chart and consider G_x the isotropy at x. It is easily seen by the definitions that, up to conjugaison, G_x is a well defined subgroup of O_n (independantly of the chart). It is called the local group at x.

   This way, X carries a natural stratification, whose stratas are the connected components of the set:
$$\mathcal{S}_H := \{x \in X, (G_x) = H\}.$$

The notion of maps between orbifols may be defined as follows:

Def. A map $f : X \to Y$ is a smooth map between orbifolds if for any $x \in X$, there exists charts $(\widetilde{U}, \Gamma, \phi), (\widetilde{V}, \Lambda, \psi)$ around x and $f(x)$ and a map $\tilde{f} : \widetilde{U} \to \widetilde{V}$ such that $\phi(\widetilde{U}) \subset \psi(\widetilde{V})$ and satisfying $\psi \circ \tilde{f} = f \circ \phi$.

1.1. **Examples.**

Example 1.4. *Global quotients*

Def. An orbifold given by a single chart (M, G, ρ) is called a **global quotient**.

A topological space X can have several orbifold structures: for instance, the topological space underlying the global quotient $(\mathbb{C}, \mathbb{Z}_2, \rho)$ ($\mathbb{Z}_2$ acting via $z \mapsto -z$) is homeomorphic to $\mathbb{C}$. Thus $\mathbb{C}$ carries at least two distinct orbifold structures (the other one is given by the trivial action).

Example 1.5. *Locally free actions of Lie groups*
Consider G a compact Lie group, and M a manifold

Def. A **locally free** action of G on M is a proper action such that isotropy subgroups are all discrete.

Assume one has a locally free action of G (compact) on a manifold M, if all isotropies G_x act faithfully on a slice at x, then by applying the slice theorem at several points of M, one sees that M/G has a naturally induced orbifold structure. As we shall see, any orbifold can be represented this way.

Remark 1.6. Note that the terminology of *global quotient* implies the discreteness of G. In the case of a locally free action of a compact group, the orbifold structure on M/G may *not* be a global quotient (in other words, one may not be able to replace M and G so that G is discrete).

Example 1.7. *Foliations*
Consider a manifold N endowed with a regular foliation $\mathcal{F}$ of codimension q, and satisfying the following properties:

- each leaf is compact;
- the holonomy group at each point is finite.

Then the quotient $N/\mathcal{F}$ has a natural structure of orbifold (again, any orbifold can be presented this way).

Remark 1.8. For explicit examples, and also a nice exposure of geometric constructions on orbifolds, I would suggest to read [1].

2. The global picture

The idea that groupoids were related with orbifolds originated in [8].

2.1. Lie groupoids.

Let me briefly recall the basic features of a groupoids. They are defined as a small catogories with all arrows are invertible. The geometric point of view goes as follows:

- $a \in \mathcal{G}$ is thought as an arrow between $x := \mathbf{s}(a)$ and $y := \mathbf{t}(a)$, the source and targets of a respectively: $y \xleftarrow{a} x$
- two arrows a', a are composable provided the source of a' coincides with the target of a.
- there are identities, inverses, and multiplication:

$$y \underset{a^{-1}}{\overset{a}{\rightleftarrows}} x \qquad\qquad z \xleftarrow{a'} y \xleftarrow{a} x, \quad z \xleftarrow{a'a} x$$

with usual properties (associativity...) for composable arrows.

The whole groupoid structure is sometimes summarized in a single diagram, as follows:

$$\begin{array}{c} \mathcal{G} \circlearrowleft^{-1} \\ u \uparrow \ \mathbf{s}\downarrow\downarrow\mathbf{t} \\ \mathcal{G}_0 \end{array}$$

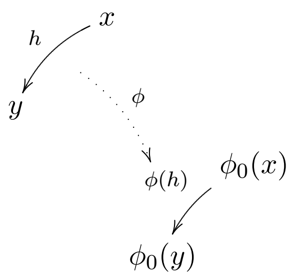
2.2. Groupoid morphisms.

Def. A groupoid morphism is a functor $\mathcal{H} \to \mathcal{G}$.

More precisely, a morphism is given by application ϕ and ϕ_0 that make all structure maps commute:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\phi} & \mathcal{G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathcal{H}_0 & \xrightarrow{\phi_0} & \mathcal{G}_0 \end{array}$$

The picture goes like this:



2.3. **From groupoids to orbifolds.** Recall our basic orbifolds came from locally free actions of compact Lie group (discrete or not). The analogous class of Lie groupoids are proper foliation (etale or not). Let us recall these notions.

Def. A Lie groupoid $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ is said to be **proper** if the map $\mathbf{s} \times \mathbf{t} : \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ is proper, *i.e.* the inverse image of any compact subset is compact.

In particular, for a proper groupoid, all isotropy groups $\mathcal{G}_x := \mathbf{s}^{-1}(x) \cap \mathbf{t}^{-1}(x)$ are *compact* Lie groups.

Def. A Lie groupoid is called a **foliation groupoid** if all isotropy groups are *discrete.*

For instance, the holonomy and monodromy groupoids of a (regular) foliation are foliation groupoids. Note also that the foliation associated to a foliation groupoid is always regular.

Def. A Lie groupoid is called **etale** if $\mathbf{s}$ and $\mathbf{t}$ are local diffeomorphisms.

A first hint why orbifolds are closely related with *proper, etale* groupoids (but in fact also proper foliation groupoids, as we shall see) may be given by the following result:

Proposition 2.1. *Let $\mathcal{G} \rightrightarrows M$ a proper etale groupoid.*

Then for any point $x_0 \in M$, there is an open neighborhood U of x_0 such that the restriction: $\mathcal{G}|_U$ identifies with $\Gamma_x \ltimes U \rightrightarrows U$.

Here Γ_x is the isotropy group at x acting "linearly" on U, via an identification $U \simeq T_xM$.

Remark 2.2. From the point of view of groupoids, there is *á priori* no reason to ask the isotropies to act *effectively.*

However, the above theorem suggests that if the action of the isotropies are all effective, then the quotient $\mathcal{G}_0/\mathcal{G}_1$, which is a topological space, may be given a orbifold structure. One may even hope that the groupoid gives orbifold charts.

This leads us to introduce the following definitions:

Def. An **orbifold groupoid** is a proper foliation groupoid (so all isotropies are finite, discrete groups).

Def. An orbifold groupoid is **reduced** or *reduced* if for any $x \in \mathcal{G}_0$, the isotropy group $\mathcal{G}_x$ acts effectively on T_xM.

3. Morita equivalences

As we already mentionned, proposition 2.1 suggests to compare the underlying topological space X of an orbifold with a quotient $\mathcal{G}_0/\mathcal{G}_1$ for some groupoid $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$. But recall that for an orbifold structure, one shall keep track of isotropies also. A natural way to do this is using the notion of Morita equivalence.

3.0.1. *Fiber-products.* Given two groupoid morphisms $\mathcal{H} \xrightarrow{\phi} \mathcal{G}, \mathcal{K} \xrightarrow{\psi} \mathcal{G}$, there is a notion of fibered product, $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$, defined as follows:

(1) objects in $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$ are triples (y, g, z) such that: g is an arrow between $\phi_0(y)$ and $\phi_0(z)$:

$$(\mathcal{H} \times_{\mathcal{G}} \mathcal{K})_0 := \left\{(y, g, z) \in \mathcal{H}_0 \ {}_{\phi_0}\!\times_{\mathbf{s}} \mathcal{G} \ {}_{\mathbf{t}}\!\times_{\psi_0} \mathcal{K}_0\right\}$$

In other words, the triples (y, g, z) lie in a diagram like this:

$$\begin{array}{ccc} z & & y \\ \downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle \psi_0} \\ \psi_0(z) & \xleftarrow{\ g\ } & \phi_0(y) \end{array}$$

(2) a morphism between (y, g, z) and (y', g', z') is a couple (h, k) such that $g'\phi(h) = \psi(k)g$:

$$\begin{array}{ccccccc} z & & \psi_0(z) & \xleftarrow{\ g\ } & \phi_0(y) & & y \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \psi(h)} & & \downarrow{\scriptstyle \psi(h)} & & \downarrow{\scriptstyle k} \\ z' & & \psi_0(z') & \xleftarrow[\ g'\ ]{} & \phi_0(y') & & y' \end{array}$$

There is an obvious multiplication, and groupoid morphisms:

$$\begin{array}{ccc} \mathcal{H} \times_{\mathcal{G}} \mathcal{K} & \dashrightarrow & \mathcal{K} \\ \downarrow & & \downarrow \\ \mathcal{H} & \longrightarrow & \mathcal{G} \end{array}$$

In general, the fibered product may not be a *Lie* groupoid. It is Lie whenever for instance:

$$\mathcal{H}_0 \times_{\mathcal{G}_0} \mathcal{G}_1 \stackrel{\mathbf{t}\circ\pi_2}{\to} \mathcal{G}_0 \text{ is a submersion.}$$

3.0.2. *Weak equivalences of Lie groupoids.* The usual notion of weak equivalence between categories can be carried through for Lie groupoids as follows.

Def. A groupoid morphism $\mathcal{H} \stackrel{\phi}{\to} \mathcal{G}$ is an (*weak*) **equivalence** if:

i) the map $\mathcal{G} \times \mathcal{H}_0 \to \mathcal{G}_0$ is a submersion,
ii) the following square is a fibered product:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\ \phi\ } & \mathcal{G} \\ \downarrow{\scriptstyle \mathbf{s}\times\mathbf{t}} & & \downarrow{\scriptstyle \mathbf{s}\times\mathbf{t}} \\ \mathcal{H}_0 \times \mathcal{H}_0 & \xrightarrow{\ \phi_0\times\phi_0\ } & \mathcal{G}_0 \times \mathcal{G}_0 \end{array}$$

Let us explain this definition:

i) says that any object x in $\mathcal{G}_0$ can be connected to an object in $\operatorname{Im}\phi_0$ by some arrow $g \in \mathcal{G}$. This condition corresponds thus to the usual condition of the functor ϕ being *"essentially surjective"*.

In particular, if one is interested in studying the quotient of $\mathcal{G}_0$ by the $\mathcal{G}$-action, then one may replace $\mathcal{G}$ by any equivalent groupoid. Indeed any equivalence induces a homeomorphism $\mathcal{H}_0/\mathcal{H} \simeq \mathcal{G}_0/\mathcal{G}$.

ii) says that for any 2 objects in the image of ϕ_0, say $x = \phi(h)$ and $y\phi(l)$, then ϕ induces an isomorphism on the space of arrows:

$$\mathcal{H}_{h,l} \stackrel{\sim}{\to} \mathcal{G}_{\phi_0(h),\phi_0(l)},$$

which corresponds to the usual notion of the functor ϕ being *"fully faithfull"*. Note in particular that isotropies are preserved by equivalence.

Now the notion of equivalence we just defined is not symetric. In fact, the good corresponding equivalence relation is the following:

Def. Two groupoids $\mathcal{G}, \mathcal{G}'$ are **Morita equivalent** if there exists a groupoid $\mathcal{H}$ and equivalences $\mathcal{H} \xrightarrow{\phi} \mathcal{G}, \mathcal{H} \xrightarrow{\phi'} \mathcal{G}'$:

$$\begin{array}{ccc} & \mathcal{H} & \\ {}^{\phi}\swarrow\sim & & \sim\searrow^{\phi'} \\ \mathcal{G} & & \mathcal{G}' \end{array}$$

Def. A **generalized morphism** $\mathcal{G} \to \mathcal{G}'$ is a usual groupoid morphism $\mathcal{H} \to \mathcal{G}'$, where $\mathcal{H}$ is equivalent to $\mathcal{G}'$.

$$\begin{array}{ccc} \mathcal{H} & & \\ \downarrow\sim & \searrow & \\ \mathcal{G} & & \mathcal{G}' \end{array}$$

One shall use fiber products in order to compose the generalized morphisms. Note that there is also a description of generalized morphisms using principal bundles. This is also closely related with stacks, see [5] for such a discussion.

Example 3.1. Let $\mathcal{G} \rightrightarrows \mathcal{G}_0$ a groupoid and $\mathcal{U} = \{U_i\}_{i \in I}$ an open cover of $\mathcal{G}_O$. Then one defines a groupoid $\mathcal{G}_\mathcal{U}$ as follows:

- the space of objects is the disjoint union: $U := \coprod U_i$. A point in U mayt be written as a pair (x, i), with $x \in U_i \subset M$ and $i \in I$.
- arrows $(x, i) \to (y, j)$ are arrows $x \to y$ in $\mathcal{G}$

Then the evident map $\mathcal{G}_\mathcal{U} \to \mathcal{G}$ is an equivalence. In fact, any generalized map can be represented by an open cover $\mathcal{U}$ and a diagram:

$$\mathcal{G} \leftarrow \mathcal{G}_\mathcal{U} \xrightarrow{\phi} \mathcal{H}$$

for some ϕ.

Example 3.2. Assume Γ is a discrete group acting properly on a manifold M, and consider action groupoid $\Gamma \ltimes M \rightrightarrows M$.

Then one can apply the slice theorem for each point $x \in M$: we obtain a Γ-invariant neighborhood $U_x \subset M$, a slice $V_x \subset M$ at x, and a Γ-equivariant diffeomorphism:

$$\Gamma \ltimes U_x \simeq \Gamma \ltimes_{\Gamma_x} V_x.$$

It makes it fairly easy to show that the injection defines an equivalence:

$$\Gamma_x \ltimes V_x \xrightarrow{\sim} \Gamma \ltimes U_x$$

Example 3.3. The example above can be carried through when replacing Γ with G a compact group acting properly on M. The only difference is now that V_x has to be a transverse to the orbit through x.

Assume now the action is locally free, then the isotropy Γ_x is discrete. Thus we still obtain an equivalence:

$$\mathcal{G} \ltimes U_x \simeq \Gamma \ltimes_{\Gamma_x} V_x.$$

Therefore, we see how the particular example of a proper foliation groupoid $G \ltimes M \rightrightarrows M$ is Morita equivalent to a proper etale groupoid.

3.1. **Properties of Morita equivalences.** There are many properties of Lie groupoids which are Morita invariant. The following ones are relevant for orbifolds.

Proposition 3.4. *The following properties are invariant by Morita equivalence:*

- *being a proper groupoid*
- *being a foliation groupoid*

Being etale is *not* Morita invariant, however:

Proposition 3.5. *A Lie groupoid $\mathcal{G}$ is a foliation groupoid iff it is Morita equivalent to an etale groupoid.*

3.2. **Orbifold structures revisited.** Let now X be a topological space, then one may redefine the notion of orbifold as follows.

Def. An orbifold structure on X is represented by an orbifold groupoid $\mathcal{G} \rightrightarrows \mathcal{G}_0$ and a homeomorphism $\phi : \mathcal{G}_0/\mathcal{G}_1 \simeq X$.

Given an equivalence $f : \mathcal{H} \to \mathcal{G}$, there is an induced homeomorphism $f : \mathcal{H}_0/\mathcal{H}_1 \simeq \mathcal{G}_0/\mathcal{G}_1$, then we say that the orbifold structures on X by $\mathcal{H}$ and $\mathcal{G}$ are equivalent.

Def. An orbifold is a space X equipped with an equivalence class of orbifold structures.

The notion of map is then defined as follows:

Def. An map between two orbifolds X and Y, represented respectively by $\mathcal{G}$ and $\mathcal{H}$, consists into a continuous map $X \to Y$ and a generalized map $\mathcal{H} \to \mathcal{G}$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \mathcal{G}_0/\mathcal{G}_1 \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\sim} & \mathcal{H}_0/\mathcal{H}_1 \end{array} .$$

3.3. **From classical orbifolds to orbifold groupoids.** Given a topological space X endowed with a (classical) orbifold structure, one obtains an effective proper etale Lie groupoid by chosing an atlas of linear charts, and patching together the corresponding local action groupoids.

Now reciprocally, one has the following:

Proposition 3.6. *Let $\mathcal{G}$ be an* effective *proper Lie groupoid, then $\mathcal{G}_0/\mathcal{G}_1$ has a natural structure of classical orbifold, that only depends on $\mathcal{G}$ up to Morita equivalence.*

In other words, classical orbifolds are the same as *effective* orbifold groupoids up to Morita equivalence.

3.4. **Presentations of orbifolds.** Since different groupoids can represent an orbifold, one may want to choose the Lie groupoid as simple as possible; the action groupoid associated to a compact group action for instance.

The following theorem gives a nice answer to this question for effective orbifolds.

Theorem 3.7. *For a Lie groupoid $\mathcal{G}$, the following are equivalent:*
- *$\mathcal{G}$ is Morita equivalent to an effective proper etale groupoid,*
- *$\mathcal{G}$ is Morita equivalent to the holonomy groupoid associated to a foliation with compact leaves and finite holonomy,*
- *$\mathcal{G}$ is Morita equivalent to a translation groupoid $K \ltimes M$, where K is a compact group acting on M, with the property that each stabilizer K_x acts effectively on the normal bundle.*

A detailed proof may be found in [8]. For $i) \iff ii)$, one may reffer to [4] for a proof using exclusively groupoids and Morita equivences. I will sketch a proof for this.

Remark 3.8. For *non-effective* orbifold groupoids, the following is an open problem:
- *Is any orbifold groupoid Morita equivalent to the action groupoid of a compact group ?*

Still there are some positive answers [8].

4. Vector bundles over orbifolds

The use of groupoids makes it particularily convenient to define vector bundles for orbifolds, as we shall see now.

4.1. **Lie groupoid representations.** Recall that the notion of an action for a groupoid goes as follows: one starts with a submersion $J : E \to \mathcal{G}_0$.

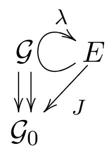
Def. A **left action** of a groupoid $\mathcal{G} \rightrightarrows \mathcal{G}_0$ on $J : E \to \mathcal{G}_0$ is a map λ:

$$\begin{array}{ccc} \mathcal{G}\ {}_{\mathbf{s}}\!\times_J E & \xrightarrow{\lambda} & \mathcal{G}\ {}_{\mathbf{t}}\!\times_J E \\ (a, x) & \longrightarrow & (a, a \cdot x) \end{array}$$

satisfying the usual relations: whenever a, a' are composable

$$\begin{aligned} a \cdot (a' \cdot x) &= (aa') \cdot x \\ 1_b \cdot x &= x, \end{aligned}$$

J is sometimes called the "moment map". A common way to picture an action is as follows:



When $E \to M$ is a vector bundle, and all maps $E_x \xrightarrow{- \cdot a} E_y$, $(a \in \mathcal{G})$ are linear, then E is a called **representation** of the groupoid $\mathcal{G}$. We will denote Rep $(\mathcal{G})$ the set of all representations of $\mathcal{G}$. Now we have the following:

Theorem 4.1. *Let $\mathcal{G}$, $\mathcal{H}$ be Morita equivalent groupoids. Then:*

$$\mathrm{Rep}\,(\mathcal{G}) \simeq \mathrm{Rep}\,(\mathcal{H}).$$

In fact, the correspondence can be made functorial with respect to Morita equivalences. This justifies to define a vector bundle over an orbifold to be a representation of $\mathcal{G}$, where $\mathcal{G}$ is any groupoid representing the orbifold structure.

There is a cohomology naturally associated to a groupoid representation. In the case of the trivial representation $E = \mathbb{R} \times M$, it is given by a double complex:

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & \Omega^{p+1}(\mathcal{G}^{(q)}) & \xrightarrow{\partial} & \Omega^{p+1}(\mathcal{G}^{(q+1)}) & \longrightarrow & \cdots \\ & & d \uparrow & & d \uparrow & & \\ \cdots & \longrightarrow & \Omega^{p}(\mathcal{G}^{(q)}) & \xrightarrow{\partial} & \Omega^{p}(\mathcal{G}^{(q+1)}) & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \end{array}$$

Here:

- $\mathcal{G}^{(q)} = \mathcal{G}_s \times_t \cdots {}_s\times_t \mathcal{G}$ consists of q-uples of composable arrows. It comes equipped with $q+1$ "face maps" $f_i : \mathcal{G}^{(q)} \to \mathcal{G}^{(q-1)}$ given by pairwise multiplication.
- $\Omega^p(\mathcal{G}^{(p)})$ are the (usual) differential p-forms on $\mathcal{G}^{(q)}$.
- d is the usual De Rham differential,
- ∂ is obtained by taking alternate sums of the pull backs f_i^*;

In fact, $\mathcal{G}^\bullet$ is a simplicial manifold. The cohomology associated to the total complex is sometimes called "De Rham cohomology" of $\mathcal{G}$, and denoted $H(\mathcal{G})$. Now we have:

Theorem 4.2. *Let $\mathcal{G}$, $\mathcal{H}$ be Morita equivalent groupoids, then $H(\mathcal{G}) = H(\mathcal{H})$.*

For instance, if one considers a manifold M seen as a groupoid over itself we obtain the De Rham cohomology of M. Let $\mathcal{U}$ be a good open cover of M. One forms the groupoid $M_{\mathcal{U}}$ as in example 3.1; since all U_i's are contractile, it is not hard to see that the above complex gives the Cech cohomology of M. Then Theorem 4.2 then amounts to the classical isomorphism between Cech and De Rham cohomology.

Applying this to orbifold groupoids, one is lead to a natural notion of sheaves over orbifolds, to which one can associated a cohomology [8].

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Depart. de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, PORTUGAL
E-mail address: `brahic@math.ist.utl.pt, rfern@math.ist.utl.pt`