# Toric Kähler-Sasaki Geometry in Action-Angle Coordinates

Miguel Abreu

Center for Mathematical Analysis, Geometry and Dynamical Systems
Instituto Superior Técnico, Lisbon

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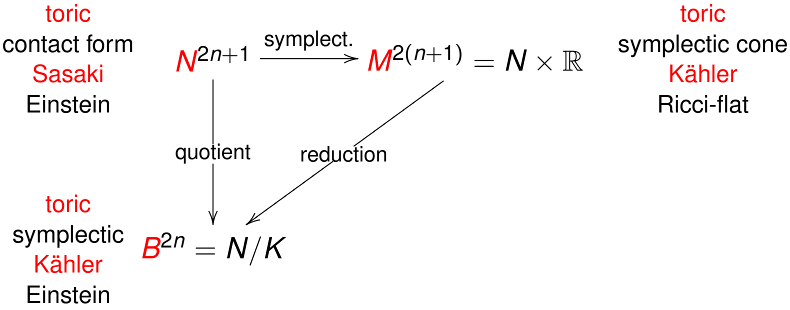
# Motivation

Understand the work of Lerman ('01-02), Gauntlett-Martelli-Sparks-Waldram ('04), Burns-Guillemin-Lerman ('05) and Martelli-Sparks-Yau ('05-06), regarding the following general geometric set-up



# Outline

# Definition of Symplectic Cone





## Definition

A symplectic cone is a triple $(M, \omega, X)$, where $(M, \omega)$ is a connected symplectic manifold and $X \in \mathcal{X}(M)$ is a vector field generating a proper $\mathbb{R}$-action $\rho_t : M \to M$, $t \in \mathbb{R}$, such that $\rho_t^*(\omega) = e^{2t}\omega$. Note that the Liouville vector field $X$ satisfies $\mathcal{L}_X \omega = 2\omega$, or equivalently

$$\omega = \frac{1}{2} d(\iota(X)\omega)\,.$$

symplectic cones $\stackrel{1:1}{\longleftrightarrow}$ co-oriented contact manifolds

In particular, $(M, \omega, X)$ is the symplectization of $(N := M/\mathbb{R}, \xi := \pi_*(\ker(\iota(X)\omega))$, where $\pi : M \to M/\mathbb{R}$.

# Definition of Kähler-Sasaki Cone





## Definition

A Kähler-Sasaki cone is a symplectic cone $(M, \omega, X)$ equipped with a compatible complex structure $J \in \mathcal{I}(M, \omega)$ such that the Reeb vector field $K := JX$ is Kähler, i.e.

$$\mathcal{L}_K \omega = 0 \quad \text{and} \quad \mathcal{L}_K J = 0 \,.$$

Note that $K$ is then also a Killing vector field for the Riemannian metric

$$g_J(\cdot, \cdot) := \omega(\cdot, J\cdot) \,.$$

Any such $J$ will be called a Sasaki complex structure on the symplectic cone $(M, \omega, X)$. The space of all Sasaki complex structures will be denoted by $\mathcal{I}_S(M, \omega, X)$.

# Properties of Kähler-Sasaki Cones



Define $r := \|K\| = \|X\| : M \to \mathbb{R}^+$. Then

- $K$ is the Hamiltonian vector field of $-r^2/2$;
- $X$ is the gradient vector field of $r^2/2$.

Define $\alpha := \iota(X)\omega/r^2 \in \Omega^1(M)$. Then

$$\omega = d(r^2\alpha)/2\,, \quad \alpha(K) \equiv 1 \quad \text{and} \quad \mathcal{L}_X \alpha = 0\,.$$

Define $N := \{r = 1\} \subset M$ and let $\xi := \ker \alpha|_N$. Then

$$(N, \xi, \alpha|_N, g_J|_N) \text{ is a Sasaki manifold.}$$

Define $B := N/K$. Then $TB \cong \xi$ and

$$(B, d\alpha|_\xi, J|_\xi) \text{ is a Kähler space,}$$

the Kähler reduction of $(M, \omega, X, J)$ by the action of $K$.

# Regular, Quasi-Regular and Irregular Kähler-Sasaki Cones



A KS cone $(M, \omega, X, J)$, with Reeb vector field $K = JX$, is said to be:

- regular if $K$ generates a free $S^1$-action.
- quasi-regular if $K$ generates a locally free $S^1$-action.
- irregular if $K$ generates an effective $\mathbb{R}$-action.

Note that the Kähler reduction $B = M//K$ is

- a smooth Kähler manifold if the KS cone is regular.
- a Kähler orbifold if the KS cone is quasi-regular.
- only a Kähler quasifold if the KS cone is irregular.

Note that the Sasaki manifold determined by a KS cone is always smooth.

# Toric Symplectic Cones



## Lemma

Let $G$ be a Lie group. Any $X$-preserving symplectic $G$-action on a symplectic cone $(M, \omega, X)$ is Hamiltonian. Moreover, its moment map $\mu : M \to \mathfrak{g}^*$ can be chosen so that

$$\mu(\rho_t(m)) = e^{2t}\rho_t(m)\,,\ \forall\, m \in M,\ t \in \mathbb{R}.$$

## Definition

A toric symplectic cone is a symplectic cone $(M, \omega, X)$ of dimension $2(n+1)$ equipped with an effective $X$-preserving $\mathbb{T}^{n+1}$-action, with moment map $\mu : M \to \mathfrak{t}^* \cong \mathbb{R}^{n+1}$ such that $\mu(\rho_t(m)) = e^{2t}\rho_t(m)\,,\ \forall\, m \in M,\ t \in \mathbb{R}$. Its moment cone is defined to be the set $C := \mu(M) \cup \{0\} \subset \mathbb{R}^{n+1}$ .

# Classification of Good Toric Symplectic Cones



## Definition (Lerman)

A cone $C \subset \mathbb{R}^{n+1}$ is good if there exists a non-empty minimal set of primitive vectors $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$ such that

(i) $C = \bigcap_{a=1}^{d} \{x \in \mathbb{R}^{n+1} \; : \; \ell_a(x) := \langle x, \nu_a \rangle \geq 0\}$.

(ii) any codimension-$k$ face $F$ of $C$, $1 \leq k \leq n$, is the intersection of exactly $k$ facets whose set of normals can be completed to an integral base of $\mathbb{Z}^{n+1}$.

## Theorem (Banyaga-Molino, Boyer-Galicki, Lerman)

For each good cone $C \subset \mathbb{R}^{n+1}$ there exists a unique toric symplectic cone $(M_C, \omega_C, X_C, \mu_C)$ with moment cone $C$.

Like for compact symplectic toric manifolds, existence follows from a symplectic reduction construction starting from a symplectic vector space.

# Boothby-Wang Cones (Lerman)



Let $P \subset \mathbb{R}^n$ be an integral Delzant polytope. Then, its standard cone

$$C := \{z(x,1) \in \mathbb{R}^n \times \mathbb{R} \,:\, x \in P\,,\ z \geq 0\} \subset \mathbb{R}^{n+1}$$

is a good cone. Moreover

(i) the toric symplectic manifold $(B_P, \omega_P, \mu_P)$ is the $S^1 \cong \{\mathbf{1}\} \times S^1 \subset \mathbb{T}^{n+1}$ symplectic reduction of the toric symplectic cone $(M_C, \omega_C, X_C, \mu_C)$ (at level one).

(ii) $(N_C := \mu_C^{-1}(\mathbb{R}^n \times \{1\}), \alpha_C := (\iota(X_C)\omega_C)|_{N_C})$ is the Boothby-Wang manifold of $(B_P, \omega_P)$. The restricted $\mathbb{T}^{n+1}$-action makes it a toric contact manifold.

(iii) $(M_C, \omega_C)$ is the symplectization of $(N_C, \alpha_C)$.

# Toric Kähler-Sasaki Cones

## Definition

A toric Kähler-Sasaki cone is a toric symplectic cone $(M, \omega, X, \mu)$ equipped with a toric Sasaki complex structure $J \in \mathcal{I}_S^{\mathbb{T}}(M, \omega)$.

- It follows from the classification theorem that any good toric symplectic cone has toric Sasaki complex structures.
- On a toric Kähler-Sasaki cone $(M, \omega, X, \mu, J)$, the Kähler action generated by the Reeb vector field $K = JX$ is part of the torus action.
- The Kähler reduction $B = M//K$ is a toric Kähler space: manifold (regular case), orbifold (quasi-regular case) or quasifold (irregular case).

# Cone Action-Angle Coordinates on $(M, \omega, X, \mu)$





$\mathbb{R}^{n+1} \supset \mu(M) = C \setminus \{0\} \supset \check{C} \equiv$ interior of $C$.

$$\check{M} \equiv \mu^{-1}(\check{C}) \equiv \{\text{points of } M \text{ where } \mathbb{T}^{n+1}\text{-action is free}\}$$

$$\cong \check{C} \times \mathbb{T}^{n+1} = \left\{(x, y) : x \in \check{C},\ y \in \mathbb{T}^{n+1} \equiv \mathbb{R}^{n+1}/2\pi\mathbb{Z}^{n+1}\right\}$$

such that $\omega|_{\check{M}} = dx \wedge dy \equiv$ standard symplectic form,

$$\mu(x, y) = x \quad \text{and} \quad X|_{\check{M}} = 2x\frac{\partial}{\partial x} = 2\sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}\,.$$

**Definition**

$(x, y) \equiv$ cone symplectic/Darboux/action-angle coordinates.

$\check{M}$ is an open dense subset of $M$.

# Toric Sasaki Complex Structures in Cone Action-Angle Coordinates



Any toric Sasaki complex structure $J \in \mathcal{I}_S^{\mathbb{T}}(\check{M}, \omega, X)$ can be written in suitable cone action-angle coordinates $(x, y)$ on $\check{M} \cong \check{C} \times \mathbb{T}^{n+1}$ as

$$J = \begin{bmatrix} 0 & -S^{-1} \\ S & 0 \end{bmatrix}$$

with

$$S = S(x) = (s_{ij}(x)) = \left( \frac{\partial^2 s}{\partial x_i \partial x_j} \right) > 0$$

for some symplectic potential $s : \check{C} \to \mathbb{R}$, such that

$$S(e^t x) = e^{-t} S(x)\,, \ \forall\, t \in \mathbb{R}\,,\ x \in \check{C}\,,$$

i.e. $S(x) = \mathrm{Hess}_x(s)$ is homogeneous of degree $-1$.

# Reeb Vector Fields in Cone Action-Angle Coordinates



The Reeb vector field $K := JX$ of such a toric Sasaki complex structure $J \in \mathcal{I}_S^{\mathbb{T}}(\check{M}, \omega, X)$ is given by

$$K = \sum_{i=1}^{n+1} b_i \frac{\partial}{\partial y_i} \quad \text{with} \quad b_i = 2 \sum_{j=1}^{n+1} s_{ij} x_j \, .$$

**Lemma (Martelli-Sparks-Yau)**

If $S(x) = (s_{ij}(x))$ is homogeneous of degree $-1$, then

$$K_s := (b_1, \ldots, b_{n+1}) \text{ is a constant vector.}$$

In other words, the action generated by $K$ is part of the torus action. Moreover,

regularity of toric KS cone $\Leftrightarrow$ rationality of $K_s \in \mathbb{R}^{n+1}$.

# Characteristic Hyperplane and Polytope

The norm of the Reeb vector field is then given by

$$\|K\|^2 = \|(\mathbf{0}, K_s)\|^2 = b_i s^{ij} b_j = b_i s^{ij}(2 s_{jk} x_k) = 2 b_i x_i = 2\langle x, K_s\rangle \,.$$

Hence

$$\|K\| > 0 \Leftrightarrow \langle x, K_s\rangle > 0 \quad \text{and} \quad \|K\| = 1 \Leftrightarrow \langle x, K_s\rangle = 1/2 \,.$$

## Definition (Martelli-Sparks-Yau)

The characteristic hyperplane $H_K$ and polytope $P_K$ of a toric Kähler-Sasaki cone $(M, \omega, X, \mu, J)$, with moment cone $C \subset \mathbb{R}^{n+1}$, are defined as

$$H_K := \{x \in \mathbb{R}^{n+1} \ : \ \langle x, K_s\rangle = 1/2\} \quad \text{and} \quad P_K := H_K \cap C \,.$$

Note that $N := \mu^{-1}(H_K)$ is a toric Sasaki manifold and $P_K$ is the moment polytope of $B = M//K$.

# Canonical Symplectic Potentials

Let $P \subset \mathbb{R}^{n+1}$ be a polyhedral set defined by

$$P = \bigcap_{a=1}^{d} \{x \in \mathbb{R}^{n+1} \ : \ \ell_a(x) := \langle x, \nu_a \rangle + \lambda_a \geq 0\}$$

where $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$ are primitive integral vectors and $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$. Assume that $P$ has a non-empty interior and the above set of defining inequalities is minimal.

Burns-Guillemin-Lerman extended Delzant's symplectic reduction construction, associating to each such polyhedral set a toric Kähler space of dimension $2(n+1)$

$$(M_P, \omega_P, \mu_P, J_P)$$

such that

$$\mu_P(M_P) = P \quad \text{and} \quad J_P \in \mathcal{I}^{\mathbb{T}}(M_P, \omega_P)\,.$$

# Canonical Symplectic Potentials



## Guillemin'94, Burns-Guillemin-Lerman'05

In appropriate action-angle coordinates $(x, y)$, the canonical symplectic potential $s_P : \breve{P} \to \mathbb{R}$ for $J_P|_{\breve{P}}$ is given by

$$s_P(x) = \frac{1}{2} \sum_{a=1}^{d} \ell_a(x) \log \ell_a(x) \,.$$

Note that $C := P$ is a cone iff $\lambda_1 = \cdots = \lambda_d = 0$. In this case, $s_C := s_P$ is the symplectic potential of a toric Sasaki complex structure $J_C \in \mathcal{I}_S^{\mathbb{T}}(M_C, \omega_C)$, since $S_C(x) = \text{Hess}_x(s_C)$ is then homogeneous of degree $-1$. The corresponding Reeb vector field $K = (\mathbf{0}, K_C)$ is given by

$$K_C = \sum_{a=1}^{d} \nu_a \,.$$

# General Cone Symplectic Potentials (Martelli-Sparks-Yau)



Given $b \in \mathbb{R}^{n+1}$, define

$$s_b(x) := \frac{1}{2}\left(\langle x, b\rangle \log\langle x, b\rangle - \langle x, K_C\rangle \log\langle x, K_C\rangle\right) .$$

Then $s := s_C + s_b$ is such that $K_s = b$. If $C$ is good, this symplectic potential $s$ defines a smooth Sasaki complex structure on the symplectic cone $(M_C, \omega_C, X_C, \mu_C)$ iff

$$\langle x, b\rangle > 0\,,\ \forall\, x \in C \setminus \{0\}\,,\ \text{i.e. } b \in \breve{C}^*$$

where $C^* \subset \mathbb{R}^{n+1}$ is the dual cone

$$C^* := \{x \in \mathbb{R}^{n+1}\ :\ \langle v, x\rangle \geq 0\,,\ \forall\, v \in C\}\,.$$

The dual cone $C^*$ can be equivalently defined as

$$C^* = \cap_\alpha \{x \in \mathbb{R}^{n+1}\ :\ \langle \eta_\alpha, x\rangle \geq 0\}\,,$$

where $\eta_\alpha \in \mathbb{Z}^{n+1}$ are the primitive generating edges of $C$.

# General Cone Symplectic Potentials (Martelli-Sparks-Yau)



Let $s, s' : \breve{C} \to \mathbb{R}$ be two symplectic potentials defined on the interior of a cone $C \subset \mathbb{R}^{n+1}$. Then

$$K_s = K_{s'} \Leftrightarrow (s - s') + \text{const. is homogeneous of degree } 1.$$

### Martelli-Sparks-Yau'05

Any toric Sasaki complex structure $J \in \mathcal{I}_S^{\mathbb{T}}$ on a toric symplectic cone $(M_C, \omega_C, X_C, \mu_C)$, associated to a good moment cone $C \in \mathbb{R}^{n+1}$, is given by a symplectic potential $s : \breve{C} \to \mathbb{R}$ of the form

$$s = s_C + s_b + h\,,$$

where $s_C$ is the canonical potential, $b \in \breve{C}^*$ and $h : C \to \mathbb{R}$ is homogeneous of degree 1 and smooth on $C \setminus \{0\}$.

# Boothby-Wang Symplectic Potentials





## Definition

Let $P \subset \mathbb{R}^n$ be an integral Delzant polytope and $C \subset \mathbb{R}^{n+1}$ its standard good cone. Given a symplectic potential $s : \check{P} \to \mathbb{R}$, define its Boothby-Wang symplectic potential $\tilde{s} : \check{C} \to \mathbb{R}$ by

$$\tilde{s}(x,z) := z\, s(x/z) + \frac{1}{2} z \log z \,,\ \forall\, x \in \check{P}\,,\ z \in \mathbb{R}^+ \,.$$

If $P = \bigcap_{a=1}^{d} \{x \in \mathbb{R}^n \ :\ \ell_a(x) := \langle x, \nu_a \rangle + \lambda_a \geq 0\}$ and

$$s(x) = \frac{1}{2} \sum_{a=1}^{d} \ell_a(x) \log \ell_a(x) - \frac{1}{2} \ell_\infty(x) \log \ell_\infty(x) \,,$$

where $\ell_\infty(x) := \sum_a \ell_a(x) = \langle x, \nu_\infty \rangle + \lambda_\infty$, then

$$\tilde{s}(x,z) = s_C(x,z) + s_b(x,z) \quad \text{with} \quad b = (0,\ldots,0,1)\,.$$

# Symplectic Potentials and Reduction



**Calderbank-David-Gauduchon'02**

Symplectic potentials restrict naturally under toric symplectic reduction.

Suppose $(M_P, \omega_P, \mu_P)$ is a toric symplectic reduction of $(M_C, \omega_C, \mu_C)$. Then there is an affine inclusion $P \subset C$ and

any $\widetilde{J} \in \mathcal{I}^{\mathbb{T}}(M_C, \omega_C)$ induces a reduced $J \in \mathcal{I}^{\mathbb{T}}(M_P, \omega_P)$.

This theorem says that if

$$\tilde{s} : \check{C} \to \mathbb{R} \text{ is a symplectic potential for } \widetilde{J}$$

then

$$s := \tilde{s}|_{\check{P}} : \check{P} \to \mathbb{R} \text{ is a symplectic potential for } J.$$

# Symplectic Potentials and Affine Transformations



## Proposition

Symplectic potentials transform naturally under affine transformations.

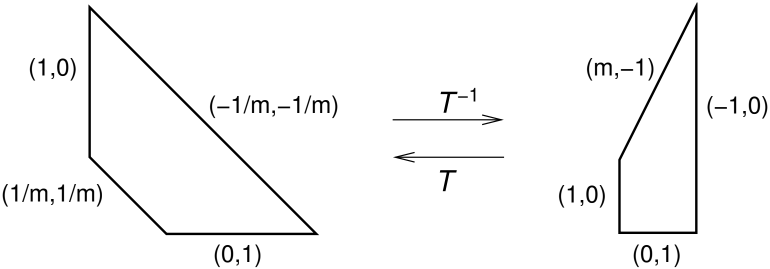
Let $T \in GL(n)$ and consider the linear symplectic change of action-angle coordinates $x' := T^{-1}x$ and $y' := T^t y$.
Then $P = \bigcap_{a=1}^{d} \{x \in \mathbb{R}^n : \ell_a(x) := \langle x, \nu_a \rangle + \lambda_a \geq 0\}$ becomes

$$P' := T^{-1}(P) = \bigcap_{a=1}^{d} \{x' \in \mathbb{R}^n : \ell'_a(x') := \langle x', \nu'_a \rangle + \lambda'_a \geq 0\}$$

with $\nu'_a = T^t \nu_a$ and $\lambda'_a = \lambda_a$, and symplectic potentials transform by $s' = s \circ T$ (in particular, $s_{P'} = s_P \circ T$). The corresponding Hessians are related by $S' = T^t (S \circ T) T$.

# Symplectic Potentials and Affine Transformations - Hirzebruch Surfaces






$$T = \begin{bmatrix} m & -1 \\ 0 & 1 \end{bmatrix}$$

# Symplectic Potentials and Scalar Curvature



- Toric Kähler metric

$$\begin{bmatrix} S & 0 \\ 0 & S^{-1} \end{bmatrix}$$

where $S = (s_{ij}) = \mathrm{Hess}_x(s)$ for a symplectic potential $s : \check{P} \subset \mathbb{R}^n \to \mathbb{R}$.

- Formula for its scalar curvature [A.'98]:

$$Sc \equiv -\sum_{j,k} \frac{\partial}{\partial x_j} \left[ s^{jk} \frac{\partial \log(\det S)}{\partial x_k} \right] = -\sum_{j,k} \frac{\partial^2 s^{jk}}{\partial x_j \partial x_k}$$

- Donaldson'02 - appropriate interpretation for this formula: view scalar curvature as moment map for action of $\mathrm{Ham}(M, \omega)$ on $\mathcal{I}(M, \omega)$.

# Scalar Curvature of Boothby-Wang Symplectic Potentials



## Proposition

Let $P \subset \mathbb{R}^n$ be a polyhedral set and $C \subset \mathbb{R}^{n+1}$ its standard cone. Given a symplectic potential $s : \check{P} \to \mathbb{R}$, let $\tilde{s} : \check{C} \to \mathbb{R}$ be its Boothby-Wang symplectic potential:

$$\tilde{s}(x,z) := z\, s(x/z) + \frac{1}{2} z \log z\,,\ \forall\, x \in \check{P}\,,\ z \in \mathbb{R}^+\,.$$

Then

$$\widetilde{Sc}(x,z) = \frac{Sc(x/z) - 2n(n+1)}{z}\,.$$

In particular

$$\widetilde{Sc} \equiv 0 \Leftrightarrow Sc \equiv 2n(n+1)\,.$$

# Boothby-Wang Symplectic Potentials and Toric Kähler-Sasaki-Einstein metrics



In compact Kähler geometry, constant scalar curvature metric plus cohomological condition $c_1 = \lambda[\omega]$ implies Kähler-Einstein metric.

## Proposition

Let $P \subset \mathbb{R}^n$ be a polyhedral set and $C \subset \mathbb{R}^{n+1}$ its standard cone. Given a symplectic potential $s : \breve{P} \to \mathbb{R}$, let $\tilde{s} : \breve{C} \to \mathbb{R}$ be its Boothby-Wang symplectic potential. Then,

$s$ defines a toric Kähler-Einstein metric with $Sc \equiv 2n(n+1)$

iff

$\tilde{s}$ defines a toric Ricci-flat Kähler metric.

When this happens, the corresponding toric Sasaki metric is Einstein.

# Calabi's Family of Extremal Kähler Metrics



- Calabi constructed in 1982 a general 4-parameter family of $U(n)$-invariant extremal Kähler metrics, which he used to put extremal Kähler metrics on

$$H^n_m := \mathbb{P}(\mathcal{O}(-m) \oplus \mathbb{C}) \longrightarrow \mathbb{P}^{n-1}\,,\ \ n, m \in \mathbb{N}\,,$$

  in any possible cohomology class. In particular, when $n = 2$, on all Hirzebruch surfaces.
- When written in action-angle coordinates, using symplectic potentials, Calabi's family can be seen to contain many other interesting Kähler metrics [A.'01].
- In particular, it contains a 1-parameter family of Kähler-Einstein metrics directly related to the Sasaki-Einstein metrics constructed in 2004 by Gauntlett-Martelli-Sparks-Waldram [A.'08].

# Calabi's Family in Action-Angle Coordinates



Consider symplectic potentials $s : (\mathbb{R}^+)^n \to \mathbb{R}$ of the form

$$s(x) = \frac{1}{2}\left(\sum_{a=1}^{n} x_a \log x_a + h(r)\right), \text{ where } r = x_1 + \cdots x_n.$$

Then

$$Sc(x) = Sc(r) = 2r^2 f''(r) + 4(n+1) r f'(r) + 2n(n+1) f(r),$$

where $f = h''/(1 + rh'')$. Moreover, extremal is equivalent to $Sc$ being an affine function, which is then equivalent to

$$h''(r) = -\frac{1}{r} + \frac{r^{n-1}}{r^n - A - Br - Cr^{n+1} - Dr^{n+2}},$$

where $A, B, C, D \in \mathbb{R}$ are the 4 parameters of the family.

# Interesting Particular Cases of Calabi's Family



$[C = D = 0]$

$$h''(r) = -\frac{1}{r} + \frac{r^{n-1}}{r^n - A - Br}$$

$[C = D = 0,\ B = 0,\ A > 0]$
Complete Ricci flat Kähler metrics on total space of $\mathcal{O}(-n) \to \mathbb{P}^{n-1}$, for any possible cohomology class. (Calabi'79)

$[C = D = 0,\ A, B > 0]$
Complete scalar flat Kähler metrics on total space of $\mathcal{O}(-m) \to \mathbb{P}^{n-1}$, for any $m > 0$ and any possible cohomology class. (LeBrun'88 ($n = 2$), Pedersen-Poon'88 and Simanca'91 ($n > 2$))

# Kähler-Sasaki-Einstein Cases in Calabi's Family



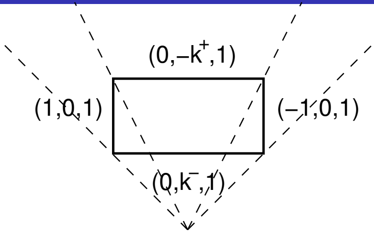
$$\left[B = D = 0,\ C = 1,\ 0 < A < n^n/(n+1)^{n+1}\right]$$

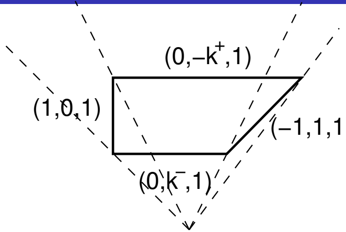
$$h''(r) = -\frac{1}{r} + \frac{r^{n-1}}{r^n(1-r) - A}$$

Kähler-Einstein quasifold metrics with $Sc = 2n(n+1)$ on certain $H_m^n := \mathbb{P}(\mathcal{O}(-m) \oplus \mathbb{C}) \longrightarrow \mathbb{P}^{n-1}$.

For a countably infinite set of values for the variable parameter $A$, the corresponding Boothby-Wang cones are $GL(n+1)$ equivalent to good cones - precisely the ones corresponding to the Sasaki-Einstein metrics of Gauntlett-Martelli-Sparks-Waldram'04 (at least when $n = 2$).

# Kähler-Sasaki-Einstein Cases in Calabi's Family - $n = 2$ examples




$p, q \in \mathbb{N}$, $0 < q < p$, $\gcd(q, p) = 1$

| | |
|---|---|
| even $p \pm q$ | odd $p \pm q$ |
| $k^{\pm} = (p \pm q)/2$ | $k^{\pm} = (p \pm (q-1))/2$ |
| $K_s = (0, -\ell/2, 3)$ | $K_s = (0, (3-\ell)/2, 3)$ |

$$\ell = \left(3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}\right)/q$$

All these Sasaki-Einstein manifolds are diffeo. to $S^2 \times S^3$.

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