On the mean Euler characteristic of Gorenstein toric contact manifolds

(joint with Leonardo Macarini, UFRJ, Brazil)

I) "Convex Hamiltonian energy surfaces and their periodic trajectories", Ekeland - Hofer, 1987

- $\mathbb{R}^{2(n+1)}$, $(x_0, \ldots, x_n, y_0, \ldots, y_n) \equiv (x, y)$

$$H: \mathbb{R}^{2(n+1)} \to \mathbb{R} \ni \begin{cases} \dot{x} = \partial H / \partial y \\ \dot{y} = -\partial H / \partial x \end{cases}$$

Problem: existence and multiplicity of periodic trajectories in a given fixed energy smooth hypersurface $M^{2n+1} := H^{-1}(\epsilon)$.

Note: $X_H = \left( \frac{\partial H}{\partial y}, - \frac{\partial H}{\partial x} \right) \in \text{Ker}(\text{d}x \wedge \text{d}y |_M)$ = characteristic foliation.

I.e.

Simple periodic trajectories of $X_H$ on $M$ $\longleftrightarrow$ Simply closed characteristics of $M$

- Non-degenerate periodic orbit $\gamma$ $\xrightarrow{\text{lin. flow}}$ index $\mu(\gamma) \in \mathbb{Z}$

$\xrightarrow{\text{mean index}}$ $\Delta(\gamma) : = \lim_{N \to \infty} \frac{\mu(\gamma^N)}{N} \in \mathbb{R}$

Note: $\Delta(\gamma) > 0$ on convex $M^{2n+1} \subset \mathbb{R}^{2(n+1)}$. 
• **Theorem (Ekeland-Hofer, 1987)**

If \( M^{2n+1} \subset T^* \mathbb{R}^{2(n+1)} \) is convex and has finitely many simple closed characteristics \( \gamma_1, \ldots, \gamma_m \), then

\[
\sum_{k=1}^{\infty} \frac{\delta(\gamma_k)}{\Delta(\gamma_k)} = \frac{1}{2}
\]

(resonance relation)

where \( \delta(\gamma_k) = \pm 1 \) or \( \pm \frac{1}{2} \) according to the parity of

\[
\mu(\gamma_k) \quad \text{even} \leftrightarrow + \quad \text{and} \quad \mu(\gamma_k^2) - \mu(\gamma_k) \quad \text{even} \leftrightarrow 1 \quad \text{odd} \leftrightarrow \frac{1}{2}.
\]

"Cor." : A generic convex \( M^{2n+1} \subset T^* \mathbb{R}^{2(n+1)} \) has \( \infty \)-many simple closed characteristics.

II) **Contact manifolds and Reeb flows**

- \([M^{2n+1}, \xi] \), \( \xi = \ker(\alpha) \) with \( \alpha \in \Omega^1(M) \) s.t. \( \alpha \wedge d\alpha \) = volume

- Reeb vector field \( R_\alpha \in \mathfrak{X}(M, \xi) : R_\alpha \in \ker(d\alpha) \) and \( \alpha(R_\alpha) = 1 \).

 Luxembourg : any \( f : M^{2n+1} \to T^* \mathbb{R} \) gives rise to another Reeb vector field \( R_\alpha f \in \mathfrak{X}(M, \xi) \) and Reeb flow.

**Examples:**

- Star-shaped Hamiltonian flows in \( T^* \mathbb{R}^{2(n+1)} \) are Reeb flows on \((S^{2n+1}, \xi_{\text{std}})\).
- Geodesic flows are Reeb flows on \((ST^*N, \xi_{\text{std}})\).
• Topological condition \( \overline{c_1}(M, \xi) = 0 \) \( \Rightarrow \) can define index \( \mu(\xi) \) and mean index \( \Delta(\xi) \) of any closed Reeb orbit.

• Theorem (Gingburg – Kerman, 2010)

If the Reeb flow of \( R_\alpha \) is non-degenerate with finitely many simple closed orbits \( \gamma_1, \ldots, \gamma_m \), with positive mean index, then

\[
\sum_{k=1}^{m} \frac{\sigma(\gamma_k)}{\Delta(\gamma_k)} = \chi(M, \xi) \quad \text{(resonance relation)}
\]

where \( \chi(M, \xi) \) is the mean Euler characteristic, an invariant of the contact manifold \( (M, \xi) \).

Notes:
• \( \chi(M, \xi) \) is the mean Euler characteristic of cylindrical contact homology \( \cong \) Morse homology for the action functional \( \mathcal{A}_\alpha(\xi) = \int_\gamma \alpha \) on the loop space of \( M \).
• Resonance relation of Ekeland-Hofer \( \Leftrightarrow \chi(S^{2n+1}, \xi_{\text{std}}) = \frac{1}{2} \).
• Kwon and van Koert, 2016: given any \( \chi \in \mathcal{A} \) there is a (Stein fillable) contact structure \( \xi_\alpha \) on \( S^5 \) such that \( \chi(S^5, \xi_\alpha) = \chi \).
III) **Gorenstein toric contact manifolds**

- **Good toric contact manifolds** (Lerman, 2003)
  
  $(\mathbb{M}^{2n+1}, \xi) \cong \mathbb{P}^{n+1}$

  are contact reductions of

  $\mathbb{C}^d \setminus \{0\} \cong (\mathbb{S}^{2d-1}, \xi_{\text{std}}) \cong \mathbb{P}^d$

  with respect to a subtorus

  $K := K_{\text{cv}} (\beta : \mathbb{P}^d \rightarrow \mathbb{P}^{n+1})$

  **Examples:** (anti-)canonical circle bundles over monotone toric symplectic manifolds (and orbifolds).

  - Surjective homomorphism $\beta$ is determined by
    
    primitive vectors $\nu_1, \ldots, \nu_d \in \mathbb{Z}^{n+1} \subset \mathbb{R}^{n+1} = \text{Lie}(\mathbb{P}^{n+1})$

  - **Gorenstein toric contact manifold** $\overset{\text{def}}{\iff} C_\perp (\mathbb{M}, \xi) = 0$

    $\iff \nu_1, \ldots, \nu_d$ can be chosen of the form

    $\nu_j := (x_j, 1)$ with $x_j \in \mathbb{Z}^n$, $j = 1, \ldots, d$.

  - $D := \text{conv}(\nu_1, \ldots, \nu_d) \subset \mathbb{R}^n$

    $\equiv$ **toric diagram of the Gorenstein toric contact manifold** $(\mathbb{M}_D^{2n+1}, \xi_D)$

- **Theorem** (A.- Macarini, 2016)

  $\chi(\mathbb{M}_D, \xi_D) = \frac{n! \cdot \text{vol}(D)}{2}$
"Proof." Any "irrational" interior point $x$ of $D$ determines a non-degenerate toric Reeb vector whose simple closed orbits are in 1-1 correspondence with the facets of $D$:

Moreover,

$$\frac{1}{\Delta(x_l)} = \frac{n!}{2} \text{vol}(\text{conv}(x_l, F_l)), \ l = 1, \ldots, m.$$ 

Result follows from Ginzburg–Kerman's resonance relation, since $\tau(x_l) = 1, l = 1, \ldots, m$. \hspace{1cm} \text{Q.E.D.}

• Several interesting applications...

A particularly nice one, using a result of Batyrev-Dais from 1996, is the following:

$$2 \chi(M_D, \xi_D) = \text{Euler characteristic of any unpunctured (i.e. $c_1 = 0$) toric symplectic filling.}$$

In particular, the 1987 resonance relation of Ekeland-Hofer reflects that

$$\chi(\text{ball}) = 1.$$

In fact, the underlying contact manifold of any convex hypersurface in $\mathbb{R}^{2(n+1)}$ is $(S^{2n+1}, \xi_{\text{std}})$, which admits the standard symplectic ball $B^{2(n+1)} \subset \mathbb{R}^{2(n+1)}$ as a unpunctured toric symplectic filling.
Examples

a) 

\[ \sim \]

\[ (0,1) \]

\[ (1,0) \]

\[ (0,1) \]

\[ (1,0) \]

\[ (0,1) \]

\[ (1,0) \]

\[ (0,1) \]

\[ (1,0) \]

b) 

\[ \sim \]

\[ (0,1,1) \]

\[ (1,0,1) \]

\[ (0,0,1) \]

c) 

\[ \sim \]

\[ (0,1) \]

\[ (1,0) \]

\[ (-1,1) \]

d) 

\[ \sim \]

\[ (S^2 \times S^3, \xi_p) \]

\[ p \in \mathbb{N} \]

e) 

\[ \sim \]

\[ (\#_5 S^2 \times S^3, \xi_p) \]

\[ p \in \mathbb{N} \]
Examples:

a) \((L(3,2), \xi_{\text{std}})\)

\[
\begin{align*}
&1! \; \text{Vol} = 3 \\
&\sim \\
&\begin{array}{c}
(3,1) \\
(0,1)
\end{array} \\
&\sim \\
&\begin{array}{c}
(3,1) \\
(0,1)
\end{array}
\]

\[\chi = 3\]

b) \((S^2 \times S^3, \xi_3)\)

\[
\begin{array}{c}
(3,3) \\
(0,1) \\
(0,0) \\
(1,0)
\end{array}
\]

\[2! \; \text{Vol} = 6\]

\[\chi = 6\]