# Toric Kähler-Sasaki Geometry

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# Motivation

Understand, through examples in action-angle coordinates, the following general geometric set-up

toric contact form Sasaki Einstein: $N^{2n+1} \xrightarrow{\text{symplect.}} M^{2(n+1)} \cong N \times \mathbb{R}^+$ : toric symplectic cone Kähler Ricci-flat

$N^{2n+1} \xrightarrow{\text{quotient}} B^{2n}$, $\quad M^{2(n+1)} \xrightarrow{\text{reduction}} B^{2n}$

toric symplectic Kähler Einstein: $B^{2n} = N/K$

# Outline

1. Toric Kähler Metrics
2. Toric Kähler-Sasaki Metrics
3. Toric Kähler-Sasaki-Einstein Metrics

# Definition of Toric Symplectic Manifolds



## Definition

A toric symplectic manifold is a connected symplectic manifold $(B^{2n}, \omega)$, equipped with an effective Hamiltonian action of the *n*-torus:

$$\tau : \mathbb{T}^n \cong \mathbb{R}^n / 2\pi \mathbb{Z}^n \hookrightarrow \mathrm{Ham}(B, \omega) \, .$$

The corresponding moment map, unique up to an additive constant, will be denoted by

$$\mu : B \to \mathrm{Lie}^*(\mathbb{T}^n) \cong (\mathbb{R}^n)^* \cong \mathbb{R}^n \, .$$

# Examples - $(\mathbb{R}^{2n}, \omega_{\mathrm{st}}, \tau_{\mathrm{st}}, \mu_{\mathrm{st}})$



Consider $(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$, with linear coordinates $(u_1, \ldots, u_n, v_1, \ldots, v_n)$ such that

$$\omega_{\mathrm{st}} = du \wedge dv := \sum_{j=1}^{n} du_j \wedge dv_j \,.$$

We will also use the usual identification with $\mathbb{C}^n$ given by

$$z_j = u_j + iv_j \,,\ j = 1, \ldots, n \,.$$

The standard $\mathbb{T}^n$-action $\tau_{\mathrm{st}}$ on $\mathbb{R}^{2n}$, given by

$$(y_1, \ldots, y_n) \cdot (z_1, \ldots, z_n) = (e^{-iy_1} z_1, \ldots, e^{-iy_n} z_n) \,,$$

is Hamiltonian, with moment map $\mu_{\mathrm{st}} : \mathbb{R}^{2n} \to \mathbb{R}^n$ given by

$$\mu_{\mathrm{st}}(u_1, \ldots, u_n, v_1, \ldots, v_n) = \frac{1}{2}(u_1^2 + v_1^2, \ldots, u_n^2 + v_n^2) \,.$$

# Examples - Projective Space $(\mathbb{P}^n, \omega_{\mathrm{FS}}, \tau_{\mathrm{FS}}, \mu_{\mathrm{FS}})$



Consider projective space $(\mathbb{P}^n, \omega_{\mathrm{FS}})$, with homogeneous coordinates $[z_0; z_1; \ldots; z_n]$.
The $\mathbb{T}^n$-action $\tau_{\mathrm{FS}}$ on $\mathbb{P}^n$ given by

$$(y_1, \ldots, y_n) \cdot [z_0; z_1; \ldots; z_n] = [z_0; e^{-iy_1} z_1; \ldots; e^{-iy_n} z_n]\,,$$

is Hamiltonian, with moment map $\mu_{\mathrm{FS}} : \mathbb{P}^n \to \mathbb{R}^n$ given by

$$\mu_{\mathrm{FS}}[z_0; z_1; \ldots; z_n] = \frac{1}{\|z\|^2}(\|z_1\|^2, \ldots, \|z_n\|^2)\,.$$

Hence, $(\mathbb{P}^n, \omega_{\mathrm{FS}}, \tau_{\mathrm{FS}}, \mu_{\mathrm{FS}})$ is an example of a compact symplectic toric manifold.
Note that the image of $\mu_{\mathrm{FS}}$ is the convex hull of the images of the $n+1$ fixed points of the action, i.e. the standard simplex in $\mathbb{R}^n$.

# Atiyah-Guillemin-Sternberg and Delzant Theorems (1982)



## Atiyah-Guillemin-Sternberg'82

Let $(B, \omega)$ be a compact toric symplectic manifold with moment map $\mu : B \to \mathbb{R}^n$.
Then, the image $\mu(B)$ of the moment map is the convex polytope given by the convex hull of the images of the fixed points of the action.
This will be usually called the moment polytope and denoted by $P$.

## Delzant'82

The moment polytope is a complete invariant of a compact toric symplectic manifold.

# Action-Angle Coordinates

- $\mathbb{R}^n \supset \mu(B) = P \supset \breve{P} \equiv$ interior of $P$

$$\breve{B} \equiv \mu^{-1}(\breve{P}) \equiv \{\text{points of } B \text{ where } \mathbb{T}^n\text{-action is free}\}$$
$$\cong \breve{P} \times \mathbb{T}^n = \left\{(x,y) \,:\, x \in \breve{P} \subset \mathbb{R}^n,\ y \in \mathbb{R}^n/2\pi\mathbb{Z}^n\right\}$$
such that $\omega|_{\breve{B}} = dx \wedge dy \equiv$ standard symplectic form

**Definition**

$(x, y) \equiv$ symplectic/Darboux/action-angle coordinates.

- $\breve{B}$ is an open dense subset of $B$.
- In these coordinates, the moment map $\mu : \breve{B} \to \breve{P}$ is given by $\mu(x, y) = x$.

# Compatible Almost Complex Structures



**Definition**

A compatible almost complex structure on a symplectic manifold $(B, \omega)$ is an almost complex structure $J$ on $B$, i.e. $J \in \Gamma(\mathrm{End}(TB))$ with $J^2 = -Id$, such that

$$g_J(\cdot, \cdot) \equiv \omega(\cdot, J\cdot)$$

is a Riemannian metric on $B$. This is equivalent to

$$\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot) \quad \text{and} \quad \omega(X, JX) > 0\,, \ \forall\, 0 \neq X \in TB\,.$$

The space of all compatible almost complex structures on a symplectic manifold $(B, \omega)$ will be denoted by $\mathcal{J}(B, \omega)$.

# Remarks



- For any symplectic manifold $(B, \omega)$, the space $\mathcal{J}(B, \omega)$ is non-empty, infinite-dimensional and contractible.
- A Kähler manifold can be defined as a symplectic manifold with an integrable compatible complex structure.
- The space of integrable compatible complex structures on a symplectic manifold $(B, \omega)$ will be denoted by $\mathcal{I}(B, \omega) \subset \mathcal{J}(B, \omega)$.
- In general, $\mathcal{I}(B, \omega)$ can be empty.

# Toric Compatible Complex Structures

## Definition

A toric compatible complex structure on a toric symplectic manifold $(B^{2n}, \omega, \tau)$ is a

$$\mathbb{T}^n\text{-invariant } J \in \mathcal{I}(B, \omega) \subset \mathcal{J}(B, \omega)\,.$$

The space of all such will be denoted by

$$\mathcal{I}^{\mathbb{T}^n}(B, \omega) \subset \mathcal{J}^{\mathbb{T}^n}(B, \omega)\,.$$

- It follows from the classification theorem of Delzant that $\mathcal{I}^{\mathbb{T}^n}(B, \omega)$ is non-empty for any compact toric symplectic manifold.

# Local Form of Toric Compatible Complex Structures



Any $J \in \mathcal{J}^{\mathbb{T}^n}(\breve{B}, \omega|_{\breve{B}})$ can be written in action-angle coordinates (x,y) on $\breve{B} \cong \breve{P} \times \mathbb{T}^n$ as

$$J = \begin{bmatrix} S^{-1}R & \vdots & -S^{-1} \\ \cdots & \cdots & \cdots \\ RS^{-1}R + S & \vdots & -RS^{-1} \end{bmatrix}$$

where $R = R(x)$ and $S = S(x)$ are real symmetric $(n \times n)$ matrices, with $S$ positive definite, i.e.

$$Z(x) \equiv R(x) + iS(x) \in \text{Siegel Upper Half Space}, \forall\, x \in \breve{P}.$$

# Local Form of Toric Compatible Complex Structures



For integrable toric compatible complex structures we have that:

$$J \in \mathcal{I}^{\mathbb{T}^n} \subset \mathcal{J}^{\mathbb{T}^n} \Leftrightarrow \frac{\partial Z_{ij}}{\partial x_k} = \frac{\partial Z_{ik}}{\partial x_j}$$

$$\Leftrightarrow \exists\, f : \check{P} \to \mathbb{C}\,,\ f(x) = r(x) + is(x)\,,\ \text{such that}$$

$$Z_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 r}{\partial x_i \partial x_j} + i \frac{\partial^2 s}{\partial x_i \partial x_j} = R_{ij} + iS_{ij}$$

# Local Form of Toric Compatible Complex Structures



Any real function

$$h : \breve{P} \to \mathbb{R}$$

is the Hamiltonian of a 1-parameter family

$$\phi_t : \breve{B} \to \breve{B}$$

of $\mathbb{T}^n$-equivariant symplectomorphisms, given in action-angle coordinates $(x, y)$ on $\breve{B} \cong \breve{P} \times \mathbb{T}^n$ by

$$\phi_t(x, y) = (x, y - t\frac{\partial h}{\partial x}) .$$

# Local Form of Toric Compatible Complex Structures



The natural action of such a $\phi_t$ on $\mathcal{J}^{\mathbb{T}^n}$, given by

$$\phi_t \cdot J := (d\phi_t)^{-1} \circ J \circ (d\phi_t)\,,$$

corresponds in the Siegel Upper Half Space parametrization to

$$\phi_t \cdot (Z = R + iS) = (R + tH) + iS\,,$$

where

$$H = (h_{ij}) = \left( \frac{\partial^2 h}{\partial x_i \partial x_j} \right)\,.$$

# Local Form of Toric Compatible Complex Structures



Hence, for any integrable $J \in \mathcal{I}^{\mathbb{T}^n}$ there exist action-angle coordinates $(x, y)$ on $\breve{B} \cong \breve{P} \times \mathbb{T}^n$ such that $R \equiv 0$ in the Siegel Upper Half Space Parametrization, i.e. such that

$$J = \begin{bmatrix} 0 & \vdots & -S^{-1} \\ \cdots & \cdots & \cdots \\ S & \vdots & 0 \end{bmatrix}$$

with

$$S = S(x) = \left(s_{ij}(x)\right) = \left(\frac{\partial^2 s}{\partial x_i \partial x_j}\right)$$

for some

real potential function $\quad s : \breve{P} \to \mathbb{R}\,.$

# Local Form of Toric Compatible Complex Structures



The corresponding Riemannian (Kähler) metric

$$g(\cdot,\cdot) \equiv \omega(\cdot, J\cdot)$$

on $\breve{M} \cong \breve{P} \times \mathbb{T}^n$ can the be written in matrix form as

$$g = \begin{bmatrix} 0 & \vdots & I \\ \cdots & \cdots & \cdots \\ -I & \vdots & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \vdots & -S^{-1} \\ \cdots & \cdots & \cdots \\ S & \vdots & 0 \end{bmatrix} = \begin{bmatrix} S & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & S^{-1} \end{bmatrix}$$

with

$$S = \left( \frac{\partial^2 s}{\partial x_i \partial x_j} \right) .$$

# Remarks

- We will call such a potential function

$$s : \check{P} \to \mathbb{R}$$

  the symplectic potential of both the complex structure $J$ and the metric $g$
- This particular way to arrive at the above form for any $J \in \mathcal{I}^{\mathbb{T}^n}$ is due to Donaldson, and illustrates a very particular part of his formal general framework for the action of the symplectomorphism group of a symplectic manifold on its space of compatible (almost) complex structures.

# Examples - $(\mathbb{R}^{2n}, \omega_{\mathrm{st}}, \tau_{\mathrm{st}}, \mu_{\mathrm{st}})$

Consider the linear complex structure $J_{\mathrm{st}} \in \mathcal{I}^{\mathbb{T}^n}(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ giving the standard identification between $\mathbb{R}^{2n}$ and $\mathbb{C}^n$. In action-angle coordinates $(x, y)$ on

$$\breve{\mathbb{R}}^{2n} = (\mathbb{R}^2 \setminus \{(0,0)\})^n \cong (\mathbb{R}^+)^n \times \mathbb{T}^n = \breve{P} \times \mathbb{T}^n ,$$

its symplectic potential is given by

$$s : \breve{P} = (\mathbb{R}^+)^n \longrightarrow \mathbb{R}$$

$$x = (x_1, \ldots, x_n) \longmapsto s(x) = \frac{1}{2} \sum_{i=1}^{n} x_i \log(x_i) .$$

# Examples - $(\mathbb{R}^{2n}, \omega_{\rm st}, \tau_{\rm st}, \mu_{\rm st})$



Hence, in these action-angle coordinates, the standard complex structure has the matrix form

$$J_{\rm st} = \begin{bmatrix} 0 & \vdots & \operatorname{diag}(-2x_i) \\ \cdots & \cdots & \cdots \\ \operatorname{diag}(1/2x_i) & \vdots & 0 \end{bmatrix}$$

while the standard flat Euclidean metric becomes

$$g_{\rm st} = \begin{bmatrix} \operatorname{diag}(1/2x_i) & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & \operatorname{diag}(2x_i) \end{bmatrix}$$

# Examples - Compact Toric Symplectic Manifolds



- To any bounded, convex, simple, integral polytope $P \subset \mathbb{R}^n$, a canonical symplectic reduction construction of Delzant associates a compact Kähler toric manifold

$$(B_P^{2n}, \omega_P, \tau_P, \mu_P, J_P) \quad \text{such that} \quad \mu_P(B_P) = P\,.$$

- Let $F_i$ denote the $i$-th facet of the polytope. The affine defining function of $F_i$ is the function

$$\begin{aligned} \ell_i : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto \ell_i(x) = \langle x, \nu_i \rangle - \lambda_i\,, \end{aligned}$$

where $\nu_i \in \mathbb{Z}^n$ is a primitive inward pointing normal to $F_i$ and $\lambda_i \in \mathbb{R}$ is such that $\ell_i|_{F_i} \equiv 0$. Note that $\ell_i|_{\breve{P}} > 0$.

# Examples - Compact Toric Symplectic Manifolds



## Guillemin'94

In appropriate action-angle coordinates $(x, y)$, the canonical symplectic potential $s_P : \breve{P} \to \mathbb{R}$ for $J_P|_{\breve{P}}$ is given by

$$s_P(x) = \frac{1}{2} \sum_{i=1}^{d} \ell_i(x) \log \ell_i(x) \,,$$

where $d$ is the number of facets of $P$.

# Examples - Projective Space $(\mathbb{P}^n, \omega_{\mathrm{FS}}, \tau_{\mathrm{FS}}, \mu_{\mathrm{FS}})$



For projective spaces $\mathbb{P}^n$ the polytope $P \subset \mathbb{R}^n$ can be taken to be the standard simplex, with defining affine functions

$$\ell_i(x) = x_i\,,\ i = 1, \ldots, n\,, \quad \text{and} \quad \ell_{n+1}(x) = 1 - r\,,$$

where $r = \sum_i x_i$. The canonical symplectic potential $s_P : \check{P} \to \mathbb{R}$, given by

$$s_P(x) = \frac{1}{2} \sum_{i=1}^{n} x_i \log x_i + \frac{1}{2}(1 - r) \log(1 - r)\,,$$

defines the standard complex structure $J_{FS}$ and Fubini-Study metric $g_{FS}$ on $\mathbb{P}^n$.

# Toric $\partial\bar{\partial}$-Lemma in Action-Angle Coordinates



**A.'01**

Let $J \in \mathcal{I}^{\mathbb{T}^n}(B_P, \omega_P)$. Then, in suitable action-angle coordinates $(x, y)$ on $\breve{B} \cong \breve{P} \times \mathbb{T}^n$, $J$ is given by a symplectic potential $s : \breve{P} \to \mathbb{R}$ of the form

$$s(x) = s_P(x) + h(x)\,,$$

where $h \in C^\infty(P)$, $\mathrm{Hess}_x(s) > 0$ in $\breve{P}$ and $\det(\mathrm{Hess}_x(s)) = (\delta(x) \prod_i \ell_i)^{-1}$, with $\delta \in C^\infty(P)$ and $\delta(x) > 0\,,\ \forall x \in P$.

Conversely, any such $s$ is the symplectic potential of a $J \in \mathcal{I}^{\mathbb{T}^n}(\breve{P} \times \mathbb{T}^n)$ that compactifies to a well defined $J \in \mathcal{I}^{\mathbb{T}^n}(B_P, \omega_P)$.

# Toric Kähler Metrics and Scalar Curvature



- Toric Kähler metric

$$g = \begin{bmatrix} (s_{ij}) & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & (s^{ij}) \end{bmatrix}$$

  where $(s_{ij}) = \mathrm{Hess}_x(s)$ for $s : \check{P} \subset \mathbb{R}^n \to \mathbb{R}$.
- Formula for its scalar curvature [A.'98]:

$$S \equiv -\sum_{j,k} \frac{\partial}{\partial x_j}\left[ s^{jk} \frac{\partial \log(\det \mathrm{Hess}_x(s))}{\partial x_k} \right] = -\sum_{j,k} \frac{\partial^2 s^{jk}}{\partial x_j \partial x_k}$$

- (Donaldson'02) Appropriate interpretation of this formula by viewing the scalar curvature as a moment map for the action of the symplectomorphism group of a symplectic manifold on its space of compatible complex structures.

# Examples - Toric Constant Scalar Curvature Metrics in Real Dimension 2



$$-\left(\frac{1}{s''(x)}\right)'' = 2k \Rightarrow s''(x) = -\frac{1}{kx^2 - 2bx - c}\,,\ k, b, c \in \mathbb{R}$$

$[k = 0, b = 0]$ (need $c > 0$)

$$s''(x) = \frac{1}{c} \Rightarrow \|\frac{\partial}{\partial y}\|^2 = c\,,\ s(x) = \frac{x^2}{2c} \Rightarrow \text{cylinder of radius } \sqrt{c}$$

$[k = 0, b > 0]$ (can assume $c = 0$, need $x > 0$)

$$s''(x) = \frac{1}{2bx} \Rightarrow s(x) = \frac{1}{b} \cdot \frac{1}{2} x \log x \Rightarrow \text{cone of angle } \pi b$$

If $b = 1/m\,,\ m \in \mathbb{N}$, get orbifold flat metric on $\mathbb{C}/\mathbb{Z}_m$.

# Examples - Toric Constant Scalar Curvature Metrics in Real Dimension 2



[$k \neq 0$] (can assume $b = 0$)

$$s''(x) = \frac{1}{c - kx^2} > 0$$

[$k > 0$] (need $c > 0$ and $-\sqrt{c/k} < x < \sqrt{c/k}$)

$$s(x) = \frac{1}{\sqrt{ck}} \cdot \frac{1}{2} \Big[ (x + \sqrt{c/k}) \log(x + \sqrt{c/k}) + (-x + \sqrt{c/k}) \log(-x + \sqrt{c/k}) \Big]$$

Singular american football metric. Smooth european football metric of Gauss curvature $k$ iff $c = 1/k$.

# Examples - Toric Constant Scalar Curvature Metrics in Real Dimension 2



$[k<0, c>0]$ $(x \in \mathbb{R})$

$$s(x) = \sqrt{\frac{-1}{ck}} \arctan\left(\sqrt{\frac{-k}{c}}\, x\right) \Rightarrow \text{hyperboloid}$$

$[k<0, c<0]$ (need $x > \sqrt{c/k}$)

$$s(x) = \frac{1}{\sqrt{ck}} \cdot \frac{1}{2}\Big[(x-\sqrt{c/k})\log(x-\sqrt{c/k}) - (x+\sqrt{c/k})\log(x+\sqrt{c/k})\Big]$$

Singular hyperbolic planes. Smooth hyperbolic planes of Gauss curvature $k$ iff $c = 1/k$.

# Definition of Symplectic Cone

## Definition

A symplectic cone is a triple $(M, \omega, X)$, where $(M, \omega)$ is a connected symplectic manifold and $X \in \mathcal{X}(M)$ is a vector field generating a proper $\mathbb{R}$-action $\rho_t : M \to M$, $t \in \mathbb{R}$, such that $\rho_t^*(\omega) = e^{2t}\omega$. Note that the Liouville vector field $X$ satisfies $\mathcal{L}_X \omega = 2\omega$, or equivalently

$$\omega = \frac{1}{2} d(\iota(X)\omega) .$$

$$\text{symplectic cones} \stackrel{1:1}{\longleftrightarrow} \text{co-oriented contact manifolds}$$

In particular, $(M, \omega, X)$ is the symplectization of $(N := M/\mathbb{R}, \xi := \pi_*(\ker(\iota(X)\omega))$, where $\pi : M \to M/\mathbb{R}$.

# Definition of Kähler-Sasaki Cone

## Definition

A Kähler-Sasaki cone is a symplectic cone $(M, \omega, X)$ equipped with a compatible complex structure $J \in \mathcal{I}(M, \omega)$ such that the Reeb vector field $K := JX$ is Kähler, i.e.

$$\mathcal{L}_K \omega = 0 \quad \text{and} \quad \mathcal{L}_K J = 0\,.$$

Note that $K$ is then also a Killing vector field for the Riemannian metric

$$g_J(\cdot, \cdot) := \omega(\cdot, J\cdot)\,.$$

Any such $J$ will be called a Sasaki complex structure on the symplectic cone $(M, \omega, X)$. The space of all Sasaki complex structures will be denoted by $\mathcal{I}_S(M, \omega, X)$.

# Properties of Kähler-Sasaki Cones



Define $r := \|K\| = \|X\| : M \to \mathbb{R}^+$. Then

- $K$ is the Hamiltonian vector field of $-r^2/2$;
- $X$ is the gradient vector field of $r^2/2$.

Define $\alpha := \iota(X)\omega/r^2 \in \Omega^1(M)$. Then

$$\omega = d(r^2\alpha)/2 \,, \quad \alpha(K) \equiv 1 \quad \text{and} \quad \mathcal{L}_X\alpha = 0 \,.$$

Define $N := \{r = 1\} \subset M$ and let $\xi := \ker \alpha|_N$. Then

$$(N, \xi, \alpha|_N, g_J|_N) \text{ is a Sasaki manifold.}$$

Define $B := N/K$. Then $TB \cong \xi$ and

$$(B, d\alpha|_\xi, J|_\xi) \text{ is a Kähler space,}$$

the Kähler reduction of $(M, \omega, X, J)$ by the action of $K$.

# Regular, Quasi-Regular and Irregular Kähler-Sasaki Cones



A KS cone $(M, \omega, X, J)$, with Reeb vector field $K = JX$, is said to be:

- regular if $K$ generates a free $S^1$-action.
- quasi-regular if $K$ generates a locally free $S^1$-action.
- irregular if $K$ generates an effective $\mathbb{R}$-action.

Note that the Kähler reduction $B = M//K$ is

- a smooth Kähler manifold if the KS cone is regular.
- a Kähler orbifold if the KS cone is quasi-regular.
- only a Kähler quasifold if the KS cone is irregular.

Note that the Sasaki manifold determined by a KS cone is always smooth.

# Toric Symplectic Cones



**Lemma**

Let $G$ be a Lie group. Any $X$-preserving symplectic $G$-action on a symplectic cone $(M, \omega, X)$ is Hamiltonian. Moreover, its moment map $\mu : M \to \mathfrak{g}^*$ can be chosen so that

$$\mu(\rho_t(m)) = e^{2t}\mu(m)\,,\ \forall\, m \in M,\ t \in \mathbb{R}.$$

**Definition**

A toric symplectic cone is a symplectic cone $(M, \omega, X)$ of dimension $2(n+1)$ equipped with an effective $X$-preserving $\mathbb{T}^{n+1}$-action, with moment map $\mu : M \to \mathfrak{t}^* \cong \mathbb{R}^{n+1}$ such that $\mu(\rho_t(m)) = e^{2t}\mu(m)\,,\ \forall\, m \in M,\ t \in \mathbb{R}$. Its moment cone is defined to be the set $C := \mu(M) \cup \{0\} \subset \mathbb{R}^{n+1}$ .

# Example - $(\mathbb{R}^{2(n+1)} \setminus \{0\}, \omega_{\rm st}, X_{\rm st})$

$(\mathbb{R}^{2(n+1)} \setminus \{0\}, \omega_{\rm st}, X_{\rm st})$, with linear coordinates $(u_1, \ldots, u_{n+1}, v_1, \ldots, v_{n+1})$ such that

$$\omega_{\rm st} = du \wedge dv := \sum_{j=1}^{n+1} du_j \wedge dv_j$$

and

$$X_{\rm st} = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} := \sum_{j=1}^{n+1} \left( u_j \frac{\partial}{\partial u_j} + v_j \frac{\partial}{\partial v_j} \right) ,$$

equipped with the standard $\mathbb{T}^{n+1}$-action, is a toric symplectic cone with moment map $\mu_{\rm st} : \mathbb{R}^{2(n+1)} \setminus \{0\} \to \mathbb{R}^{n+1}$ given by

$$\mu_{\rm st}(u_1, \ldots, u_{n+1}, v_1, \ldots, v_{n+1}) = \frac{1}{2}(u_1^2 + v_1^2, \ldots, u_{n+1}^2 + v_{n+1}^2) .$$

Its moment cone is $C = (\mathbb{R}_0^+)^{n+1} \subset \mathbb{R}^{n+1}$ .

# Classification of Good Toric Symplectic Cones

## Definition (Lerman)

A cone $C \subset \mathbb{R}^{n+1}$ is good if there exists a non-empty minimal set of primitive vectors $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$ such that

(i) $C = \bigcap_{a=1}^{d} \{x \in \mathbb{R}^{n+1} \ : \ \ell_a(x) := \langle x, \nu_a \rangle \geq 0\}$.

(ii) any codimension-$k$ face $F$ of $C$, $1 \leq k \leq n$, is the intersection of exactly $k$ facets whose set of normals can be completed to an integral base of $\mathbb{Z}^{n+1}$.

## Theorem (Banyaga-Molino, Boyer-Galicki, Lerman)

For each good cone $C \subset \mathbb{R}^{n+1}$ there exists a unique toric symplectic cone $(M_C, \omega_C, X_C, \mu_C)$ with moment cone $C$.

Like for compact symplectic toric manifolds, existence follows from a symplectic reduction construction starting from a symplectic vector space.

# Boothby-Wang Cones (Lerman)

Let $P \subset \mathbb{R}^n$ be an integral Delzant polytope. Then, its standard cone

$$C := \{z(x,1) \in \mathbb{R}^n \times \mathbb{R} \,:\, x \in P\,,\ z \geq 0\} \subset \mathbb{R}^{n+1}$$

is a good cone. Moreover

(i) the toric symplectic manifold $(B_P, \omega_P, \mu_P)$ is the $S^1 \cong \{\mathbf{1}\} \times S^1 \subset \mathbb{T}^{n+1}$ symplectic reduction of the toric symplectic cone $(M_C, \omega_C, X_C, \mu_C)$ (at level one).

(ii) $(N_C := \mu_C^{-1}(\mathbb{R}^n \times \{1\}), \alpha_C := (\iota(X_C)\omega_C)|_{N_C})$ is the Boothby-Wang manifold of $(B_P, \omega_P)$. The restricted $\mathbb{T}^{n+1}$-action makes it a toric contact manifold.

(iii) $(M_C, \omega_C)$ is the symplectization of $(N_C, \alpha_C)$.

# Example: Boothby-Wang Cone of the Standard Simplex



Let $P \subset \mathbb{R}^n$ be the standard simplex, i.e. $B_P = \mathbb{P}^n$. Its standard cone $C \subset \mathbb{R}^{n+1}$ is the moment cone of $(M_C = \mathbb{C}^{n+1} \setminus \{0\}, \omega_{\mathrm{st}}, X_{\mathrm{st}})$ equipped with the $\mathbb{T}^{n+1}$-action given by

$$
\begin{aligned}
&(y_1, \ldots, y_n, y_{n+1}) \cdot (z_1, \ldots, z_n, z_{n+1}) \\
&= (e^{-i(y_1+y_{n+1})} z_1, \ldots, e^{-i(y_n+y_{n+1})} z_n, e^{-iy_{n+1}} z_{n+1})
\end{aligned}
$$

The moment map $\mu_C : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1}$ is given by

$$\mu_C(z) = \frac{1}{2}(|z_1|^2, \ldots, |z_n|^2, |z_1|^2 + \cdots + |z_n|^2 + |z_{n+1}|^2)\,.$$

and

$$N_C := \mu_C^{-1}(\mathbb{R}^n \times \{1\}) = \left\{ z \in \mathbb{C}^{n+1} \;:\; \|z\|^2 = 2 \right\} \cong S^{2n+1}\,.$$

# Toric Kähler-Sasaki Cones

## Definition

A toric Kähler-Sasaki cone is a toric symplectic cone $(M, \omega, X, \mu)$ equipped with a toric Sasaki complex structure $J \in \mathcal{I}_S^{\mathbb{T}}(M, \omega)$.

- It follows from the classification theorem that any good toric symplectic cone has toric Sasaki complex structures.
- On a toric Kähler-Sasaki cone $(M, \omega, X, \mu, J)$, the Kähler action generated by the Reeb vector field $K = JX$ is part of the torus action.
- The Kähler reduction $B = M//K$ is a toric Kähler space: manifold (regular case), orbifold (quasi-regular case) or quasifold (irregular case).

# Cone Action-Angle Coordinates on $(M, \omega, X, \mu)$



$\mathbb{R}^{n+1} \supset \mu(M) = C \setminus \{0\} \supset \check{C} \equiv$ interior of $C$.

$$\check{M} \equiv \mu^{-1}(\check{C}) \equiv \{\text{points of } M \text{ where } \mathbb{T}^{n+1}\text{-action is free}\}$$

$$\cong \check{C} \times \mathbb{T}^{n+1} = \left\{(x, y) : x \in \check{C},\ y \in \mathbb{T}^{n+1} \equiv \mathbb{R}^{n+1}/2\pi\mathbb{Z}^{n+1}\right\}$$

such that $\omega|_{\check{M}} = dx \wedge dy \equiv$ standard symplectic form,

$$\mu(x, y) = x \quad \text{and} \quad X|_{\check{M}} = 2x\frac{\partial}{\partial x} = 2\sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}\,.$$

**Definition**

$(x, y) \equiv$ cone symplectic/Darboux/action-angle coordinates.

$\check{M}$ is an open dense subset of $M$.

# Toric Sasaki Complex Structures in Cone Action-Angle Coordinates



Any toric Sasaki complex structure $J \in \mathcal{I}_S^{\mathbb{T}}(\check{M}, \omega, X)$ can be written in suitable cone action-angle coordinates $(x, y)$ on $\check{M} \cong \check{C} \times \mathbb{T}^{n+1}$ as

$$J = \begin{bmatrix} 0 & -S^{-1} \\ S & 0 \end{bmatrix}$$

with

$$S = S(x) = (s_{ij}(x)) = \left( \frac{\partial^2 s}{\partial x_i \partial x_j} \right) > 0$$

for some symplectic potential $s : \check{C} \to \mathbb{R}$, such that

$$S(e^t x) = e^{-t} S(x) \, , \; \forall \, t \in \mathbb{R} \, , \; x \in \check{C} \, ,$$

i.e. $S(x) = \text{Hess}_x(s)$ is homogeneous of degree $-1$.

# Reeb Vector Fields in Cone Action-Angle Coordinates



The Reeb vector field $K := JX$ of such a toric Sasaki complex structure $J \in \mathcal{I}_S^{\mathbb{T}}(\check{M}, \omega, X)$ is given by

$$K = \sum_{i=1}^{n+1} b_i \frac{\partial}{\partial y_i} \quad \text{with} \quad b_i = 2 \sum_{j=1}^{n+1} s_{ij} x_j \, .$$

**Lemma (Martelli-Sparks-Yau)**

If $S(x) = (s_{ij}(x))$ is homogeneous of degree $-1$, then

$$K_s := (b_1, \ldots, b_{n+1}) \text{ is a constant vector.}$$

In other words, the action generated by $K$ is part of the torus action. Moreover,

$$\text{regularity of toric KS cone} \Leftrightarrow \text{rationality of } K_s \in \mathbb{R}^{n+1}.$$

# Characteristic Hyperplane and Polytope

The norm of the Reeb vector field is then given by

$$\|K\|^2 = \|(\mathbf{0}, K_s)\|^2 = b_i s^{ij} b_j = b_i s^{ij}(2 s_{jk} x_k) = 2 b_i x_i = 2\langle x, K_s\rangle \,.$$

Hence

$$\|K\| > 0 \Leftrightarrow \langle x, K_s\rangle > 0 \quad \text{and} \quad \|K\| = 1 \Leftrightarrow \langle x, K_s\rangle = 1/2\,.$$

**Definition (Martelli-Sparks-Yau)**

The characteristic hyperplane $H_K$ and polytope $P_K$ of a toric Kähler-Sasaki cone $(M, \omega, X, \mu, J)$, with moment cone $C \subset \mathbb{R}^{n+1}$, are defined as

$$H_K := \{x \in \mathbb{R}^{n+1} \;:\; \langle x, K_s\rangle = 1/2\} \quad \text{and} \quad P_K := H_K \cap C\,.$$

Note that $N := \mu^{-1}(H_K)$ is a toric Sasaki manifold and $P_K$ is the moment polytope of $B = M//K$.

# Canonical Symplectic Potentials

Let $P \subset \mathbb{R}^{n+1}$ be a polyhedral set defined by

$$P = \bigcap_{a=1}^{d} \{x \in \mathbb{R}^{n+1} \ : \ \ell_a(x) := \langle x, \nu_a \rangle + \lambda_a \geq 0\}$$

where $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1}$ are primitive integral vectors and $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$. Assume that $P$ has a non-empty interior and the above set of defining inequalities is minimal.

Burns-Guillemin-Lerman extended Delzant's symplectic reduction construction, associating to each such polyhedral set a toric Kähler space of dimension $2(n+1)$

$$(M_P, \omega_P, \mu_P, J_P)$$

such that

$$\mu_P(M_P) = P \quad \text{and} \quad J_P \in \mathcal{I}^{\mathbb{T}}(M_P, \omega_P)\,.$$

# Canonical Symplectic Potentials

**Guillemin'94, Burns-Guillemin-Lerman'05**

In appropriate action-angle coordinates $(x, y)$, the canonical symplectic potential $s_P : \breve{P} \to \mathbb{R}$ for $J_P|_{\breve{P}}$ is given by

$$s_P(x) = \frac{1}{2} \sum_{a=1}^{d} \ell_a(x) \log \ell_a(x) \, .$$

Note that $C := P$ is a cone iff $\lambda_1 = \cdots = \lambda_d = 0$. In this case, $s_C := s_P$ is the symplectic potential of a toric Sasaki complex structure $J_C \in \mathcal{I}_S^{\mathbb{T}}(M_C, \omega_C)$, since $S_C(x) = \text{Hess}_x(s_C)$ is then homogeneous of degree $-1$. The corresponding Reeb vector field $K = (\mathbf{0}, K_C)$ is given by

$$K_C = \sum_{a=1}^{d} \nu_a \, .$$

# Example: canonical symplectic potential of the standard cone over the standard simplex

The standard cone over the standard simplex is given by

$$C = \bigcap_{i=1}^{n+1} \{ x \in \mathbb{R}^{n+1} \ : \ \ell_i(x) := \langle x, \nu_i \rangle \geq 0 \} \,,$$

where

$$\nu_i = e_i \,, \ i = 1, \ldots, n, \quad \text{and} \quad \nu_{n+1} = (-1, \ldots, -1, 1) \,.$$

Hence, using $r = \sum_{i=1}^n x_i$, we have that

$$s_C(x) = \frac{1}{2} \left( \sum_{i=1}^n x_i \log x_i + (x_{n+1} - r) \log(x_{n+1} - r) \right)$$

and

$$K_C = \sum_{i=1}^{n+1} \nu_i = (0, \ldots, 0, 1) \,.$$

# General Cone Symplectic Potentials (Martelli-Sparks-Yau)



Let $s, s' : \check{C} \to \mathbb{R}$ be two symplectic potentials defined on the interior of a cone $C \subset \mathbb{R}^{n+1}$. Then

$$K_s = K_{s'} \Leftrightarrow (s - s') + \text{const. is homogeneous of degree 1}.$$

Given $b \in \mathbb{R}^{n+1}$, define

$$s_b(x) := \frac{1}{2} \left( \langle x, b \rangle \log \langle x, b \rangle - \langle x, K_C \rangle \log \langle x, K_C \rangle \right) .$$

Then $s := s_C + s_b$ is such that $K_s = b$. If $C$ is good, this symplectic potential $s$ defines a smooth Sasaki complex structure on the symplectic cone $(M_C, \omega_C, X_C, \mu_C)$ iff

$$\langle x, b \rangle > 0 \, , \ \forall \, x \in C \setminus \{0\} \, , \ \text{i.e. } b \in \check{C}^*$$

where $C^* \subset \mathbb{R}^{n+1}$ is the dual cone

$$C^* := \{ x \in \mathbb{R}^{n+1} \ : \ \langle v, x \rangle \geq 0 \, , \ \forall \, v \in C \} \, .$$

# General Cone Symplectic Potentials (Martelli-Sparks-Yau)



## Martelli-Sparks-Yau'05

Any toric Sasaki complex structure $J \in \mathcal{I}_S^{\mathbb{T}}$ on a toric symplectic cone $(M_C, \omega_C, X_C, \mu_C)$, associated to a good moment cone $C \in \mathbb{R}^{n+1}$, is given by a symplectic potential $s : \check{C} \to \mathbb{R}$ of the form

$$s = s_C + s_b + h\,,$$

where $s_C$ is the canonical potential, $b \in \check{C}^*$ and $h : C \to \mathbb{R}$ is homogeneous of degree 1 and smooth on $C \setminus \{0\}$.

The dual cone $C^*$ can be equivalently defined as

$$C^* = \cap_\alpha \{x \in \mathbb{R}^{n+1} \ : \ \langle \eta_\alpha, x\rangle \geq 0\}\,,$$

where $\eta_\alpha \in \mathbb{Z}^{n+1}$ are the primitive generating edges of $C$.

# Boothby-Wang Symplectic Potentials

## Definition

Let $P \subset \mathbb{R}^n$ be an integral Delzant polytope and $C \subset \mathbb{R}^{n+1}$ its standard good cone. Given a symplectic potential $s : \check{P} \to \mathbb{R}$, define its Boothby-Wang symplectic potential $\tilde{s} : \check{C} \to \mathbb{R}$ by

$$\tilde{s}(x,z) := z\, s(x/z) + \frac{1}{2} z \log z\,, \ \forall\, x \in \check{P}\,,\ z \in \mathbb{R}^+\,.$$

If $P = \bigcap_{a=1}^{d} \{x \in \mathbb{R}^n \ :\ \ell_a(x) := \langle x, \nu_a \rangle + \lambda_a \geq 0\}$ and

$$s(x) = \frac{1}{2} \sum_{a=1}^{d} \ell_a(x) \log \ell_a(x) - \frac{1}{2} \ell_\infty(x) \log \ell_\infty(x)\,,$$

where $\ell_\infty(x) := \sum_a \ell_a(x) = \langle x, \nu_\infty \rangle + \lambda_\infty$, then

$$\tilde{s}(x,z) = s_C(x,z) + s_b(x,z) \quad \text{with} \quad b = (0, \ldots, 0, 1)\,.$$

# Example: Boothby-Wang symplectic potential of the standard cone over the standard simplex



If $P \subset \mathbb{R}^n$ is the standard simplex and $r = \sum_i x_i$, then

$$s_P(x) = \frac{1}{2}\left(\sum_{i=1}^{n} x_i \log x_i + (1-r)\log(1-r)\right)$$

and

$$\begin{aligned} 2\tilde{s}_P(x,z) &= 2\left(z s_P(x/z) + \frac{1}{2} z \log z\right) \\ &= \sum_{i=1}^{n} x_i \log(x_i/z) + (z-r)\log((z-r)/z) + z\log z \\ &= \sum_{i=1}^{n} x_i \log x_i + (z-r)\log(z-r) = 2s_C(x,z)\,, \end{aligned}$$

where $C$ is the standard cone over $P$.

# Symplectic Potentials and Reduction

**Calderbank-David-Gauduchon'02**

Symplectic potentials restrict naturally under toric symplectic reduction.

Suppose $(B_P, \omega_P, \mu_P)$ is a toric symplectic reduction of $(M_C, \omega_C, \mu_C)$. Then there is an affine inclusion $P \subset C$ and

$$\text{any } \widetilde{J} \in \mathcal{I}^{\mathbb{T}}(M_C, \omega_C) \text{ induces a reduced } J \in \mathcal{I}^{\mathbb{T}}(B_P, \omega_P).$$

This theorem says that if

$$\tilde{s} : \breve{C} \to \mathbb{R} \text{ is a symplectic potential for } \widetilde{J}$$

then

$$s := \tilde{s}|_{\breve{P}} : \breve{P} \to \mathbb{R} \text{ is a symplectic potential for } J.$$

# Symplectic Potentials and Affine Transformations



**Proposition**

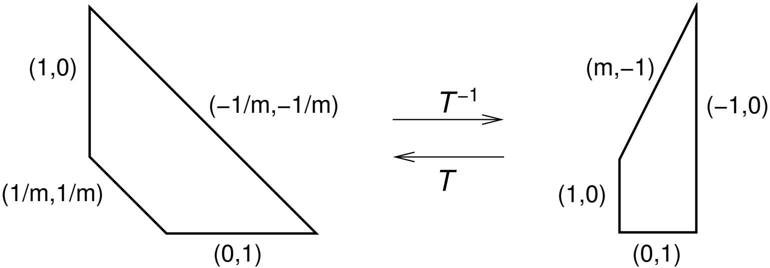
Symplectic potentials transform naturally under affine transformations.

Let $T \in GL(n)$ and consider the linear symplectic change of action-angle coordinates $x' := T^{-1}x$ and $y' := T^t y$.
Then $P = \bigcap_{a=1}^d \{x \in \mathbb{R}^n \,:\, \ell_a(x) := \langle x, \nu_a \rangle + \lambda_a \geq 0\}$ becomes

$$P' := T^{-1}(P) = \bigcap_{a=1}^{d} \{x' \in \mathbb{R}^n \,:\, \ell'_a(x') := \langle x', \nu'_a \rangle + \lambda'_a \geq 0\}$$

with $\nu'_a = T^t \nu_a$ and $\lambda'_a = \lambda_a$, and symplectic potentials transform by $s' = s \circ T$ (in particular, $s_{P'} = s_P \circ T$). The corresponding Hessians are related by $S' = T^t (S \circ T) T$.

# Symplectic Potentials and Affine Transformations - Hirzebruch Surfaces




$$T = \begin{bmatrix} m & -1 \\ 0 & 1 \end{bmatrix}$$

# Symplectic Potentials and Scalar Curvature



- Toric Kähler metric

$$\begin{bmatrix} S & 0 \\ 0 & S^{-1} \end{bmatrix}$$

  where $S = (s_{ij}) = \text{Hess}_x(s)$ for a symplectic potential $s : \check{P} \subset \mathbb{R}^n \to \mathbb{R}$.

- Formula for its scalar curvature [A.'98]:

$$Sc \equiv -\sum_{j,k} \frac{\partial}{\partial x_j} \left[ s^{jk} \frac{\partial \log(\det S)}{\partial x_k} \right] = -\sum_{j,k} \frac{\partial^2 s^{jk}}{\partial x_j \partial x_k}$$

- Donaldson'02 - appropriate interpretation for this formula: view scalar curvature as moment map for action of $\text{Ham}(M, \omega)$ on $\mathcal{I}(M, \omega)$.

# Scalar Curvature of Boothby-Wang Symplectic Potentials



## Proposition

Let $P \subset \mathbb{R}^n$ be a polyhedral set and $C \subset \mathbb{R}^{n+1}$ its standard cone. Given a symplectic potential $s : \check{P} \to \mathbb{R}$, let $\tilde{s} : \check{C} \to \mathbb{R}$ be its Boothby-Wang symplectic potential:

$$\tilde{s}(x,z) := z\, s(x/z) + \frac{1}{2} z \log z\,, \ \forall\, x \in \check{P}\,,\ z \in \mathbb{R}^+\,.$$

Then

$$\widetilde{Sc}(x,z) = \frac{Sc(x/z) - 2n(n+1)}{z}\,.$$

In particular

$$\widetilde{Sc} \equiv 0 \Leftrightarrow Sc \equiv 2n(n+1)\,.$$

# Boothby-Wang Symplectic Potentials and Toric Kähler-Sasaki-Einstein metrics



In compact Kähler geometry, constant scalar curvature metric plus cohomological condition $c_1 = \lambda[\omega]$ implies Kähler-Einstein metric.

## Proposition

Let $P \subset \mathbb{R}^n$ be a polyhedral set and $C \subset \mathbb{R}^{n+1}$ its standard cone. Given a symplectic potential $s : \breve{P} \to \mathbb{R}$, let $\tilde{s} : \breve{C} \to \mathbb{R}$ be its Boothby-Wang symplectic potential. Then,

$s$ defines a toric Kähler-Einstein metric with $Sc \equiv 2n(n+1)$

iff

$\tilde{s}$ defines a toric Ricci-flat Kähler metric.

When this happens, the corresponding toric Sasaki metric is Einstein.

# Calabi's Family of Extremal Kähler Metrics



- Calabi constructed in 1982 a general 4-parameter family of $U(n)$-invariant extremal Kähler metrics, which he used to put extremal Kähler metrics on

$$H_m^n := \mathbb{P}(\mathcal{O}(-m) \oplus \mathbb{C}) \longrightarrow \mathbb{P}^{n-1}\,,\ \ n, m \in \mathbb{N}\,,$$

  in any possible cohomology class. In particular, when $n = 2$, on all Hirzebruch surfaces.
- When written in action-angle coordinates, using symplectic potentials, Calabi's family can be seen to contain many other interesting Kähler metrics [A.'01].
- In particular, it contains a 1-parameter family of Kähler-Einstein metrics directly related to the Sasaki-Einstein metrics constructed in 2004 by Gauntlett-Martelli-Sparks-Waldram [A.'08].

# Calabi's Family in Action-Angle Coordinates

Consider symplectic potentials $s : (\mathbb{R}^+)^n \to \mathbb{R}$ of the form

$$s(x) = \frac{1}{2}\left(\sum_{a=1}^{n} x_a \log x_a + h(r)\right), \text{ where } r = x_1 + \cdots x_n.$$

Then

$$Sc(x) = Sc(r) = 2r^2 f''(r) + 4(n+1) r f'(r) + 2n(n+1) f(r),$$

where $f = h''/(1 + rh'')$. Moreover, extremal is equivalent to $Sc$ being an affine function, which is then equivalent to

$$h''(r) = -\frac{1}{r} + \frac{r^{n-1}}{r^n - A - Br - Cr^{n+1} - Dr^{n+2}},$$

where $A, B, C, D \in \mathbb{R}$ are the 4 parameters of the family.

# Interesting Particular Cases of Calabi's Family



$[C = D = 0]$

$$h''(r) = -\frac{1}{r} + \frac{r^{n-1}}{r^n - A - Br}$$

$[C = D = 0,\ B = 0,\ A > 0]$
Complete Ricci flat Kähler metrics on total space of $\mathcal{O}(-n) \to \mathbb{P}^{n-1}$, for any possible cohomology class. (Calabi'79)

$[C = D = 0,\ A, B > 0]$
Complete scalar flat Kähler metrics on total space of $\mathcal{O}(-m) \to \mathbb{P}^{n-1}$, for any $m > 0$ and any possible cohomology class. (LeBrun'88 ($n = 2$), Pedersen-Poon'88 and Simanca'91 ($n > 2$))

# Kähler-Sasaki-Einstein Cases in Calabi's Family



$$\left[B = D = 0,\ C = 1,\ 0 < A < n^n/(n+1)^{n+1}\right]$$

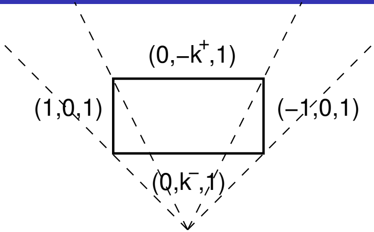
$$h''(r) = -\frac{1}{r} + \frac{r^{n-1}}{r^n(1-r) - A}$$

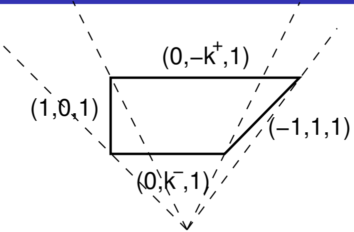
Kähler-Einstein quasifold metrics with $Sc = 2n(n+1)$ on certain $H^n_m := \mathbb{P}(\mathcal{O}(-m) \oplus \mathbb{C}) \longrightarrow \mathbb{P}^{n-1}$.

For a countably infinite set of values for the variable parameter $A$, the corresponding Boothby-Wang cones are $GL(n+1)$ equivalent to good cones - precisely the ones corresponding to the Sasaki-Einstein metrics of Gauntlett-Martelli-Sparks-Waldram'04 (at least when $n = 2$).

# Kähler-Sasaki-Einstein Cases in Calabi's Family - $n = 2$ examples






$$p, q \in \mathbb{N}, \quad 0 < q < p, \quad \gcd(q,p) = 1$$

| | |
|---|---|
| even $\quad p \pm q$ | odd $\quad p \pm q$ |
| $k^{\pm} = (p \pm q)/2$ | $k^{\pm} = (p \pm (q-1))/2$ |
| $K_s = (0, -\ell/2, 3)$ | $K_s = (0, (3-\ell)/2, 3)$ |

$$\ell = \left(3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}\right)/q$$

All these Sasaki-Einstein manifolds are diffeo. to $S^2 \times S^3$.